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On the domination number of the cartesian product of the cycle of length n and any graph

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Abstract

Let $\gamma(G)$ denote the domination number of a graph G and let $C_n \Box G$ denote the cartesian product of C_n, the cycle of length $n \geqslant 3$, and G. In this paper, we are mainly concerned with the question: which connected nontrivial graphs satisfy $\gamma(C_n \Box G) = \gamma(C_n)\gamma(G)$? We prove that this equality can only hold if $n \equiv 1 \pmod 3$. In addition, we characterize graphs which satisfy this equality when $n = 4$ and provide infinite classes of graphs for general $n \equiv 1 \pmod 3$.

MSC: 05C69

Keywords: Domination number; Cartesian product; Vizing's conjecture

1. Introduction

Let G be a simple finite graph whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, the *open neighborhood* of v is $N_G(v) = \{u : uv \in E(G)\}$ and the *open neighborhood of a subset* $X \subseteq V(G)$ is $N_G(X) = \bigcup_{v \in X} N_G(v) - X$. The respective *closed neighborhoods* are $N_G[v] = N_G(v) \cup \{v\}$ and $N_G[X] = N_G(X) \cup X$. A subset D of $V(G)$ is called *a dominating set* of G if $N_G[D] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Any dominating set of G with cardinality $\gamma(G)$ will be called a $\gamma(G)$-set or simply a γ-set if the graph is clear from the context. For graphs G and H, the *cartesian product* $G \Box H$ is the graph with vertex set $V(G \Box H) = V(G) \times V(H)$ and edge set $E(G \Box H) = \{(x_1, y_1)(x_2, y_2) : x_1x_2 \in E(G)$ and $y_1 = y_2$ or $x_1 = x_2$ and $y_1y_2 \in E(G)\}$.

In 1963, Vizing [10] conjectured that $\gamma(G \Box H) \geqslant \gamma(G)\,\gamma(H)$. This conjecture remains open despite numerous results proving its validity in special cases; see the survey article [5] and the more recent articles [1,9]. It seems natural to ask then how sharp this conjecture, if true, is? In [5], Hartnell and Rall pointed out the existence of several infinite families of graphs G, H which satisfy the equation $\gamma(G \Box H) = \gamma(G)\gamma(H)$. Clearly, if G or H is the trivial graph $\overline{K_n}$; the *complement* of the complete graph K_n of order n, then this equation is satisfied. So, if we are looking for solutions of this equation, we should concentrate on nontrivial connected graphs. In this paper, nontrivial solutions of the equation $\gamma(C_n \Box G) = \gamma(C_n)\gamma(G)$ are considered where C_n is the cycle of length n. In this direction, little progress has been

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achieved. The case $n = 3$ is easy and follows from a result of [5]. Hartnell and Rall [5] mentioned without proof that there is no solution when $n = 6$. Further, they asked for a characterization of graphs that are solutions for the case $n = 4$.

The main result of this paper is that the equation $\gamma(C_n \Box G) = \gamma(C_n)\gamma(G)$ has a nontrivial solution G if and only if $n \equiv 1 \pmod 3$. All solutions for the case of C_4 are characterized. For $n \equiv 1 \pmod 3$; $n \geqslant 7$, infinite classes of solutions are given. Then we show that any other solution, if exists, must be one of those that arise in the case $n = 4$. The paper is organized as follows: in Section 2, some previous results and notations which will be needed are given. Section 3 is devoted to an investigation of graphs arising when $n = 4$. We prove in Section 4 the nonexistence of nontrivial solution when $n \equiv 0$ or $2 \pmod 3$. Finally, in Section 5, infinite classes of graphs for general $n \equiv 1 \pmod 3$ are described.

2. Preliminaries

The demonstration of the original work of the paper depends on several previous results and some notations which are given in the following paragraphs.

The *corona* of two graphs G and H is the graph $G \circ H$ formed from one copy of G and $|G|$ copies of H, where the ith vertex of G is adjacent to every vertex in the ith copy of H, see [7]. The corona $G \circ K_1$, traditionally known as the *generalized comb*, has even order and domination number half its order. Graphs having no isolated vertices and domination number half their order were characterized independently by Payan and Xuong [8], and Fink et al. [4] as follows:

Theorem 2.1 (*Fink et al. [4], Payan and Xuong [8]*). *For any graph G with even order and no isolated vertices, $\gamma(G) = |G|/2$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

In the following theorem, Hartnell and Rall [6] gave a characterization of graphs which satisfy $\gamma(G \Box K_2) = \gamma(G)$.

Theorem 2.2 (*Hartnell and Rall [6]*). *For any connected graph G, $\gamma(G \Box K_2) = \gamma(G)$ if and only if G has a γ-set D that can be partitioned into two nonempty subsets D_1 and D_2 such that $V(G) - N_G[D_1] = D_2$ and $V(G) - N_G[D_2] = D_1$.*

The next theorem, which is due to Fink and Jacobson [3], plays a crucial role in proving some results obtained in the paper.

Theorem 2.3 (*Fink and Jacobson [3]*). *Let D be a γ-set for G. Then there is a vertex $v \in V(G) - D$ such that v is adjacent to at most two vertices of D.*

The following proposition is an immediate consequence of the proof of Theorem 8 in [2]. Here and in what follows the vertices of the cycle C_n will be denoted by $1, 2, \ldots, n$ and interpreted modulo n.

Proposition 2.4. *Let D be a dominating set for $C_n \Box G$. Define the sets*

$$D' = \{x \in V(G) : (n-1, x) \vee (n, x) \in D\} \cup \{x \in V(G) : (1, x) \wedge (n-2, x) \in D\},$$

and

$$D'' = \{(i, x) \in D : 1 \leqslant i \leqslant n-3\} \cup \{(1, x) : (n-2, x) \in D\} \cup \{(1, x) : (n-1, x) \wedge (n, x) \in D\}.$$

Then,

(i) *for $n \geqslant 4$, D' is a dominating set for G.*
(ii) *for $n \geqslant 6$, D'' is a dominating set for $C_{n-3} \Box G$.*

As an immediate result of Proposition 2.4, we have

Corollary 2.5. *For $n \geqslant 6$, $\gamma(C_n \Box G) \geqslant \gamma(C_{n-3} \Box G) + \gamma(G)$.*

Throughout the rest of this paper, the following terminology will be used: G will denote a connected nontrivial graph. For simplicity, we use $N(\,) = N_G(\,)$ and $N[\,] = N_G[\,]$. For $x \in V(G)$, denote

$$(C_n)_x = \{(i, x) : 1 \leqslant i \leqslant n\} \subset V(C_n \Box G).$$

If D is a dominating set of $C_n \Box G$, define

$$D_i = \{x \in V(G) : (i, x) \in D\} \quad \text{for each } i; \quad 1 \leqslant i \leqslant n.$$

For $0 \leqslant j \leqslant n$, let

$$B_j = \{x \in V(G) : |(C_n)_x \cap D| = j\}$$

and

$$B_j^* = \{x \in V(G) : |(C_n)_x \cap D| \geqslant j\}.$$

For each i; $1 \leqslant i \leqslant n$, we have $B_0 \subseteq N(D_i)$. Using the above notation and Proposition 2.4, one can easily conclude that $(D_{n-1} \cup D_n) \cup (D_1 \cap D_{n-2})$ is a dominating set for G. More generally, for each i; $1 \leqslant i \leqslant n$,

$$|(D_i \cup D_{i+1}) \cup (D_{i-1} \cap D_{i+2})| \geqslant \gamma(G). \tag{1}$$

3. The graph $C_4 \Box G$

In this section, we characterize those graphs G which satisfy $\gamma(C_4 \Box G) = 2\gamma(G)$.

Let us assume that D is a dominating set of $C_4 \Box G$ with $|D| = 2\gamma(G)$. Summing inequality (1) for $i = 1$ up to 4, we get

$$\sum_{i=1}^{4} |(D_i \cup D_{i+1}) \cup (D_{i-1} \cap D_{i+2})| \geqslant 4\gamma(G). \tag{2}$$

Consider $x \in V(G)$. If $x \in B_1$, then $(C_4)_x$ contains exactly one vertex of D which contributes 2 to the sum on the left-hand side of inequality (2). If $x \in B_2^*$, then $(C_4)_x$ contributes exactly 4 to this sum. Thus, $2|B_1| + 4|B_2^*| \geqslant 4\gamma(G)$.

On the other hand, $|D| = \sum_{i=1}^{4} i|B_i|$ which implies that $2\gamma(G) = |D| \geqslant |B_1| + 2|B_2^*|$. Consequently, $|D| = |B_1| + 2|B_2^*|$. In particular, both B_3 and B_4 are empty.

Now, we classify the resulting graphs, according to whether some of B_0, B_1, B_2 are also empty, as follows: obviously, there exists eight cases, some of which are ruled out immediately. Clearly, the two cases in which $B_1 = B_2 = \emptyset$ cannot occur since $D \neq \emptyset$. Also, if $B_0 = \emptyset$ and $B_2 \neq \emptyset$, then $\gamma(G) > \frac{1}{2}|G|$ which contradicts the fact that G is connected. So, there remain four cases to be considered. The resulting graphs in these cases will be called types 1–4 as indicated below. Although the first three types can be considered as special cases of the fourth one. It is preferable to discuss them separately first since their special structures are simple.

Type 1: $B_0 = \emptyset$, $B_1 \neq \emptyset$, *and* $B_2 = \emptyset$. In this case, we have $\gamma(G) = \frac{1}{2}|G|$ which implies by Theorem 2.1 that G is either isomorphic to C_4 or a generalized comb. It can easily be checked that $\gamma(C_4 \Box C_4) = 4 = \gamma(C_4)^2$. Suppose that G is a generalized comb, say $G = H \circ K_1$. Denote $U = V(G) - V(H)$. In addition, let $U = U' \cup U''$ be any partition of U, $W' = N(U')$ and $W'' = N(U'')$. Define

$$D = \{(1, x) : x \in U'\} \cup \{(2, x) : x \in W''\} \cup \{(3, x) : x \in W'\} \cup \{(4, x) : x \in U''\}.$$

Obviously, D is a dominating set of $C_4 \Box G$ with cardinality $2\,\gamma(G)$.

Type 2: $B_0 \neq \emptyset$, $B_1 = \emptyset$, *and* $B_2 \neq \emptyset$. Note that B_2 is a γ-set for G. Let $F_1 = D_1$ and $F_2 = B_2 - F_1$. We shall now prove that $\gamma(G \Box K_2) = \gamma(G)$ by showing that the partition $F_1 \cup F_2$ of B_2 satisfies the condition of Theorem 2.2. As mentioned earlier, $B_0 \subseteq N(D_1)$ which implies that $V(G) - N[F_1] = F_2$. Define $U = \{x \in B_0 : N(x) \cap B_2 \subset F_1\}$. Suppose that $U \neq \emptyset$. For any vertex $x \in U$ and each $i = 2, 3, 4$, there exists a vertex $y_i \in D_i$ such that $xy_i \in E(G)$. By the definition of U, $y_i \in D_1$; $i = 2, 3, 4$. One can conclude that the three vertices y_2, y_3, y_4 are distinct since $y_i = y_j$

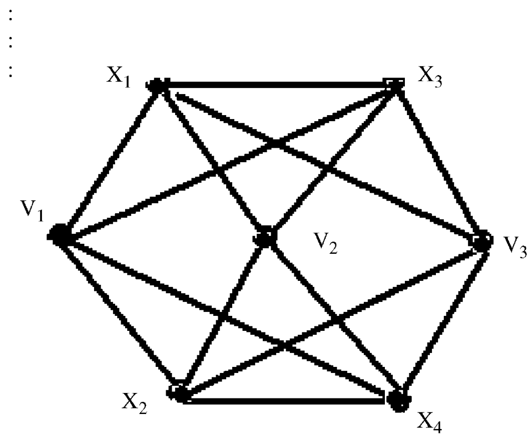

Fig. 1. An example of type 3.

would imply that $D_1 \cap D_i \cap D_j \neq \emptyset$. Now, fix some $x \in U$ and let y_2, y_3, y_4 be the corresponding adjacent vertices. The set $(B_2 - \{y_2, y_3\}) \cup \{x\}$ is a dominating set for G with smaller cardinality. This contradiction shows that $U = \emptyset$. Since $U = \emptyset$, we have $V(G) - N[F_2] = F_1$. Thus, $\gamma(G \Box K_2) = \gamma(G)$ as required. Clearly, $\gamma(G \Box K_2) = \gamma(G)$ implies that $\gamma(C_4 \Box G) = 2\gamma(G)$.

Type 3: $B_0 \neq \emptyset$, $B_1 \neq \emptyset$, *and* $B_2 = \emptyset$. In this case, G contains four mutually disjoint subsets $D_1, \ldots, D_4$ such that for each i; $1 \leqslant i \leqslant 4$, we have $V(G) - D_{i-1} - D_{i+1} \subset N[D_i]$ and $D_i \cup D_{i+1}$ is a $\gamma(G)$-set. Obviously, any graph G having four subsets with these properties satisfies $\gamma(C_4 \Box G) = 2\gamma(G)$. An example of such graphs is illustrated in Fig. 1, in which $B_0 = \{v_1, v_2, v_3\}$, and $D_i = \{x_i\}$, $1 \leqslant i \leqslant 4$.

Type 4: $B_0 \neq \emptyset$, $B_1 \neq \emptyset$ *and* $B_2 \neq \emptyset$. The study of this case is complicated since graphs of this type may involve, in some way or another, graphs of previous types. First, the graph $C_4 \Box G$ may have two γ-sets of different specified types. For example, this occurs if $|B_0| = |B_2|$, which implies that $\gamma(G) = \frac{1}{2}|G|$, and hence G is already of type 1. Moreover, $C_4 \Box G$ has a dominating set for which $B_1 = V(G)$. This situation occurs for $C_4 \Box G_m$; $m \geqslant 4$, which are defined and discussed in Section 5.

Second, let G' and G'' be two graphs of types 1 and 2, respectively. Suppose that G is obtained from $G' \cup G''$ by adding an edge from a vertex in G' to another one in G'' such that the domination number does not decrease, that is, $\gamma(G) = \gamma(G') + \gamma(G'')$. Then $\gamma(C_4 \Box G) = 2\gamma(G)$ and G will be of the fourth type, see Fig. 2 in which $B_0 = \{v_1, v_2, v_3\}$, $B_1 = \{x_1, x_2, x_3, x_4, y_1, y_3\}$, and $B_2 = \{y_{12}, y_{34}\}$.

However, there are graphs which cannot be constructed directly from the previous three types, such graphs will be called *pure type* 4. To illustrate the structure of those graphs in more detail, let us introduce some further related notations. For $1 \leqslant i, j \leqslant 4$, we define the following:

$$F_i = D_i - \bigcup_{i \neq j} D_j, \quad F = \bigcup_{i=1}^{4} F_i$$

$$Y_{ij} = Y_{ji} = D_i \cap D_j \;\; i \neq j, \quad Y = \bigcup_{i=1}^{4} \bigcup_{j=i+1}^{4} Y_{ij},$$

$$X = B_0, \quad X_i = X \cap N(F_i),$$

and

$$X_{ij} = X \cap N(Y_{ij}).$$

Observe that:

(a) $F_i \subseteq N(D_{i+2})$. Suppose on the contrary that $x \in F_i$ and $x \notin N(D_{i+2})$. This implies that the vertex $(i+2, x)$ is not dominated by D and hence the result follows.

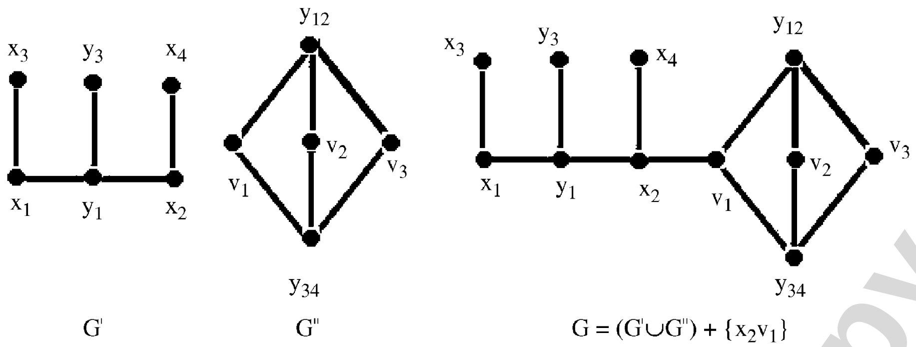

Fig. 2. A graph G of type 4 with $\gamma(G) = \gamma(G') + \gamma(G'')$.

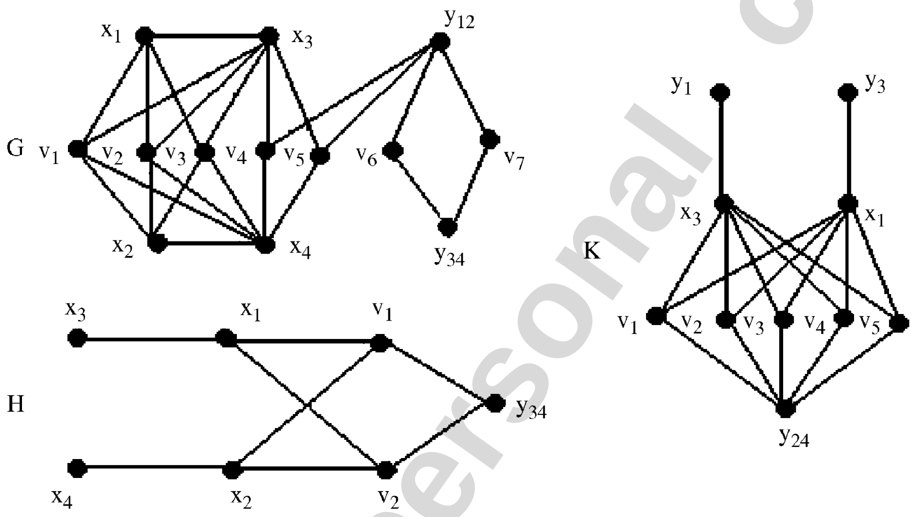

Fig. 3. Examples of pure type 4.

(b) If each $F_i \neq \emptyset$ and $Z = \bigcap_{i=1}^{4} X_i \neq \emptyset$, then the subgraph induced by $F \cup Z$, is of type 3.
(c) If $X_{ij} \cap X_{kl} \neq \emptyset$; for $\{i, j\} \cap \{k, l\} = \emptyset$, then the subgraph induced by $(Y_{ij} \cup Y_{kl}) \cup (X_{ij} \cap X_{kl})$, is of type 2.

The main aspect of pure type 4 graphs is that at least one of the sets $X_i \cap X_j \cap X_{kl} \neq \emptyset$ for some distinct i, j, k, and l. For further illustration, consider the graphs G, H, and K in Fig. 3, where we labeled the vertices such that each of x_i and $y_i \in F_i$, F_i does not contain any other vertex and $Y_{ij} = \{y_{ij}\}$; $1 \leqslant i, j \leqslant 4$.

The graph G satisfies:

(a) $X_3 \cap X_4 \cap X_{12} \subset X$.
(b) the subgraph induced by $F \cup Z$ is of type 3.
(c) the subgraph induced by $(Y_{12} \cup Y_{34}) \cup (X_{12} \cap X_{34})$ is of type 2.

On the other hand, in the graph H of Fig. 3, the set $X_1 \cap X_2 \cap X_{34} = X$. While in the graph K of this same Fig. 3, $F_2 = F_4 = \emptyset$, and $X_1 \cap X_3 \cap X_{24} = X$.

Now, we give a characterization of all nontrivial solutions of the equation $\gamma(C_4 \Box G) = 2\gamma(G)$ of type 4. Since, as we mentioned above, the previous three types are special cases of the fourth one, this characterization is also valid for any nontrivial solution.

Theorem 3.1. *For any connected graph G, $\gamma(C_4 \Box G) = 2\gamma(G)$ if and only if $V(G)$ has four (not necessarily distinct) subsets D_1, D_2, D_3, and D_4 such that for $1 \leqslant i \leqslant 4$, the following conditions are satisfied:*

(i) *Each of the sets $A_i = (D_i \cup D_{i+1}) \cup (D_{i-1} \cap D_{i+2})$ is a $\gamma(G)$-set.*

(ii) $V(G) - N[D_i] \subseteq D_{i-1} \cup D_{i+1}$.
(iii) $V(G) - N[D_i \cup D_{i+1}] = D_{i-1} \cap D_{i+2}$.

Proof. (i) The proof of the first statement follows directly from the above discussion. (ii) Assume that $x \notin N[D_i]$, then either $(i-1, x)$ or $(i+1, x) \in D$, otherwise, (i, x) would not be dominated by D. Thus, $x \in D_{i-1} \cup D_{i+1}$.

(iii) It is obvious that if $x \notin N[D_i \cup D_{i+1}]$, then both $(i-1, x)$ and $(i+2, x) \in D$. To prove the reverse inclusion, let $x \in D_{i-1} \cap D_{i+2}$ and assume that $x \in N[D_i \cup D_{i+1}]$. Since any vertex $z \notin A_i$ which is dominated by x, is also dominated by $D_i \cup D_{i+1}$, the set $A_i - \{x\}$ would be a $\gamma(G)$-set with a smaller cardinality.

Conversely, assume $V(G)$ has four (not necessarily distinct) subsets D_1, D_2, D_3, and D_4 which satisfy statements (i)–(iii). We claim that the set $D = \bigcup_{i=1}^{4} (\{i\} \times D_i)$ is a dominating set of $C_4 \Box G$ with cardinality $2\gamma(G)$. To show this, let $(i, x) \in V(C_4 \Box G) - D$. Set $X = V(G) - \bigcup_{i=1}^{4} D_i$, $Y = \bigcup_{i \neq j} Y_{ij}$, and $F_i = D_i - \bigcup_{i \neq j} D_j$. If $x \in X \cup F_{i+2}$, then by statement (ii) the edge $(i, x)\,(i, y_i) \in E(C_4 \Box G)$ for some $y_i \in D_i$. Since G is of type 4 then $F_j \neq \emptyset$ for some j, and $Y \neq \emptyset$. Thus, if $x \in F_{i-1} \cup F_{i+1}$, then (i, x) is dominated either by $(i-1, x)$ or by $(i+1, x)$. Finally, if $x \in Y$, say $x \in Y_{kl}$ for some k, l; $1 \leqslant k, l \leqslant 4$, then $(C_4)_x$ is dominated by (k, x) and (l, x). Therefore, D is a dominating set of $C_4 \Box G$.

Now, $|D| = \sum_{i=1}^{4} |\{i\} \times D_i| = \sum_{i=1}^{4} |D_i|$. As a consequence of statements (i) and (iii) we get $2\gamma(G) = |A_1| + |A_3| = \sum_{i=1}^{4} |D_i|$ as required. $\Box$

It is interesting at this point to note that if $V(G) = \bigcup_{i=1}^{4} D_i = B_1$, then G is of type 1. If each $D_i \subseteq B_2$, then the result is a graph of type 2. Moreover, if the sets D_1, D_2, D_3, and D_4 are mutually disjoint and $\bigcup_{i=1}^{4} D_i \subset V(G)$, then we get a graph of type 3.

4. The graph $C_n \Box G$; $n \equiv 0$ or $2 \pmod 3$

In this section, we consider first the cases $n = 3$ and 5. Although the next lemma is a result of [5], we give a different proof based on Theorem 2.3.

Lemma 4.1. *Let D be a dominating set for $C_3 \Box G$, where G is a connected nontrivial graph. Then $|D| > \gamma(G)$.*

Proof. Suppose that $|D| = \gamma(G)$. Define $A = \{x \in V(G) : (C_3)_x \cap D \neq \emptyset\}$. Clearly, A is a dominating set for G. It follows that A is a $\gamma(G)$-set and $|(C_3)_x \cap D| = 1$ for each $x \in A$. Let $y \in V(G) - A$. In order that $(C_3)_y$ be dominated, there must exist three distinct vertices: $x_1, x_2, x_3 \in A$, adjacent to y such that $(i, x_i) \in D$, which contradicts Theorem 2.3. $\Box$

For the case $n = 5$, we prove the following lemma.

Lemma 4.2. *Let D be a dominating set for $C_5 \Box G$, where G is a connected nontrivial graph. Then $|D| > 2\gamma(G)$.*

Proof. Suppose that $|D| = 2\gamma(G)$. Summing inequality (1) for $i = 1$–5, we get

$$\sum_{i=1}^{5} |(D_i \cup D_{i+1}) \cup (D_{i-1} \cap D_{i+2})| \geqslant 5\gamma(G). \tag{3}$$

Consider a vertex $x \in V(G)$. If $|(C_5)_x \cap D| = 1$, then $(C_5)_x$ contributes 2 to the sum on the left-hand side of (3). If $|(C_5)_x \cap D| = 2$, then this $(C_5)_x$ contributes 3 or 5 depending on whether the two vertices of D on $(C_5)_x$ are adjacent or not. Finally, if $|(C_5)_x \cap D| \geqslant 3$, then $(C_5)_x$ contributes 5. Therefore,

$$2|B_1| + 5|B_2^*| \geqslant 5\gamma(G) \tag{4}$$

and also,

$$|B_1| + 2|B_2^*| \leqslant |D| = 2\gamma(G).$$

Then we conclude that $B_1 = \emptyset$ and $|B_2^*| = \gamma(G)$. Hence, $|(C_5)_x \cap D| = 0$ or 2 for every $x \in V(G)$.

Now any $y \in V(G) - B_2$ must be adjacent to at least three vertices: $x_1, x_2, x_3 \in B_2$, otherwise, not all the vertices of $(C_5)_y$ would be dominated. Thus, B_2 is a $\gamma(G)$-set which contradicts Theorem 2.3. □

The above lemma implies that $\gamma(C_5 \Box G) \geqslant 2\gamma(G) + 1$. To show that this lower bound is sharp consider the following example: let G^k be the graph with

$$V(G^k) = \{x, y_1, y_2, \dots, y_{k-1}\} \cup X,$$

the edge set of G^k is determined so that

(i) X is the open neighborhood of x with cardinality at least $2(k-1)$.
(ii) $\{N(y_1), N(y_2), \dots, N(y_{k-1})\}$ is a partition of X which do not contain any singleton.
(iii) G^k has no other edges joining its vertices.

Observe that $\gamma(G^k) = k$. Let U be any γ-set for C_5. The set

$$D = (U \times \{y_1, y_2, \dots, y_{k-1}\}) \cup (V(C_5) - U) \times \{x\}$$

is a γ-set for $C_5 \Box G^k$ with cardinality $2k + 1$.

Finally, based on Lemmas 4.1 and 4.2 as well as the recursive inequality given in Corollary 2.5, we can easily deduce the following theorem.

Theorem 4.3. *For any connected nontrivial graph G and all $n \equiv 0$ or* 2 (mod 3); $n \geqslant 3$,

$$\gamma(C_n \Box G) > \gamma(C_n)\gamma(G).$$

5. The graph $C_n \Box G$; $n \equiv 1 \pmod 3$, $n \geqslant 7$

Throughout this section assume that $n \equiv 1 \pmod 3$, say, $n = 3k + 1$. We may assume that $k \geqslant 1$, since the obtained result is also applicable to C_4. Let G_m denote the generalized comb $K_m \circ K_1$ whose core is the complete graph K_m, $m \geqslant 3$. Denote the vertices of G_m by $v_1, v_2, \dots, v_{2m}$, where v_i; $1 \leqslant i \leqslant m$, has degree 1 and is adjacent to v_{m+i}. While the vertices $v_{m+1}, \dots, v_{2m}$ induce K_m.

Theorem 5.1. *Suppose that $m \geqslant n = 3k + 1$, where $k \geqslant 1$. Then $\gamma(C_n \Box G_m) = \gamma(C_n)\gamma(G_m)$.*

Proof. Define the following subsets of $V(C_n \Box G_m)$:

$$A_1 = \bigcup_{i=1}^{n} \bigcup_{j=0}^{k-1} \{(2 + i + 3j, v_i)\}, \quad A_2 = \bigcup_{i=n+1}^{m} \bigcup_{j=0}^{k} \{(1 + 3j, v_i)\},$$

$$A_3 = \bigcup_{i=1}^{n} \{(i, v_{i+m})\} \quad \text{and} \quad D = A_1 \cup A_2 \cup A_3.$$

We shall prove that D is a dominating set for $C_n \Box G_m$. First note that for each i; $1 \leqslant i \leqslant n$, the cycle $(C_n)_{v_i}$ contains k vertices from A_1 situated in such a way so that they dominate all of its vertices except (i, v_i). The vertex (i, v_i) is dominated by $(i, v_{i+m}) \in A_3$. Each cycle $(C_n)_{vi}$, $n + 1 \leqslant i \leqslant m$, contains $k + 1$ vertices from A_2 which dominate all of its vertices. The set of vertices $\{(i, v_j) : 1 \leqslant i \leqslant n, m + 1 \leqslant j \leqslant m + n\}$ is dominated by A_3 since the core of the comb is isomorphic to K_m. Now $|D| = m(k + 1) = \gamma(C_n)\gamma(G_m)$. This completes the proof of the theorem. □

Let us remark here that if m is a multiple of n then we can take the core of the comb to be $\overline{C_m}$, the complement of C_m, instead of K_m and we still have the required equality. Another family of graphs could be obtained if we add r vertices, where $r \geqslant 1$, and more edges to the core K_m to produce K_{m+r}. In addition, more edges joining a pendant vertex v_i, $1 \leqslant i \leqslant m$, of G_m to a subset of the newly added r vertices could also be added provided that $N[v_i] \cap N[v_j] = \emptyset$ for all i, I s.t., $1 \leqslant i \neq j \leqslant m$.

If G is a nontrivial graph which satisfies $\gamma(C_n \Box G) = \gamma(C_n)\gamma(G)$; $n \equiv 1 \pmod 3$ and $n \geqslant 7$, then Corollary 2.5 implies that $\gamma(C_4 \Box G) = \gamma(C_4)\gamma(G)$, hence G is one of the four types considered in Section 3.

6. Conclusion

We have studied nontrivial solutions of the equation $\gamma(C_n \Box G) = \gamma(C_n)\gamma(G)$. We think that the equation $\gamma(G \Box H) = \gamma(G)\gamma(H)$ deserves more attention, probably it interferes with the long-awaited proof of Vizing's conjecture. We conclude with a conjecture about such graphs. Call a minimum dominating set D of G *excessive* if there exists a vertex $v \in D$ such that $N[D - \{v\}] = V(G) - \{v\}$.

Conjecture. *If G and H are two connected nontrivial graphs such that $\gamma(G \Box H) = \gamma(G)\gamma(H)$, then each of G and H is either K_2 or else has an excessive dominating set.*

This conjecture is not only true for cycles, but also for all pairs of graphs known to satisfy equality in Vizing's conjecture. If D is an excessive γ-set of G then there exist some $v \in D$ such that $N[D - \{v\}] = V(G) - \{v\}$, this vertex plays the role of an "absorbant" to the vertices in the dominating set of the Cartesian product which minimizes the cardinality of such sets. In some sense, graphs with excessive dominating sets represent "threshold" to all graphs satisfying Vizing's conjecture.

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