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PERTURBATION OF AN INFINITE NETWORK OF IDENTICAL CAPACITORS

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The capacitance between any two arbitrary lattice sites in an infinite square lattice is studied when one bond is removed (i.e. perturbed). A connection is made between the capacitance and the lattice Green's function of the perturbed network, where they are expressed in terms of those of the perfect network. The asymptotic behavior of the perturbed capacitance is investigated as the separation between the two sites goes to infinity. Finally, numerical results are obtained along different directions and a comparison is made with the perfect capacitances.

Keywords: Lattice Green's function; capacitors; perturbed lattice.

1. Introduction

The calculation of the resistance of an infinite network of identical resistors is a classic well-studied problem in the electric circuit theory.[1–3] Despite this, the above-mentioned problem is still of interest and it brings the attention of many authors[4–6] to study and investigate it.

At the beginning of this century, this problem arose again in many publications.[7,8] The methods used in studying this problem vary from superposition of current distribution,[4,5] random walk theory[6] and lattice Green's function (LGF) method.[7,8]

In a recent work, the problem for many lattices for both the perfect and perturbed cases using Cserti's[7,8] method has been studied:

(i) Asad[9] and Asad *et al.*[10] studied the perfect and perturbed (i.e. where one bond is removed) square and simple cubic (SC) infinite lattices mathematically and experimentally. There was a good agreement between the mathematical and the experimental results, especially for the bulk values. Also, there was a good

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agreement between our calculated values and those calculated by other previous authors.[4–8]

(ii) Asad *et al.*[11] studied the infinite perturbed (i.e. where two bonds are removed) square lattice mathematically. Numerical results are obtained and a comparison with those obtained in Ref. 9. is carried out.

The LGF defined in our work is related to the Green's function (GF) of the tight-binding Hamiltonian (TBH).[12] The properties of the LGF for SC lattice have been studied in detail especially when impurities are often introduced,[13–17] where in these references some numerical results are presented and some recurrence formulae are used to calculate other values of the LGF.

The analysis of the capacitance of an infinite network of identical capacitors has been arising recently.[18–20] Asad *et al.*[19] used the LGF method to calculate the capacitance between arbitrary lattice sites in a perfect infinite square lattice consisting of identical capacitors. While Asad *et al.*[20] used the superposition of charge distribution in calculating the capacitance between two points in an infinite square grid of identical capacitors. There was excellent agreements between the calculated results in these two papers.

In this work, we shall use the LGF approach to determine the capacitance for the so-called perturbed lattice obtained by removing one bond (capacitor) from the perfect infinite lattice. The content of this paper is helpful for electric circuit design and the method is instructive. As an example (see Fig. 1), consider an infinite square lattice consisting of identical capacitances C. Removing one bond from this perfect lattice results in a perturbed lattice.

2. Perfect Lattice

We reviewed the formalism of the perfect infinite network using Dirac's notation.

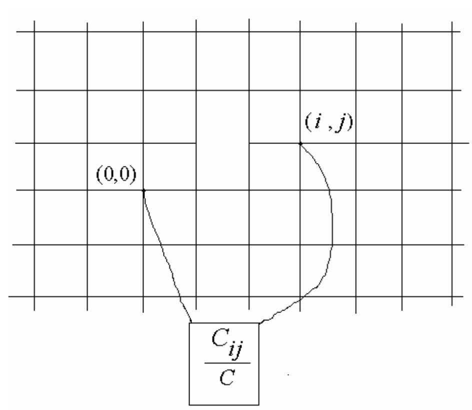

Fig. 1. Perturbation of an infinite square lattice consisting of identical capacitances C by removing the bond between the sites $\vec{r}_{io}$ and $\vec{r}_{jo}$. The capacitance $C(i, j)$ is calculated between arbitrary lattice sites $\vec{r}_i$ and $\vec{r}_j$.

Consider a perfect d-dimensional infinite lattice consisting of identical capacitors each of capacitance C. All lattice points are specified by the position vector $\vec{r}$ given in the form

$$\vec{r} = l_1\vec{a}_1 + l_2\vec{a}_2 + \cdots + l_d\vec{a}_d \,, \tag{2.1}$$

where

$$l_1, l_2, \ldots, l_d \text{ are integers (positive, negative or zero),}$$

and

$$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_d \text{ are independent primitive translation vectors.}$$

Let the potential at the site $\vec{r}_i$ be $V(\vec{r}_i)$ and assuming a charge Q enters the site $\vec{r}_i$ and a charge $-Q$ exits the site $\vec{r}_j$, while the charges are zero at all other lattice sites. Thus, one may write:

$$Q_m = Q[\delta_{mi} - \delta_{mj}] \text{ for all } m. \tag{2.2}$$

Then, according to Ohm's and Kirchhoff's laws, we may write:

$$\frac{Q(\vec{r}_i)}{C} = \sum_{\vec{n}}[V(\vec{r}_i) - V(\vec{r}_i + \vec{n})] \,, \tag{2.3}$$

where $\vec{n}$ are the vectors from site $\vec{r}$ to its nearest neighbors ($\vec{n} = \pm a_i, i = 1, 2, \ldots, d$). One can form two state vectors, V and Q at the site $\vec{r}_i$ such that:

$$\begin{aligned} V &= \sum_i |i\rangle V_i \,; \\ Q &= \sum_i |i\rangle Q_i \,, \end{aligned} \tag{2.4}$$

where $V_i = V(\vec{r}_i)$ and $Q_i = Q(\vec{r}_i)$.

Here we assumed that $|i\rangle$, associated with the site $\vec{r}_i$, forms a complete orthonormal set, i.e. $\langle i|k\rangle = \delta_{ik}$ and $\sum_i |i\rangle\langle i| = 1$. Using Eqs. (2.4) and (2.3) one gets:

$$\sum_j (z\delta_{ij} - \Delta_{ij})\langle j|V = \frac{\langle i|Q}{C} \,. \tag{2.5}$$

z is the number of neighbors of each lattice site (e.g. $z = 2d$ for a d-dimensional hypercubic lattice) and

$$\Delta_{kl} \text{ is defined as:}$$

$$\Delta_{kl} = \begin{cases} 1, \vec{r}_k, \vec{r}_l & \text{are nearest neighbors,} \\ \text{zero}\,, & \text{otherwise.} \end{cases} \tag{2.6}$$

The summation in Eq. (2.5) is taken over all lattice sites. Multiplying Eq. (2.5) by $|i\rangle$ and summing over i, one gets:

$$\sum_{i,j} |i\rangle(\Delta_{ij} - z\delta_{ij})\langle j|V = \frac{-Q}{C} \,. \tag{2.7}$$

Or one may write:

$$L_o V = \frac{-Q}{C}\,, \tag{2.8}$$

where $L_o = \sum_{i,j} |i\rangle(\Delta_{ij} - z\delta_{ij})\langle j|$.

L_o is the so-called lattice Laplacian.

Similar to the definition used in Economou,[12] the LGF for an infinite perfect lattice can be defined as:

$$L_o G_o = -1\,. \tag{2.9}$$

The solution of Eq. (2.9), which is a Poisson-like equation, in its simple form, can be given as:

$$V = -\frac{L_o^{-1} Q}{C} = \frac{G_o Q}{C}\,. \tag{2.10}$$

Inserting Eq. (2.2) into Eq. (2.10), we obtain:

$$\begin{aligned} V_k = \langle k|V &= \frac{\langle k|G_o Q}{C}\,; \\ &= \frac{1}{C}\sum_m \langle k|G_o|m\rangle Q_m\,; \\ &= \frac{Q}{C}[G_o(k,i) - G_o(k,j)]\,, \end{aligned} \tag{2.11}$$

where $G_o(l,m) = \langle l|G_o|m\rangle$ is the matrix element of the operator G_o in the basis $|l\rangle$.

Finally, the capacitance between the sites $\vec{r}_i$ and $\vec{r}_j$ is then given as:

$$\frac{1}{C_o(i,j)} = \frac{V_i - V_j}{Q} = \frac{2}{C}[G_o(i,i) - G_o(i,j)]\,. \tag{2.12}$$

The above formula can be rewritten as:

$$C_o(i,j) = \frac{C}{2[G_o(i,i) - G_o(i,j)]}\,, \tag{2.13}$$

where we have used the fact that the LGF is symmetric (i.e. $G_o(l,m) = G_o(m,l)$).

It is interesting to study the asymptotic form of the capacitance for large separation between the sites $\vec{r}_i$ and $\vec{r}_j$. Using Eq. (2.30) in Refs. 19 and 20, one can write

$$\frac{C_o(i,j)}{C} = \frac{1}{\frac{1}{\pi}\left(Ln\sqrt{(j_x - i_x)^2 + (j_y - i_y)^2} + \gamma + \frac{Ln8}{2}\right)}\,. \tag{2.14}$$

As the separation between the sites $\vec{r}_i$ and $\vec{r}_j$ goes to infinity, one finds that

$$\frac{C_o(i,j)}{C} \to 0\,. \tag{2.15}$$

3. Perturbed Lattice

The charge contribution δQ_i at the site $\vec{r}_i$ due to the bond $(i_o j_o)$ is given by

$$\begin{aligned}\frac{\delta Q_i}{C} &= \delta_{ii_o}(V_{io} - V_{jo}) + \delta_{ij_o}(V_{jo} - V_{io})\,;\\ &= \langle i|i_o\rangle(\langle i_o| - \langle j_o|)V + \langle i|j_o\rangle(\langle j_o| - \langle i_o|)\,;\\ &= \langle i|(|i_o\rangle - |j_o\rangle)(\langle i_o| - \langle j_o|)V\,;\\ &= \langle i|L_1 V\,,\end{aligned} \tag{3.1}$$

where the operator L_1 has the form

$$L_1 = (|i_o\rangle - |j_o\rangle)(\langle i_o| - \langle j_o|)\,. \tag{3.2}$$

Now, removing the bond $(i_o j_o)$ from the perfect lattice the charge Q_i at the site $\vec{r}_i$ is given as:

$$(-L_o V)_i - \frac{1}{C}\delta Q_i = \frac{-Q_i}{C}\,. \tag{3.3}$$

Thus, Ohm and Kirchhoff's laws for the perturbed lattice may be written as:

$$L_{o1} V = \frac{-Q}{C}\,, \tag{3.4}$$

where

$$L_{o1} = L_o + L_1\,. \tag{3.5}$$

Similar to the perfect lattice, the LGF G_{o1} for the perturbed lattice is defined as:

$$L_{o1} G_{o1} = -1\,. \tag{3.6}$$

Therefore, Eq. (3.4) becomes

$$V = \frac{G_{o1} Q}{C}\,. \tag{3.7}$$

Note that Eq. (3.7) is similar to Eq. (2.10). Here the operator L_{o1} is now a sum of L_o associated with the perfect lattice and a perturbation given by L_1.

To calculate the capacitance between the sites $\vec{r}_i$ and $\vec{r}_j$, we assume the charge to be given as in Eq. (2.2). Inserting Eq. (2.2) into Eq. (3.7) we get:

$$\begin{aligned}V_k = \langle k|V &= \frac{\langle k|G_{o1}Q}{C}\,;\\ &= \frac{1}{C}\sum_m \langle k|G_{o1}|m\rangle Q_m\,;\\ &= \frac{Q}{C}\left[G_{o1}(k,i) - G_{o1}(k,j)\right].\end{aligned} \tag{3.8}$$

Thus, the capacitance between the lattice sites $\vec{r}_i$ and $\vec{r}_j$ is

$$\frac{1}{C_{o1}(i,j)} = \frac{V_i - V_j}{Q}\,;$$
$$= \frac{1}{C}\left[G_{o1}(i,i) - G_{o1}(i,j) + G_{o1}(j,j) - G_{o1}(j,i)\right]. \tag{3.9}$$

The above formula can be rewritten as:

$$C_{o1}(i,j) = \frac{C}{[G_{o1}(i,i) - G_{o1}(i,j) + G_{o1}(j,j) - G_{o1}(j,i)]}\,. \tag{3.10}$$

Note here that $G_{o1}(i,i) \neq G_{o1}(j,j)$ since the translational symmetry is broken, but $G_{o1}(i,j) = G_{o1}(j,i)$. Our problem of finding the capacitances reduces to the calculation of the perturbed LGF, because once we calculate it then from Eq. (3.10) we find the perturbed capacitances on the networks.

Instead of the above idea, in the following we write the perturbed capacitance of the networks in terms of the perfect ones. Using Eqs. (2.5) and (2.6) we get:

$$(L_o + L_1)G_{o1} = -1\,. \tag{3.11a}$$

Or it can be rewritten as:

$$G_{o1} = -(L_o + L_1)^{-1}\,. \tag{3.11b}$$

Multiplying both sides of Eq. (3.11a) by G_o we obtain the so-called Dyson's equation:

$$G_{o1} = G_o + G_o L_1 G_{o1} = G_o + G_o L_1 G_o + G_o L_1 G_o L_1 G_o + \cdots. \tag{3.12}$$

To solve the above formula, one can use the method presented by Economou.[12] Inserting Eq. (3.2) into the above equation, one gets:

$$G_{o1}(i,j) = \langle i|G_{o1}|j\rangle$$
$$= G_o(i,j) + \frac{[G_o(i,i_o) - G_o(i,j_o)][G_o(i_o,j) - G_o(j_o,j)]}{1 - 2[G_o(i_o,i_o) - G_o(i_o,j_o)]}\,. \tag{3.13}$$

Finally, the capacitance between $\vec{r}_i$ and $\vec{r}_j$ can be obtained in terms of the perfect capacitances by using Eqs. (3.13), (3.10) and (2.13). Thus, we get after some simply straightforward algebra

$$\frac{C_{o1}(i,j)}{C} = \frac{1}{\dfrac{1}{C_o(i,j)} + \dfrac{\left[\dfrac{1}{C_o(i,j_o)} + \dfrac{1}{C_o(j,i_o)} - \dfrac{1}{C_o(i,i_o)} - \dfrac{1}{C_o(j,j_o)}\right]^2}{4\left[1 - \dfrac{1}{C_o(i_o,j_o)}\right]}}\,. \tag{3.14}$$

This is our final result for the perturbed capacitance between the sites $\vec{r}_i$ and $\vec{r}_j$ in which the bond $(i_o j_o)$ is removed.

To study the asymptotic behavior of the capacitance in an infinite perturbed square lattice for large separation between the sites $\vec{r}_i$ and $\vec{r}_j$, one can easily show from Eq. (3.14) and the fact that $C_o \to 0$ (see Refs. 19 and 20) that also $C_{o1} \to 0$.

For symmetry reasons, the capacitance between $\vec{r}_{i_o}$ and $\vec{r}_{j_o}$ in an infinite perfect d-dimensional lattice is $\frac{C_o(i_o,j_o)}{C} = d$, if $d \succ 1$. Then, using Eq. (3.14) the capacitance between the two ends of the removed bond is $\frac{C_{o1}(i_o,j_o)}{C} = d - 1$. For a square lattice ($d = 2$) the capacitance is C as mentioned in the introduction.

4. Numerical Results

In this section, numerical results are presented for an infinite square lattice, including both the perfect and the perturbed case. The capacitance between the origin

Table 1. The calculated capacitance between the origin and the site $j = (j_x, j_y)$, for an infinite square lattice in which the bond between the sites $i_o = (0,0)$ and $j_o = (1,0)$ is broken.

$j = (j_x, j_y)$	$\frac{C_{01}(i,j)}{C}$	$j = (j_x, j_y)$	$\frac{C_{01}(i,j)}{C}$
$(1,0)$	1	$(0,1)$	1.7667
$(2,0)$	1.0092	$(0,2)$	1.2054
$(3,0)$	0.9421	$(0,3)$	1.0249
$(4,0)$	0.8849	$(0,4)$	0.9314
$(5,0)$	0.8408	$(0,5)$	0.8714
$(6,0)$	0.8063	$(0,6)$	0.8285
$(7,0)$	0.7785	$(0,7)$	0.7956
$(8,0)$	0.7556	$(0,8)$	0.7692
$(9,0)$	0.7362	$(0,9)$	0.7475
$(10,0)$	0.7195	$(0,10)$	0.7291
$(11,0)$	0.7050	$(0,11)$	0.7132
$(12,0)$	0.6921	$(0,12)$	0.6993
$(13,0)$	0.6807	$(0,13)$	0.6870
$(14,0)$	0.6703	$(0,14)$	0.6760
$(15,0)$	0.6610	$(0,15)$	0.6661
$(0,0)$	∞	$(0,0)$	∞
$(-1,0)$	1.8610	$(0,-1)$	1.7667
$(-2,0)$	1.2597	$(0,-2)$	1.2054
$(-3,0)$	1.0601	$(0,-3)$	1.0249
$(-4,0)$	0.9563	$(0,-4)$	0.9314
$(-5,0)$	0.8902	$(0,-5)$	0.8714
$(-6,0)$	0.8433	$(0,-6)$	0.8285
$(-7,0)$	0.8076	$(0,-7)$	0.7956
$(-8,0)$	0.7793	$(0,-8)$	0.7692
$(-9,0)$	0.7561	$(0,-9)$	0.7475
$(-10,0)$	0.7365	$(0,-10)$	0.7291
$(-11,0)$	0.7198	$(0,-11)$	0.7132
$(-12,0)$	0.7051	$(0,-12)$	0.6993
$(-13,0)$	0.6923	$(0,-13)$	0.6870
$(-14,0)$	0.6808	$(0,-14)$	0.6760
$(-15,0)$	0.6704	$(0,-15)$	0.6661

and the lattice site (l, m) in an infinite perfect square network has been calculated in Refs. 19 and 20 (Asad *et al.*).

On the perturbed square lattice the capacitance can be calculated from Eq. (3.14), and the calculated values of the perfect infinite square lattice mentioned above. In this work, the site $\vec{r}_i$ is fixed while the site $\vec{r}_j$ is moved along the line of the removed bond. Here we considered two cases: first, when the removed bond is between $i_o = (0, 0)$ and $j_o = (1, 0)$, where, our calculated values of the capacitances are arranged in Table 1. In the second case, the removed bond is shifted to be between $i_o = (1, 0)$ and $j_o = (2, 0)$, the calculated values of the capacitances are arranged in Table 2.

The capacitance between the sites of the removed bond is equal to C as shown in Table 1. This capacitance can be considered as connected in parallel with a single

Table 2. The calculated capacitance between the origin and the site $j = (j_x, j_y)$, for an infinite square lattice in which the bond between the sites $i_o = (1, 0)$ and $j_o = (2, 0)$ is broken.

$j = (j_x, j_y)$	$\frac{C_{01}(i,j)}{C}$	$j = (j_x, j_y)$	$\frac{C_{01}(i,j)}{C}$
$(1, 0)$	1.8610	$(0, 1)$	1.9838
$(2, 0)$	1.0092	$(0, 2)$	1.3546
$(3, 0)$	1.0379	$(0, 3)$	1.1398
$(4, 0)$	0.9813	$(0, 4)$	1.0267
$(5, 0)$	0.9284	$(0, 5)$	0.9545
$(6, 0)$	0.8858	$(0, 6)$	0.9032
$(7, 0)$	0.8516	$(0, 7)$	0.8642
$(8, 0)$	0.8235	$(0, 8)$	0.8331
$(9, 0)$	0.7999	$(0, 9)$	0.7920
$(10, 0)$	0.7923	$(0, 10)$	0.7861
$(11, 0)$	0.7624	$(0, 11)$	0.7677
$(12, 0)$	0.7470	$(0, 12)$	0.7516
$(13, 0)$	0.7334	$(0, 13)$	0.7374
$(14, 0)$	0.7212	$(0, 14)$	0.7247
$(15, 0)$	0.7102	$(0, 15)$	0.7133
$(0, 0)$	∞	$(0, 0)$	∞
$(-1, 0)$	1.9828	$(0, -1)$	1.9838
$(-2, 0)$	1.3593	$(0, -2)$	1.3546
$(-3, 0)$	1.1460	$(0, -3)$	1.1398
$(-4, 0)$	1.0328	$(0, -4)$	1.0267
$(-5, 0)$	0.9601	$(0, -5)$	0.9545
$(-6, 0)$	0.9081	$(0, -6)$	0.9032
$(-7, 0)$	0.8685	$(0, -7)$	0.8642
$(-8, 0)$	0.8369	$(0, -8)$	0.8331
$(-9, 0)$	0.8110	$(0, -9)$	0.7920
$(-10, 0)$	0.7891	$(0, -10)$	0.7861
$(-11, 0)$	0.7704	$(0, -11)$	0.7677
$(-12, 0)$	0.7534	$(0, -12)$	0.7516
$(-13, 0)$	0.7396	$(0, -13)$	0.7374
$(-14, 0)$	0.7268	$(0, -14)$	0.7247
$(-15, 0)$	0.7152	$(0, -15)$	0.7133

capacitor of capacitance C, which gives a result of $2C$ (the capacitance between two adjacent sites in an infinite perfect network of identical capacitances C).

In Figs. 2–5 the capacitance is plotted against the site, for both the perfect and perturbed cases along [10] and [01] directions. Figures 2 and 3 show the capacitance against the site along [10] direction. It is seen from those two figures that the

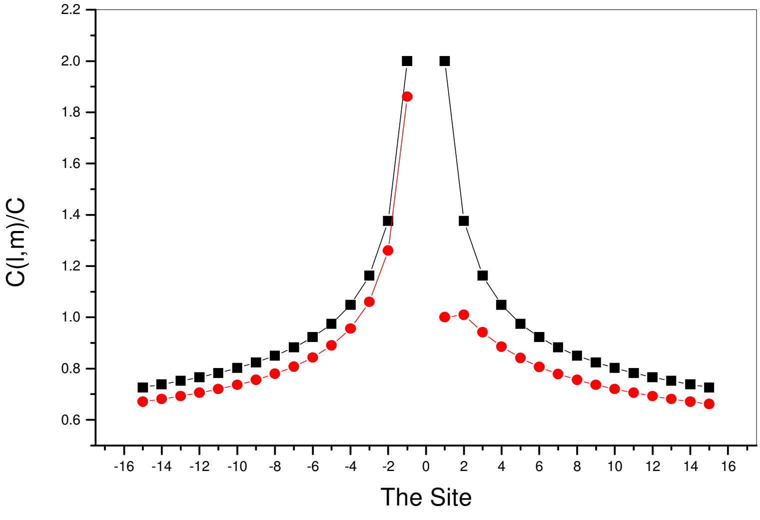

Fig. 2. The capacitance on the perfect (squares) and the perturbed (circles) square infinite lattice between $i = (0, 0)$ and $j = (j_x, 0)$ along the [10] direction as a function of j_x. The ends of the removed bond are $i_o = (0, 0)$ and $j_o = (1, 0)$.

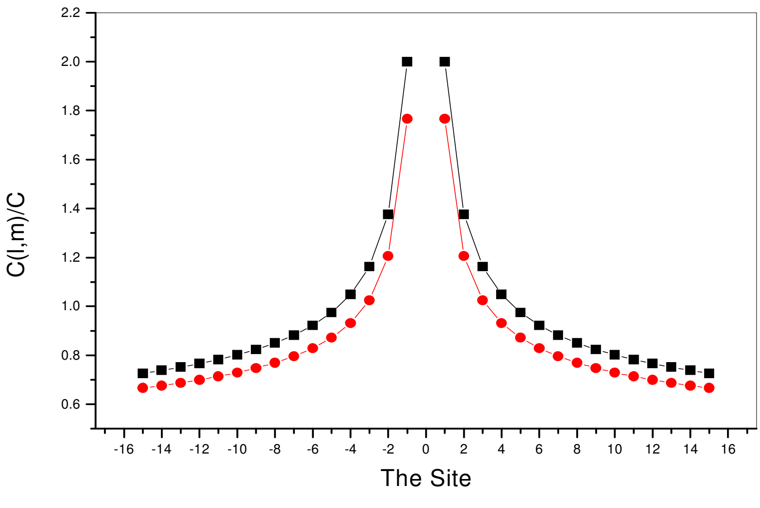

Fig. 3. The capacitance on the perfect (squares) and the perturbed (circles) square infinite lattice between $i = (0, 0)$ and $j = (j_x, 0)$ along the [10] direction as a function of j_x. The ends of the removed bond are $i_o = (1, 0)$ and $j_o = (2, 0)$.

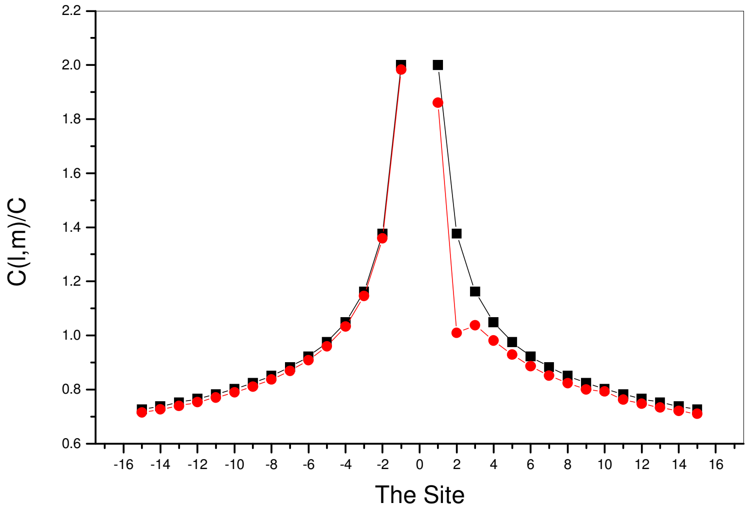

Fig. 4. The capacitance on the perfect (squares) and the perturbed (circles) square infinite lattice between $i = (0, 0)$ and $j = (0, j_y)$ along the [01] direction as a function of j_y. The ends of the removed bond are $i_o = (0, 0)$ and $j_o = (1, 0)$.

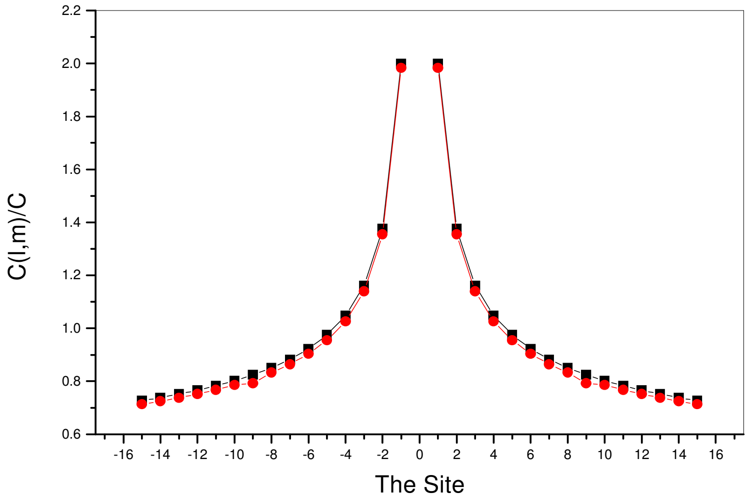

Fig. 5. The capacitance on the perfect (squares) and the perturbed (circles) square infinite lattice between $i = (0, 0)$ and $j = (0, j_y)$ along the [01] direction as a function of j_y. The ends of the removed bond are $i_o = (1, 0)$ and $j_o = (2, 0)$.

perturbed capacitance is not symmetric along [10] due to the fact that the inversion symmetry of the lattice is broken along this direction. Also, as the broken bond is shifted away from the origin, the perturbed capacitance approaches the perfect one more rapidly. Figures 4 and 5 display the capacitance against the site along [01] direction. As shown from these two figures, the inversion symmetry of the perturbed capacitance is not affected because there is no broken bond along this direction. From Figs. 4 and 5 one can see that as the broken bond along [10] direction is shifted away from the origin, the perturbed capacitance along [01] direction approaches the perfect one more rapidly.

Finally, one can see from Figs. 2–5 that for large separation between the sites $\vec{r}_i$ and $\vec{r}_j$, the perturbed capacitance goes to a finite value (i.e. $C_{o1} \to 0$) as mentioned before in Sec. 3.

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