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THE EFFECTIVE INTERACTION AND SOME SOUND PHENOMENA IN DILUTE NEUTRAL FERMI SYSTEMS

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TO THOSE WHOM I LOVE

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Abstract

This Thesis is mainly concerned with the effective interaction and its relation to sound phenomena in dilute neutral Fermi systems, which comprise nucleonic matter, ^{3}He in Vycor glass, and ^{3}He–^{4}He mixtures.

Following a brief, but comprehensive, description of the basic ingredients of the formalism, including the systems of interest, the input potential, and the Galitskii–Migdal–Feynman (GMF) T–matrix, a general expression for the expansion of the input NN potential, which is nonlocal, noncentral, and state–dependent, is derived in terms of the two nucleon eigenstates (channels). This, in turn, is used as an input to derive the full equations for the state–dependent T–matrix, hence the effective interaction and the corresponding (proper) self–energy; whereas the simpler central case is obtained by simply switching off the state dependence. In addition, the orthogonality and the completeness properties of the T–matrices are derived for the first time.

The various sound modes in these systems are then studied. The corresponding equations are obtained starting from the traditional approach of the induced density fluctuations. Especial attention is devoted to the intimate relation between sound propagation and such quantities as the static structure factor and the acoustic impedance. In particular, a complementary expansive framework is established which links this microscopic study with macroscopic manifestations, thereby laying the ground for various applications whose basic input is the T–matrix.

Finally, the Thesis concludes with a summary and a list of some open problems for future work.

Contents

CHAPTER ONE

I N T R O D U C T I O N

Chapter 1

Introduction

1.1 Statement of the Problem

The central theme of this work is twofold: First, the effective interaction in dilute neutral Fermi systems; and, secondly, the role it plays in shedding some light on sound phenomena in these systems.

More precisely, a general expression for the effective interaction in these systems will be derived in terms of the Galitskii–Migdal–Feynman (GMF) T–matrix [1 − 5], starting with a noncentral input two–body potential, from which the simpler central–potential equations can be obtained at once as a special case. This T–matrix will then be used to compute the proper self–energy. In passing, new derivations concerning the orthogonality and completeness of the T–matrix will be presented. The general theoretical framework thereby obtained lends itself naturally to a variety of applications, including the study of sound propagation in these systems, which will be examined here rather cursorily in the hope of laying the ground for more thorough future studies. In particular, it will be attempted to understand within a unified picture such quantities and phenomena as the static structure factor, the acoustic impedance, the various sound modes, and so forth.

Needless to say, the present work lies in the general line of the age–old problem of establishing a link between the ***microscopic*** and the ***macroscopic***. To be specific, attention here is confined to ***dilute neutral Fermi*** systems. The key qualifiers just mentioned will be elaborated presently.

1.2 General Background

Table 1 lists the three basic *dilute neutral Fermi* systems with which this Thesis is concerned.

System	**Elementary Constituents**
Nucleonic Matter	Nucleons
^{3}He in Vycor Glass	^{3}He Quasiparticles
^{3}He–HeII Mixtures	^{3}He Quasiparticles

Table 1: The basic dilute neutral Fermi systems of interest.

First, they are all *many–fermionic* systems since the elementary constituent in each case is a spin–half particle/excitation (quasiparticle) [6]. In nucleonic matter this elementary constituent is the nucleon (neutron or proton): If the numbers of neutrons and protons are equal, the system is called *nuclear matter proper* ; if they are not, it is *nucleonic matter.* Occasionally, in astrophysics, one is interested in neutron matter which consists purely of neutrons [7, 8]. It should be added that the nucleon is the elementary constituent in nucleonic matter provided that the energy range considered $\leq$ 300 MeV; i.e., in the low–energy nonrelativistic limit [8 − 11] – otherwise other excitations should be taken into account, including the pion condensate [12, 13]. As for the other two systems, the elementary constituent is, of course, the ^{3}He atom/quasiparticle. In this respect, it should be noted that, at temperatures less than the Fermi degeneracy temperature ($\simeq$ 0.6 K at zero pressure), the predominant excitations in dilute ^{3}He–HeII mixtures are the ^{3}He quasiparticles, the Bose–excitations (phonons and rotons) being negligibly small [14].

Secondly, the above systems are all *neutral*, since the Coulomb interaction in nucleonic matter is switched off, and since ^{3}He is neutral.

Finally, these systems are all dilute and low–dense, in the sense that the average interparticle spacing exceeds the interaction range [15 − 17]. While this is clear for nucleonic matter and dilute ^{3}He–HeII mixtures, it is interesting to note that the intricate network of capillaries in Vycor glass seems to suppress the ^{3}He interparticle correlations. This, in turn, renders the ^{3}He liquid dilute [18], although in ordinary circumstances this system is strongly interacting and extremely dense [3].

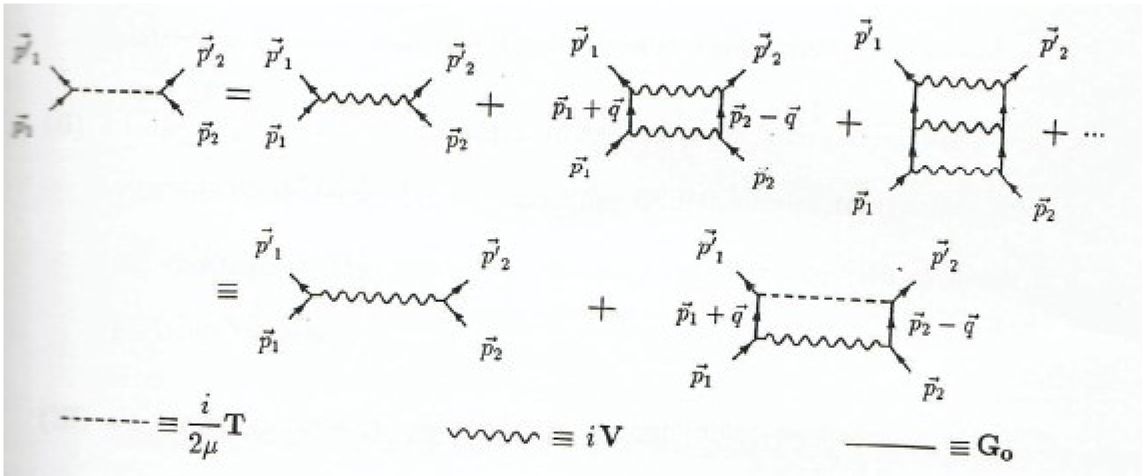
The implication, then, is that an independent–pair model is ideally valid for these systems, the many–body medium being incorporated into the picture in some average sense (for example, through an effective interaction and an effective mass) $\left[5, 19\right]$.

A most suitable framework which fits the foregoing considerations to a large extent is the GMF T–matrix formalism which is amazingly rich in content. Not only does it take into account the scattering of pairs inside the Fermi sea, in addition to particle–particle scattering outside the sea (thereby surpassing its main competitor in this field, the Brueckner–Bethe–Goldstone formalism $\left[20\right]$), but also it represents a generalized scattering amplitude that encompasses the environment; i.e., the pressure and temperature. The key point here is that this is essentially a low–dense formalism; its application to strongly interacting systems, such as liquid ^{3}He is, in principle, dubious, to say the least. Further, in the latter systems it should be complemented by a suitable formalism which embraces long–range correlations $\left[1-5, 21\right]$.

Mathematically speaking, the GMF T–matrix is an integral equation which represents an infinite number of binary (hole–hole and particle–particle) collisions inside and outside the vacuum (in the quantum–field–theoretic language) $\left[\,4\,\right]$.

Diagrammatically, it is simply a series expansion that sums the so–called ladder diagrams (Fig.1) $\left[1-5, 19, 22\right]$:

Fig.1: The diagrammatic representation of the GMF T–matrix .



The wavy line represents the nonrelativistic two–body potential, which is a basic input in the formalism $\left[3, 23\right]$. In the low–density, low–energy limit, this is clearly an excellent representation of the interaction of the system $\left[7, 15-17, 24\right]$, which is usually extracted by well-known methods from both scattering– and bound–state data $\left[7-11, 16, 24, 25\right]$. In view of the overall complexity of microscopic formalisms, this is often simulated by a suitable simplified analytical model – e.g., the pure hard–core potential, a Herzfield potential (a hard core plus a square attractive well outside), the pure boundary condition model (which may be expressed as the limiting form of a local interaction given by the sum of a repulsive unit–step function and an attractive delta function), and so forth $\left[26\right]$.

For the above systems, the general features of the input potential are essentially the same – namely,

(i) a short–range repulsive part, which arises from the Pauli exclusion principle in atomic systems, including ^{3}He, and from the exchange of quarks in nucleonic matter (or of vector mesons in the traditional meson–theoretic approach);

(ii) a long–range attractive part which, in helium, arises from the relatively weak Van der Waals forces (or, in quantum–field–theoretic terminology, from the exchange of long–wavelength phonons) and from one–pion exchange in nucleonic systems;

(iii) a minimum in between representing the equilibrium separation of the interacting pair $\left[8, 25, 27, 28\right]$.

As is well–known, however, there exist fundamental differences between He–He and NN potentials. The former, interhelium potential is purely central $\left[28\right]$; this remains the case even in dilute ^{3}He–HeII mixtures, although it should be modified there so as to include the induced effects of the He "ether–background" $\left[29\right]$, as will be elaborated in Chapter Two. Conversely, the NN potential is a state–dependent noncentral potential; it depends not only on the relative separation of the interacting nucleons, but also on their relative angular momentum as well as their spins, isospins and relative orientation. Its overall form is restricted by symmetry principles dictated by certain conservation laws $\left[8, 25, 30 - 32\right]$.

Our strategy in this work, as will be seen in Chapter Two, is to derive all our results for the most general NN potential, so that on switching off the state–dependence, in all its ramifications, the corresponding results for the central case are immediately restored.

1.3 Motivation

This Thesis is inspired, in the first place, by the new–old problem in the many–fermionic theory – namely, the derivation of the macroscopic properties of the system from the properties of its microscopic constituents. This, of course, is the fundamental problem in statistical mechanics and many–body theory [33]. The emphasis here is on the effective interaction in momentum space, whose inverse Fourier transform yields its counterpart in real space, and which constitutes the basic input for computing the (proper) self–energy. In its turn, this is the basic quantity from which the bulk and thermodynamic properties of the system are derived – including its effective mass, compressibility, and so forth [5, 19, 34, 35]. An attempt is also made in this work to link the effective interaction derived with sound modes and phenomena in view of the longstanding interest in their bizarre properties [14].

The present work is further motivated by the recent renewed interest in dilute Fermi systems, especially those listed in Table 1 [18, 36 − 38]. That this is the case in nucleonic matter is evident from the sustained interest in the astrophysical applications of the system [25, 39 − 41], including the implications of possible pairing and enhanced ordering in general [5, 12, 21]. That this is also clear in liquid helium–3 can be detected in the intriguing possibility of obtaining a realistic, dilute ^{3}He system in Vycor glass, as well as the ever–renewed interest in detecting a new ordered phase [18], based either on molecular [42] or Cooper pairing [43], in dilute ^{3}He–He II mixtures. While such a state has so far proved to be elusive, the quest is definitely still on [44]. In short, then, there is still enough exciting physics in these systems to make them so appealing as both theoretical and experimental laboratories, so to speak.

A yet third motivation is the rich physics involved in the sound phenomena occurring in these systems. True, much has already been done in this domain [45, 46]; but there is still much more that needs further illumination, especially regarding the relation of the various sound modes and properties to the effective interaction of the system and bringing together the various aspects involved in a unified whole.

Finally, the present work is motivated by the loopholes and lacunas in the T–matrix theory, insofar as the properties of the matrix itself are concerned, such as completeness and orthogonality. The aim here is to illuminate these loopholes.

1.4 Synopsis of the Thesis

The structure of this Thesis should by now be quite clear. Following the present introductory chapter, there comes Chapter Two which is concerned with all the mathematical aspects of the GMF T–matrix just mentioned. In particular, the full equations for the state–dependent T–matrix and the corresponding (proper) self–energy are derived meticulously in such a manner as to be reduced immediately to their counterparts in ^{3}He–systems on switching off the state–dependence of the input potential. The chapter is concluded with the derivation of the orthogonality and completeness properties of the T–matrix.

Next, in Chapter Three, it is attempted to shed some light on the foregoing sound phenomena, with especial emphasis on the unifying themes. This is meant to be a preliminary attempt, which can be pursued further in future work.

Other extensions and elaborations are among the open problems reserved for the concluding chapter of this Thesis, Chapter Four.

CHAPTER TWO

THE EFFECTIVE INTERACTION

Chapter 2

The Effective Interaction

This Chapter is devoted to the mathematical aspects of the effective interaction described qualitatively in Chapter One. We begin, in Section 2.1, with the two–body NN potential expansion in terms of the two–nucleon eigenstates. The corresponding T–matrix expansion is then derived in Section 2.2. The effective interaction thereby obtained is used, in Section 2.3, to compute the (proper) self–energy. In Section 2.4, novel derivations concerning the orthogonality and completeness of the GMF T–matrix are presented. Throughout, the relevant expressions for the central (say, He–He) potential are obtained as especial cases by simply switching off the state-dependence. Finally, some concluding remarks round up this Chapter in Section 2.5.

2.1 The Input Potential

The NN potential, then, will be taken as the starting point. A realistic NN potential is, of course, state-dependent. It may be nonlocal, with spin–orbit and (or) tensor components. Its general form must conserve the total angular momentum J, its projection M_J (the third component), the total spin S, the total isospin T, its projection M_T (charge), and parity $\Big[31, 32, 47\Big]$.

The set of quantum numbers JST are constants of motion. Together with L , the orbital angular momentum, which, of course, is not conserved, these define a set of orthonormal eigenstates or channels of the two–nucleon system, $|LSJT\rangle$ $\Big[23, 47\Big]$.

It is convenient to express the potential by a relative–partial wave expansion in terms of these channels.

Our complete set is given by

$$\begin{aligned} |LSJM_J;TM_T\rangle \quad &\equiv \quad |LSJT\rangle = |LSJM_J\rangle|\tfrac{1}{2}\tfrac{1}{2}TM_T\rangle \\ &= \sum_{M_L M_S} \langle LSM_LM_S|LSJM_J\rangle|LM_L\rangle \; |\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle \; |\tfrac{1}{2}\tfrac{1}{2}TM_T\rangle. \qquad (2.1) \end{aligned}$$

This simply defines a two–nucleon state in which the total orbital angular momentum L of the two nucleons, with projection M_L, couples to their total spin S, with projection M_S, to give a total angular momentum J, with projection M_J, and a total isospin T, with projection M_T, where $|LM_L\rangle$ is the relative orbital angular momentum state; $|\frac{1}{2}\frac{1}{2}SM_S\rangle$ is the two–nucleon spin state; $|\frac{1}{2}\frac{1}{2}TM_T\rangle$ is the two–nucleon isospin state, and $\langle LSM_LM_S|LSJM_J\rangle$ are the relevant Clebsh–Gordan (CG) coefficients $\left[31, 47, 48\right]$.

The total angular momentum function is defined by $\left[9, 31, 47\right]$

$$\langle\hat{r}|LSJM_J\rangle \equiv \mathcal{Y}^{LSJ}_{M_J}(\hat{r}) = \sum_{M_LM_S}\langle LSM_LM_S|LSJM_J\rangle\mathcal{Y}_{LM_L}(\hat{r})\ |\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle, \tag{2.2}$$

where $\mathcal{Y}_{LM_L}(\hat{r})$ is the spherical harmonic function, such that

$$\langle\hat{r}l|LM_L\rangle = \mathcal{Y}_{LM_L}(\hat{r})\delta_{lL}.$$

Our set can be expressed in terms of a new function, defined as:

$$\langle\hat{r}|LSJM_J;TM_T\rangle \equiv \langle\hat{r}|LSJT\rangle = \mathcal{Y}^{LSJ}_{M_J}(\hat{r})|\tfrac{1}{2}\tfrac{1}{2}TM_T\rangle \equiv \mathcal{Y}^{LSJT}_{M_JM_T}(\hat{r}) \equiv \mathcal{Y}^{L\alpha}_{M_JM_T}(\hat{r}). \tag{2.3}$$

Throughout this Chapter the set SJT will be denoted by α, although it will occasionally also appear explicitly .

Let us , first, expand the matrix elements of the potential in relative coordinates by partial–wave decomposition, in terms of the $|LSJT\rangle$ channels:

$$\begin{aligned}\langle\vec{r}|V|\vec{r'}\rangle &= \langle\hat{r}|\langle r|V|r'\rangle|\hat{r'}\rangle \\ &\equiv \langle\hat{r}|V(r,r')|\hat{r'}\rangle.\end{aligned} \tag{2.4}$$

On inserting the closure relation of the complete set $|L\alpha\rangle$,

$$\sum_{L\alpha} |L\alpha\rangle\langle L\alpha| \ = 1; \tag{2.5}$$

or, equivalently,

$$\sum_{\substack{L\alpha \\ MM_T}} |LSJM_J\rangle\langle LSJM_J|TM_T\rangle\langle TM_T| = 1, \tag{2.6}$$

Eq.(2.4) becomes

$$\begin{aligned}
\langle\hat{r}|V(r,r')|\hat{r'}\rangle &= \sum_{LL'\alpha}\langle\hat{r}|L\alpha\rangle\langle L\alpha|V(r,r')|L'\alpha\rangle\langle L'\alpha|\hat{r'}\rangle \\
&= \sum_{\substack{LL'\alpha \\ M_J M_T}} \langle\hat{r}|LSJM_J\rangle|\tfrac{1}{2}\tfrac{1}{2}TM_T\rangle \ V^{\alpha}_{LL'}(r,r') \ \langle\tfrac{1}{2}\tfrac{1}{2}TM_T|\langle L'SJM_J|\hat{r'}\rangle \\
&= \sum_{\substack{LL'\alpha \\ M_J M_T}} \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{r}) \ V^{\alpha}_{LL'}(r,r') \ \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{r'})^{\dagger}.
\end{aligned} \tag{2.7}$$

Hence,

$$V(\vec{r},\vec{r'}) = \sum_{\substack{LL'\alpha \\ M_J M_T}} \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{r}) \ V^{\alpha}_{LL'}(r,r') \ \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{r'})^{\dagger}, \tag{2.8}$$

where

$$\begin{aligned}
V^{\alpha}_{LL'}(r,r') \quad &= \langle L\alpha|V(r,r')|L'\alpha\rangle \\
&= \int\int d\hat{r}d\hat{r'} \ \langle L\alpha|\hat{r}\rangle \ \langle\hat{r}|V(r,r')|\hat{r'}\rangle \ \langle\hat{r'}|L'\alpha\rangle \\
&= \int\int d\hat{r}d\hat{r'} \ \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{r})^{\dagger} \ \langle\vec{r}|V|\vec{r'}\rangle \ \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{r'}).
\end{aligned} \tag{2.9}$$

Now,

$$
\begin{aligned}
V &= \sum_{LL'\alpha} |L\alpha\rangle\langle L\alpha|V|L'\alpha\rangle\langle L'\alpha| \\
&= \sum_{LL'\alpha} \int\int d\vec{r} d\vec{r'}\ |L\alpha\rangle\langle L\alpha|\vec{r}\rangle\ \langle\vec{r}|V|\vec{r'}\rangle\ \langle\vec{r'}|L'\alpha\rangle\langle L'\alpha| \\
&= \sum_{LL'\alpha} \int\int d\vec{r} d\vec{r'}\ |L\alpha\rangle\langle L\alpha|\vec{r}\rangle\ V(\vec{r},\vec{r'})\ \langle\vec{r'}|L'\alpha\rangle\langle L'\alpha|,
\end{aligned}
$$

where we have used

$$\int d\vec{r}|\vec{r}\rangle\langle\vec{r}| = 1, \tag{2.10}$$

together with Eq.(2.5).

Accordingly,

$$\langle\vec{p}|V|\vec{p'}\rangle = \sum_{LL'\alpha} \int\int d\vec{r} d\vec{r'} \langle\vec{p}|L\alpha\rangle\langle L\alpha|\vec{r}\rangle\ V(\vec{r},\vec{r'})\ \langle\vec{r'}|L'\alpha\rangle\langle L'\alpha|\vec{p'}\rangle. \tag{2.11}$$

The state vector $|\vec{a}\rangle$ can be written as:

$$|\vec{a}\rangle = \sum_{l} |\hat{a}l\rangle|al\rangle; \tag{2.12}$$

and

$$\langle\hat{p}l|L\alpha\rangle = \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{p})\delta_{lL}. \tag{2.13}$$

Equation (2.11) then takes the form

$$\begin{aligned}
\langle\vec{p}|V|\vec{p'}\rangle &= \sum_{LL'\alpha}\sum_{ll'}\int\int d\vec{r}d\vec{r'}\ \langle\hat{p}l|L\alpha\rangle\langle L\alpha|\hat{r}l\rangle\ \langle pl|rl\rangle\ V(\vec{r},\vec{r'}) \\
&\quad \times\langle\hat{r'}l'|L'\alpha\rangle\langle L'\alpha|\hat{p'}l'\rangle\ \langle r'l'|p'l'\rangle \\
&= \sum_{\substack{LL'\alpha \\ M_J M_T}}\int\int d\vec{r}d\vec{r'}\mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{p})\ \langle pL|rL\rangle\ \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{r})^{\dagger}\ \langle\vec{r}|V|\vec{r'}\rangle \\
&\quad \times\mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{r'})\ \langle r'L'|p'L'\rangle\ \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{p'})^{\dagger}. \qquad (2.14)
\end{aligned}$$

But $\langle\vec{p}|\vec{r}\rangle$ can be written as:

$$\begin{aligned}
\langle\vec{p}|\vec{r}\rangle &= \sum_{l}\langle pl|rl\rangle\langle\hat{p}l|\hat{r}l\rangle \\
&= \sum_{LM_L l}\langle pl|rl\rangle\langle\hat{p}l|LM_L\rangle\langle LM_L|\hat{r}l\rangle \\
&= \sum_{LM_L}\langle pL|rL\rangle\mathcal{Y}_{LM_L}(\hat{p})\mathcal{Y}^{*}_{LM_L}(\hat{r}). \qquad (2.15)
\end{aligned}$$

Invoking Euler's expansion $\Big[50, 51\Big]$,

$$\langle\vec{p}|\vec{r}\rangle \equiv e^{i\vec{p}.\vec{r}} = 4\pi\sum_{LM_L} i^{L} j_L(pr)\mathcal{Y}_{LM_L}(\hat{p})\mathcal{Y}^{*}_{LM_L}(\hat{r}), \qquad (2.16)$$

we then obtain

$$\langle pL|rL\rangle = 4\pi i^{L} j_L(pr). \qquad (2.17)$$

Equation (2.14) now reads

$$\langle\vec{p}|V|\vec{p'}\rangle = (4\pi)^2 \sum_{\substack{LL'\alpha \\ M_J M_T}} \int\int r^2 dr\ {r'}^2 dr' i^{L-L'} \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{p}) j_L(pr) V^{\alpha}_{LL'}(r,r') j_{L'}(p'r') \times \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{p}')^{\dagger}, \tag{2.18}$$

which is the Fourier–Bessel transform of $V^{\alpha}_{LL'}(r,r')$ given by Eq.(2.9).

Thus,

$$V(\vec{p},\vec{p'}) = 4\pi \sum_{\substack{LL'\alpha \\ M_J M_T}} i^{L-L'}\ \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{p})\ V^{\alpha}_{LL'}(p,p')\ \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{p}')^{\dagger}, \tag{2.19}$$

where

$$\begin{aligned} V^{\alpha}_{LL'}(p,p') &= 4\pi \int\int r^2 dr\ {r'}^2 dr'\ j_L(pr)\ V^{\alpha}_{LL'}(r,r')\ j_{L'}(p'r') \\ &= 4\pi \int\int r^2 dr\ {r'}^2 dr'\ j_L(pr) j_{L'}(p'r')\ \int\int d\hat{r} d\hat{r'}\ \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{r})^{\dagger} \\ &\quad \times V(\vec{r},\vec{r'})\ \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{r'}). \end{aligned} \tag{2.20}$$

Next let us examine the case where the potential is local; i.e.,

$$V(\vec{r},\vec{r'}) = V(\vec{r})\delta(\vec{r}-\vec{r'}). \tag{2.21}$$

Then,

$$V^{\alpha}_{LL'}(r,r') = \int\int d\hat{r} d\hat{r'}\ \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{r})^{\dagger}\ V(\vec{r})\delta(\vec{r}-\vec{r'})\ \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{r'}). \tag{2.22}$$

But [49, 50]

$$\delta(\vec{r}-\vec{r'}) = \frac{1}{r^2}\delta(r-r')\delta(\hat{r}-\hat{r'}). \qquad (2.23)$$

Thus,

$$\begin{aligned}
V^{\alpha}_{LL'}(r,r') &= \int\int d\hat{r}d\hat{r'}\ \langle L\alpha|\hat{r}\rangle\ V(\vec{r})\ \langle\hat{r'}|L'\alpha\rangle\ \frac{1}{r^2}\delta(r-r')\delta(\hat{r}-\hat{r'}) \\
&= \int d\hat{r}\ \langle L\alpha|V(\vec{r})|L'\alpha\rangle\ |\hat{r}\rangle\langle\hat{r}|\ \frac{1}{r^2}\delta(r-r') \\
&= \langle L\alpha|V(\vec{r})|L'\alpha\rangle\ \frac{1}{r^2}\delta(r-r'), \qquad (2.24)
\end{aligned}$$

where the Kronecker–delta property and the completeness of the set $|\hat{r}\rangle$ have been used.

Hence, Eq.(2.20) becomes

$$\begin{aligned}
V^{\alpha}_{LL'}(p,p') &= 4\pi\int\int r^2dr\ r'^2dr'\ j_L(pr)j_{L'}(p'r')\ \frac{1}{r^2}\delta(r-r')\langle L\alpha|V(\vec{r})|L'\alpha\rangle \\
&= 4\pi\int r^2dr\ j_L(pr)\ j_{L'}(p'r)\ \ \langle L\alpha|V(\vec{r})|L'\alpha\rangle, \qquad (2.25)
\end{aligned}$$

where

$\langle L\alpha|V(\vec{r})|L'\alpha\rangle$ is a matrix element of $V(\vec{r})$. For, say, Reid's potential [32]

$$V(\vec{r}) = V_C(r) + V_{LS}(r)\vec{L}.\vec{S} + V_T(r)\hat{S}_{12}, \qquad (2.26)$$

this is given by

$$\langle L\alpha|V(\vec{r})|L'\alpha\rangle = V_c^\alpha(r)\langle L\alpha|L'\alpha\rangle + V_{LS}^\alpha(r)\langle L\alpha|\vec{L}.\vec{S}|L'\alpha\rangle + V_T^\alpha(r)\langle L\alpha|\hat{S}_{12}|L'\alpha\rangle, \qquad (2.27)$$

where $\vec{r}$ is the relative separation vector of the two nucleons. $V_c^\alpha(r)$, $V_{LS}^\alpha(r)$, and $V_T^\alpha(r)$ are the central , spin–orbit, and tensor central parts of the potential, respectively, pertaining to the channel α. Further, $\hat{S}_{12}$ and $\vec{L}.\vec{S}$ are the tensor and spin–orbit operators; these will be defined later.

If the potential is purely central, as for the case of the He–He interaction, only the first term in Eq.(2.27) is needed; so we write

$$\begin{aligned}
\langle L\alpha|V(\vec{r})|L'\alpha\rangle &= V_c^\alpha(r)\langle LSJT|L'SJT\rangle \\
&= V_c^\alpha(r)\sum_{M_L M_S}\sum_{M_L'}\int d\hat{r}\ \langle LSJM|LSM_LM_S\rangle\langle L'SM_L'M_S|L'SJM_J\rangle \\
&\qquad \times\langle \tfrac{1}{2}\tfrac{1}{2}SM_S|\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle\langle \tfrac{1}{2}\tfrac{1}{2}TM_T|\tfrac{1}{2}\tfrac{1}{2}TM_T\rangle\ \mathcal{Y}_{LM_L}^*(\hat{r})\mathcal{Y}_{L'M_L'}(\hat{r}) \\
&= V_c^\alpha(r)\sum_{M_L M_S}\langle LSJM|LSM_LM_S\rangle\langle LSM_LM_S|LSJM_J\rangle\delta_{LL'}\delta_{M_LM_L'} \\
&= V_c^\alpha(r)\delta_{LL'} \ = \ V(r)\delta_{LL'}; \qquad (2.28)
\end{aligned}$$

and Eq.(2.25) then reads

$$V_L^\alpha(p,p') = V_L(p,p') = 4\pi\int r^2 dr j_L(pr)V(r)j_L(p'r). \qquad (2.29)$$

Therefore Eq.(2.19), which gives the matrix elements of V, takes the form

$$
\begin{aligned}
V(\vec{p},\vec{p'}) &= 4\pi \sum_{\substack{LL'\alpha \\ M_J M_T}} i^{L-L'} \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{p})\ V^{\alpha}_{LL'}(p,p')\ \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{p'})^{\dagger} \\
&= 4\pi \sum_{L\alpha} \langle \hat{p}|L\alpha\rangle\ V_L(p,p')\langle L\alpha|\hat{p'}\rangle \\
&= 4\pi \sum_{LM_L} \langle \hat{p}|LM_L\rangle\langle LM_L|\hat{p'}\rangle\ V_L(p,p') \\
&= 4\pi \sum_{LM_L} \mathcal{Y}_{LM_L}(\hat{p})\mathcal{Y}^{*}_{LM_L}(\hat{p'})V_L(p,p'). \qquad (2.30)
\end{aligned}
$$

Finally, we get

$$
V(\vec{p},\vec{p'}) = 4\pi \sum_{\mathrm{L}} (2L+1)P_L(\hat{p}.\hat{p'})V_L(p,p'), \qquad (2.31)
$$

where the following relations have been used $\Big[49, 50\Big]$:

$$
\sum_{M_L} \mathcal{Y}_{LM_L}(\hat{p})\mathcal{Y}^{*}_{LM_L}(\hat{p'}) = \frac{2L+1}{4\pi} P_L(\hat{p}.\hat{p'}); \qquad (2.32)
$$

$$
\sum_{LM_L} |LM_L\rangle\langle LM_L| = 1; \qquad (2.33)
$$

$$
\sum_{L\alpha} |L\alpha\rangle\langle L\alpha| = 1. \qquad (2.34)
$$

It is obvious from Eqs.(2.36,2.39) below that the presence of the spin–orbit and tensor operators in the NN potential couples channels of different angular momentum. These two operators are non-operative for spin–singlet states (S=0), as well as for spin–triplet states with J=L, and the $^3P_0$ state, since $\vec{L}.\vec{S}$ and $\hat{S_{12}}$ are both zero in these cases. However, they are operative between triplet states (S=1) with $J = L \pm 1$. Further, only states (channels) with the same parity will be admixed, these being constrained by the parity rule $(-)^L$.

For uncoupled states; i.e., all states except spin–triplet states with $J = L \pm 1$, the interaction can be replaced by an effective central potential, and the momentum matrix elements are obtainable from Eq.(2.31). But for coupled channels, it is necessary to calculate the matrix elements $< \vec{L}.\vec{S} >$ and $< \hat{S}_{12} >$, which are not zero in these states.

The operator $\vec{L}.\vec{S}$ is given by $\left[8, 31, 48\right]$:

$$J^2 = \left[\vec{L} + \vec{S}\right]^2 = L^2 + S^2 + 2\vec{L}.\vec{S} \; ; \tag{2.35}$$

so that

$$\vec{L}.\vec{S} = \frac{1}{2}[J(J+1) - L(L+1) - S(S+1)], \tag{2.36}$$

where

$\vec{J} \equiv$ the total angular momentum operator of the two–nucleon system;

$\vec{L} \equiv$ the corresponding orbital angular momentum operator;

$\vec{S} \equiv$ the total spin operator;

and

$\vec{S} = \frac{1}{2}\vec{\sigma}$; $\vec{\sigma} \equiv$ the Pauli spin operator.

Thus,

$< \vec{L}.\vec{S} > \equiv \langle LSJT|\vec{L}.\vec{S}|L'SJT\rangle = \langle L1JT|\vec{L}.\vec{S}|L'1JT\rangle;$

or

$$< \vec{L}.\vec{S} >= \begin{cases} 0 & S = 0 \\ 0 & L \neq L' \\ J-1 & L = L' = J-1 \\ -J-2 & L = L' = J+1, \end{cases} \tag{2.37}$$

where we have used the orthonormality of the set $|LSJT\rangle$, Eq.(2.28):

$$\langle L1JT|L'1JT\rangle = \delta_{LL'}. \tag{2.38}$$

In its turn, the operator $\hat{S}_{12}$ is the standard tensor operator defined by $\left[31, 48, 51\right]$:

$$\begin{aligned} \hat{S}_{12} &= 3(\vec{\sigma_1}.\hat{r})(\vec{\sigma_2}.\hat{r}) - \vec{\sigma_1}.\vec{\sigma_2} \\ &= 2\left[\frac{3}{r^2}(\vec{S}.\hat{r})^2 - \vec{S}^2\right], \end{aligned} \tag{2.39}$$

where

$$\vec{S} = \frac{1}{2}(\vec{\sigma_1} + \vec{\sigma_2}); \tag{2.40}$$

$$2S^2 = (3 + \vec{\sigma_1}.\vec{\sigma_2}). \tag{2.41}$$

Thus,

$$< \hat{S_{12}} > = \langle L1JT|\hat{S_{12}}|L'1JT\rangle = \int d\Omega \mathcal{Y}^{L1JT}_{MM_T}(\Omega)^\dagger \hat{S_{12}} \mathcal{Y}^{L'1JT}_{MM_T}(\Omega), \tag{2.42}$$

where $\int d\Omega$ stands for integration over the solid angle, as well as for the summation over the spins and isospins. The matrix elements $< \hat{S}_{12} >$ can then be readily computed. These are given in Table 2 below.

L \ L'	J+1	J	J-1
J+1	$-\frac{2(J+2)}{2J+1}$	0	$+\frac{6\sqrt{J(J+1)}}{2J+1}$
J	0	+2	0
J-1	$+\frac{6\sqrt{J(J+1)}}{2J+1}$	0	$-\frac{2(J-1)}{2J+1}$

Table 2: Matrix elements of the tensor operator $\hat{S}_{12}$.

2.2 The Galitskii-Migdal-Feynman T–Matrix

The GMF T–matrix integral equation, already defined physically and diagrammatically in Section 1.2, is given, in the relative–momentum representation, by $\left[2-5\right]$:

$$T(\vec{p},\vec{p'};s,\vec{P}) = u(\vec{p}-\vec{p'}) - \int \frac{d\vec{k}}{(2\pi)^3} u(\vec{p}-\vec{k}) \left[Q(\vec{k},\vec{P})g_o(k,s) - \overline{Q}(\vec{k},\vec{P})g_o(k,s)^\dagger\right] \times T(\vec{k},\vec{p'};s,\vec{P}). \qquad (2.43)$$

We define the relative initial, intermediate, and final momenta $\vec{p}$, $\vec{k}$, and $\vec{p'}$ in terms of the incoming and outgoing momenta of the interacting pair, $\vec{p_1}$, $\vec{p_2}$; $\vec{k_1}$, $\vec{k_2}$; and $\vec{p'_1}$, $\vec{p'_2}$, respectively , such that

$$\vec{p} \equiv \frac{1}{2}(\vec{p_1} - \vec{p_2})\,; \quad \vec{k} \equiv \frac{1}{2}(\vec{k_1} - \vec{k_2})\,; \quad \vec{p'} \equiv \frac{1}{2}(\vec{p'_1} - \vec{p'_2})\,; \tag{2.44}$$

and the average or centre of mass (c m) momentum:

$$\vec{P} \equiv \frac{1}{2}(\vec{p_1} + \vec{p_2}) \;=\; \frac{1}{2}(\vec{p'_1} + \vec{p'_2}) \;=\; \frac{1}{2}(\vec{k_1} + \vec{k_2}). \tag{2.45}$$

Throughout this work, we shall use a system of units such that $\hbar = 2m = 1$; so that μ, the reduced mass, $= m/2 = 1/4$ and the energy will have the dimension $[\text{Length}]^{-2}$ $\big[3, 52\big]$. The operator $u \equiv 2\mu V = V/2$. The free two–body Green function, the propagator $g_o(k, s)$, is given by

$$g_o(k, s) \;=\; (\tfrac{1}{2}H_\circ - s - i\eta)^{-1} = (k^2 - s - i\eta)^{-1}, \tag{2.46}$$

where $H_\circ$ is the relative kinetic energy operator of the pair, and η is a positive infinitesimal in the scattering region $(s > 0)$ and zero elsewhere. Here s is the relative total energy of the pair in the c m frame:

$$s = 2P_\circ - P^2, \tag{2.47}$$

$2P_\circ$ being the total energy of the pair; so that P^2 is the energy carried by the c m. The operator $Q(\overline{Q})$ is the product of particle–particle (hole–hole) occupation probabilities defined as:

$$Q \equiv \theta(|\vec{P}+\vec{k}| - k_F)\theta(|\vec{P}-\vec{k}| - k_F); \tag{2.48}$$

$$\overline{Q} \equiv \theta(k_F - |\vec{P}+\vec{k}|)\ \theta(k_F - |\vec{P}-\vec{k}|), \tag{2.49}$$

where $\theta(k)$ is the unit–step function.

In momentum space , the hole–occupation probability is just the Fermi–Dirac distribution , which reduces to the unit step function at zero temperature . When subtracted from unity , this yields the particle–occupation probability . $Q(\overline{Q})$ is equal to one if both particles are outside (inside) the Fermi-sea.

With the angle–averaging approximation for Q and $\overline{Q}$, the T– matrix becomes a function of $\vec{p},\ \vec{p'}$ and the magnitude of $\vec{P}$, which makes it possible to treat the T–matrix as the two–body potential operator and to decompose it, on an equal and symmetric footing, into relative–partial eigenwave channels. Hence,

$$T(\vec{p},\vec{p'};s,\vec{P}) \;=\; 4\pi \sum_{\substack{LL'\alpha \\ M_J M_T}} i^{L-L'} \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{p})\ T^{\alpha}_{LL'}(p,p';s,P) \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{p'})^{\dagger}, \tag{2.50}$$

where $T^{\alpha}_{LL'}(p,p';s,P)$ is the relevant component of the T–matrix.

Equation (2.43) now becomes

$$4\pi \sum_{\substack{LL'\alpha \\ M_J M_T}} i^{L-L'} \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{p}) \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{p'})^{\dagger} \left[\ T^{\alpha}_{LL'}(p,p';s,P) - u^{\alpha}_{LL'}(p,p')\ \right] =$$

$$-(4\pi)^2 \sum_{\substack{Ll\alpha \\ M_J M_T}} \sum_{\substack{\alpha' L' \\ M_J' M_T'}} \int \frac{k^2\, dk\, d\hat{k}}{(2\pi)^3} i^{L-L'} u^{\alpha}_{Ll}(p,k)\ \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{p}) \mathcal{Y}^{l\alpha}_{M_J M_T}(\hat{k})^{\dagger}$$

$$\times \Big[g_o(k,s) Q(k,P) - g_o(k,s)^{\dagger} \overline{Q}(k,P) \Big] \mathcal{Y}^{l\alpha'}_{M_J' M_T'}(\hat{k}) \mathcal{Y}^{L'\alpha'}_{M_J' M_T'}(\hat{p}')^{\dagger} T^{\alpha}_{lL'}(k,p';s,P). \quad (2.51)$$

Using the orthonormality of the function $\mathcal{Y}^{L\alpha}_{M_J M_T}$; i.e.,

$$\int d\hat{k} \mathcal{Y}^{l\alpha}_{M_J M_T}(\hat{k})^{\dagger} \mathcal{Y}^{l'\alpha'}_{M_J' M_T'}(\hat{k}) = \sum_{\substack{M_L M_S \\ M_L' M_S'}} \langle \tfrac{1}{2} \tfrac{1}{2} S' M_S' | \tfrac{1}{2} \tfrac{1}{2} S M_S \rangle \langle \tfrac{1}{2} \tfrac{1}{2} T M_T | \tfrac{1}{2} \tfrac{1}{2} T' M_T' \rangle$$

$$\times \int d\hat{k} \mathcal{Y}^{\star}_{lM_l}(\hat{k}) \mathcal{Y}_{l'M_l'}(\hat{k}) \langle lSJM_J | lSM_l M_S \rangle \langle l'S'J'M_J' | l'S'M_l' M_S' \rangle = \delta_{ll'} \delta_{M_l M_l'} \delta_{SS'}$$

$$\times \delta_{M_S M_S'} \delta_{TT'} \delta_{M_T M_T'} \sum_{M_L M_S} \langle lSJM_J | lSM_l M_S \rangle \langle lSJ'M_J' | lSM_l M_S \rangle = \delta_{ii'} \delta_{M_i M_i'}, \quad (2.52)$$

where $ii' = JJ',\ ll',\ SS'$ and TT', it follows that the rhs of Eq.(2.51) becomes

$$-4\pi \sum_{\substack{LL'\alpha \\ M_J M_T}} \mathcal{Y}^{L\alpha}_{M_J M_T}(\hat{p})\ \mathcal{Y}^{L'\alpha}_{M_J M_T}(\hat{p}')^{\dagger} \ \sum_l \int \frac{k^2 dk}{2\pi^2}\ u^{\alpha}_{Ll}(p,k)$$

$$\times \Big[g_o(k,s) Q(k,P) - g_o(k,s)^{\dagger} \overline{Q}(k,P) \Big]\ T^{\alpha}_{lL'}(k,p';s,P). \quad (2.53)$$

Thus, the T–matrix partial–coupled-channels equation takes the form

$$T^{\alpha}_{LL'}(p,p';s,P) \ = \ u^{\alpha}_{LL'}(p,p') \ - \ \sum_l \int \frac{k^2 dk}{2\pi^2} u^{\alpha}_{Ll}(p,k)$$

$$\times \Big[g_o(k,s) Q(k,P) - g_o(k,s)^{\dagger} \overline{Q}(k,P) \Big]\ T^{\alpha}_{lL'}(k,p';s,P). \quad (2.54)$$

The GMF T–matrix integral equation has thereby been reduced to a set of (coupled) channels in the magnitudes of momenta. These channels are a complete set of nucleonic states. Moreover, in channels where the potential state-dependence is absent (uncoupled channels or the central case), from (2.29),

$$V^{\alpha}_{LL}(p,p') \to V_L(p,p').$$

Consequently,

$$T^{\alpha}_{LL}(p,p';s,P) \to T_L(p,p';s,P).$$

Thus, Eq.(2.54) is reduced to:

$$\begin{aligned} T_L(p,p';s,P) \;&=\; u_L(p,p') \;-\; \int \frac{k^2 dk}{2\pi^2} u_L(p,k) \\ &\times \left[g_o(k,s)Q(k,P) - g_o(k,s)^{\dagger}\overline{Q}(k,P) \right] \; T_L(k,p';s,P), \end{aligned} \tag{2.55}$$

which is applicable to these uncoupled channels as well as the central-potential case.

The solution of Eqs.(2.54,2.55) represents the fully–off–shell GMF T–matrix. In this respect, three kinds of the T–matrix may be defined. When $\vec{p} = \vec{p'}$ and $s = p^2$, we have the on–shell T–matrix; but when $\vec{p} \neq \vec{p'}$ and $s = p^2$ or p'^2, we have the half–shell T–matrix. Finally, the fully-off-shell T–matrix is obtained when neither p^2 nor p'^2 is equal to s. In particular, the on-energy shell T–matrix can be used to describe the elastic scattering of pairs in the many–body medium. In general, in the language of formal scattering theory, one is interested in the scattering matrix which is intimately related to the scattering amplitude or transition probability of the system from an initial to a final state. Conservation of flux in the scattering problem reflects the unitarity of the scattering matrix , which, in turn, leads to its parametrization in terms of real phase shifts $\left[53 - 55\right]$.

On the other hand, the on–energy–shell T–matrix channel $T^{\alpha}_{LL}(p, p, p^2; P) \equiv T^{\alpha}_{L}(p^2; P)$ can be parametrized in terms of real *effective* many–body eigenphase shifts according to $\left[2 - 4\right]$:

$$T^{\alpha}_{L}(p^2; P) = -\frac{4\pi}{p}\left[Q \quad + \quad \overline{Q}\right]^{-1} \exp(i\delta^{\alpha}_{L}(p; P))\ \sin\delta^{\alpha}_{L}(p; P)\ , \tag{2.56}$$

where $\delta^{\alpha}_{L}(p; P)$ is the *effective* phase shift pertaining to the channel $L\alpha$ and is defined as the difference in phase between the asymptotic ($r \rightarrow \infty$) forms of the perturbed and unperturbed (V=0) wave functions, such that:

$$\tan\delta^{\alpha}_{L}(p; P) \equiv \frac{Im\ T^{\alpha}_{L}(p^2; P)}{Re\ T^{\alpha}_{L}(p^2; P)}. \tag{2.57}$$

For convenience, while the T–matrix is complex, one can define a real K–matrix associated with the principal value of the integral in Eq.(2.43). In the bound-state region ($s < 0$), where T is real, this is given by

$$Re\ T(\vec{p}, \vec{p'}; s, \vec{P}) = K(\vec{p}, \vec{p'}; s, \vec{P}). \tag{2.58}$$

Similarly, this real K–matrix can be parametrized in terms of $\delta_L^\alpha(p;P)$. Following Blatt and Biedenharn, we can write [56, 57]:

$$\tan\delta_L^\alpha(p;P) \;=\; -\frac{p}{4\pi}K_{LL}^\alpha(p,p;s,P) \tag{2.59}$$

for uncoupled channels; and

$$\begin{aligned}\tan\delta_{J\pm1}^\alpha(p;P) = -\frac{p}{4\pi}\Big[K_{J-1,J-1}^\alpha(p,p;s,P) + K_{J+1,J+1}^\alpha(p,p;s,P)\\ \mp\frac{K_{J-1,J-1}^\alpha(p,p;s,P) - K_{J+1,J+1}^\alpha(p,p;s,P)}{\cos2\epsilon_J^\alpha}\Big]\end{aligned} \tag{2.60}$$

for coupled channels, where

$$\tan2\epsilon_J^\alpha = \frac{2K_{J-1,J+1}^\alpha(p,p;s,P)}{K_{J-1,J-1}^\alpha(p,p;s,P) - K_{J+1,J+1}^\alpha(p,p;s,P)} \tag{2.61}$$

is the mixing parameter that gives the the proportions into which an incoming beam in one channel (partial wave) divides between the outgoing channels, $K_{J\pm1,J\pm1}^\alpha$ being the K–matrix element in the channel $L(=J\pm1)\alpha \equiv \langle J\pm1\,\alpha|K|J\pm1\,\alpha\rangle$.

An alternative expression for these matrix elements is given in terms of the barred phase shifts [58]. These are connected to the Blatt and Biedenharn phase shifts through:

$$\delta_+ + \delta_- = \overline{\delta}_+ + \overline{\delta}_-; \tag{2.62}$$

$$\sin\overline{\Delta} \equiv \sin(\overline{\delta}_- - \overline{\delta}_+) \;=\; \frac{\tan2\overline{\epsilon}}{\tan2\epsilon}; \tag{2.63}$$

$$\sin\Delta \equiv \sin(\delta_- - \delta_+) = \frac{\sin 2\overline{\epsilon}}{\sin 2\epsilon}, \tag{2.64}$$

where $\delta_\pm \equiv \delta^\alpha_{J\pm1}$, $\overline{\delta}_\pm \equiv \overline{\delta}^\alpha_{J\pm1}$, $\epsilon \equiv \epsilon^\alpha_J$, and $\overline{\epsilon} \equiv \overline{\epsilon}^\alpha_J$. From (2.63) and (2.64)

$$\cos 2\overline{\epsilon} = \frac{\sin\Delta}{\sin\overline{\Delta}} \cos 2\epsilon; \tag{2.65}$$

$$\sin 2\overline{\epsilon} = \sin\Delta \sin 2\epsilon. \tag{2.66}$$

Squaring and adding the above two equations yield:

$$\cos^2 2\overline{\epsilon} + \sin^2 2\overline{\epsilon} = 1 = \sin^2\Delta(1 - \cos^2 2\epsilon) + \frac{\cos^2 2\epsilon \, \sin^2\Delta}{\sin^2\overline{\Delta}}; \tag{2.67}$$

$$\begin{aligned} \cos^2\Delta \equiv 1 - \sin^2\Delta &= \cos^2 2\epsilon \, \sin^2\Delta \left(\frac{1}{\sin^2\overline{\Delta}} - 1 \right) \\ &= \cos^2 2\epsilon \, \sin^2\Delta \cot^2\overline{\Delta}. \end{aligned} \tag{2.68}$$

Finally, we get

$$\tan\overline{\Delta} = \cos 2\epsilon \, \tan\Delta; \tag{2.69}$$

$$\overline{\delta}_- - \overline{\delta}_+ = \tan^{-1}\left[\cos 2\epsilon \, \tan(\delta_- - \delta_+)\right]. \tag{2.70}$$

2.3 The Proper self–Energy $\Sigma^*(P_\mu)$

Diagrammatically $\left[1,4\right]$, this is represented by Fig.(2) below:

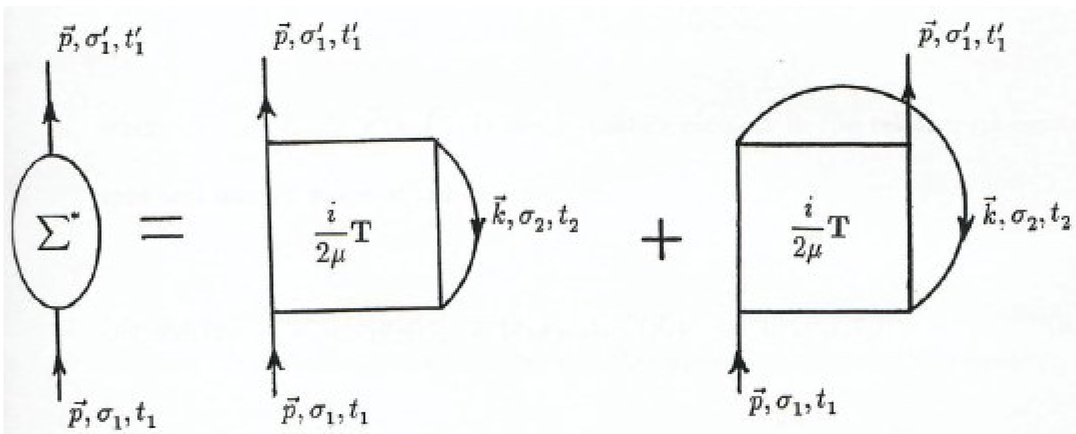


Fig.2: The proper self–energy in the ladder approximation.

Mathematically,

$$\begin{aligned}\sum\nolimits^*_{\substack{\sigma_1\sigma_1'\\ t_1t_1'}}(\mathbf{P}_\mu) &= \frac{i}{2\mu}\int\frac{d^4k}{(2\pi)^4}G_\circ(\mathbf{k})\\ &\times\sum_{\sigma_2t_2}\left[-T_{\substack{\sigma_1\sigma_2;\sigma_1'\sigma_2\\ t_1t_2;t_1't_2}}(\vec{pk};\vec{pk})+T_{\substack{\sigma_1\sigma_2;\sigma_2\sigma_1'\\ t_1t_2;t_2t_1'}}(\vec{pk};\vec{kp})\right], \qquad (2.71)\end{aligned}$$

where P_μ is the four–dimensional vector $(\vec{P},P_o)$, and σ_i , t_i are the spin and isospin of particle i. The minus sign in the first T–term results from the closed Fermi loop; and the two terms in the sum represent both the direct and exchange contributions, respectively.

The relative and c m wavevectors are defined by

$$\vec{q} = \frac{1}{2}(\vec{p}-\vec{k});$$

$$\vec{P} \;=\; \frac{1}{2}(\vec{p}+\vec{k}). \tag{2.72}$$

The proper self–energy can then be written as:

$$\begin{aligned}\sum{}^{*}_{\substack{\sigma_1\sigma_1'\\ t_1t_1'}}(P_\mu) \;=\; \frac{i}{2\mu}\int\frac{d^4k}{(2\pi)^4}G_o(\mathbf{k})\sum_{\sigma_2t_2}\Bigg[&-T_{\substack{\sigma_1\sigma_2;\sigma_1'\sigma_2\\ t_1t_2;t_1't_2}}(\vec{q},\vec{q};s,P)\\ &-T_{\substack{\sigma_1\sigma_2;\sigma_2\sigma_1'\\ t_1t_2;t_2t_1'}}(\vec{q},-\vec{q};s,P)\Bigg],\end{aligned} \tag{2.73}$$

where $T_{\substack{\sigma_1\sigma_2;\sigma_1'\sigma_2'\\ t_1t_2;t_1't_2'}}(\vec{p},\vec{p}';s,\vec{P})$ is the T–matrix element in the relative momentum, spin and isospin space of the two nucleons:

$$\langle\vec{p}\sigma_1\sigma_2t_1t_2|T(s,\vec{P})|\vec{p}'\sigma_1'\sigma_2't_1't_2'\rangle \equiv \langle\sigma_1\sigma_2t_1t_2|T(\vec{p},\vec{p}';s,\vec{P})|\sigma_1'\sigma_2't_1't_2'\rangle, \tag{2.74}$$

σ and t being the respective third components of the spin and isospin of a particle. The states $|\sigma_1\sigma_2\rangle \equiv |\frac{1}{2}\frac{1}{2}\sigma_1\sigma_2\rangle$ and $|t_1t_2\rangle \equiv |\frac{1}{2}\frac{1}{2}t_1t_2\rangle$ are the two–particle states of spins $\frac{1}{2}\frac{1}{2}$ with projections $\sigma_1\sigma_2$ in the spin space, and isospins $\frac{1}{2}\frac{1}{2}$ with projections t_1t_2, respectively.

Defining

$$\mathcal{J}_{\substack{\sigma_1\sigma_1'\\ t_1t_1'}}(\vec{q};s,P) \equiv \sum_{\sigma_2t_2}\Bigg[T_{\substack{\sigma_1\sigma_2;\sigma_1'\sigma_2\\ t_1t_2;t_1't_2}}(\vec{q},\vec{q};s,P) - T_{\substack{\sigma_1\sigma_2;\sigma_2\sigma_1'\\ t_1t_2;t_2t_1'}}(\vec{q},-\vec{q};s,P)\Bigg], \tag{2.75}$$

we get

$$\sum{}^{*}_{\substack{\sigma_1\sigma_1'\\ t_1t_1'}}(P_\mu) = -\frac{i}{2\mu}\int\frac{d^4k}{(2\pi)^4}G_o(\mathbf{k})\mathcal{J}_{\substack{\sigma_1\sigma_1'\\ t_1t_1'}}(\vec{q};s,P). \tag{2.76}$$

In relative–partial waves, the T–matrix is written as:

$$T_{\substack{\sigma_1\sigma_2;\sigma_1'\sigma_2' \\ t_1t_2;t_1't_2'}}(\vec{p},\vec{p'};s,\vec{P}) = 4\pi \sum_{\substack{LL'\alpha\alpha' \\ M_JM_J' \\ M_TM_T'}} i^{L-L'} T^{\alpha}_{LL'}(p,p';s,P)$$

$$\times\langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1\sigma_2|\langle \tfrac{1}{2}\tfrac{1}{2}t_1t_2|\mathcal{Y}^{L\alpha}_{M_JM_T}(\hat{p})\mathcal{Y}^{L'\alpha'}_{M_J'M_T'}{}^{\dagger}(\hat{p'})|\tfrac{1}{2}\tfrac{1}{2}\sigma_1'\sigma_2'\rangle|\tfrac{1}{2}\tfrac{1}{2}t_1't_2'\rangle, \qquad (2.77)$$

where the prescribed pairwise potential conserves J, M, S, T and M_T; conversely, $L_i \neq L_f$, $M_L \neq M_L'$, $M_S \neq M_S'$. Thus,

$$T_{\substack{\sigma_1\sigma_2;\sigma_1'\sigma_2 \\ t_1t_2;t_1't_2}}(\vec{p},\vec{p'};s,P) = 4\pi \sum_{\substack{LL'\alpha \\ M_JM_T}} \sum_{\substack{M_LM_S \\ M_L'M_S'}} i^{L-L'} T^{\alpha}_{LL'}(p,p';s,P)\mathcal{Y}_{LM_L}(\hat{p})\mathcal{Y}^{*}_{L'M_L'}(\hat{p'})$$

$$\langle LSM_LM_S|LSJM_J\rangle\langle L'SJM_J|L'SM_L'M_S'\rangle\langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1\sigma_2|\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle\langle \tfrac{1}{2}\tfrac{1}{2}SM_s'|\tfrac{1}{2}\tfrac{1}{2}\sigma_1'\sigma_2\rangle$$

$$\langle \tfrac{1}{2}\tfrac{1}{2}t_1t_2|\tfrac{1}{2}\tfrac{1}{2}TM_T\rangle\langle \tfrac{1}{2}\tfrac{1}{2}TM_T|\tfrac{1}{2}\tfrac{1}{2}t_1't_2\rangle. \qquad (2.78)$$

Substitution in Eq.(2.75) yields

$$\mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1t_1'}}(\vec{p};s,P) = 4\pi \sum_{\substack{LL'\alpha \\ M_JM_T}} \sum_{\substack{M_LM_S \\ M_L'M_S'}} \sum_{\sigma_2t_2} i^{L-L'} T^{\alpha}_{LL'}(p,p;s,P)\mathcal{Y}_{LM_L}(\hat{p})$$

$$\langle LSM_LM_S|LSJM_J\rangle\langle L'SJM_J|L'SM_L'M_S'\rangle\langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1\sigma_2|\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle\langle \tfrac{1}{2}\tfrac{1}{2}t_1t_2|\tfrac{1}{2}\tfrac{1}{2}TM_T\rangle$$

$$\times\Bigg[\ \mathcal{Y}^{*}_{LM_L}(\hat{p})\langle \tfrac{1}{2}\tfrac{1}{2}SM_s'|\tfrac{1}{2}\tfrac{1}{2}\sigma_1'\sigma_2\rangle\langle \tfrac{1}{2}\tfrac{1}{2}TM_T|\tfrac{1}{2}\tfrac{1}{2}t_1't_2\rangle$$

$$-\mathcal{Y}^{*}_{L'M_L'}(-\hat{p})\langle \tfrac{1}{2}\tfrac{1}{2}SM_s'|\tfrac{1}{2}\tfrac{1}{2}\sigma_2\sigma_1'\rangle\langle \tfrac{1}{2}\tfrac{1}{2}TM_T|\tfrac{1}{2}\tfrac{1}{2}t_2t_1'\rangle\ \Bigg]. \qquad (2.79)$$

Invoking the relations $\Big[31, 59, 60\Big]$:

$$i)\ \mathcal{Y}^*_{LM_L}(-\hat{p}) = (-)^L \mathcal{Y}^*_{LM_L}(\hat{p}); \tag{2.80}$$

$$ii)\ \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle = (-)^{j_1+j_2-j}\langle j_2 j_1 m_2 m_1 | j_2 j_1 j m\rangle, \tag{2.81}$$

we can write

$$i) \qquad \langle \tfrac{1}{2}\tfrac{1}{2} S M'_s | \tfrac{1}{2}\tfrac{1}{2}\sigma_2\sigma'_1\rangle \;=\; (-)^{1-S}\langle \tfrac{1}{2}\tfrac{1}{2} S M'_s | \tfrac{1}{2}\tfrac{1}{2}\sigma'_1\sigma_2\rangle; \tag{2.82}$$

$$ii) \qquad \langle \tfrac{1}{2}\tfrac{1}{2} T M_T | \tfrac{1}{2}\tfrac{1}{2} t_2 t'_1\rangle \;=\; (-)^{1-T}\langle \tfrac{1}{2}\tfrac{1}{2} T M_T | \tfrac{1}{2}\tfrac{1}{2} t'_1 t_2\rangle; \tag{2.83}$$

so that

$$\mathcal{J}_{\substack{\sigma_1\sigma'_1\\ t_1 t'_1}}(\vec{p}; s, P) = 4\pi \sum_{LL'\alpha} i^{L-L'}\Bigg[\;1 - (-)^{L'-S-T+2}\;\Bigg] T^{\alpha}_{LL'}(p, p; s, P)$$
$$\sum_{\substack{M_L M'_L\\ M_J}} \;\; \sum_{\substack{M_s M'_s\\ \sigma_2}} \mathcal{Y}_{L'M'_L}(\hat{p})\mathcal{Y}^*_{L'M_{L'}}(\hat{p})\langle LSM_L M_S|LSJM_J\rangle\langle L'SJM_J|L'SM'_L M'_S\rangle$$
$$\langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1\sigma_2|\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle\langle \tfrac{1}{2}\tfrac{1}{2}SM'_s|\tfrac{1}{2}\tfrac{1}{2}\sigma'_1\sigma_2\rangle \sum_{M_T t_2}\langle \tfrac{1}{2}\tfrac{1}{2}t_1 t_2|\tfrac{1}{2}\tfrac{1}{2}TM_T\rangle\langle \tfrac{1}{2}\tfrac{1}{2}TM_T|\tfrac{1}{2}\tfrac{1}{2}t'_1 t_2\rangle. \tag{2.84}$$

(1) Let us, first, evaluate

$$\sum_{M_T t_2}\langle \tfrac{1}{2}\tfrac{1}{2}t_1 t_2|\tfrac{1}{2}\tfrac{1}{2}TM_T\rangle\langle \tfrac{1}{2}\tfrac{1}{2}TM_T|\tfrac{1}{2}\tfrac{1}{2}t'_1 t_2\rangle.$$

The symmetry relation of the CG coefficients,

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \;=\; (-)^{j_2+m_2}(\frac{2j+1}{2j_2+1})^{1/2}\langle j_2 j - m_2 m | j_2 j j_1 m_1\rangle, \tag{2.85}$$

together with orthonormality $\Big[31, 59, 60\Big]$, yield at once:

$$\sum_{M_T t_2} \langle \tfrac{1}{2}\tfrac{1}{2} t_1 t_2 | \tfrac{1}{2}\tfrac{1}{2} T M_T \rangle \langle \tfrac{1}{2}\tfrac{1}{2} T M_T | \tfrac{1}{2}\tfrac{1}{2} t_1' t_2 \rangle$$

$$= \sum_{M_T t_2} \left(\frac{2T+1}{2} \right) \langle \tfrac{1}{2} T - t_2 M_T | \tfrac{1}{2} T \ \tfrac{1}{2} t_1 \rangle \langle \tfrac{1}{2} T - t_2 M_T | \tfrac{1}{2} T \tfrac{1}{2} t_1' \rangle$$

$$= \left(\frac{2T+1}{2} \right) \delta_{t_1 t_1'}. \tag{2.86}$$

(2) Secondly, we consider

$$\sum_{\substack{M_L M_L' \\ M_J}} \langle L S M_L M_S | L S J M_J \rangle \langle L' S J M_J | L' S M_L' M_S' \rangle \mathcal{Y}_{L M_L}(\hat{p}) \mathcal{Y}^*_{L' M_L'}(\hat{p}). \tag{2.87}$$

The product of the two spherical harmonics is obtained using the familiar Clebsh–Gordan series $\Big[59 - 61\Big]$:

$$D^{j_1}_{m_1' m_1}(w) D^{j_2}_{m_2' m_2}(w) = \sum_{\substack{j_3 \\ m_3 m_3'}} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j_3 m_3 \rangle \langle j_1 j_2 m_1' m_2' | j_1 j_2 j_3 m_3' \rangle$$

$$\times D^{j_3}_{m_3' m_3}(w). \tag{2.88}$$

But

$$\mathcal{Y}_{lm'}(w) = \sum_m D^l_{mm'}(w) \ \mathcal{Y}_{lm}(w); \tag{2.89}$$

$$\mathcal{Y}_{l0}(w) = (\frac{4\pi}{2l+1})^{1/2} \sum_m \mathcal{Y}^*_{lm}(w) \mathcal{Y}_{lm}(w). \tag{2.90}$$

It follows that

$$D^l_{m0}(w) = \left(\frac{4\pi}{2l+1}\right)^{1/2} \mathcal{Y}^*_{lm}(w). \tag{2.91}$$

In Eq.(2.88), we have

$j_1 = L$; $j_2 = L'$; $j_3 = l$; $m'_1 = -M_L$; $m'_2 = M'_L$; $m_1 = m_2 = m_3 = 0$; $j_3 = l$,

and $m'_3 = m$;

so that

$$D^{L'}_{M'_L 0}(w) = \left(\frac{4\pi}{2L'+1}\right)^{1/2} \mathcal{Y}^*_{L'M'_L}(w); \tag{2.92}$$

$$D^{L}_{-M_L 0}(w) = \left(\frac{4\pi}{2L+1}\right)^{1/2} \mathcal{Y}^*_{L-M_L}(w) = (-)^{M_L}\left(\frac{4\pi}{2L+1}\right)^{1/2} \mathcal{Y}_{LM_L}(w); \tag{2.93}$$

$$\left[\mathcal{Y}_{L-M_L}(w) = (-)^{M_L}\mathcal{Y}^*_{LM_L}(w)\right]. \tag{2.94}$$

Consequently,

$$\mathcal{Y}_{LM_L}(\hat{p})\mathcal{Y}^*_{L'M'_L}(\hat{p}) = \sum_{lm}(-)^{M_L}\left[\frac{(2L+1)(2L'+1)}{4\pi(2l+1)}\right]^{1/2} \mathcal{Y}^*_{lm}(\hat{p}) \times\langle LL'-M_LM'_L|LL'lm\rangle\langle LL'00|LL'l0\rangle. \tag{2.95}$$

Now Eq.(2.87) becomes

$$\sum_{lm}\sum_{\substack{M_L M'_L\\ M_J}} (-)^{M_L}\left[\frac{(2L+1)(2L'+1)}{4\pi(2l+1)}\right]^{1/2} \mathcal{Y}^*_{lm}(\hat{p})\langle LL'00|LL'l0\rangle \langle LSM_LM_S|LSJM_J\rangle\langle L'SJM_J|L'SM'_LM'_S\rangle\langle LL'-M_LM'_L|LL'lm\rangle. \tag{2.96}$$

To evaluate

$$\sum_{\substack{M_L M_L' \\ M_J}} (-)^{M_L} \langle LSM_LM_S|LSJM_J\rangle \langle L'SJM_J|L'SM_L'M_S'\rangle \langle LL'-M_LM_L'|LL'lm\rangle, \quad (2.97)$$

the following relation [59, 60] is used:

$$\sum_{\substack{m_2 m_3 \\ m_{12}}} \langle j_1j_2m_1m_2|j_1j_2j_{12}m_{12}\rangle \langle j_{12}j_3m_{12}m_3|j_{12}j_3j'm'\rangle \langle j_2j_3m_2m_3|j_2j_3j_{23}m_{23}\rangle$$

$$= (-)^{j_1+j_2+j_3+j'} \sqrt{(2j_{12}+1)(2j_{23}+1)} \langle j_1j_{23}m_1m_{23}|j_1j_{23}j'm'\rangle \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j' & j_{23} \end{array} \right\}, \quad (2.98)$$

where $\vec{j_i}$ $(i = 1, 2, 3)$ are three different angular momenta; $\vec{j_1}+\vec{j_2} = \vec{j_{12}}$, $\vec{j_2}+\vec{j_3} = \vec{j_{23}}$, $\vec{j'} = \vec{j_1} + \vec{j_{23}} = \vec{j_{12}} + \vec{j_3}$; and the quantity in the braces { } is the 6j-symbol.

With the substitutions

$$\left(\begin{array}{ccc} j_1 = S & j_2 = L & j_{12} = J \\ j_3 = L' & j' = S & j_{23} = l \\ & & \\ m_1 = M_S & m_2 = M_L & m_{12} = M_J \\ m_3 = M_L' & m' = M_S' & m_{23} = m \end{array} \right),$$

together with the symmetry relations [31, 59, 60]:

$$a)\ \langle j_2j_1m_2m_1|j_2j_1j_{12}m_{12}\rangle = (-)^{j_1+j_2-j_{12}} \langle j_1j_2m_1m_2|j_1j_2j_{12}m_{12}\rangle \quad (2.99)$$

$$\langle LSM_LM_S|LSJM_J\rangle = (-)^{S+L-J} \langle SLM_SM_L|SLJM_J\rangle; \quad (2.100)$$

$$b)\ \langle j_3j'm_3m'|j_3j'j_{12}m_{12}\rangle = (-)^{j_3-m_3} \left(\frac{2j_{12}+1}{2j'+1} \right)^{1/2} \langle j_{12}j_3m_{12}-m_3|j_{12}j_3j'm'\rangle \quad (2.101)$$

$$\langle L'SM_L'M_S'|L'SJM_J\rangle = (-)^{L'-M_L'} \left(\frac{2J+1}{2S+1} \right)^{1/2} \langle JL'M_J-M_L'|JL'SM_S'\rangle; \quad (2.102)$$

$$c)\langle j_2 j_3 m_2 m_3 | j_2 j_3 j_{23} m_{23}\rangle = (-)^{j_2+j_3-j_{23}}\langle j_2 j_3 - m_2 - m_3 | j_2 j_3 j_{23} - m_{23}\rangle \tag{2.103}$$

$$\langle LL' - M_L M_L' | LL'lm\rangle = (-)^{L+L'-l}\langle LL'M_L - M_L' | LL'l - m\rangle; \tag{2.104}$$

and the change $-M_L' \to M_L'$, Eq.(2.97) becomes

$$\sum_{\substack{M_L M_L' \\ M_J}} (-)^{M_L}\langle LSM_L M_S | LSJM_J\rangle\langle L'SJM_J | L'SM_L' M_S'\rangle\langle LL' - M_L M_L' | LL'lm\rangle$$

$$= \sum_{\substack{M_L M_L' \\ M_J}} (-)^{m+S-J-l}\left(\frac{2J+1}{2S+1}\right)^{1/2}\langle SLM_S M_L | SLJM_J\rangle$$

$$\times\langle JL'M_J M_L' | JL'SM_S'\rangle\langle LL'M_L M_L' | LL'l - m\rangle$$

$$= (-)^{m+l+S-J}(-)^{S+L+L'+S}\left(\frac{2J+1}{2S+1}\right)^{1/2}\langle SlM_S - m | SlSM_S'\rangle$$

$$\times\Big[\ (2J+1)(2l+1)\ \Big]^{1/2}\left\{\begin{array}{ccc} S & L & J \\ L' & S & l \end{array}\right\}, \tag{2.105}$$

noting that, L , L' , l and $m = -M_L + M_L'$ are integers.

The function $\mathcal{J}$ then reads

$$\mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1 t_1'}}(p,s;P) = 4\pi\sum_{LL'\alpha} i^{L-L'}\Big[\ 1 - (-)^{L'-S-T}\ \Big]T^{\alpha}_{LL'}(p,p;s,P)\frac{2T+1}{2}\delta_{t_1 t_1'}$$

$$\times\sum_{lm}(-)^{m+l-J+L+L'+S}\sqrt{\frac{(2L+1)(2L'+1)}{4\pi}}\frac{2J+1}{(2S+1)^{1/2}}\left\{\begin{array}{ccc} S & L & J \\ L' & S & l \end{array}\right\}$$

$$\times\mathcal{Y}^*_{lm}(\hat{p})\langle LL'00 | LL'l0\rangle\sum_{\substack{M_s M_s' \\ \sigma_2}}\langle\tfrac{1}{2}\tfrac{1}{2}\sigma_1\sigma_2 | \tfrac{1}{2}\tfrac{1}{2}SM_S\rangle\langle\tfrac{1}{2}\tfrac{1}{2}SM_s' | \tfrac{1}{2}\tfrac{1}{2}\sigma_1'\sigma_2\rangle$$

$$\times\langle SlM_S - m | SlSM_S'\rangle. \tag{2.106}$$

(3) Thirdly, we evaluate the sum

$$\sum_{\substack{M_s M_s' \\ \sigma_2}} \langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1\sigma_2|\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle\langle \tfrac{1}{2}\tfrac{1}{2}SM_s'|\tfrac{1}{2}\tfrac{1}{2}\sigma_1'\sigma_2\rangle\langle SlM_S - m|SlSM_S'\rangle. \tag{2.107}$$

Here again the substitutions

$$\begin{pmatrix} j_1 = \frac{1}{2} & j_2 = \frac{1}{2} & j_{12} = S \\ j_3 = S & j' = l & j_{23} = \frac{1}{2} \\ & & \\ m_1 = \sigma_1 & m_2 = \sigma_2 & m_{12} = M_S \\ m_3 = M_S' & m' = -m & m_{23} = \sigma_1' \end{pmatrix}$$

are used, together with the symmetry relations:

$$a)\ \langle j_{12}j'm_{12}m'|j_{12}j'j_3m_3\rangle = (-)^{j_{12}-m_{12}}\left(\tfrac{2j_3+1}{2j'+1}\right)^{1/2}\langle j_{12}j_3m_{12}-m_3|j_{12}j_3j'm'\rangle, \tag{2.108}$$

$$\langle SlM_S - m|SlSM_S'\rangle = (-)^{S-M_S}\left(\frac{2S+1}{2l+1}\right)^{1/2}\langle SSM_S - M_S'|SSl-m\rangle; \tag{2.109}$$

$$b)\ \langle j_{23}j_2m_{23}m_2|j_{23}j_2j_3m_3\rangle = (-)^{j_2+m_2}\left(\tfrac{2j_3+1}{2j_{23}+1}\right)^{1/2}\langle j_2j_3-m_2m_3|j_2j_3j_{23}m_{23}\rangle; \tag{2.110}$$

$$c)\ \langle j_2j_3-m_2m_3|j_2j_3j_{23}+m_{23}\rangle = (-)^{j_2+j_3-j_{23}}\langle j_2j_3-m_2-m_3|j_2j_3j_{23}-m_{23}\rangle. \tag{2.111}$$

From (b) and (c):

$$\begin{aligned}\langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1'\sigma_2|\tfrac{1}{2}\tfrac{1}{2}SM_S'\rangle &= (-)^{\frac{1}{2}+\sigma_2}\left(\frac{2S+1}{2}\right)^{1/2}\langle \tfrac{1}{2}S-\sigma_2M_S'|\tfrac{1}{2}S\tfrac{1}{2}\sigma_1'\rangle \\ &= (-)^{\frac{1}{2}+\sigma_2+S}\left(\frac{2S+1}{2}\right)^{1/2}\langle \tfrac{1}{2}S\sigma_2-M_S'|\tfrac{1}{2}S\tfrac{1}{2}-\sigma_1'\rangle.\end{aligned} \tag{2.112}$$

The sum (2.107) then becomes

$$\sum_{\substack{M_s M_s' \\ \sigma_2}} \langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1\sigma_2|\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle\langle \tfrac{1}{2}\tfrac{1}{2}SM_s'|\tfrac{1}{2}\tfrac{1}{2}\sigma_1'\sigma_2\rangle\langle SlM_S-m|SlSM_S'\rangle$$

$$= \sum_{\substack{M_s M_s' \\ \sigma_2}} (-)^{S-M_S+\frac{1}{2}+\sigma_2+S}\left(\frac{2S+1}{2}\right)^{1/2}\left(\frac{2S+1}{2l+1}\right)^{1/2}\langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1\sigma_2|\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle$$

$$\times\langle SSM_S-M_S'|SSl-m\rangle\langle \tfrac{1}{2}S\sigma_2-M_S'|\tfrac{1}{2}S\tfrac{1}{2}-\sigma_1'\rangle. \tag{2.113}$$

With the change of dummies:

$$-M_S' \to M_S';$$

$$(\sigma_2 - M_S) \to -\sigma_1,$$

where

$$\sigma_1+\sigma_2 = M_S,$$

we obtain

$$\sum_{\substack{M_s M_s' \\ \sigma_2}} \langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1\sigma_2|\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle\langle \tfrac{1}{2}\tfrac{1}{2}SM_s'|\tfrac{1}{2}\tfrac{1}{2}\sigma_1'\sigma_2\rangle\langle SlM_S-m|SlSM_S'\rangle =$$

$$(-)^{\frac{1}{2}-\sigma_1}(-)^{1+S+l}\left[\frac{(2S+1)^{2/3}}{(2l+1)^{1/2}}\right]\left\{\begin{array}{ccc} 1/2 & 1/2 & S \\ S & l & 1/2 \end{array}\right\}\langle \tfrac{1}{2}\tfrac{1}{2}\sigma_1-\sigma_1'|\tfrac{1}{2}\tfrac{1}{2}l-m\rangle; \tag{2.114}$$

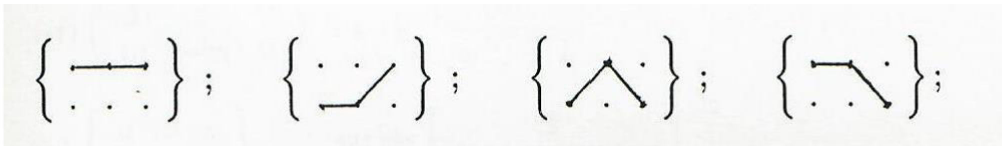
so that

$$\mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1t_1'}}(p,s;P) = 4\pi\sum_{LL'\alpha} i^{L-L'}\left[\ 1-(-)^{L'+S+T}\ \right]T^{\alpha}_{LL'}(p,p;s,P)\frac{2T+1}{2}\delta_{t_1t_1'}$$

$$\times\sum_{lm}\left[\frac{(2L+1)(2L'+1)}{4\pi(2l+1)}\right]^{1/2}(2J+1)\,(2S+1)\,\langle LL'00|LL'l0\rangle(-)^{1+m+L+L'-J}$$

$$\mathcal{Y}^*_{lm}(\hat{p})\langle\tfrac{1}{2}\tfrac{1}{2}\sigma_1 - \sigma_1'|\tfrac{1}{2}\tfrac{1}{2}l - m\rangle \left\{ \begin{array}{ccc} 1/2 & 1/2 & S \\ S & l & 1/2 \end{array} \right\} \left\{ \begin{array}{ccc} S & L & J \\ L' & S & l \end{array} \right\} (-)^{1/2-\sigma_1}. \quad (2.115)$$

Since the 6j–symbols must add vectorialy in the following ways:



then

$i)$ $\left\{ \begin{array}{ccc} S & L & J \\ L' & S & l \end{array} \right\}$ requires $\vec{S} + \vec{S} = \vec{l};$ $\vec{L'} + \vec{L} = \vec{l},$

which implies that $l = (0, 1, 2)$ since $S = (0, 1)$; and

$l \equiv$ even $\equiv (0, 2)$ since $L + L' \equiv$ even to conserve parity of the state;

$ii)$ $\left\{ \begin{array}{ccc} 1/2 & 1/2 & S \\ S & l & 1/2 \end{array} \right\}$ requires $\vec{\tfrac{1}{2}} + \vec{\tfrac{1}{2}} = \vec{l},$

which implies that $l = 0$, 1 only; and in order to be even, $l = 0$ and m $= 0$; i.e., $L = L'$.

Therefore,

$$\mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1t_1'}}(p, s; P) = 4\pi \sum_{L\alpha} T^{\alpha}_{LL}(p, p; s, P) \left[1 - (-)^{L+S+T} \right] \frac{2T+1}{2} \delta_{t_1t_1'} (2L+1)$$

$$\times (2J+1)(2S+1)\langle LL00|LL00\rangle \frac{1}{\sqrt{4\pi}} \mathcal{Y}^*_{00}(\hat{p})(-)^{1-J+L+L}$$

$$\times \left\{ \begin{array}{ccc} S & L & J \\ L & S & 0 \end{array} \right\} \left\{ \begin{array}{ccc} 1/2 & 1/2 & S \\ S & 0 & 1/2 \end{array} \right\} (-)^{1/2-\sigma_1} \langle\tfrac{1}{2}\tfrac{1}{2}\sigma_1 - \sigma_1'|\tfrac{1}{2}\tfrac{1}{2}00\rangle. \quad (2.116)$$

From the relations $\Big[50, 59, 60\Big]$:

$$i)\ \mathcal{Y}^{*}_{00}(\hat{p}) = \frac{1}{\sqrt{4\pi}}; \tag{2.117}$$

$$ii) \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle = (-)^{j_2 - j_1 - m} (2j+1)^{1/2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}; \tag{2.118}$$

$$iii) \begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = (-)^{j-m} \frac{1}{\sqrt{2j+1}}; \tag{2.119}$$

$$iv) \left\{ \begin{matrix} a & b & c \\ 0 & c & b \end{matrix} \right\} = (-)^{a+b+c} \left[\frac{1}{(2b+1)(2c+1)} \right]^{1/2}, \tag{2.120}$$

we get

$$\langle LL00 | LL00 \rangle = \begin{pmatrix} L & L & 0 \\ 0 & 0 & 0 \end{pmatrix} = (-)^{L} \frac{1}{\sqrt{2L+1}}; \tag{2.121}$$

$$\langle \tfrac{1}{2} \tfrac{1}{2} \sigma_1 - \sigma_1' | \tfrac{1}{2} \tfrac{1}{2} 00 \rangle = \begin{pmatrix} 1/2 & 1/2 & 0 \\ \sigma_1 & -\sigma_1' & 0 \end{pmatrix} = (-)^{1/2 - \sigma_1} \frac{1}{\sqrt{2}} \delta_{\sigma_1 \sigma_1'}. \tag{2.122}$$

Since the 6j–symbols are invariant under column interchange, we can write:

$$\left\{ \begin{matrix} S & L & J \\ L & S & 0 \end{matrix} \right\} = \left\{ \begin{matrix} J & S & L \\ 0 & L & S \end{matrix} \right\} = (-)^{J+S+L} \left[\frac{1}{(2L+1)(2S+1)} \right]^{1/2}; \tag{2.123}$$

$$\left\{ \begin{matrix} 1/2 & 1/2 & S \\ 1/2 & 0 & 1/2 \end{matrix} \right\} = \left\{ \begin{matrix} 1/2 & 1/2 & S \\ 0 & S & 1/2 \end{matrix} \right\} = (-)^{1+S} \left[\frac{1}{2(2S+1)} \right]^{1/2}. \tag{2.124}$$

Hence,

$$\mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1t_1'}}(p,s,P) = 4\pi \sum_{L\alpha} T^{\alpha}_{LL}(p,p;s,P)\left[1-(-)^{L+S+T}\right]\delta_{\sigma_1\sigma_1'}\delta_{t_1t_1'}$$
$$\times(2T+1)(2J+1)(2L+1)(2S+1)\left[\frac{1}{(2L+1)(2S+1)}\right]^{1/2}\left[\frac{1}{2(2S+1)}\right]^{1/2}$$
$$\times\frac{1}{\sqrt{2L+1}}(\frac{1}{4\pi})\left[\frac{1}{2}\right]^{1/2}(-)^{1/2-\sigma_1}(-)^{1/2-\sigma_1}. \quad (2.125)$$

Noting that $(-)^{1+2\sigma_1} = 1$, $\sigma_1 = \pm 1/2$, we finally obtain

$$\mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1t_1'}}(p,s,P)$$
$$= \sum_{L\alpha}\frac{(2T+1)(2J+1)}{4}\; T^{\alpha}_{LL}(p,p;s,P)\left[1-(-)^{L+S+T}\right]\delta_{\sigma_1\sigma_1'}\delta_{t_1t_1'}. \quad (2.126)$$

The above matrix element, which is diagonal in both spin and isospin spaces, is applicable to the general case, the two–component nuclear matter, where both isospin states $T = 0, 1$ must be included. Further, in the two–nucleon channels where the potential is spin–isospin independent (uncoupled channels), V has a central component only and Eq.(2.29) implies:

$$V^{\alpha}_{LL} = V_L. \quad (2.127)$$

Consequently,

$$T^{\alpha}_{LL} \to T_L. \tag{2.128}$$

In this case, the factor

$$\left[1-(-)^{L+S+T}\right] = \begin{cases} 2 & L+S+T = \text{odd}; \\ 0 & \text{otherwise}; \end{cases} \tag{2.129}$$

which reflects the fact that the Pauli exclusion principle restricts $(L+S+T)$ to be an odd integer, such that the two–nucleon state is asymmetric under exchange of particle coordinates. Here we have two possibilities:

i) L $\equiv$ an odd integer; therefore $T+S$ must be even, such that

$$\left.\begin{array}{llcl} T=1\ ,\ S=1 & J & = & L-1,\ L,\ L+1; \\ T=0\ ,\ S=0 & J & = & L; \end{array}\right\} \tag{2.130}$$

ii) L $\equiv$ an even integer; therefore $T+S$ must be odd, such that

$$\left.\begin{array}{llcl} T=0\ ,\ S=1 & J & = & L-1,\ L,\ L+1; \\ T=1\ ,\ S=0 & J & = & L. \end{array}\right\} \tag{2.131}$$

All these four combinations (for uncoupled channels) are needed in nuclear matter. Thus, in this case, we have

$$\mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1t_1'}}(p,s,P)$$
$$= \sum_{JSTL} \frac{(2J+1)(2T+1)}{4} T^{JST}_{LL}(p,p;s,P) \left[1-(-)^{L+S+T}\right] \delta_{\sigma_1\sigma_1'}\delta_{t_1t_1'}$$

$$= \sum_{L\ odd} T_L(p,p;s,P)\ 2\left[\frac{(2L+3+2L+1+2L-1)\ 3}{4} + \frac{(2L+1)}{4}\right]$$

$$+ \sum_{L\ even} T_L(p,p;s,P)\ 2\left[\frac{(2L+3+2L+1+2L-1)}{4} + \frac{(2L+1)\ 3}{4}\right]$$

$$= \sum_{L\ odd} 5(2L+1)\ T_L(p,p;s,P) + \sum_{L\ even} 3(2L+1)\ T_L(p,p;s,P)$$

$$= \sum_{L} \left[\ 4-(-)^L\ \right]\ (2L+1)\ T_L(p,p;s,P). \tag{2.132}$$

Conversely, only the $T = 1, M_T = -1$ term is required in neutron matter $T_n = \frac{1}{2}$, $t_n = -\frac{1}{2}$. In this case, it is obvious that the sum in Eq.(2.86) is

$$\sum_{M_T t_2} \langle \tfrac{1}{2}\tfrac{1}{2} t_1 t_2 | \tfrac{1}{2}\tfrac{1}{2} T M_T \rangle \langle \tfrac{1}{2}\tfrac{1}{2} T M_T | \tfrac{1}{2}\tfrac{1}{2} t_1' t_2 \rangle$$

$$= \langle \tfrac{1}{2}\tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2} | \tfrac{1}{2}\tfrac{1}{2} 1 - 1 \rangle \langle \tfrac{1}{2}\tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2} | \tfrac{1}{2}\tfrac{1}{2} 1 - 1 \rangle$$

$$= 1. \tag{2.133}$$

This follows from the fact that the contribution from both isospin states in Eq.(2.127) is restricted to one, since only one state is available.

Then for neutron matter:

$$\mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1 t_1'}}(p,s,P) = \sum_{LSJ} \left[1-(-)^{L+S+1}\right] \frac{(2J+1)}{2} T^{SJ1}_{LL}(p,p;s,P). \tag{2.134}$$

Only two combinations are needed – namely,

$$\left.\begin{array}{llcl} L = \text{odd} \quad T = 1\ ,\ S = 1 & J & = & L-1,\ L,\ L+1; \\ L = \text{even} \quad T = 1\ ,\ S = 0 & J & = & L; \end{array}\right\} \tag{2.135}$$

and Eq.(2.129) becomes(for uncoupled channels)

$$\begin{aligned} \mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1 t_1'}}(p,s,P) \\ &= \sum_{LSJ} \frac{(2J+1)}{2} \left[1-(-)^{L+S+1}\right] T_{LL}^{SJ1}(p,p;s,P) \\ &= \sum_{L\ odd} 2\frac{(6L+3)}{2} T_L(p,p;s,P) + \sum_{L\ odd} T_L(p,p;s,P) 2\frac{(2L+1)}{2} \\ &= \sum_{L\ odd} 3(2L+1)\, T_L(p,p;s,P) + \sum_{L\ even} (2L+1)\, T_L(p,p;s,P) \\ &= \sum_{L} \left[\ 2-(-)^L\ \right]\ (2L+1)\ T_L(p,p;s,P). \end{aligned} \tag{2.136}$$

We turn to the other fermionic systems considered in this work – namely, pure liquid ^{3}He in Vycor glass and dilute ^{3}He–He II mixtures. These interact via a state–independent, central potential. Besides the SJT independence, the isotopic concept is inapplicable here. In this case, the factor $\left[1-(-)^{L+S+T}\right]$, which results originally from the combinations $(-)^L, (-)^{1-T}$ and $(-)^{1-S}$ in Eq.(2.84), takes the form:

$$\left[1-(-)^{L+S+1}\right]. \tag{2.137}$$

$L + S + 1$ must be odd; so that

$$\left.\begin{array}{llcl} L = \text{odd} & S = 1 & J = & L-1,\ L,\ L+1; \\ L = \text{even} & S = 0 & J = & L. \end{array}\right\} \tag{2.138}$$

Therefore,

$$\begin{aligned} \mathcal{J}_{\substack{\sigma_1\sigma_1' \\ t_1t_1'}}(p, s, P) &= \sum_{L\ odd} 3(2L+1)\, T_L(p,p;s,P) + \sum_{L\ even} (2L+1)\, T_L(p,p;s,P) \\ &= \sum_L \left[\ 2 - (-)^L\ \right]\ (2L+1)\ T_L(p,p;s,P). \end{aligned} \tag{2.139}$$

One last remark is in order here concerning dilute ^{3}He–He II mixtures. Below the Fermi degeneracy temperature, where the quasiparticle picture is valid and where the Fermi-type excitations (^{3}He quasiparticles) dominate the Bose-type excitations (phonons and rotons) , the effective interaction between two ^{3}He quasiparticles, assumed to be instantaneously fixed so that retardation effects may be neglected, will be local with the following features$\left[29, 42\right]$:

$$V_{eff}(r) \approx \begin{cases} V(r), & r < r_c; \\ \alpha^2 V(r), & r > 2r_c, \end{cases} \tag{2.140}$$

where r_c is some "effective" core radius $\approx 2.5\mathring{A}$, and α is the so-called volume differential coefficient $\approx$0.3; it is just the relative fraction by which the volume occupied by the ^{3}He atom is larger than that of ^{4}He, as a result of the greater zero-point energy of the ^{3}He, by virtue of its smaller mass.

A more accurate form for this interaction has been derived in configuration space [29, 62]. In addition, the important properties of the interaction have been extensively discussed with especial emphasis on its binding properties [63]. For the present purpose, however, the main point is that the relevant T–matrix in this case is defined as that obtained between two ^{3}He–quasiparticles; i.e., with $\alpha \approx 0.3$, minus that obtained with two ^{4}He atoms instead of ^{3}He atoms; i.e, with $\alpha = 0$ in the calculations. This definition follows from our interest in the properties of the ^{3}He dilute mixture resulting solely from the presence of the ^{3}He atoms (regarded as impurities) [42, 63].

2.4 Orthogonality and Completeness of the T–matrices

2.4.1 Orthogonality

In operator notation, in the center–of–mass frame of the interacting pair, the GMF T–matrix equation is given by:

$$T^G(s, \vec{P}) = u - u\left[g_o(s)Q(\vec{P}) - g_o(s)^\dagger \overline{Q}(\vec{P})\right] T^G(s, \vec{P}). \tag{2.141}$$

It is convenient to define an equivalent T–matrix, $T^A(z, \vec{P})$, which is analytic in the entire positive imaginary (upper) s-plane:

$$T^A(s, \vec{P}) = u - u\left[Q(\vec{P}) - \overline{Q}(\vec{P})\right] g_o(s)T^A(s, \vec{P}). \tag{2.142}$$

On switching off the hole–hole interaction in Eq.(2.142) the Brueckner–Gammel T–matrix is obtained:

$$\begin{aligned} T^B(s,\vec{P}) &= u - u\ Q(\vec{P})g_o(s)\ T^B(s,\vec{P}) \\ &= u - T^B(s,\vec{P})\ Q(\vec{P})g_o(s)\ u. \end{aligned} \tag{2.143}$$

Further, in the absence of the many–body medium, $Q = 1$ and $\overline{Q} = 0$; so that (2.136) reduces at once to the Lippmann–Schwinger T–matrix:

$$\begin{aligned} t &= u - ug_o t \\ &= u - tg_o u. \end{aligned} \tag{2.144}$$

The integral equation corresponding to Eq.(2.142) has already been obtained using the operator multiplication in the relative–momentum representation:

$$\langle\vec{p}|AB|\vec{p'}\rangle = \int \frac{d\vec{k}}{(2\pi)^3}\langle\vec{p}|A|\vec{k}\rangle\langle\vec{k}|B|\vec{p'}\rangle; \tag{2.145}$$

and noting that

$$T(\vec{p},\vec{p'};s,\vec{P}) = \langle\vec{p}|T(s,\vec{P})|\vec{p'}\rangle; \tag{2.146}$$

$$\begin{aligned} \langle\vec{k}|Q(\vec{P})|\vec{k'}\rangle &= (2\pi)^3\delta(\vec{k}-\vec{k'})\theta(|\vec{k}+\vec{P}|-k_F)\theta(|\vec{k}-\vec{P}|-k_F) \\ &= (2\pi)^3\delta(\vec{k}-\vec{k'})Q(\vec{k},\vec{P}); \end{aligned} \tag{2.147}$$

$$\langle\vec{k}|\overline{Q}(\vec{P})|\vec{k'}\rangle = (2\pi)^3\delta(\vec{k}-\vec{k'})\theta(k_F-|\vec{k}+\vec{P}|)\theta(k_F-|\vec{k}-\vec{P}|)$$

$$= \quad (2\pi)^3\delta(\vec{k}-\vec{k'})\overline{Q}(\vec{k},\vec{P}); \tag{2.148}$$

$$\langle\vec{k}|g_o(s)|\vec{k'}\rangle \quad = \quad (2\pi)^3\delta(\vec{k}-\vec{k'})g_o(\vec{k},s) = (2\pi)^3\delta(\vec{k}-\vec{k'})\left[k^2-s-i\eta\right]^{-1}; \tag{2.149}$$

$$s \quad = \quad 2mP_o - P^2 \equiv 2P_o - P^2. \tag{2.150}$$

Take

$$z = s + i\eta.$$

Then

$$\langle\vec{k}|g_o(z)|\vec{k'}\rangle = (2\pi)^3\delta(\vec{k}-\vec{k'})g_o(\vec{k},z) = (2\pi)^3\delta(\vec{k}-\vec{k'})\left[k^2-z\right]^{-1}. \tag{2.151}$$

Let

$$\Gamma(z,\vec{P}) \equiv \begin{cases} Q(\vec{P})g_o(z) - \overline{Q}(\vec{P})g_o(z)^\dagger & (G) \\ \left[Q(\vec{P}) - \overline{Q}(\vec{P})\right]g_o(z) & (A) \\ Q(\vec{P})g_o(z) & (B) \end{cases} \tag{2.152}$$

and define the half–shell T–matrix:

$$F(\vec{p},\vec{p'};\vec{P}) \equiv \langle\vec{p}|T(p'^2+i\eta;\vec{P}|\vec{p'}\rangle = T(\vec{p},\vec{p'};p'^2+i\eta,\vec{P}). \tag{2.153}$$

Now,

$$\begin{aligned} T^*(z,\vec{P}) \quad &= \quad u \quad - u\,\Gamma(z^*,\vec{P})\;T(z^*,\vec{P}) \\ &= \quad u \quad - T(z^*,\vec{P})\;\Gamma(z^*,\vec{P})\;u; \end{aligned} \tag{2.154}$$

$$
\begin{aligned}
T^{\dagger}(z,\vec{P}) &= u^{\dagger} \;-\; \left[u\,\Gamma(z,\vec{P})\;T(z,\vec{P})\right]^{\dagger} \\
&= u \;-\; T^{\dagger}(z,\vec{P})\;\Gamma^{\dagger}(z,\vec{P})\;u. \qquad (2.155)
\end{aligned}
$$

Since

$$
\Gamma^{\dagger}(z,\vec{P}) = \Gamma(z^*,\vec{P}); \qquad (2.156)
$$

$$
T^{\dagger}(z,\vec{P}) = T^*(z,\vec{P}) = T(z^*,\vec{P}), \qquad (2.157)
$$

we have

$$
\begin{aligned}
T^{\dagger}(\vec{p},\vec{p'};p'^2+i\eta,\vec{P}) &= \langle\vec{p}|T^{\dagger}(p'^2+i\eta;\vec{P}|\vec{p'}\rangle \;=\; \langle\vec{p}|F^{\dagger}(\vec{P})|\vec{p'}\rangle \\
&= \langle\vec{p'}|T^{*}(p'^2+i\eta;\vec{P}|\vec{p}\rangle \;=\; T^{*}(\vec{p'},\vec{p};p'^2+i\eta,\vec{P}) \\
&= T(\vec{p'},\vec{p};p'^2-i\eta,\vec{P}) = F^{\dagger}(\vec{p},\vec{p'};\vec{P}). \qquad (2.158)
\end{aligned}
$$

Let us now define a generalized wave operator W:

$$
T(z,\vec{P}) \equiv u\,W(z,\vec{P}), \qquad (2.159)
$$

$$
W(z,\vec{P}) \equiv 1-\Gamma(z,\vec{P})T(z,\vec{P}), \qquad (2.160)
$$

such that

$$
\langle\vec{p}|W(z,\vec{P})|\vec{p'}\rangle = \langle\vec{p}|1|\vec{p'}\rangle - \int\frac{d\vec{k}}{(2\pi)^3}\langle\vec{p}|\Gamma(z,\vec{P})|\vec{k}\rangle\langle\vec{k}|T(z,\vec{P})|\vec{p'}\rangle; \qquad (2.161)
$$

$$
\langle\vec{p}|\Gamma(z,\vec{P})|\vec{k}\rangle = \langle\vec{p}|Q(\vec{P})g_o(z)-\overline{Q}(\vec{P})g_o^{\dagger}(z)|\vec{k}\rangle. \qquad (2.162)
$$

But

$$
\begin{aligned}
\langle\vec{p}|Q(\vec{P})g_o(z)|\vec{k}\rangle &= \int \frac{d\vec{q}}{(2\pi)^3}\langle\vec{p}|Q(\vec{P})|\vec{q}\rangle\langle\vec{q}|g_o(z)|\vec{k}\rangle \\
&= (2\pi)^3\int d\vec{q}\; Q(\vec{p},\vec{P})\delta(\vec{p}-\vec{q})\; g_o(\vec{q},z)\delta(\vec{q}-\vec{k}) \\
&= (2\pi)^3 Q(\vec{p},\vec{P})\; g_o(\vec{k},z)\delta(\vec{p}-\vec{k}); \qquad (2.163)
\end{aligned}
$$

and

$$
\langle\vec{p}|\overline{Q}(\vec{P})g_o^{\dagger}(z)|\vec{k}\rangle = (2\pi)^3\overline{Q}(\vec{p},\vec{P})\; g_o^{\dagger}(\vec{k},z)\delta(\vec{p}-\vec{k}). \qquad (2.164)
$$

Thus,

$$
\begin{aligned}
\langle\vec{p}|\Gamma(z,\vec{P})|\vec{k}\rangle &= (2\pi)^3\delta(\vec{p}-\vec{k})\left[Q(\vec{p},\vec{P})\; g_o(\vec{k},z)-\overline{Q}(\vec{p},\vec{P})g_o^{\dagger}(\vec{k},z)\right] \\
&= (2\pi)^3\delta(\vec{p}-\vec{k})\;\; \Gamma(\vec{p};z,\vec{P}). \qquad (2.165)
\end{aligned}
$$

Therefore, the full–off shell wave operator is given by

$$
\begin{aligned}
W(\vec{p},\vec{p}';z,\vec{P}) &= (2\pi)^3\delta(\vec{p}-\vec{p}') - \int d\vec{k}\delta(\vec{p}-\vec{k})\left[Q(\vec{p},\vec{P})\; g_o(\vec{k},z)\right. \\
&\qquad \left.\times\,\overline{Q}(\vec{p},\vec{P})g_o^{\dagger}(\vec{k},z)\right] T(\vec{k},\vec{p}';z,\vec{P}) \\
&= (2\pi)^3\delta(\vec{p}-\vec{p}')\; - \;\Gamma(\vec{p};z;\vec{P})\; T(\vec{p},\vec{p}';z,\vec{P}). \qquad (2.166)
\end{aligned}
$$

In the scattering region,($s > 0$: the positive energy-solution), the particles interacting by a two–body potential are scattered outside the range of the potential, giving scattering solutions in the many–body medium which behave asymptotically as free states – of outgoing waves for particles and incoming waves for holes. The corresponding eigenstates $|\Psi_{\vec{k}}(\vec{P})\rangle$ in the relative–momentum representation are related to the half–shell value of the wave operator as follows:

$$\begin{aligned}
\langle\vec{p}|\Psi_{\vec{p}'}(\vec{P})\rangle &= \Psi_{\vec{p}'}(\vec{p},\vec{P}) \\
&= \langle\vec{p}|W(p'^2+i\eta,\vec{P})|\vec{p}'\rangle = W(\vec{p},\vec{p}';p'^2+i\eta,\vec{P}) \\
&= \langle\vec{p}|\chi(p'^2,\vec{P})|\vec{p}'\rangle = \chi(\vec{p},\vec{p}';\vec{P}); \\
&= (2\pi)^3\delta(\vec{p}-\vec{p}') - \Gamma(\vec{p};p'^2+i\eta;\vec{P})\ F(\vec{p},\vec{p}';\vec{P}). \qquad (2.167)
\end{aligned}$$

Now, the orthonormality condition is written as

$$\begin{aligned}
\langle\Psi_{\vec{k}}(\vec{P})|\Psi_{\vec{k}'}(\vec{P})\rangle &= \int\frac{d\vec{p}}{(2\pi)^3}\ \langle\Psi_{\vec{k}}(\vec{P})|\vec{p}\rangle\langle\vec{p}|\Psi_{\vec{k}'}(\vec{P})\rangle \\
&= \int\frac{d\vec{p}}{(2\pi)^3}\ \chi^\dagger(\vec{p},\vec{k};\vec{P})\ \chi(\vec{p},\vec{k}';\vec{P}). \qquad (2.168)
\end{aligned}$$

Substituting for χ from Eq.(2.168), we obtain

$$\int\frac{d\vec{p}}{(2\pi)^3}\ \chi^\dagger(\vec{p},\vec{k};\vec{P})\ \chi(\vec{p},\vec{k}';\vec{P})$$

$$= \int\frac{d\vec{p}}{(2\pi)^3}\left[\ (2\pi)^3\delta(\vec{p}-\vec{k}) - \Gamma(\vec{p};k^2+i\eta;\vec{P})F(\vec{p},\vec{k};\vec{P})\ \right]^\dagger$$

$$\times \left[\; (2\pi)^3 \delta(\vec{p} - \vec{k'}) \; - \; \Gamma(\vec{p}; k'^2 + i\eta; \vec{P}) F(\vec{p}, \vec{k'}; \vec{P}) \; \right]$$

$$= (2\pi)^3 \delta(\vec{k} - \vec{k'}) - \int d\vec{p} \left[\; \delta(\vec{p} - \vec{k'}) \; F^\dagger(\vec{p}, \vec{k}; \vec{P}) \; \Gamma^\dagger(\vec{p}; k^2 + i\eta; \vec{P}) \right.$$

$$\left. + \delta(\vec{p} - \vec{k}) \; \Gamma(\vec{p}; k'^2 + i\eta; \vec{P}) \; F(\vec{p}, \vec{k'}; \vec{P}) \; \right] + \int \frac{d\vec{p}}{(2\pi)^3} \; F^\dagger(\vec{p}, \vec{k}; \vec{P}) \times$$

$$\Gamma^\dagger(\vec{p}; k^2 + i\eta; \vec{P}) \; \Gamma(\vec{p}; k'^2 + i\eta; \vec{P}) F(\vec{p}, \vec{k'}; \vec{P}); \tag{2.169}$$

or

$$\langle \Psi_{\vec{k}}(\vec{P}) | \Psi_{\vec{k'}}(\vec{P}) \rangle \; =$$

$$(2\pi)^3 \delta(\vec{k} - \vec{k'}) - \left[\; F^\dagger(\vec{k'}, \vec{k}; \vec{P}) \Gamma^\dagger(\vec{k'}; k^2 + i\eta; \vec{P}) + \Gamma(\vec{k}; k'^2 + i\eta; \vec{P}) F(\vec{k}, \vec{k'}; \vec{P}) \; \right]$$

$$+ \int \frac{d\vec{p}}{(2\pi)^3} \; F^\dagger(\vec{p}, \vec{k}; \vec{P}) \; \Gamma^\dagger(\vec{p}; k^2 + i\eta; \vec{P}) \; \Gamma(\vec{p}; k'^2 + i\eta; \vec{P}) \; F(\vec{p}, \vec{k'}; \vec{P}). \tag{2.170}$$

But

$$\Gamma^\dagger(\vec{q}; z; \vec{P}) = \Gamma(\vec{q}; z^*; \vec{P}).$$

Therefore, we need to evaluate integrals of the form:

$$\int \frac{d\vec{p}}{(2\pi)^3} \; F^\dagger(\vec{p}, \vec{k}; \vec{P}) \; \left[\; \Gamma(\vec{p}; k^2 - i\eta; \vec{P}) \; \Gamma(\vec{p}; k'^2 + i\eta; \vec{P}) \; \right] F(\vec{p}, \vec{k'}; \vec{P}). \tag{2.171}$$

Now, $u = u^\dagger$. Consequently, a generalized unitarity of the T–matrix can be expressed in the following way:

$$T(z',\vec{P}) - T(z,\vec{P}) = \Big(u - T(z',\vec{P})\Gamma(z',\vec{P})\ u\Big) - \Big(u - u\ \Gamma(z,\vec{P})T(z,\vec{P})\Big)$$

$$= -T(z',\vec{P})\Gamma(z',\vec{P})\ u + u\ \Gamma(z,\vec{P})T(z,\vec{P}). \qquad (2.172)$$

But

$$u = T(z,\vec{P}) + u\ \Gamma(z,\vec{P})\ T(z,\vec{P});$$

so that

$$T(z',\vec{P}) - T(z,\vec{P}) =$$

$$-T(z',\vec{P})\Gamma(z',\vec{P})T(z,\vec{P}) - T(z',\vec{P})\Gamma(z',\vec{P})\ \ u\ \ \Gamma(z,\vec{P})T(z,\vec{P})$$

$$+T(z',\vec{P})\Gamma(z,\vec{P})T(z,\vec{P}) + T(z',\vec{P})\Gamma(z',\vec{P})\ \ u\ \ \Gamma(z,\vec{P})T(z,\vec{P})$$

$$= T(z',\vec{P})\left[\ \Gamma(z,\vec{P})\ -\ \Gamma(z',\vec{P})\ \right]T(z,\vec{P}). \qquad (2.173)$$

On using Eq. (2.146), this may be written as

$$T(\vec{p},\vec{p'};z';\vec{P}) - T(\vec{p},\vec{p'};z;\vec{P}) =$$

$$\int \frac{d\vec{q}}{(2\pi)^3}\ T(\vec{p},\vec{q};z';\vec{P})\left[\Gamma(\vec{q};z;\vec{P}) - \Gamma(\vec{q};z';\vec{P})\right]T(\vec{q},\vec{p'};z;\vec{P}). \qquad (2.174)$$

With

$$z' = (p^2 + i\eta)^* = p^2 - i\eta \quad \text{and} \quad z = p'^2 + i\eta, \qquad (2.175)$$

$$T(\vec{p}, \vec{p'}; z'; \vec{P}) = T^*(\vec{p'}, \vec{p}; p^2 + i\eta; \vec{P}) = F^*(\vec{p'}, \vec{p}; \vec{P}) = F^\dagger(\vec{p'}, \vec{p}; \vec{P}); \tag{2.176}$$

$$T(\vec{p}, \vec{p'}; z; \vec{P}) = T(\vec{p}, \vec{p'}; p'^2 + i\eta; \vec{P}) = F(\vec{p}, \vec{p'}; \vec{P}); \tag{2.177}$$

so that

$$F^\dagger(\vec{p'}, \vec{p}; \vec{P}) - F(\vec{p}; \vec{p'}; \vec{P}) =$$
$$\frac{1}{(2\pi)^3} \int d\vec{q}\; F^\dagger(\vec{q}, \vec{p}; \vec{P}) \left[\Gamma(\vec{q}; p'^2 + i\eta; \vec{P}) - \Gamma(\vec{q}; p^2 - i\eta; \vec{P}) \right] F(\vec{q}, \vec{p'}; \vec{P}). \tag{2.178}$$

For both (B) and (A), $\Gamma(z, \vec{P})$ is of the form $A(\vec{P})g(z)$. In these two cases, the term in square brackets is equal to:

$$A(\vec{P}) \left[\frac{1}{q^2 - p'^2 - i\eta} - \frac{1}{q^2 - p^2 + i\eta} \right]$$
$$= A(\vec{P}) \frac{1}{q^2 - p'^2 - i\eta} \quad \frac{1}{q^2 - p^2 + i\eta} \; (p'^2 - p^2 + i\eta')$$
$$= (p'^2 - p^2 + i\eta) \quad \Gamma(\vec{q}; p'^2 + i\eta; \vec{P}) \quad \Gamma(\vec{q}; p^2 - i\eta; \vec{P}). \tag{2.179}$$

For (B) and (A):

$$\Gamma(\vec{q}; p'^2 + i\eta; \vec{P}) \; - \; \Gamma(\vec{q}; p^2 - i\eta; \vec{P}) =$$
$$(p'^2 - p^2 + i\eta)\, \Gamma(\vec{q}; p'^2 + i\eta; \vec{P})\, \Gamma(\vec{q}; p^2 - i\eta; \vec{P}). \tag{2.180}$$

It is noted that:

i) $\eta' \longrightarrow \eta \equiv$ an infinitesimal number;

ii) strictly, the rhs of the above equation(s) should be divided by $A(\vec{P}) \sim O(1)$ ... .

For (G):

$$\Gamma(\vec{q};p'^2+i\eta;\vec{P}) \;-\; \Gamma(\vec{q};p^2-i\eta;\vec{P}) =$$

$$Q(\vec{P})\left[\frac{1}{q^2-p'^2-i\eta} - \frac{1}{q^2-p^2+i\eta}\right] \;-\; \overline{Q}(\vec{P})\left[\frac{1}{q^2-p'^2+i\eta} - \frac{1}{q^2-p^2-i\eta}\right]$$

$$= (p'^2-p^2+i\eta)\left(\frac{1}{q^2-p'^2-i\eta}\right)\left(\frac{1}{q^2-p^2+i\eta}\right)Q(\vec{P}) \;-\; \overline{Q}(\vec{P})(p'^2-p^2-i\eta)$$

$$\times\left(\frac{1}{q^2-p^2-i\eta}\right)\left(\frac{1}{q^2-p'^2+i\eta}\right)$$

$$= (p'^2-p^2+i\eta)\left[\frac{Q(\vec{P})}{(q^2-p'^2-i\eta)(q^2-p^2+i\eta)} \;+\; \frac{\overline{Q}(\vec{P})}{(q^2-p^2-i\eta)(q^2-p'^2+i\eta)}\right]$$

$$-\overline{Q}(\vec{P})\left[\frac{(p'^2-p^2+i\eta)+(p'^2-p^2-i\eta)}{(q^2-p^2-i\eta)(q^2-p'^2+i\eta)}\right]$$

$$= (p'^2-p^2+i\eta)\Gamma(\vec{q};p'^2+i\eta;\vec{P})\Gamma(\vec{q};p^2-i\eta;\vec{P}) - \frac{2(p'^2-p^2)\,\overline{Q}(\vec{P})}{(q^2-p^2-i\eta)(q^2-p'^2+i\eta)}. \quad (2.181)$$

It follows that, for (G),

$$\int\frac{d\vec{p}}{(2\pi)^3}F^\dagger(\vec{p},\vec{k};\vec{P})\;\;\Gamma(\vec{p};k^2-i\eta;\vec{P})\;\;\Gamma(\vec{p};k'^2+i\eta;\vec{P})\;F(\vec{p},\vec{k'};\vec{P}) =$$

$$\left(\frac{1}{k'^2-k^2+i\eta}\right)\int\frac{d\vec{p}}{(2\pi)^3}F^\dagger(\vec{p},\vec{k};\vec{P})\left[\Gamma(\vec{p};k'^2+i\eta;\vec{P}) \;-\; \Gamma(\vec{p};k^2-i\eta;\vec{P})\right]F(\vec{p},\vec{k'};\vec{P})$$

$$+\frac{2(k'^2-k^2)}{k'^2-k^2+i\eta}\int\frac{d\vec{p}}{(2\pi)^3}\frac{F^\dagger(\vec{p},\vec{k};\vec{P})\overline{Q}(\vec{P})F(\vec{p},\vec{k'};\vec{P})}{(p^2-k^2-i\eta)(p^2-k'^2+i\eta)}$$

$$= \frac{F^\dagger(\vec{k'},\vec{k};\vec{P})-F(\vec{k},\vec{k'};\vec{P})}{k'^2-k^2+i\eta}+\frac{2(k'^2-k^2)}{k'^2-k^2+i\eta}\int\frac{d\vec{p}}{(2\pi)^3}\frac{F^\dagger(\vec{p},\vec{k};\vec{P})\overline{Q}(\vec{P})F(\vec{p},\vec{k'};\vec{P})}{(p^2-k^2-i\eta)(p^2-k'^2+i\eta)}.$$

(2.182)

To sum up, then,

$$\begin{aligned}
\langle\Psi_{\vec{k}}(\vec{P})|\Psi_{\vec{k}'}(\vec{P})\rangle &= (2\pi)^3\delta(\vec{k}-\vec{k}') + \frac{F^\dagger(\vec{k}',\vec{k};\vec{P}) - F(\vec{k},\vec{k}';\vec{P})}{k'^2 - k^2 + i\eta} \\
&- \left[F^\dagger(\vec{k}',\vec{k};\vec{P})\Gamma^\dagger(\vec{k}';k^2+i\eta;\vec{P}) \ + \ \Gamma(\vec{k};k'^2+i\eta;\vec{P})F(\vec{k},\vec{k}';\vec{P}) \right] \\
&+ \frac{2(k'^2-k^2)}{k'^2-k^2+i\eta}\int \frac{d\vec{p}}{(2\pi)^3}\,\frac{F^\dagger(\vec{p},\vec{k};\vec{P})\overline{Q}(\vec{P})F(\vec{p},\vec{k}';\vec{P})}{(p^2-k^2-i\eta)(p^2-k'^2+i\eta)}. \qquad (2.183)
\end{aligned}$$

Clearly, the last term on the rhs of this equation reduces to zero, except for (G).

2.4.2 Completeness

We use contour Γ defined as shown in Fig.(3):

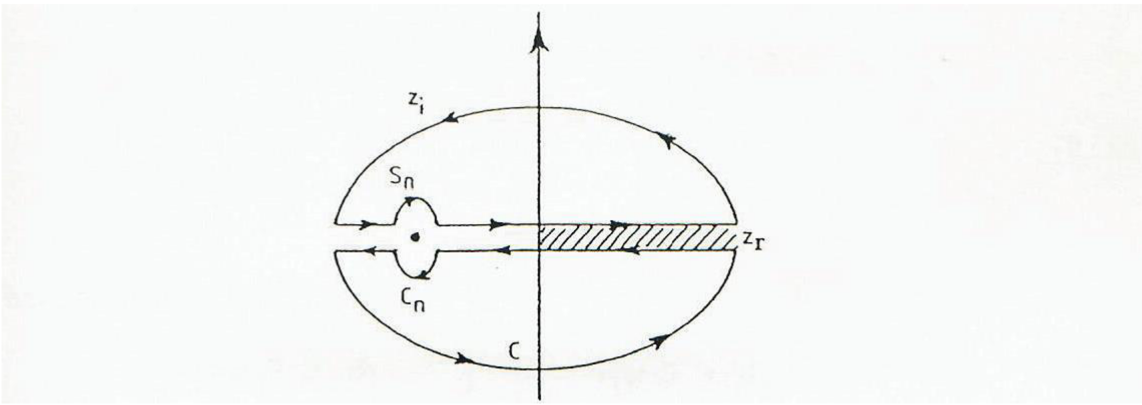


Fig.3: The contour used in the manipulations below.

Let us also define

$$F_n(z) \equiv \begin{cases} T(\vec{p},\vec{p'};z;\vec{P}) : & n=0; \\ (z-p'^2-i\eta)^{-1}\ F_o(z) : & n=1, \end{cases} \tag{2.184}$$

∀ z is " interior" to Γ :

$$F_n(z) \;=\; \frac{1}{2\pi i}\oint_\Gamma dz' \frac{F_n(z')}{z'-z}. \tag{2.185}$$

For n=0:

$$T(\vec{p},\vec{p'};z;\vec{P}) \;=\; \frac{1}{2\pi i}\oint_\Gamma dz' \frac{T(\vec{p},\vec{p'};z';\vec{P})}{z'-z}. \tag{2.186}$$

We assume bound states of the form:

$$\lim_{z \to S_n} \quad T(\vec{p}, \vec{p'}; z; \vec{P}) \ = \ \frac{R_n(\vec{p}, \vec{p'}; \vec{P})}{z - S_n(\vec{P})}, \tag{2.187}$$

where S_n is real (< 0).

Further, we recall that

$$\lim_{|z| \to \infty} \quad T(\vec{p}, \vec{p'}; z; \vec{P}) \longrightarrow \quad u(\vec{p} - \vec{p'}). \tag{2.188}$$

Then, the rhs of Eq.(2.187) becomes:

$$\begin{aligned}
&\frac{1}{2\pi i} \sum_n \oint_{C_n} dz'_n \frac{T(\vec{p}, \vec{p'}; z'_n; \vec{P})}{z'_n - z} \ + \ \frac{1}{2\pi i} \int_C dz' \frac{T(\vec{p}, \vec{p'}; z'; \vec{P})}{z' - z} \\
&\quad + \frac{1}{2\pi i} \int_0^\infty ds' \left[\frac{T(\vec{p}, \vec{p'}; s' + i\eta; \vec{P})}{s' + i\eta - z} \ - \ \frac{T(\vec{p}, \vec{p'}; s' - i\eta; \vec{P})}{s' - i\eta - z} \right] \\
&\qquad = - \sum_n \frac{R_n(\vec{p}, \vec{p'}; \vec{P})}{S_n - z} \ + \ u(\vec{p} - \vec{p'}) \\
&\qquad\quad + \frac{1}{2\pi i} \int_0^\infty ds' \ \frac{T(\vec{p}, \vec{p'}; s' + i\eta; \vec{P}) \ - \ T(\vec{p}, \vec{p'}; s' - i\eta; \vec{P})}{s' - z},
\end{aligned} \tag{2.189}$$

z being interior to Γ; so that $s' - z$ is not singular.

Now,

$$T(\vec{p'}, \vec{p}; s' - i\eta; \vec{P}) \ = \ T^\dagger(\vec{p'}, \vec{p}; s' + i\eta; \vec{P}); \tag{2.190}$$

$$\begin{aligned}
T(\vec{p}, \vec{p'}; s' + i\eta; \vec{P}) \ - \ T^\dagger(\vec{p'}, \vec{p}; s' + i\eta; \vec{P}) \ &= \ 2\pi i \int \frac{d^3 q}{(2\pi)^3} \\
\times T(\vec{p}, \vec{q}; s + i\eta; \vec{P}) \left[Q(\vec{q}, \vec{P}) \ + \overline{Q}(\vec{q}, \vec{P}) \right] T^\dagger(\vec{p'}, \vec{q}; s + i\eta; \vec{P}) \ \delta(q^2 - s).
\end{aligned} \tag{2.191}$$

Therefore,

$$\frac{1}{2\pi i}\int_0^\infty ds'\ \frac{T(\vec{p},\vec{p'};s'+i\eta;\vec{P})\ -\ T^\dagger(\vec{p'},\vec{p};s'+i\eta;\vec{P})}{s'-z} = \frac{1}{2\pi i}\int_0^\infty ds' \int \frac{d^3q}{(2\pi)^3}(-2\pi i)$$

$$\times\frac{T(\vec{p},\vec{q};s'+i\eta;\vec{P})\left[Q(\vec{q},\vec{P})\ +\overline{Q}(\vec{q},\vec{P})\right]T^\dagger(\vec{p'},\vec{q};s'+i\eta;\vec{P})}{s'-z}\ \delta(s-q^2)$$

$$= \int \frac{d^3q}{(2\pi)^3}\frac{T(\vec{p},\vec{q};q^2+i\eta;\vec{P})\left[Q(\vec{q},\vec{P})\ +\overline{Q}(\vec{q},\vec{P})\right]T^\dagger(\vec{p'},\vec{q};q^2+i\eta;\vec{P})}{q^2-z}$$

$$= \int \frac{d^3q}{(2\pi)^3}\frac{F(\vec{p},\vec{q};\vec{P})\left[Q(\vec{q},\vec{P})\ +\overline{Q}(\vec{q},\vec{P})\right]F^\dagger(\vec{q},\vec{p'};\vec{P})}{q^2-z}. \tag{2.192}$$

Thus, for (G),

$$T(\vec{p},\vec{p'};z;\vec{P})\ =\ u(\vec{p}-\vec{p'})\ -\ \sum_n \frac{R_n(\vec{p},\vec{p'};\vec{P})}{S_n-z}$$

$$+\int \frac{d^3q}{(2\pi)^3}\frac{F(\vec{p},\vec{q};\vec{P})\left[Q(\vec{q},\vec{P})\ +\overline{Q}(\vec{q},\vec{P})\right]F^\dagger(\vec{p'},\vec{q};\vec{P})}{q^2-z}. \tag{2.193}$$

For (B), $\overline{Q}=0;$

For (A), $Q \longrightarrow\ Q-\overline{Q}.$

This is for n=0; but for n=1:

$$\frac{T(\vec{p},\vec{p'};z;\vec{P})}{z-p'^2-i\eta} = \frac{1}{2\pi i}\int_\Gamma dz' \frac{T(\vec{p},\vec{p'};z';\vec{P})}{(z'-z)(z'-p'^2-i\eta)} =$$

$$-\sum_n \frac{R_n(\vec{p},\vec{p'};\vec{P})}{(S_n-z)(S_n-p'^2-i\eta)} + \frac{1}{2\pi i}\int_0^\infty ds'\left[\frac{T(\vec{p},\vec{p'};s'+i\eta;\vec{P})}{s'-p'^2-i\eta} - \frac{T(\vec{p},\vec{p'};s'-i\eta;\vec{P})}{s'-p'^2+i\eta}\right]$$

$$\times\left(\frac{1}{s'-z}\right) + \frac{F(\vec{p},\vec{q};\vec{P})}{z-p'^2-i\eta}$$

$$= -\sum_n \frac{R_n(\vec{p},\vec{p'};\vec{P})}{(S_n-z)(S_n-p'^2-i\eta)} + \int \frac{d^3q}{(2\pi)^3}\frac{F(\vec{p},\vec{q};\vec{P})\left[Q(\vec{q},\vec{P})-\overline{Q}(\vec{q},\vec{P})\right]F^\dagger(\vec{p'},\vec{q};\vec{P})}{(q^2-z)(q^2-p'^2-i\eta)}$$

$$+\frac{F(\vec{p},\vec{q};\vec{P})}{z-p'^2-i\eta}. \qquad (2.194)$$

It follows that:

$$\frac{T(\vec{p},\vec{p'};z;\vec{P})-F(\vec{p},\vec{p'},\vec{P})}{z-p'^2-i\eta} = -\sum_n \frac{R_n(\vec{p},\vec{p'};\vec{P})}{(S_n-z)(S_n-p'^2-i\eta)}$$
$$+\int\frac{d^3q}{(2\pi)^3}\frac{F(\vec{p},\vec{q};\vec{P})\left[Q(\vec{q},\vec{P})\ +\overline{Q}(\vec{q},\vec{P})\right]F^{\dagger}(\vec{p'},\vec{q};\vec{P})}{(q^2-z)(q^2-p'^2-i\eta)}. \qquad (2.195)$$

2.5 Conclusion

The basic results of this Chapter are summarized in Eqs.(2.54, 2.76, 2.127, 2.184 and 2.196).

The first of these gives the explicit expression for the effective interaction which constitutes the central theme of this Thesis.

Equation (2.76) represents the basic application of this effective interaction, as embodied by the proper self–energy, from which the bulk and other properties of the system can be obtained according to standard recipes$\left[1,4\right]$.

For convenience, the problem has been formulated such that the general case is derived first [Eq.(2.127)]; whereas the other simpler (purely central) cases are obtained as especial cases at once by simply switching off the state–dependence.

Finally, Eqs.(2.184 and 2.196) can be considered as a sort of sum rules imposed on the effective interaction, as required by the underlying physics.

CHAPTER THREE

SOME SOUND PHENOMENA IN DILUTE NEUTRAL FERMI SYSTEMS

Chapter 3

Some Sound Phenomena in Dilute Neutral Fermi Systems

The purpose of this Chapter is to explore some aspects concerning the propagation of sound in dilute neutral Fermi systems. The subject will not be explored at length; rather it will be treated within a self-consistent framework that aims at a new understanding of the various sound modes and related phenomena in these systems. In Section 3.1, we shall present a general look at these modes and how they are related to such apparently disparate quantities as the static structure factor and the acoustic impedance, among others. In Section 3.2, we shall attempt to establish a connection between the macroscopic behavior of the system and its microscopic properties. The Chapter is concluded with a summary and discussion (Section 3.3).

3.1 Sound Phenomena in Fermi Systems

The significance of sound propagation in a many–body quantum system stems from the fact that it is intimately related to the excitation spectrum of the system [6, 38, 64 − 68]. This topic remains one of current interest in view of its implications as a potent probe of the properties of the system. Not only does it supply us with information about both the static and dynamic behaviour of the system [46, 67, 68 − 70]; but also it provides an indirect method to measure the correlations among its constituents and their distribution [71, 72]. Normal sound modes, as well as spin and isospin sound modes, are collective quantal modes of vibrations which are common features of Fermi systems [65, 73]. In the normal case, the total particle density varies with space and time periodically as a consequence of an external periodic field or internal fluctuation of the system (free oscillations), when an inhomogeneous particle distribution function exists, measured by the departure of the Fermi surface from its equilibrium value. Spin and isospin waves are collective modes of oscillations which represent a second kind of possible excitations for the Fermi system in terms of the associated density fluctuations, which exist even in the absence of an external perturbation [45, 68].

At zero temperature, T=0, the system is in its ground (vacuum) state. In this case collisions between excited quasiparticles play no role, thanks to the exclusion principle since no empty states exist. One is, therefore, in the collisionless regime where the mean time ν^{-1} is very large compared to the period of oscillation of the mode w^{-1}. The restoring force for these oscillations, in this regime, is the averaged self-consistent field of all quasiparticles which involves the coherent motion of the system as a whole in its own self-consistent field, governed by the global interaction between quasiparticles in response to the induced density fluctuations. In this regime, a well–defined mode of sound propagation, called by Landau zero sound, exists $\Big[6, 64, 74, 75\Big]$.

On the other hand, as T is raised, a number of the quasiparticles are excited within $k_B T$ of the Fermi surface. Collisions become sufficiently frequent, so that the mean free path is small and the relaxation collision time is short compared to the period of oscillation; i.e,

$$w \quad \ll \quad \nu. \tag{3.1}$$

In this hydrodynamic (collision–dominated) regime, the system will sustain another mode of collective density oscillation – namely, the hydrodynamic ordinary or first sound. The collisions act to bring a state of local thermodynamic equilibrium – a necessary condition that provides the restoring force for the system to construct a well defined first sound; at the same time these collisions act to disrupt thoroughly a zero-sound mode $\Big[45, 64, 74\Big]$.

At low T, a propitious starting point is Landau's transport equation that governs the flow of quasiparticles in the system $\Big[\,6\,\Big]$:

$$\begin{aligned} \frac{\partial \delta n_{\vec{k}}(\vec{r},t)}{\partial t} + \vec{v}_{\vec{k}} . \bigtriangledown_{\vec{r}} \left[\delta n_{\vec{k}}(\vec{r},t) - \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} \sum_{\vec{k}'\vec{\sigma}'} f(\vec{k}\vec{\sigma},\vec{k}'\vec{\sigma}') \delta n_{\vec{k}'}(\vec{r},t) \right] \\ + \vec{F}_{\vec{k}} . \vec{v}_{\vec{k}} \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} = I(\delta n_{\vec{k}}(\vec{r},t)). \end{aligned} \tag{3.2}$$

Here $\delta n_{\vec{k}}(\vec{r},t) = n^o_{\vec{k}}(\vec{r},t) - n_{\vec{k}}(\vec{r},t)$,which measures the departure of the quasiparticle–distribution function in the state $\vec{k}\vec{\sigma}$, $n_{\vec{k}}(\vec{r},t)$, from its ground–state (equilibrium) value, $n^o_{\vec{k}}(\vec{r},t)$, as a result of the density fluctuations due to an applied external field, or to an internal field generated when the average force on the quasiparticle due to the others is not zero, the latter field corresponding to the free oscillations in the system and can be approximated by a polarization potential [68]; $\vec{v}_{\vec{k}}$ is the velocity of the quasiparticle of wavevector $\vec{k}$; $f(\vec{k}\vec{\sigma},\vec{k}'\vec{\sigma}')$ is the effective interaction, called Landau's f–function, between two–excited quasiparticles; actually it is the quasiparticle–quasihole (qp–qh) interaction in the long–wavelength limit, in momentum space, as we shall see in Section 3.2 [76, 77]; $F_{\vec{k}}$ is the force felt by the quasiparticle; in fact, it is what the bare particle feels, under the application of the external potential; and $I(\delta n_{\vec{k}})$ is a collision integral taking into account the change of $\delta n_{\vec{k}}$ due to collisions. Consider the response of the system to a scalar potential that is periodic in space and time:

$$\phi(\vec{r},t) = \phi(\vec{q},w)e^{i(\vec{q}.\vec{r}-wt)}, \tag{3.3}$$

which produces a force

$$\vec{F}_{\vec{k}} = -i\vec{q}\phi(\vec{q},w)e^{i(\vec{q}.\vec{r}-wt)}. \tag{3.4}$$

The system will respond at the same wavevector and frequency; therefore,

$$\delta n_{\vec{k}}(\vec{r},t) = \delta n_{\vec{k}}(\vec{q},w)e^{i(\vec{q}.\vec{r}-w)}, \tag{3.5}$$

where $\phi(\vec{q},w)$ and $\delta n_{\vec{k}}(\vec{q},w)$ are the Fourier components of the $\phi(\vec{r},t)$ and $\delta n_{\vec{k}}(\vec{r},t)$, respectively.

Take a given Fourier component of Eq.(3.2):

$$(\vec{q}.\vec{v}_{\vec{k}} - w)\delta n_{\vec{k}} - \vec{q}.\vec{v}_{\vec{k}} \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} \sum_{\vec{k}'\vec{\sigma}} f(\vec{k}\vec{\sigma}, \vec{k}'\vec{\sigma}')\delta n_{\vec{k}'} = \vec{q}.\vec{v}_{\vec{k}}\ \phi(\vec{q},w) \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}}. \qquad (3.6)$$

At low T, both $\vec{k}$ and $\vec{k}'$ are at the Fermi surface where the quasiparticle concept is well–defined. Hence, the f–function is only a function of the angle between $\vec{k}$ and $\vec{k}'$ and can be expanded in Legendre polynomials. The variation $\delta n_{\vec{k}}$ is determined by its value at the Fermi surface in the direction of $\vec{k}$. We may write, suppressing the spin dependence of f for the moment [73],

$$\begin{aligned} f(\vec{k},\vec{k}') &= \sum_L f_L P_L(\hat{k}.\hat{k}') = \sum_{LM_L} \left(\frac{4\pi}{2L+1}\right) f_L \mathcal{Y}_{LM_L}(\hat{k}) \mathcal{Y}_{LM_L}(\hat{k}'); \\ \delta n_{\vec{k}}(\vec{q},w) &= -\mathcal{V}(\hat{k},\vec{q},w)\phi(\vec{q},w)\frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}}. \end{aligned} \qquad (3.7)$$

The function $\mathcal{V}(\hat{k},\vec{q},w)$ measures the displacement of the Fermi surface in the direction of $\hat{k}$. Taking $\vec{q}$ as the polar axis, and letting

$$\frac{\vec{q}.\vec{v}_{\vec{k}}}{qv_F} = \hat{q}.\hat{k} = \cos\theta_k \equiv x; \qquad \frac{w}{qv_F} \equiv s, \qquad (3.8)$$

where $\vec{q}.\vec{v}_{\vec{k}}$ is the velocity of the quasiparticle of wavevector $\vec{k}$ in the direction of $\vec{q}$ and $\vec{v}_{\vec{k}} = \hbar\vec{k}/m^* = \hbar k_F\hat{k}/m^*$; so that s is the ratio of the propagation velocity

$u = w/q$ to the velocity of the quasiparticle at the Fermi surface v_F, we obtain

$$(s-x)\mathcal{V}(\hat{k},\vec{q},w) + x\sum_{\vec{k}'\sigma'} f(\vec{k},\vec{k}')\mathcal{V}(\hat{k}',\vec{q},w)\frac{\partial n^o_{\vec{k}'}}{\partial \epsilon_{\vec{k}'}} = x, \tag{3.9}$$

and

$$\begin{aligned}\sum_{\vec{k}'\vec{\sigma}'} f(\vec{k},\vec{k}')\mathcal{V}(\hat{k}',\vec{q},w)\frac{\partial n^o_{\vec{k}'}}{\partial \epsilon_{\vec{k}'}} &= \frac{-g\Omega}{(2\pi)^3}\int 4\pi k'^2 \frac{dk'}{d\epsilon_{\vec{k}'}} d\epsilon_{\vec{k}'}\delta(\epsilon_{\vec{k}'} - \epsilon_F)\mathcal{V}(\hat{k}',\vec{q},w) \\ &\times \sum_{LM_L} \frac{f_L}{2L+1}\mathcal{Y}_{LM_L}(\hat{k})\mathcal{Y}_{LM_L}(\hat{k}')d\hat{k} \\ &= -N(0)\int \sum_{LM_L} \frac{f_L}{2L+1}\mathcal{Y}_{LM_L}(\hat{k})\mathcal{Y}_{LM_L}(\hat{k}')\mathcal{V}(\hat{k}',\vec{q},w)d\hat{k}',\end{aligned} \tag{3.10}$$

g being the degeneracy of the momentum state, Ω the volume of the system, and

$$N(0) \equiv \left[\frac{dn}{d\epsilon_{\vec{k}}}\right]_{\epsilon_F} = \frac{g\Omega\, m^* k_F}{2\pi^2\hbar^2} \tag{3.11}$$

being the density of states at the Fermi surface. Equation (3.9), then, takes the form

$$\mathcal{V}(\hat{k},\vec{q},w) = \frac{x}{s-x}\left[1 + \int \sum_{LM_L} \frac{F_L}{2L+1}\mathcal{Y}_{LM_L}(\hat{k})\mathcal{Y}_{LM_L}(\hat{k}')\mathcal{V}(\hat{k}',\vec{q},w)d\hat{k}'\right], \tag{3.12}$$

where $F_L = N(0)\, f_L$ is the dimensionless Landau parameter.

It should be remarked here that the disturbances produced in the liquid, which travel in the form of density, spin–density, and isospin–density excitations, may interact with the quasiparticles in the medium and excite qp–qh pairs, primarily single qp–qh pairs in the dilute case, through the exchange of virtual quanta (phonons) which can carry momentum up to $2k_F$. This interaction corresponds to the induced part of the quasiparticle interaction which is reduced to the qp–qh interaction itself in the long–wavelength limit $\left[67, 76, 77\right]$. To generalize Eq. (3.12) and include values of $\vec{q}$ up to $2k_F$, we notice that

$$\frac{x}{s-x}\frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} = \frac{\vec{q}.\vec{v}_{\vec{k}}}{w-\vec{q}.\vec{v}_{\vec{k}}}\frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} = \lim_{q\to 0}\frac{n_{\vec{k}+\vec{q}}-n_{\vec{k}}}{w-w_{\vec{k}\vec{q}}+i\eta}$$
$$= \lim_{q\to 0} \propto(\vec{k},\vec{q}), \tag{3.13}$$

where

$$w_{\vec{k}\vec{q}} \equiv \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}} = \frac{\hbar^2}{m^*}\left[\frac{q^2}{2}+\vec{k}.\vec{q}\right]$$

is the excitation energy of the qp–qh pair, $n_{\vec{k}}$ is the occupation number of the state $\vec{k}$, and

$$\begin{aligned} n_{\vec{k}+\vec{q}} &= n_{\vec{k}} + \vec{q}.\nabla_{\vec{k}} n_{\vec{k}} + ... \\ &= n_{\vec{k}} + \vec{q}.\nabla_{\vec{k}}\epsilon_{\vec{k}}\left(\frac{\partial n_{\vec{k}}}{\partial \epsilon_{\vec{k}}}\right) + ... \\ &= n_{\vec{k}} + \vec{q}.\hbar\vec{v}_{\vec{k}}\frac{\partial n_{\vec{k}}}{\partial \epsilon_{\vec{k}}} + ... \\ &= n_{\vec{k}} + \vec{q}.\vec{v}_{\vec{k}}\frac{\partial n_{\vec{k}}}{\partial \epsilon_{\vec{k}}} + ... \, . \end{aligned} \tag{3.14}$$

Thus, we write Eq.(3.12) in the form

$$\frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}}\mathcal{V}(\hat{k},\vec{q},w) = \propto(\vec{k},\vec{q})\left[1+\int\sum_{LM_L}\frac{F_L}{2L+1}\mathcal{Y}_{LM_L}(\hat{k})\mathcal{Y}_{LM_L}(\hat{k}')\mathcal{V}(\hat{k}',\vec{q},w)d\hat{k}'\right]. \tag{3.15}$$

Let us confine our attention, for the moment, to the long–wavelength limit. Usually $\mathcal{V}(\hat{k},\vec{q},w)$ is separated into two components, the spin symmetric and asymmetric modes, such that $\Big[64, 75\Big]$

$$\mathcal{V}^S = \frac{1}{2}(\mathcal{V}_\uparrow + \mathcal{V}_\downarrow)\,; \qquad\qquad \mathcal{V}^A = \frac{1}{2}(\mathcal{V}_\uparrow - \mathcal{V}_\downarrow), \tag{3.16}$$

where S(A) and ↑(↓) stand for symmetric (asymmetric) and spin up(down) situations, respectively. In $\mathcal{V}^S$ modes, opposite–spin orientations oscillate in phase, whereas in $\mathcal{V}^A$ they oscillate out of phase. This is closely related to the quasiparticle interactions (Section 3.2). The amplitude $\mathcal{V}$ can be written as a sum of multipoles; i.e. $\Big[78\Big]$,

$$\mathcal{V}(\hat{k},\vec{q},w) = \sum_{LM_L} \mathcal{V}_{LM_L}(\vec{q},w)\mathcal{Y}_{LM_L}(\hat{k}). \tag{3.17}$$

It follows that

$$\sum_{L'M_{L'}}\left[\mathcal{V}_{L'M'_L}\mathcal{Y}_{L'M'_L}(\hat{k})(s-x) + x\int\sum_{L'M_{L'}}\frac{F_{L'}}{2L'+1}\mathcal{Y}_{L'M'_L}(\hat{k})\mathcal{Y}_{L'M'_L}(\hat{k}')\mathcal{V}_{LM_L}\mathcal{Y}_{LM_L}(\hat{k}')d\hat{k}'\right] = x;$$
$$\sum_{L'M_{L'}}\left[\mathcal{V}_{L'M'_L}\mathcal{Y}_{L'M'_L}(\hat{k})(s-x) + x\frac{F_{L'}}{2L'+1}\mathcal{V}_{L'M'_L}\mathcal{Y}_{L'M'_L}(\hat{k})\right] = x. \tag{3.18}$$

Multiplication by $\mathcal{Y}^*_{LM_L}(\hat{k})$ and integration over $d\hat{k}$ yield:

$$s\mathcal{V}_{LM_L} - \sum_{L'M'_L} < LM_L|x|L'M'_L > \mathcal{V}_{L'M'_L}\left[1 + \frac{F_{L'}}{2L'+1}\right] = \int x\,\mathcal{Y}^*_{LM_L}(\hat{k})d\hat{k}, \tag{3.19}$$

where

$$< LM_L|x|L'M'_L > \;=\; \int \mathcal{Y}^*_{LM_L}(\hat{k})\left(\hat{q}.\hat{k}\right)\mathcal{Y}_{L'M'_L}(\hat{k})d\hat{k}. \tag{3.20}$$

On using [49]

$$\cos\theta_k \mathcal{Y}_{L'M_L{}'}(\hat{k}) = \left[\frac{(L'-M'_L)(L'+M'_L)}{(2L'-1)(2L'+1)}\right]^{1/2}\mathcal{Y}_{L'\,-\,1\,M_{L'}}(\hat{k}) + \left[\frac{(L'-M'_L+1)(L'+M'_L+1)}{(2L'+1)(2L'+3)}\right]^{1/2}\mathcal{Y}_{L'\,+\,1\,M_{L'}}(\hat{k}), \tag{3.21}$$

$$< LM_L|\cos\theta_k|L'M'_L >= \left[\frac{(L'-M'_L)(L'+M'_L)}{(2L'-1)(2L'+1)}\right]^{1/2}\delta_{L\,L'\,-\,1M_LM'_L} + \left[\frac{L'-M'_L+1)(L'+M'_L+1)}{(2L'+1)(2L'+3)}\right]^{1/2}\delta_{L\,L'\,+\,1M_LM'_L}, \tag{3.22}$$

we have

$$s\mathcal{V}_{LM_L} - \mathcal{V}_{L\,+\,1\,M_L}G_{L\,+\,1}\left[\frac{(L-M_L+1)(L+M_L+1)}{(2L+1)(2L+3)}\right]^{1/2} - \mathcal{V}_{L\,-\,1\,M_L}G_{L\,-\,1} \times\left[\frac{(L-M_L)(L+M_L)}{(2L-1)(2L+1)}\right]^{1/2} = \sqrt{\frac{4\pi}{3}}\delta_{L1\,M_L0}, \tag{3.23}$$

where the right term is obtained using $x \equiv \cos\theta_k = \sqrt{\frac{4\pi}{3}}\mathcal{Y}_{10}(\hat{k})$. For free oscillations we put $\phi = 0$, and

$$\delta n_{\vec{k}}(\vec{q}, w) \;=\; -\mathcal{V}(\hat{k}, \vec{q}, w)\frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}}, \tag{3.24}$$

to get

$$s\mathcal{V}_{LM_L} - \mathcal{V}_{L+1\,M_L} G_{L+1} \left[\frac{(L - M_L + 1)(L + M_L + 1)}{(2L+1)(2L+3)} \right]^{1/2}$$
$$- \mathcal{V}_{L-1\,M_L} G_{L-1} \left[\frac{(L - M_L)(L + M_L)}{(2L-1)(2L+1)} \right]^{1/2} = 0, \qquad (3.25)$$

where

$$G_L = 1 + \frac{F_L}{2L+1}. \qquad (3.26)$$

This result defines an infinite set of linear equations for the coupled amplitudes $\mathcal{V}_{LM_L}$ grouped according to the M_L quantum number, where different values of M_L are completely decoupled but each is a mixture of L's. For $M_L = 0$ we have the longitudinal mode; For $M_L = 1$ we have the transverse mode; and so forth. Amongst these is the longitudinal symmetric mode which involves the particle–density fluctuations (zero sound), and the longitudinal asymmetric mode (zero–spin waves) which involves spin–density oscillations. It should be noted that the zero sound mode amplitude $\mathcal{V}^S(\hat{k}, \vec{q}, w)$ is the full solution of Eq.(3.12). Many harmonics contribute to it; whereas only a finite number of the multipoles are considered in the other modes since higher–order multipole distortions undergo a rapid relaxation through collisions (collision damping) $\left[6, 73\right]$.

For zero sound ($M_L = 0, \mathcal{V} = \mathcal{V}^S$), Eq.(3.12) takes the form

$$(s - x)\mathcal{V}^S(x, s) = x + \frac{x}{2} \int_{-1}^{1} F_L^S P_L(x) P_L(y) \mathcal{V}^S(y, s) dy, \qquad (3.27)$$

where $x = \vec{q}.\hat{k}$, $y = \vec{q}.\hat{k}'$; so that

$$\mathcal{V}^S(x,s) = \frac{x}{s-x}\left[1 + \frac{1}{2}\int_{-1}^{1} R^S(x,y)\mathcal{V}^S(y,s)dy\right]. \tag{3.28}$$

Consider the simple case $F_L^S = F_o^S$; then

$$\mathcal{V}^S(x,s) = \frac{x}{s-x} + A\frac{x}{s-x} = C\frac{x}{s-x}, \tag{3.29}$$

A and C being constants. Substitution back in Eq.(3.28) gives

$$\frac{1}{F_o^S} = \frac{1}{2}s\ln\frac{s+1}{s-1} - 1 = Q_1(s), \tag{3.30}$$

where $Q_1(s)$ is (first–order) Legendre's polynomial of the second kind, and the speed of zero sound is given by $u_o = sv_F$. A real solution exists so long as $s > 1$ or $u > v_F$. Since $Q_1(s)$ is an increasing function of s, it is always > 0 [79]; F_o must, then, be greater than zero, which ensures that the interaction is repulsive. On the other hand, when $x = s$, $\mathcal{V}$ is singular; this gives rise to Landau's damping of the zero sound through the excitation of the qp–qh pairs, which manifests itself as a peak in the dynamic structure factor [67].

Consider now the other regime, the hydrodynamic, where collisional damping acts to disturb the mean field picture, and hence zero sound; while it represents the restoring force for first sound to propagate. Here, high–order multiple distortion undergoes collisional damping through collisions which maintain such disturbance at a negligibly small value. Thus, $\mathcal{V}_{LM_L}$ contains only the $L = 0, 1$ components.

These two terms conserve the total number of particles as well as the total energy and momentum; but they contribute nothing to $I(\delta n_{\vec{k}})$, which makes the truncated set $L \leq 1$ converge toward the exact solution [78]. From Eq.(3.25), we have

$$
\begin{aligned}
L = 0, &\quad s\mathcal{V}^S_{oo} - G^S_1\sqrt{\tfrac{1}{3}}\mathcal{V}^S_{1o} = 0; \\
L = 1, &\quad s\mathcal{V}^S_{1o} - G^S_o\sqrt{\tfrac{1}{3}}\mathcal{V}^S_{oo} = 0.
\end{aligned}
\tag{3.31}
$$

Using Eqs.(3.26) and (3.31), it follows that

$$
s^2 = \frac{1}{3}G^S_oG^S_1 \;=\; \frac{1}{3}\left[1 + F^S_o\right]\left[1 + \frac{F^S_1}{3}\right]. \tag{3.32}
$$

But

$$
1 + \frac{F^S_1}{3} \;=\; \frac{m^*}{m}. \tag{3.33}
$$

Thus,

$$
u_1^2 \;=\; \frac{P_F^2}{3m^*m}\left[1 + F^S_o\right] = \frac{k_F^2}{3m^*m}\left[1 + F^S_o\right], \tag{3.34}
$$

recalling that this is the usual result obtained from the compressibility definition of first sound:

$$
u_1^2 \;=\; \frac{1}{\kappa\rho_m} = -\frac{\Omega}{\rho_m}\left(\frac{\partial P}{\partial \Omega}\right)_S = \left(\frac{\partial P}{\partial \rho_m}\right)_S, \tag{3.35}
$$

where P is the pressure, $\rho_m \equiv m\frac{N}{\Omega}$ is the mass density, S is the entropy, and u_1 is the first–sound velocity with which pressure waves (particle–density oscillations) travel in Fermi systems.

The first experimental observation of sound velocities in ^{3}He was made by Abel *et al.* $\left[74, 80\right]$. The measurements corresponded to an average first–sound velocity of 187.9 m/s and an average zero–sound velocity of 194.4 m/s for a pressure of 0.32 atm, at frequencies 15.4 and 45.5 Hz, and in the temperature range 0.002-0.1 K.

If this is the case for a pure system of ^{3}He quasiparticles, the introduction of a small quantity of ^{3}He atoms into He II modifies the first-sound velocity due to the interaction of ^{3}He with ^{4}He and the decrease in the ^{4}He mass density by the amount $m_3^* - m_3\rho_3(\vec{r})$ $\left[81\right]$. A general formula for the velocity of first sound has been derived by Khalatnikov, to first order in the ^{3}He concentration x ($x = \frac{N_3}{N}$, where N_3 and N are, respectively, the number of ^{3}He atoms and the total number of atoms per unit volume), by thermodynamic and Galilean invariance, in the low–frequency limit; this formula is given by $\left[14, 82, 83\right]$

$$u_1^2 = [1+d]\,\frac{\partial P}{\partial \rho(x)}\,, \tag{3.36}$$

where

$$d = \frac{\rho_s}{\rho_n}\left(\frac{x}{\rho(x)}\frac{\partial \rho(x)}{\partial x}\right)^2 = \frac{\rho_s}{\rho_n}\left(\frac{\partial \ln \rho(x)}{\partial \ln x}\right)^2 .$$

Here $\rho(x)$ is the overall concentration–dependent mass density; ρ_s and ρ_n are the superfluid and normal ^{4}He densities, respectively:

$$\rho_n \simeq \rho(x)\left(\frac{m_3^*}{m_4}\right) x\,;$$

$$\rho(x) = \left[x m_3 + (1-x) m_4\right] / \Omega_4 (1+\alpha x)\,,$$

$\Omega_4(1+\alpha x)$ representing the molar volume of the solution as related to that of pure ^{4}He. In the limit $x \to 0$, $d \to 0$, Eq.(3.36) reduces to the expression for the first–sound velocity in pure ^{4}He, as it should.

On the other hand, second sound is analogous to first sound in ^{3}He where the effective interactions are strong. In the hydrodynamic limit a rather complicated equation, which also depends on thermodynamics and Galilean invariance, was given by Khalatnikov as

$$m_4 u_2^2 = (1 - f\xi)\left[-\left(\frac{\partial \mu_4}{\partial \ln \xi}\right)_{TP} + \left(\frac{\xi T}{N_3 C_P}\right)\left(\frac{\partial S}{\partial \ln \xi}\right)_{TP}^2\right]\left[\frac{\rho_n}{\rho_s} + f^2\xi^2\right]^{-1}, \qquad (3.37)$$

where $\xi = n_3\Omega_{40} = x/(1+\alpha x)$; $f = 1 + \alpha - m_3/m_4 \simeq 0.53$ at $P = 0$; N_3, C_P are the number of ^{3}He atoms and heat capacity at constant pressure; and μ_4 is the chemical potential of ^{4}He.

For low concentrations ($\xi \ll 1$) and, writing $\mu_4(PTx) = \mu_4(PT0) - \pi\Omega_{40}(P)$, where π is the osmotic pressure of the solution, Eq.(3.37) can be transformed to

$$\rho_n u_2^2 \;\cong\; \left(\frac{\partial \pi}{\partial \ln \xi}\right)_{SP}. \qquad (3.38)$$

This last equation demonstrates the analogy between second sound in the mixture and first (ordinary) sound in the quasiparticle-gas. The derivative $\partial\pi/\partial \ln \xi$ can be regarded as the "osmotic bulk modulus" of the mixture $\Big[14, 84, 85\Big]$.

In this discussion it is useful to consider the density response to the density fluctuation operator induced by ϕ_{ext} or the polarization potential. In the linear response, the change in the density $\delta\rho(\vec{q},w)$ is proportional to ϕ_{ext} according to

$$\delta < \rho(\vec{q},w) > \;=\; \chi(\vec{q},w)\,\phi_{ext}(\vec{q},w), \tag{3.39}$$

where $\chi(\vec{q},w)$ is the density-density response function. The importance of this quantity stems from the fact that it is related to the dynamic structure factor, which is a direct measure of the real transitions of the system and, hence, the static structure factor which , in turn, provides a direct measure of the correlations between positions and momenta. These two quantities can be easily extracted using neutron and X-ray scattering $\left[6, 80\right]$.

The calculation of $\chi(\vec{q},w)$ requires the calculation of $\delta\rho$:

$$\begin{aligned}
\delta < \rho(\vec{r},t) > &= \sum_{\vec{k}\vec{\sigma}} \delta n_{\vec{k}}(\vec{r},t) \\
&= \sum_{\vec{k}\vec{\sigma}} \delta n_{\vec{k}}(\vec{q},w) e^{i(\vec{q}.\vec{r}-wt)}, \\
&= \delta < \rho(\vec{q},w) > e^{i(\vec{q}.\vec{r}-wt)}; \\
\delta < \rho(\vec{q},w) > &= \sum_{\vec{k}\vec{\sigma}} \delta n_{\vec{k}}(\vec{q},w) \\
&= -\sum_{\vec{k}\vec{\sigma}} \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} \mathcal{V}^S(\hat{k},\vec{q},w)\phi_{ext}(\vec{q},w). \qquad (3.40)
\end{aligned}$$

From Eqs. (3.15) and (3.39):

$$\chi(\vec{q},w) \;=\; -\sum_{\vec{k}\vec{\sigma}} \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} \mathcal{V}^S(\hat{k},\vec{q},w)$$

$$= -\sum_{\vec{k}\vec{\sigma}} \propto(\vec{k},\vec{q}) \left[1 + \int \sum_{LM_L} \frac{F_L}{2L+1} \mathcal{Y}_{LM_L}(\hat{k}) \mathcal{Y}_{LM_L}(\hat{k}') \mathcal{V}^S(\hat{k}',\vec{q},w) d\hat{k}'\right]. \quad (3.41)$$

Particle–density fluctuations involve only the $M_L = 0$ component, which is isotropic in the plane perpendicular to $\hat{k}$. It emerges that

$$\begin{aligned} \mathcal{V}(\hat{k},\vec{q},w) &= \mathcal{V}^S(\hat{k},\vec{q},w) = \sum_{LM_L} \mathcal{V}^S_{LM_L}(\vec{q},w) \mathcal{Y}_{LM_L}(\hat{k})\,; \\ \mathcal{V}^S(x,w) &= \sum_L \mathcal{V}^S_{Lo} \mathcal{Y}_{L0}(\hat{k}.\hat{q}) = \sum_L \mathcal{V}^S_{Lo} \mathcal{Y}_{L0}(x). \end{aligned} \quad (3.42)$$

From this we note that

$$\begin{aligned} \chi(\vec{q},w) &= -\sum_{\vec{k}\vec{\sigma}} \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} \mathcal{V}^S(\hat{k},\vec{q},w) \\ &= N(0) \int \sum_L \mathcal{V}^S_{Lo} \mathcal{Y}_{L0}(\hat{k}) d\hat{k}/4\pi \\ &= \frac{N(0)}{\sqrt{4\pi}} \mathcal{V}_{oo}; \end{aligned} \quad (3.43)$$

and

$$\begin{aligned} \chi(\vec{q},w) &= -\sum_{\vec{k}\vec{\sigma}} \propto(\vec{k},\vec{q}) \left[1 + \int \sum_{Ll} \frac{F_L}{2L+1} \mathcal{Y}_{LM_L}(\hat{k}) \mathcal{Y}_{LM_L}(\hat{k}') \mathcal{V}^S_{lo} \mathcal{Y}_{lo}(\hat{k}') d\hat{k}'\right] \\ &= -\sum_{\vec{k}\vec{\sigma}} \propto(\vec{k},\vec{q}) \left[1 + \sum_L \frac{F_L}{2L+1} \mathcal{Y}_{Lo}(\hat{k}) \mathcal{V}^S_{Lo}\right]. \end{aligned} \quad (3.44)$$

In the long–wavelength limit:

$$\lim_{q\to 0} \chi(\vec{q},w) = -\sum_{\vec{k}\vec{\sigma}} \frac{x}{s-x} \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} \left[1 + \sum_L \frac{F_L}{2L+1} \mathcal{Y}_{Lo}(\hat{k}) \mathcal{V}^S_{Lo}\right]$$

$$
\begin{aligned}
&= \left[N(0)\frac{1}{2}\int \frac{x}{s-x}\,dx\left(1+\sum_{L}\frac{F_{L}}{2L+1}\mathcal{Y}_{Lo}(x)\mathcal{V}^{S}_{Lo}\right)\right] \\
&= \left[N(0)\left(Q_1(s)+\frac{1}{2}\int \frac{x}{s-x}\,dx\sum_{L}\frac{F_{L}}{2L+1}\mathcal{Y}_{Lo}(x)\mathcal{V}^{S}_{Lo}\right)\right]. \qquad (3.45)
\end{aligned}
$$

Using [49]

$$
\begin{aligned}
\mathcal{Y}_{Lo}(x) &= \left(\frac{2L+1}{4\pi}\right)^{1/2} P_{L}(x); \\
x\,P_{L}(x) &= \frac{L+1}{2L+1}P_{L+1}(x)+\frac{L}{2L+1}P_{L-1}(x), \qquad (3.46)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\lim_{q\to 0}\chi(\vec{q},w) = N(0)\Bigg(Q_1(s)+\frac{1}{2}\int\sum_{L}\frac{F_{L}}{(4\pi)^{1/2}}\mathcal{V}^{S}_{Lo}\Bigg[&\frac{L+1}{(2L+1)^{3/2}}\frac{P_{L+1}(x)}{s-x} \\
&+\frac{L}{(2L+1)^{3/2}}\frac{P_{L-1}(x)}{s-x}\Bigg]dx\Bigg) \\
= \left(\psi_1(s)+\sum_{L}\frac{F_{L}}{(4\pi)^{1/2}}\mathcal{V}^{S}_{Lo}\left[\frac{L+1}{(2L+1)^{3/2}}\psi_{L+1}(s)+\frac{L}{(2L+1)^{3/2}}\psi_{L-1}(s)\right]\right), \qquad (3.47)
\end{aligned}
$$

where [49]

$$
\begin{aligned}
\psi_{L}(s) &= N(0)Q_{L}(s)\,; \\
Q_L(s) &= \frac{1}{2}\int\frac{P_{L}(x)}{s-x}\,dx\,; \\
Q_1(s) &= \frac{s}{2}\ln\frac{s+1}{s-1}-1\,. \qquad (3.48)
\end{aligned}
$$

This enables us to calculate the response function to all orders in L. Assume that only $F_o \neq 0$; then

$$\lim_{q\to 0} \chi(\vec{q}, w) = \psi_1(s) + \frac{F_o}{(4\pi)^{1/2}} \mathcal{V}^S_{oo} N(0) Q_1(s). \tag{3.49}$$

From Eq.(3.43):

$$\begin{aligned} \lim_{q\to 0} \chi(\vec{q}, w) &= \frac{\psi_1(s)}{1 + F_o Q_1(s)}; \\ \lim_{q\to 0} \chi^0(\vec{q}, w) &= \psi_1(s), \end{aligned} \tag{3.50}$$

where the superscript (0) applies for the noninteracting response function, obtained by putting $F^S_L = 0$. From Eq. (3.41):

$$\chi^0(\vec{q}, w) = -\sum_{\vec{k}\vec{\sigma}} \propto^0(\vec{k}, \vec{q}) = -\sum_{\vec{k}\vec{\sigma}} \frac{n^o_{\vec{k}+\vec{q}} - n^o_{\vec{k}}}{w - w_{\vec{k}\vec{q}} + i\eta}; \tag{3.51}$$

$$\begin{aligned} \chi^0(q) &= \sum_{\vec{k}\vec{\sigma}} \frac{n^o_{\vec{k}+\vec{q}} - n^o_{\vec{k}}}{\epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}} + i\eta} \\ &= \frac{3n}{4\epsilon_F} \left(\frac{1}{2} + \frac{4k_F^2 - q^2}{8k_F q} \ln \left| \frac{2k_F + q}{2k_F - q} \right| \right), \end{aligned} \tag{3.52}$$

which gives the familiar Lindhard static function, n being the number density :

$$n = \frac{N}{\Omega} \frac{g}{2} \frac{k_F^3}{3\pi^2}, \tag{3.53}$$

and ϵ_F the Fermi energy such that $N(0) = 3n/2\epsilon_F$.

The dynamic structure factor is obtained from the imaginary part of the response function [80]:

$$
\begin{aligned}
S(\vec{q}, w) &= \frac{-Im\, \chi^0(\vec{q}, w)}{\pi} \\
\lim_{q \to 0} S(\vec{q}, w) &= \frac{1}{\pi} \sum_{\vec{k}\vec{\sigma}} Im\, \mathcal{V}(\hat{k}, \vec{q}, w + i\eta) \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}};
\end{aligned}
\tag{3.54}
$$

for longitudinal oscillations:

$$
\begin{aligned}
\lim_{q \to 0} S(\vec{q}, w) &= \frac{1}{\pi} \sum_{\vec{k}\vec{\sigma}} Im\, \mathcal{V}^S(x, w + i\eta) \frac{\partial n^o_{\vec{k}}}{\partial \epsilon_{\vec{k}}} \\
&= -\frac{1}{\pi} \frac{N(0)}{2} \int_{-1}^{1} dx Im\, \mathcal{V}^S(x, w + i\eta).
\end{aligned}
\tag{3.55}
$$

The static structure factor, therefore, is given by

$$
\begin{aligned}
N \lim_{q \to 0} S(\vec{q}) &= \frac{-1}{\pi} \frac{N(0)}{2} \int dw \int_{-1}^{1} dx\, Im\, \mathcal{V}^S(x, w + i\eta) \\
&= \frac{-qv_F}{\pi} \frac{N(0)}{2} \int ds \int_{-1}^{1} dx\, Im\, \mathcal{V}^S(x, s + i\eta)\,.
\end{aligned}
\tag{3.56}
$$

With Eqs. (3.11) and (3.53):

$$
\lim_{q \to 0} S(q) = \frac{3}{2} \beta \frac{q}{2k_F}\,, \tag{3.57}
$$

where

$$
\beta = \frac{-2}{\pi} \int ds \int_{-1}^{1} dx\, Im\, \mathcal{V}^S(x, w + i\eta). \tag{3.58}
$$

Equation (3.57) represents an integral equation which can be solved numerically for any order in F_L; for the noninteracting case, $F_L = 0$, we get

$$\begin{aligned} \beta &= \frac{-2}{\pi} \int ds \int_{-1}^{1} dx \, \pi x \delta(s-x) \\ &= 2 \int s \, ds \; = 1, \qquad\qquad 0 < s < 1. \end{aligned} \tag{3.59}$$

The results (3.57) and (3.59) have also been obtained from entirely different considerations [80, 86].

Of particular interest is the induced particle density:

$$\begin{aligned} \delta < \rho(\vec{r}, t) > &= \sum_{\vec{q}\vec{\sigma}} \delta < \rho(\vec{q}, w) > e^{i(\vec{q}.\vec{r} - wt)}, \\ &= \sum_{\vec{q}\vec{\sigma}} \chi(\vec{q}, w) \phi_{ext}(\vec{q}, w) e^{i(\vec{q}.\vec{r} - wt)}. \end{aligned} \tag{3.60}$$

$\chi(\vec{q}, w)$ represents the response to any scalar potential which does not depend on $(\vec{q}, w)$; for simplicity we take

$$\phi(\vec{r}, t) = \phi_o \delta(t), \tag{3.61}$$

which corresponds to an impulsive perturbation. Further, if we approximate $\chi(\vec{q}, w)$ by $\chi^0(\vec{q}, w)$, and then consider the static case, we get

$$\begin{aligned} \delta < \rho(\vec{r}) > &= \phi_o \sum_{\vec{q}\vec{\sigma}} \chi^0(q) e^{i(\vec{q}.\vec{r})} \\ &= \phi_o \frac{g\Omega}{(2\pi)^2} \int 4\pi q^2 dq \chi^0(q) e^{i(qr\cos\theta_{\vec{r}})} d\hat{q}/4\pi. \end{aligned} \tag{3.62}$$

By simple analogy, this integral is similar to that obtained by Ghassib and Baskaran in a different context [87]. We borrow the result and write

$$\delta < \rho(\vec{r}) >= \frac{g\Omega}{2}\frac{3n\phi_o}{2^5\pi\epsilon_F k_F}\left(\frac{\sin Qr}{(Qr)^4} - \frac{\cos Qr}{(Qr)^3}\right); \qquad Q = 2k_F. \tag{3.63}$$

From the above equation we see that the induced density does not go to zero exponentially; rather it oscillates at large values of r, and behaves in the $r \to \infty$ limit as

$$\lim_{r\to\infty} \delta < \rho(\vec{r}) >\sim \frac{\cos 2k_F r}{(2k_F)^3}. \tag{3.64}$$

To see how this long–range oscillations come from, we remark that

$$\left(\frac{\partial \chi^0(q)}{\partial x}\right)_{x=1} \to \infty; \qquad x = q/2k_F. \tag{3.65}$$

This logarithmic singularity is not difficult to trace: It is a consequence of the sharpness of the Fermi surface, and is a direct reflection of the discontinuity in the momentum distribution function at $\epsilon = \epsilon_F$ [71, 88]. It is a general property of a Fermi system, and is not affected by the interaction [89]. Information about $n(\vec{k})$ and the singularity may be obtained from the observed $S(\vec{q}, w)$ in elastic neutron scattering [90]. The underlying interpretation is that the virtual excitation brought about by the density operator $\hat{\rho}_{\vec{q}}$ can excite quasiparticles up to $2k_F$ with energy conservation; while for $q > 2k_F$ we must supply an external energy to excite such quasiparticles.

An interesting quantity which is intimately related to sound propagation in Fermi systems is the so–called acoustic impedance. This is generally defined as the product $Z \equiv \rho u$, where ρ is the density of the liquid and u is either u_o or u_1, depending on whether we are in the low–temperature, collisionless regime or the higher–temperature, hydrodynamic regime, respectively. Physically, it is a measure of the sonic energy radiated into an acoustic medium by a vibrating surface (for example, a quartz crystal). In a normal liquid this energy appears as waves of first sound. In liquid ^{3}He, however, this is the case only at relatively high temperatures; at sufficiently low temperatures the sonic energy flows primarily as zero sound (together with a small contribution from single–particle modes which vanishes altogether at elevated pressures).

To our knowledge, the only available measurements of Z for this system date from the mid–sixties: Z was measured as a function of temperature under both the saturated vapour pressure [91, 92] and higher pressures [93]. A thorough analysis of these measurements was undertaken [94 − 96], following the earlier work [97] carried out on the basis of Landau's theory in connection with a calculation of thermal boundary (Kapitza) resistance between solids and ^{3}He. The prime reason for this intense interest was that, prior to the first direct observation of zero sound [74], the acoustic–impedance measurements provided the sole experimental evidence on this mode of sound. As T is reduced, the transition from the first- to the zero-sound regime appears as a sharp increase in Z (which diminishes with increasing pressure). Curiously, however, the increase observed experimentally is somewhat larger than accounted for theoretically, and the discrepancy has not been satisfactorily resolved so far [46]. This may be attributed either to the lack of accuracy in the determination of the Landau f-factors involved, or to the experimental accuracy itself, which may be not adequate [94, 98], or even to some other hitherto unsuspected factors (such as the effect of ^{4}He impurities, few as these might be) [99].

3.2 3.2 Connection with Macroscopic Studies

The key link between the microscopic and the macroscopic is the scattering amplitude of the interacting pair in the Fermi sea [14, 38, 100, 101].

In a scattering process the quantity of interest is the transition probability of the pair from an initial state $(\vec{k}_1\vec{\sigma_1}, \vec{k}_2\vec{\sigma_2})$ to a final state $(\vec{k}_1'\vec{\sigma_1}', \vec{k}_2'\vec{\sigma_2}')$. At low temperatures, the final states available are within $k_{\rm B}T$ of the Fermi surface: The four vectors$(\hat{k}_1\hat{k}_2\hat{k}_1'\hat{k}_2')$ nearly have k_F in magnitude and practically depend only on the relative orientation; i.e., the two angles (θ, ϕ), θ being the angle between the initial wavevectors of the colliding pair $(\hat{k}_1\hat{k}_2)$ and ϕ the angle formed between the planes of the initial and final pairs of the vectors. The transition probability is given by [6, 14, 38]:

$$w(\theta, \phi) = \frac{2\pi}{\hbar} \left| a_i(\theta, \phi) \right|^2 , \tag{3.66}$$

where $a_i(\theta, \phi)$ is the corresponding scattering amplitude. The scattering process involves scattering of parallel spins as well as antiparallel spins. We denote parallel-spin states by $|\uparrow\uparrow>$, which is a mixture of the two $|11>$ and $|1-1>$ triplet states, and antiparallel–spin states by $|\uparrow\downarrow>=|\downarrow\uparrow>$, which is a mixture of the two $|10>$ triplet and $|00>$ singlet states, represented symbolically by

$$
\begin{aligned}
|\uparrow\uparrow> &= \frac{1}{\sqrt{2}}\left[\ |11>+|1-1>\ \right]; \\
|\uparrow\downarrow> &= \frac{1}{\sqrt{2}}\left[\ |10>+|00>\ \right]. \qquad (3.67)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
w^{\uparrow\uparrow}(\theta,\phi) &= \frac{2\pi}{\hbar}\,|a_i^{\uparrow\uparrow}(\theta,\phi)|^2 \\
&= \frac{2\pi}{\hbar}\,|<11|(a\,\theta,\phi)|11>+<1-1|a\,(\theta,\phi)|1-1>|^2 \\
&= \frac{2\pi}{\hbar}\,[2a_o]^2\,; \qquad (3.68) \\
w^{\uparrow\downarrow}(\theta,\phi) &= \frac{2\pi}{\hbar}\,[a_o+a_e]^2\,, \qquad (3.69)
\end{aligned}
$$

where o(e) stands for odd (even) angular–momentum states; spin–singlet states can have only even L values, but the triplet state has only odd L values, so as to preserve the asymmetry of the two–fermion state.

As emphasized in Chapter Two, the T-matrix is, in effect, a generalized scattering amplitude of the interacting pair. In both spin and isospin spaces, this matrix may be separated into a spin or spin-isospin independent component (the direct part) and a spin or spin–isospin dependent component (the exchange part). The latter exists even when the interaction is state–independent so as to satisfy the requirement of asymmetry of the particle state, the Pauli principle. Suppressing the isospin dependence, we write :

$$T = T_D - P_\sigma T_E; \tag{3.70}$$

$$T_D = T(12, 1'2')^{\vec{\sigma_1}\vec{\sigma_2}}; \tag{3.71}$$

$$T_E = P_{\vec{k}'} T(12, 1'2')^{\vec{\sigma_1}\vec{\sigma_2}}. \tag{3.72}$$

D, E denotes direct and exchange parts of T; $12, 1'2' = \vec{k}_1\, \vec{k}_2, \vec{k}_1'\vec{k}_2'$; $P_{\vec{k}'}$, and $P_{\vec{\sigma}}$ are the momentum– and spin–exchange operators :

$$P_{\vec{k}'}|\vec{k}_1'\vec{k}_2' > = |\vec{k}_2'\vec{k}_1' >= \pm|\vec{k}_1'\vec{k}_2' >; \tag{3.73}$$

$$P_\sigma|\vec{\sigma_1}\vec{\sigma_2} > = \frac{1}{2}(1 + \vec{\sigma_1}.\vec{\sigma_2})|\vec{\sigma_1}\vec{\sigma_2} >= \pm|\boldsymbol{\sigma_2\sigma_1} >, \tag{3.74}$$

where +(–) in the first equation applies for even (odd) L, and in the second for S=1(0) spin states. In terms of the total and relative, initial and final, wavevectors:

$$\vec{K} = \vec{k}_1 + \vec{k}_2 = \vec{k}_1' + \vec{k}_2'; \tag{3.75}$$

$$\vec{k} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2); \qquad \vec{k}' = \frac{1}{2}(\vec{k}_1' - \vec{k}_2'), \tag{3.76}$$

$$\Longrightarrow$$

$$T_D = T(\vec{k}, \vec{k}'; s, \vec{K})^{\vec{\sigma_1}\vec{\sigma_2}}; \tag{3.77}$$

$$T_E = T(\vec{k}, -\vec{k}'; s, \vec{K})^{\vec{\sigma_1}\vec{\sigma_2}}. \tag{3.78}$$

But

$$\begin{aligned}
T_{DT} = T_T(\vec{k},\vec{k}'; s,\vec{K})^{\vec{\sigma_1}\vec{\sigma_2}} &= \sum_{\text{odd } L}(2L+1)T_L P_L(\hat{k}.\hat{k}') = a_o; \\
T_{DS} = T_S(\vec{k},\vec{k}'; s,\vec{K})^{\vec{\sigma_1}\vec{\sigma_2}} &= \sum_{\text{even} L}(2L+1)T_L P_L(\hat{k}.\hat{k}') = a_e; \\
T_{ET} = T_T(\vec{k},-\vec{k}'; s,\vec{K})^{\vec{\sigma_1}\vec{\sigma_2}} &= \sum_{\text{odd } L}(2L+1)T_L (-)^L P_L(\hat{k}.\hat{k}') = -a_o; \\
T_{ES} = T_S(\vec{k},-\vec{k}'; s,\vec{K})^{\vec{\sigma_1}\vec{\sigma_2}} &= \sum_{\text{even} L}(2L+1)T_L (-)^L P_L(\hat{k}.\hat{k}') = a_e. \qquad (3.79)
\end{aligned}$$

From Eqs.(3.77, 3.78 and 3.79), respectively,

$$\begin{aligned}
T^{\uparrow\uparrow} &= T_{DT} - P_\sigma T_{ET} = 2a_o; \\
T^{\uparrow\downarrow} &= T_{DT} - T_{DS} = a_o + a_e; \qquad (3.80)
\end{aligned}$$

and Eq. (3.70) becomes

$$\begin{aligned}
T &= T_{DT} + T_{DS} - P_\sigma \left(T_{ET} + T_{ES}\right) \\
&= \left(a_o + a_e\right) - P_\sigma \left(a_e - a_o\right) \\
&= \frac{1}{2}\left(3a_o + a_e\right) + \frac{1}{2}\left(\vec{\sigma_1}.\vec{\sigma_2}\right)\left(a_o - a_e\right) \\
&= \frac{1}{2}\left(T^{\uparrow\uparrow} + T^{\uparrow\downarrow}\right) + \frac{1}{2}\left(\vec{\sigma_1}.\vec{\sigma_2}\right)\left(T^{\uparrow\uparrow} - T^{\uparrow\downarrow}\right), \qquad (3.81)
\end{aligned}$$

which can be rewritten as:

$$T = T^S + \left(\vec{\sigma_1}.\vec{\sigma_2}\right)\; T^A, \qquad (3.82)$$

where

$$T^S \equiv \frac{1}{2}(T^{\uparrow\uparrow} + T^{\uparrow\downarrow}) = \frac{1}{2}(3a_o + a_e); \tag{3.83}$$

$$T^A \equiv \frac{1}{2}(T^{\uparrow\uparrow} - T^{\uparrow\downarrow}) = \frac{1}{2}(a_o - a_e) \tag{3.84}$$

are the spin-symmetric and spin-asymmetric components of the T-matrix, respectively.

With the above considerations, the transition probability used by Hone [102] in terms of a_i to calculate the transport coefficients can readily be rewritten in terms of the generalized (T–matrix) amplitude; for instance, the spin-diffusion coefficient is proportional to

$$\begin{aligned} w^{\uparrow\downarrow}(\theta,\phi) &= \frac{2\pi}{\hbar}\left|T^{\uparrow\uparrow}(\theta,\phi)\right|^2 = \frac{2\pi}{\hbar}\left[a_o + a_e\right]^2 \\ &= \frac{2\pi}{\hbar}\left|T_{DT} + T_{DS}\right|^2; \end{aligned} \tag{3.85}$$

likewise, for viscosity η and thermal conductivity κ :

$$\begin{aligned} w_\eta(\theta,\phi) &= w_\kappa(\theta,\phi) = \frac{2\pi}{\hbar}\sum_{\substack{\vec{\sigma_1}\vec{\sigma_2},\\ \vec{\sigma_1}'\vec{\sigma_2}'}} \frac{1}{2}\left|< \vec{\sigma_1}\vec{\sigma_2}|T|\vec{\sigma_1}'\vec{\sigma_2}' >\right|^2 \\ &= \frac{2\pi}{\hbar}\frac{1}{2}\left[3|a_o|^2 + |a_e|^2\right] \end{aligned} \tag{3.86}$$

An alternative approach hinges on the Landau f-function, which can be obtained from the total energy of the system computed microscopically, say, within the framework of Brueckner's well–known theory [100, 101].

The Landau f–function, $f(\vec{k}_1, \vec{k}_2)^{\vec{\sigma_1}\vec{\sigma_2}} \equiv f(\vec{k}\vec{\sigma}, \vec{k}'\vec{\sigma}')$, is the key quantity in his Fermi-liquid theory; it is defined as the second derivative of the total energy with respect to the occupation numbers of the quasiparticle states $\Big[103, 104\Big]$:

$$
\begin{aligned}
f(\vec{k}_1, \vec{k}_2)^{\vec{\sigma_1}\vec{\sigma_2}} &= \frac{\delta^2 E}{\delta n(\vec{k}_1, \vec{\sigma_1})\delta n(\vec{k}_2, \vec{\sigma_2})} \\
&= \frac{\delta\epsilon(\vec{k}_1, \vec{\sigma_1})}{\delta n(\vec{k}_2, \vec{\sigma_2})},
\end{aligned}
\tag{3.87}
$$

where $\epsilon(\vec{k}_1, \vec{\sigma_1})$ is the energy of the quasiparticle $(\vec{k}_1, \vec{\sigma_1})$. Physically, f may be interpreted as the change in the energy of the quasiparticle (or state) $(\vec{k}_1, \vec{\sigma_1})$ due to the addition (removal) of the quasiparticles $(\vec{k}_2, \vec{\sigma_2})$, or the change of the system energy when the two excited quasiparticles $(\vec{k}_1, \vec{\sigma_1})$, $(\vec{k}_2, \vec{\sigma_2})$ are present or removed. It is obvious that the removal of the quasiparticle $(\vec{k}_1\vec{\sigma_1})$, $(\vec{k}_2\vec{\sigma_2})$ will create the quasiholes $(-\vec{k}_1\vec{\sigma_1}, -\vec{k}_2\vec{\sigma_2})$. Actually, the Landau function is the resulting qp–qh interaction. Let us construct the lowest–order matrix element of the potential connecting two qp–qh states of total momentum $\vec{q}$:

$$
< U_{ph} > = < \Phi^{\vec{k}_2}_{\vec{k}_2 - \vec{q}} | V_{ph} | \Phi^{\vec{k}_1}_{\vec{k}_1 - \vec{q}} > \ . \tag{3.88}
$$

The qp–qh state is given by $\Big[1\Big]$

$$
|\Phi^{\vec{k}}_{\vec{k}-\vec{q}} > = a^{\dagger}_{\vec{k}} a'^{\dagger}_{\vec{k}-\vec{q}} |0>, \tag{3.89}
$$

where $a^{\dagger}_{\vec{k}} a'^{\dagger}_{\vec{k}-\vec{q}}$ are the quasiparticle and the quasihole creation operators, such that $a'^{\dagger}_{\vec{k}-\vec{q}} = a_{\vec{k}-\vec{q}}$, and $|0>$ is the ground state of the system. U_{ph} can be written as:

$$\begin{aligned} U_{ph} &= \sum_{\substack{\vec{k}_1\vec{k}_2 \\ \vec{q}}} |\Phi^{\vec{k}_2}_{\vec{k}_2-\vec{q}}><\Phi^{\vec{k}_2}_{\vec{k}_2-\vec{q}}|U_{ph}|\Phi^{\vec{k}_1}_{\vec{k}_1-\vec{q}}><\Phi^{\vec{k}_1}_{\vec{k}_1-\vec{q}}| \\ &= \sum_{\substack{\vec{k}_1\vec{k}_2 \\ \vec{q}}} <U_{ph}> a^{\dagger}_{\vec{k}_2} a_{\vec{k}_2-\vec{q}} a^{\dagger}_{\vec{k}_1-\vec{q}} a_{\vec{k}_1}. \end{aligned} \tag{3.90}$$

In terms of the qp–qh matrix elements, $<U_{ph}>$ is given by

$$<\Phi^{\vec{k}_2}_{\vec{k}_2-\vec{q}}|U_{ph}|\Phi^{\vec{k}_1}_{\vec{k}_1-\vec{q}}>=<\vec{k}_2\vec{k}_1-\vec{q}|V|\vec{k}_1\vec{k}_2-\vec{q}>-<\vec{k}_1-\vec{q}\vec{k}_2|V|\vec{k}_1\vec{k}_2-\vec{q}>, \tag{3.91}$$

and the associated diagrams are shown in Fig.4.

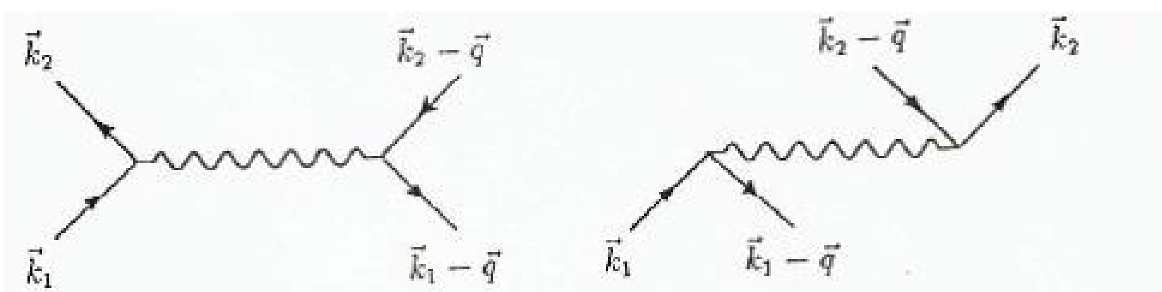


Fig.4: Lowest–order matrix elements of the qp–qh interaction.

In the limit $q \to 0$, both the quasiparticle and the quasihole are at the Fermi surface, and

$$\begin{aligned} U_{ph} &= \sum_{\substack{\vec{k}_1\vec{k}_2 \\ \vec{q}}} <U_{ph}> \left(a^{\dagger}_{\vec{k}_2} a_{\vec{k}_2}\right)\left(a^{\dagger}_{\vec{k}_1} a_{\vec{k}_1}\right) \\ &= \sum_{\substack{\vec{k}_1\vec{k}_2 \\ \vec{q}}} <U_{ph}> \hat{n}_{\vec{k}_2}\hat{n}_{\vec{k}_1}, \end{aligned} \tag{3.92}$$

$\hat{n}_{\vec{k}}$ being the number operator. The expectation value of U_{ph} in the ground state is

$$< 0|U_{ph}|0 > \;=\; \sum_{\substack{\vec{k}_1\vec{k}_2\\ \vec{q}}} < U_{ph} > n_{\vec{k}_2} n_{\vec{k}_1}, \tag{3.93}$$

where $n_{\vec{k}}$ is the occupation number of the state $\vec{k}$. If we take the second functional derivative of this last equation we obtain f. The implication is that f appears as the qp–qh interaction in the long–wavelength limit.

Bearing this in mind, one can write the proper–self energy (2.71) as:

$$\sum\nolimits^{*}_{\vec{\sigma_1}}(1, k_{o1}) = -\frac{i}{\Omega}\sum_{2\vec{\sigma_2}}\int \frac{dk_{o1}}{2\pi} G_o(2k_{o1}) \Big[\; T(12,12) - T(12,21) \;\Big]. \tag{3.94}$$

In this expression the T-matrix element depends on k_{o1} and k_{o2}; i.e., it is energy–dependent. We can remove this dependence by approximating T by its on energy-shell value, say $k_{o1} = \epsilon_1$ and $k_{o2} = \epsilon_2$. Integration of Eq.(3.94), using the contour in the upper-half plane, and [1]

$$\begin{aligned} G_o(2k_{o1}) &\equiv \frac{\theta(|2| - k_F)}{k_{o2} - k^2 + i\eta} + \frac{\theta(k_F - |2|)}{k_{o2} + k^2 - i\eta} \\ &\equiv \frac{1 - n(2)}{k_{o2} - k^2 + i\eta} + \frac{n(2)}{k_{o2} + k^2 - i\eta}, \end{aligned} \tag{3.95}$$

yields at once

$$\sum\nolimits^{*}_{\vec{\sigma_1}}(1, \epsilon_1) = \frac{1}{V}\sum_{2\vec{\sigma_2}} \Big[\; T(12,12) - T(12,21) \;\Big] n(2). \tag{3.96}$$

The single-quasiparticle energy is simply

$$\epsilon(1\sigma_1) = \epsilon^o(1\sigma_1) + {\sum}^*_{\vec{\sigma_1}}(1, \epsilon_1). \tag{3.97}$$

The variation of the system energy from the ground-state due to one excitation (quasiparticle) is given by

$$\delta E \;=\; \sum_{1\vec{\sigma_1}} \epsilon(1\sigma_1)\delta n(1); \tag{3.98}$$

so that

$$\begin{aligned} E &= \sum_{1\vec{\sigma_1}} \int \epsilon(1\sigma_1)\delta n(1) \\ &= \sum_{1\vec{\sigma_1}} \left[\epsilon^o(1\sigma_1) + {\sum}^*_{\vec{\sigma_1}}(1, \epsilon_1)\right] n(1) \\ &= \sum_{1\vec{\sigma_1}} \epsilon^o(1\sigma_1) n(1) + \frac{1}{\Omega} \sum_{\substack{1\vec{\sigma_1} \\ 2\vec{\sigma_2}}} \left[\; T(12,12) - T(12,21) \;\right] n(1)n(2), \end{aligned} \tag{3.99}$$

and the quasiparticle energy, in a modified more practical form, taking the first derivative of E with respect to $n(1)$, becomes:

$$\epsilon(1\sigma_1) = \epsilon^o(1\sigma_1) + {\sum}^{*S}_{\vec{\sigma_1}}(1, \epsilon_1) \,+\, {\sum}^{*R}_{\vec{\sigma_1}}(1, \epsilon_1), \tag{3.100}$$

where

$$\begin{aligned} {\sum}^{*S}_{\vec{\sigma_1}}(1, \epsilon_1) &\equiv \frac{1}{\Omega} \sum_{2\vec{\sigma_2}} \left[\; T(12,12) - T(12,21) \;\right] n(2); \\ {\sum}^{*R}_{\vec{\sigma_1}}(1, \epsilon_1) &\equiv \frac{1}{2\Omega} \sum_{\substack{2\vec{\sigma_2} \\ 3\sigma_3}} \frac{\partial}{\partial n(1)} \left[\; T(23,23) - T(23,32) \;\right] n(2)n(3). \end{aligned} \tag{3.101}$$

The first term on the rhs of Eq.(3.99) is the variation in energy due to the addition (removal) of the quasiparticle. The second is the rearrangement energy that corresponds to the change in the correlations between other quasiparticles; it takes into account the redistribution effects in the medium and the dependence of T on the off–energy–shell values [19, 46]. The Landau f–function is

$$\begin{aligned} f(1\vec{\sigma_1}, 2\vec{\sigma_2}) &= \frac{\partial^2 E}{\partial n(1)\partial n(2)} \\ &= \frac{1}{\Omega}\Bigg(\Big[\ T(12,12) - T(12,21)\ \Big] + \sum_{3\vec{\sigma_3}} \frac{\partial}{\partial n(2)}\Big[\ T(13,13) - T(13,31)\ \Big] n(3) \\ &+ \sum_{3\vec{\sigma_3}} \frac{\partial}{\partial n(1)}\Big[\ T(23,23) - T(23,32)\ \Big] n(3) + \frac{1}{2}\sum_{\substack{3\vec{\sigma_3}\\ 4\vec{\sigma_4}}} \frac{\partial}{\partial n(1)}\frac{\partial}{\partial n(2)} \\ &\times \Big[\ T(34,34) - T(34,43)\ \Big] n(3)n(4)\Bigg). \end{aligned} \tag{3.102}$$

Thus, the lowest-order contribution to f is the T-matrix itself; the other terms are associated with rearrangement effects and correspond to multipair excitations, which are absent in the low-density limit [77].

Alternatively, it is useful to consider another microscopic treatment based on the Landau f-function. In the above formulation, the T–matrix appears as the generalized scattering amplitude without any further specification. This time the f–function will be related to T through the forward–scattering amplitude. That amplitude in which two quasiparticles exchange momentum $\vec{q}$, or equivalently, a qp–qh pair with respective wavevectors $\vec{k}_1 \pm \frac{\vec{q}}{2}$ are scattered into the states $\vec{k}_2 \pm \frac{\vec{q}}{2}$, is said to be forward when the angle ϕ between the two planes $(\vec{k}_1\vec{k}_2)$ and $(\vec{k}_1'\vec{k}_2')$ is zero and $\vec{q}$ is very small so that the scattering is completely in the forward direction. The Landau parameters are related to the forward–scattering amplitude according to $\left[75, 105\right]$:

$$\mathcal{A}(\vec{k}_1, \vec{k}_2)\,{}^{\vec{\sigma}_1\vec{\sigma}_2}_{\vec{\tau}_1\vec{\tau}_2} = \mathcal{F}(\vec{k}_1, \vec{k}_2)\,{}^{\vec{\sigma}_1\vec{\sigma}_2}_{\vec{\tau}_1\vec{\tau}_2} - \sum_{\vec{k}_3\vec{\sigma}_3} \int \mathcal{F}(\vec{k}_1, \vec{k}_3)\,{}^{\vec{\sigma}_1\vec{\sigma}_2}_{\vec{\tau}_1\vec{\tau}_2}\, \mathcal{A}(\vec{k}_3, \vec{k}_2)\,{}^{\vec{\sigma}_1\vec{\sigma}_2}_{\vec{\tau}_1\vec{\tau}_2}\, \frac{d\hat{k}_3}{4\pi}\,. \tag{3.103}$$

In the general nucleonic matter case, the quasiparticle interaction, suppressing tensor and spin–orbit forces, is parametrized by $\left[106, 107\right]$

$$\begin{aligned} \mathcal{F}(\vec{k}_1, \vec{k}_2)\,{}^{\sigma_1\sigma_2}_{\tau_1\tau_2} \quad &= \quad F^{\,1}(\vec{k}_1, \vec{k}_2) + F^{\,2}(\vec{k}_1, \vec{k}_2)\; \vec{\sigma_1}.\vec{\sigma_2}\; + F^{\,3}(\vec{k}_1, \vec{k}_2)\; \vec{\tau_1}.\vec{\tau_2} \\ &+ \quad F^{\,4}(\vec{k}_1, \vec{k}_2)\; \vec{\sigma_1}.\vec{\sigma_2}\;\; \vec{\tau_1}.\vec{\tau_2}\,; \end{aligned} \tag{3.104}$$

and the corresponding forward–scattering amplitude $\left[104, 105\right]$ by

$$\begin{aligned} \mathcal{A}(\vec{k}_1, \vec{k}_2)\,{}^{\vec{\sigma}_1\vec{\sigma}_2}_{\vec{\tau}_1\vec{\tau}_2} \quad &= \quad A^{\,1}(\vec{k}_1, \vec{k}_2) + A^{\,2}(\vec{k}_1, \vec{k}_2)\; \vec{\sigma_1}.\vec{\sigma_2} + A^{\,3}(\vec{k}_1, \vec{k}_2)\; \vec{\tau_1}.\vec{\tau_2} \\ &\quad\;\; + A^{\,4}(\vec{k}_1, \vec{k}_2)\;\; \vec{\sigma_1}.\vec{\sigma_2}\;\; \vec{\tau_1}.\vec{\tau_2}. \end{aligned} \tag{3.105}$$

The dimensionless T-matrix ,

$$\mathcal{T} = \left(\frac{dn}{d\epsilon_{\vec{k}}}\right)_{\epsilon_F} T, \tag{3.106}$$

can be parametrized accordingly. It is easy to show that, in the present case,

$$k = k' = k_F \sin\frac{\theta}{2}; \qquad \text{and } K = 2k_F \cos\frac{\theta}{2}, \tag{3.107}$$

where we have used the condition $k_1 = k_2 = k_F$, θ being the angle between $\vec{k}_1$ and $\vec{k}_2$. In this limit, all the parameters mentioned above depend on the variable θ and can be expanded in Legendre polynomials:

$$
\begin{aligned}
F^{\,i}(\vec{k}_1, \vec{k}_2) &= \sum_L F^i_L P_L(\hat{k}_1.\hat{k}_2)\,; \\
A^{\,i}(\vec{k}_1, \vec{k}_2) &= \sum_L A^i_L P_L(\hat{k}_1.\hat{k}_2)\,; \\
\mathcal{T}^{\,i}(\vec{k}, \vec{k}'; s, \vec{K}) &= \sum_L \mathcal{T}^i_L P_L(\hat{k}_1.\hat{k}_2)\,.
\end{aligned}
\tag{3.108}
$$

Using Eqs.(3.103 ,3.104 and 3.105), together with the addition theorem of spherical harmonics,we get the following relations between $\mathcal{F}^{\,i}_{\,L}$ and $\mathcal{A}^{\,i}_{\,L}$ (or $\mathcal{T}^{\,i}_{\,L}$):

$$
A^i_L = F^i_L / \left(1 + F^i_L/(2L+1)\right). \tag{3.109}
$$

The asymmetry of the two-body wave function requires that the forward–scattering amplitude must be asymmetric under the exchange of fermionic coordinates; i.e.,

$$
\begin{aligned}
\mathbf{P}\mathcal{A}(\vec{k}_1, \vec{k}_2)_{\,\substack{\vec{\sigma}_1\vec{\sigma}_2 \\ \vec{\tau}_1\vec{\tau}_2}} &= -\mathcal{A}(\vec{k}_1, \vec{k}_2)_{\,\substack{\vec{\sigma}_1\vec{\sigma}_2 \\ \vec{\tau}_1\vec{\tau}_2}} \\
&= \mathcal{A}(\vec{k}_2, \vec{k}_1)^{\,\substack{\vec{\sigma}_2\vec{\sigma}_1 \\ \vec{\tau}_2\vec{\tau}_1}}\,,
\end{aligned}
\tag{3.110}
$$

where

$$
\mathbf{P} \equiv P_{\vec{k}} P_{\vec{\sigma}} P_{\vec{\tau}} = -1 \tag{3.111}
$$

is the exchange operator. Letting $k_1 = k_2$, $P_{\vec{k}} = 1$, then, in the S=1 and T=1 states where $P_{\vec{\sigma}} P_{\vec{\tau}} = 1$,

$$\begin{aligned} \mathbf{P} A(\vec{k}_1, \vec{k}_1)\ {}^{\sigma_1 \sigma_2}_{\tau_1 \tau_2} &= A(\vec{k}_1, \vec{k}_1)\ {}^{\vec{\sigma}\vec{\sigma}}_{\vec{\tau}\vec{\tau}} \\ &= -A(\vec{k}_1, \vec{k}_1)\ {}^{\vec{\sigma}\vec{\sigma}}_{\vec{\tau}\vec{\tau}}\ . \end{aligned} \tag{3.112}$$

Thus,

$$A(\vec{k}_1, \vec{k}_1)\ {}^{\vec{\sigma}\vec{\sigma}}_{\vec{\tau}\vec{\tau}} = 0, \tag{3.113}$$

which guarantees that the forward–scattering amplitude for two quasiparticles of parallel spins and isospins in the same momentum state vanishes. This allows us, using Eqs.(3.105, 3.108, and 3.113), to obtain the following sum rule:

$$\sum_L A_L^1 + A_L^2 + A_L^3 + A_L^4 = 0\,. \tag{3.114}$$

In the S=0, T=0 states, $P_{\vec{\sigma}} P_{\vec{\tau}} = 1$, which implies that

$$\sum_L A_L^1 - 3A_L^2 - 3A_L^3 + 9A_L^4 = 0\,. \tag{3.115}$$

In the two cases above, L=odd; i.e, the T-amplitude vanishes in odd partial waves. No such rules exist for even parity since $\mathbf{P} = P_{\vec{k}} P_{\vec{\sigma}} P_{\vec{\tau}} = -1$ (S=1,T=0 or S=0,T=1). For ^{3}He, there is only one sum rule:

$$\sum_L A_L^1 + A_L^2 = 1, \tag{3.116}$$

which corresponds to

$$\begin{aligned}\mathbf{P}A(\vec{k}_1,\vec{k}_2)^{\uparrow\uparrow} &= A(\vec{k}_1,\vec{k}_2)^{\uparrow\uparrow}\\ &= -A(\vec{k}_1,\vec{k}_2)^{\uparrow\uparrow}.\end{aligned} \tag{3.117}$$

In this latter case 1, 2 refer to S, A parts of A or $\mathcal{T}$, respectively.

3.3 Summary and Discussion

The aim of this Chapter has been to study some sound phenomena in dilute neutral Fermi systems within a unified framework. In Section 3.1, the rudiments of this framework have been developed starting from Landau's transport equation and its general solution for collective modes of vibrations. Various sound modes have then been examined.

In Section 3.2, an appropriate scheme has been introduced that links the T-matrix with macroscopic phenomena. In particular, the effective interaction has been related to the transition probabilities, Eqs. (3.68, 3.69), on the one hand, and to Landau's parameters, Eq.(3.102), on the other. Finally, the (diagonal) T-matrix has been expressed in terms of the forward–scattering amplitude, with the associated sum rules, Eqs. (3.109, 3.114, 3.115, and 3.117).

Clearly, the framework presented here is general enough to embrace many fundamental aspects of sound phenomena in dilute neutral Fermi systems. Although we have set out to lay out a mere outline of a suitable theoretical scheme, the net result has evidently turned out to be much more than this. The next step is presumably to invoke our arsenal of numerical techniques and realistic input potentials to convert the foregoing abstract scheme into concrete numbers and figures. This is one possible extension of the present work; other possible extensions will be suggested in the next chapter.

CHAPTER FOUR

CONCLUSION

Chapter 4

Conclusion

This concluding Chapter aims primarily at suggesting some possible extensions of the present work, as has just been mentioned. These will be listed briefly in Section 4.2. Prior to this, however, it is in order to summarize (in Section 4.1) the main highlights of our Thesis.

4.1 General Summary

Motivated by the recent renewed interest in dilute neutral Fermi systems, as well as by the long–standing problem of deriving their macroscopic properties from their microscopic constituents, we have attempted to calculate a general effective interaction for these systems. To this end, we have invoked the Galitskii–Migdal–Feynman (GMF) T–matrix. Our strategy has been to derive this matrix, which is in effect the required effective interaction in momentum space, for the general noncentral input two–body potential, from which the simpler central case can be obtained at once by switching off the noncentral aspects of the problem. In passing, we have presented new derivations, published here for the first time, concerning orthogonality and completeness of the T–matrix, which can be regarded as useful sum rules imposed on this matrix by physical considerations.

We have then used our general effective interaction to compute the proper self–energy, from which the macroscopic properties of the system can be immediately computed according to standard recipes.

Our effective interaction has then been used within an ambitious theoretical framework for shedding more light on the rich physics involved in the sound phenomena occurring in dilute neutral Fermi systems, the idea being to bring together the multifarious aspects involved in a unified whole. Among the elements discussed in this context have been various sound modes propagating in these systems and their relation with the density–response function, the static structure factor and the acoustic impedance.

Throughout this work our main emphasis has been on the connection between the *microscopic* and the *macroscopic*, which has been realized in a crystal–clear manner within the framework of the sound phenomena studied.

4.2 Possible Extensions of the Present Work

As mentioned in the concluding remarks of Chapter Three, our theoretical framework lends itself to immediate numerical extensions through a judicious choice of realistic input potentials. The list of physical quantities which can be then computed seems to be endless: It could include the effective mass, compressibility, and many other bulk properties; in addition to all sorts of acoustic properties.

Another type of extension lies in the age–old quest of improving the independent–pair model used here to embrace long–range correlations, which are vital for denser systems – not to mention the question of three– and higher–body forces.

Yet, a third class of possible extensions comprises other interesting quantum systems, such as spin–polarized systems, so–called new quantum systems and low–dimensional systems.

Needless to say, all these extensions should ultimately incorporate the explicit temperature dependence into the picture.

By now it should be clear that the determination of a sound effective interaction for the systems under present consideration, among others, remains the central problem in many–body theory!

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مُـلـخّـــــص

تُعْنى هذِه الرِّسالةُ - فـي الدَّرَجَةِ الأولـى - بالتَّفاعـُلِ الفعَّالِ وعَلاقَـتِـهِ بالظَّـواهِـرِ الصَّوْتِيَّـةِ في أنْظِمَةِ فيرمِـي الـمتَعادِلـَةِ المـُخَفَّفـةِ، التي تَشْملُ المادّةَ النـوَوِيَّة، والـهيليـوم3 فـي زُجـاجِ فايْكـر، ومخاليـطَ الـهيليـوم3-الـهيليـوم4.

فَبَعْدَ وَصْفٍ مُخْتَصَر، لكنْ شـامل، لِلْمُقَوِّماتِ الأساسِـيَّةِ للصّياغَةِ الرِّياضِيّـة، بما فـي ذلك الأنْظمةُ المعنـيّة، والجُهْدُ المُدْخَل، ومَصْفوفَةُ غاليتْسكي-مِغْدال-فاينْمـان، اشتُـقَّ تَعْبـيرٌ رِياضيٌّ عامٌ لمفكوك جُهْدِ النّيوكليُـون-النّيوكليُـون المُدْخَـل الذي هولامَحَلِّـيٌّ ولامَرْكَـزِيٌّ ويَعْتَـمِدُ عَلـى الحالـةِ، بِدِلالـة الحالاتِ الصّحيحة (القنـوات) للنّيوكلُـونَيْن. وهذا، بدَوْره، اسْتُـخْدِمَ كمُدْخَل لاشتِـقاقِ مُعـادلاتِ المصْفوفَـةكامِلـةً، ومِنْ ثَمَّ التَّـفاعُل الفعّـال والطّاقةِ الذّاتيّــةِ (الصّحيحــةِ) الـمُناظِـرة؛ بيْنَما حُصِلَ على الحالـةِ الـمَرْكَزِيّةِ الأسهل- ببسـاطة - بإبطالِ اعتمادِ الحالةِ. إضافةً الى ذلك، فإنَّ الخصائصَ التّعامُدِيَّـةَ والتّكْميليّةَ لِهذهِ المَصْفوفَـةِ اشتُقّتْ لأوّل مرّة.

بَعْدَ ذلك، تمّـتْ دراسـةُ أنْمـاطِ الصّوْتِ المُتـعَدِّدةِ فـي هذه الأنْظمَـة. فحُصِلَ على الـمُعادلاتِ المـُناظِرَةِ ابتـداءً مِـنَ النـّهْجِ التّـقْلـيديّ لتَـراوُحـاتِ الكثافَـةِ الـمُسْتَـحَثَّة. وَوُجّـهَ اهـتمـامٌ خـاصٌ لِلْصِّـلَـةِ الوَثـيـقَـةِ بَيْنَ انْتِـشـارِ الصَّوْتِ والكمّـيّـات مِـن مِثْـل عـامِـل البِنْـيَــةِ السّـكونيّـةِ و الـمُعاوَقَــةِ الصَّوْتِـيَّـة. وبصورةٍ خـاصّة، أُسّـِسَ إطارٌ شـامـلٌ تكميلي يَرْبِـطُ هـذه الدِّراسـةَ المِجْـهَرِيّـةَ بالـمظـاهِرِ الجـاهِـرِيَّة، مُرْسِـياً بـذلـك الأرْضِيّـةَ لتَـطْبيـقـاتٍ عِـدّةٍ مُدْخَلُها الأساسـيّ مَصفـوفَـةُ غاليتسْــكي.

أخـيراً، تَخْـتَـتِمُ الرِّسـالـةُ بـخُـلاصَـةٍ و بقائـمـةٍ بِبَعْـضِ الـمَسـائِـلِ الـمُعَلّقـة للمتابعـةِ فـي المسـتـقبـَل .