

INFINITE NETWORKS OF IDENTICAL CAPACITORS

J. H. ASAD*

*Department of Physics, Tabuk University,
P.O. Box 741, Tabuk 71491, Kingdom of Saudi Arabia
jhasad1@yahoo.com*

R. S. HIJJAWI

*Department of Physics, Mutah University, Jordan
hijjawi@mutah.edu.jo*

A. J. SAKAJI

*Department of Basic Sciences, Ajman University, UAE
a_sakaji@yahoo.com*

J. M. KHALIFEH

*Department of Physics, University of Jordan, Amman 11942, Jordan
jkalifa@ju.edu.jo*

Received 7 May 2009

Revised 3 August 2009

The capacitance between the origin and any other lattice site in an infinite square lattice of identical capacitors each of capacitance C is calculated. The method is generalized to infinite Simple Cubic (SC) lattice of identical capacitors each of capacitance C . We make use of the superposition principle and the symmetry of the infinite grid.

Keywords: Capacitance; infinite networks; adjacent nodes; charge distribution; difference equation.

PACS Number(s): 61.50.Ah, 61.50.+f, 61.66.f, 84.37.+q, 05.50.+q, 04.60.Nc, 63.90.+t, 84.30.Bv, 02.90.+p

1. Introduction

A classic problem in electric circuit theory studied by many authors over many years is the computation of the resistance between two nodes in a resistor network. Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walk,^{1,2} theory of harmonic functions³ to first-passage processes⁴ to Lattice Green's Functions⁵

*Corresponding author.

(LGF). The connection with these problems originates from the fact that electrical potentials on a grid are governed by the same difference equations as those occurring in the other problems. For this reason, the resistance problem is often studied from the point of view of solving the difference equations, which is most conveniently carried out for infinite networks. Kirchhoff⁶ formulated the study of electric networks more than 150 years ago. The electric-circuit theory is discussed in detail in a classic text by Van der Pol and Bremmer⁷ where they derived the resistance between nearby points on the square lattice.

In the previous 60 years, many efforts focused on analyzing infinite networks of identical resistors. In these efforts, scientists used different methods. For example, F. J. Bartis⁸ introduced how complex systems can be treated at the elementary level and showed how to calculate the effective resistance between adjacent nodes of different lattices of one-ohm resistors. His note is interesting and educational. Venezian,⁹ Atkinson and Van Steenwijk¹⁰ used the superposition of current distribution to calculate the effective resistance between adjacent sites on infinite networks (i.e. square, SC, hexagonal, ...). The mathematical problem involves the solution of an infinite set of linear, inhomogeneous difference equations which are solved by the method of separation of variables. Numerical results for the resistances between arbitrary sites are presented.

Monwhea Jeng¹¹ introduced a mapping between random walk problems and resistor network problems, where his method was used to calculate the effective resistance between any two sites in an infinite two-dimensional square lattice of unit resistors and the superposition principle was used to find the effective resistances on toroidal- and cylindrical-square lattices. In the last decade, many papers^{12–18} have been published using an alternative method based on the LGF. The resistance between two arbitrary lattice sites for different infinite networks of identical resistors were studied for both the perfect and the perturbed networks. Finally, Wu¹⁹ obtained the resistance between two arbitrary nodes in a resistor network in terms of the eigenvalues and eigenfunctions of the Laplacian matrix associated with the network. Explicit formulas for two point resistances are deduced in his paper for regular lattices in one, two, and three-dimensions under various boundary conditions.

Little attention has been paid to infinite networks consisting of identical capacitances C . Van Enk²⁰ studied the behavior of the impedance of a standard ladder network of capacitors and inductors where he analyzed it as a function of the size of the network. Recently, Asad *et al.*^{21–23} and Hijjawi *et al.*²⁴ investigated many infinite lattices of identical capacitors using the LGF method and charge distribution method. In these papers, numerical results for the equivalent capacitance between the origin and any other lattice site was presented — using the above two methods — for the perfect infinite square network. Numerical results was also presented for the so-called perturbed infinite square network which results by removing one or two bonds from the perfect network.

In this paper, we investigate analytically and numerically the capacitance between arbitrary lattice sites in an infinite square and SC grids using the charge distribution method which is based upon the superposition principle. The asymptotic behavior is also studied for large separation between the two sites. The basic approach used here is similar to that followed by Atkinson and Steenwijk.¹⁰

Finally, it is important to mention that problems involving large or infinite electrical resistive networks are interesting and educational.^{25,26} Applications include geophysical exploration with electrical currents, petroleum flow in oil wells, and random walks.²⁷

2. Infinite Square Lattice

In this section, we consider an infinite square network consisting of identical capacitances C . Let us define the voltage at the node (l, m) to be given by $V_{l,m}$, and suppose that a charge of $Q_{l,m}$ enters that node from an external source.

Now using Ohm's and Kirchhoff's laws, we can write

$$\begin{aligned}
 Q_{l,m} &= (V_{l,m} - V_{l+1,m})C + (V_{l,m} - V_{l-1,m})C \\
 &\quad + (V_{l,m} - V_{l,m+1})C + (V_{l,m} - V_{l,m-1})C \\
 &= 4V_{l,m}C - V_{l+1,m}C - V_{l-1,m}C - V_{l,m+1}C - V_{l,m-1}C.
 \end{aligned}
 \tag{1}$$

We shall look for an integral representation for $V_{l,m}$, and take it to be in the form,

$$V_{l,m} = \int_0^{2\pi} d\beta F(\beta) V_{l,m}(\beta), \tag{2}$$

with

$$V_{l,m}(\beta) = \exp(i|l|\alpha + im\beta). \tag{3}$$

Here, α is a function of β .

The above representation is a modified Fourier transform. For $l > 0$, we get

$$4V_{l,m}(\beta) - V_{l+1,m}(\beta) - V_{l,m+1}(\beta) - V_{l,m-1}(\beta) = 2 \exp(il\alpha + im\beta) [2 - \cos \alpha - \cos \beta]. \tag{4}$$

From the above equation, we require (i.e. in order that the contributions of the potential vanishes) α to be related to β as

$$\cos \alpha + \cos \beta = 2. \tag{5}$$

In a similar way, we find zero for this contribution if $l < 0$. Thus, for any integrable $F(\beta)$, then $Q_{l,m}$ given in Eq. (1) goes to zero, unless $l = 0$.

For $0 < \beta < 2\pi$, then Eq. (5) has only an imaginary solution given by

$$\alpha = i \log [2 - \cos \beta + \sqrt{3 - 4 \cos \beta + \cos^2 \beta}]. \tag{6}$$

For the case $l = 0$, we may write:

$$\begin{aligned}
 Q_{0,m} &= C \int_0^{2\pi} d\beta F(\beta) \exp(im\beta) [4 - 2 \exp(i\alpha) - 2 \cos \beta]; \\
 &= 2C \int_0^{2\pi} d\beta F(\beta) \exp(im\beta) [\cos \alpha - \exp(i\alpha)]; \\
 &= -2iC \int_0^{2\pi} d\beta F(\beta) \sin \alpha \exp(im\beta). \tag{7}
 \end{aligned}$$

The above charges may be construed as the coefficients of the Fourier series

$$-2iF(\beta) \sin \alpha = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} Q_{0,m} \exp(-im\beta). \tag{8}$$

Now, take $Q_{0,m} = \delta_{0m}$, i.e. $Q_{0,0} = 1$, and $Q_{0,m} = 0$. This situation corresponds to the case where a charge Q enters at the node $(0, 0)$ and leaves at infinity. Note that no charges leave the lattice at any other finite node. Hence, with this choice,

$$F(\beta) = \frac{i}{4\pi \sin \alpha}. \tag{9}$$

Thus, we can write

$$V_{l,m} = \frac{i}{4\pi} \int_0^{2\pi} \frac{d\beta}{\sin \alpha} \exp(i|l|\alpha + im\beta). \tag{10}$$

One may ask, what the capacitance between the origin and the site (l, m) is? First of all, it is clear that the potential difference between these two sites is $(V_{0,0} - V_{l,m})$. Now, imagine that a charge of one micro-Coulomb enters the network at the node (l, m) instead of $(0, 0)$, allowing it to leave at infinity. The new potential at (l, m) will now be what we called $V_{0,0}$ and the new potential at $(0, 0)$ will be, by symmetry, what we called $V_{l,m}$. Thus, the new potential difference between the origin and (l, m) is just minus the previous one.

If we choose to imagine that one micro-Coulomb leaves the node (l, m) , all the potential difference will be reversed in sign. Therefore, the new potential difference between the origin and the node (l, m) is again given by $(V_{0,0} - V_{l,m})$.

Exploiting the linearity of Ohm's law and superpose all the charges and potentials appertaining to the configuration in which one micro Coulomb enters at $(0, 0)$ and leaves at (l, m) , one can write

$$2[V_{0,0} - V_{l,m}]C_{l,m} = 1, \tag{11}$$

or,

$$C_{l,m} = \frac{1}{2[V_{0,0} - V_{l,m}]}. \tag{12}$$

Thus, $C_{l,m}$ can be written as

$$C_{l,m} = \frac{1}{\frac{i}{2\pi} \int_0^{2\pi} \frac{d\beta}{\sin \alpha} [1 - \exp(i|l|\alpha + im\beta)]}. \tag{13}$$

Table 1. Numerical values of $C_{l,m}$ in units of C for an infinite square grid.

l, m	$C_{l,m}$
0	∞
1,0	2
2,0	1.37597
3,0	1.16203
4,0	1.04823
5,0	0.974844
1,1	1.5708
2,1	1.29326
3,1	1.13539
4,1	1.03649
5,1	0.968523
2,2	1.1781
3,2	1.08177
4,2	1.00814
5,2	0.951831
3,3	1.02443
4,3	0.972869
5,3	0.929041
4,4	0.937123
5,4	0.90391
5,5	0.878865

It is obvious that $C_{l,m} = C_{m,l}$ due to the symmetry of the lattice. Finally, we may transform Eq. (13) into the manifestly real form,

$$C_{l,m} = \frac{1}{\frac{1}{\pi} \int_0^\pi \frac{d\beta}{\sinh|\alpha|} [1 - \exp(-|l||\alpha|) \cos m\beta]}. \tag{14}$$

As $l \rightarrow \infty$, $C_{\infty,m} \rightarrow \frac{1}{\frac{1}{\pi} \int_0^\pi \frac{d\beta}{\sinh|\alpha|}} \rightarrow 0$.

Using Eq. (14), one can calculate the capacitance $C_{l,m}$ by means of Mathematica. Table 1 shows some calculated values. Similar results were obtained by us²² using a similar approach, and by using the LGF approach.⁷

3. Infinite SC Network

The above method can be generalized to an infinite three-dimensional SC network of identical capacitors. Here we consider the three indices and a charge entering the site (l, m, n) is related to the potentials by

$$Q_{l,m,n} = (6V_{l,m,n} - V_{l+1,m,n} - V_{l-1,m,n} - V_{l,m+1,n} - V_{l,m-1,n} - V_{l,m,n+1} - V_{l,m,n-1})C. \tag{15}$$

Choose $V_{l,m,n}$ to be given as

$$V_{l,m,n} = \int_0^{2\pi} d\beta \int_0^{2\pi} d\gamma F(\beta, \gamma) v_{l,m,n}(\beta, \gamma), \tag{16}$$

where $v_{l,m,n}(\beta, \gamma) = \exp(i|l|\alpha + im\beta + in\gamma)$, and with $\cos \alpha + \cos \beta + \cos \gamma = 3$, i.e. $\alpha = \cos^{-1}(3 - \cos \beta - \cos \gamma)$.

For $l \neq 0$, one can easily show that

$$Q_{l,m,n} = 2 \int_0^{2\pi} d\beta \int_0^{2\pi} d\gamma F(\beta, \gamma) \exp(i|l|\alpha + im\beta + in\gamma) x [3 - \cos \alpha - \cos \beta - \cos \gamma], \tag{17}$$

whereas

$$Q_{0,m,n} = -2i \int_0^{2\pi} d\beta \int_0^{2\pi} d\gamma F(\beta, \gamma) \sin \alpha \cos m\beta \cos n\gamma. \tag{18}$$

The inverse of this double Fourier series gives

$$-2iF(\beta, \gamma) \sin \alpha = \frac{4}{\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} Q_{0,m,n} \exp(-im\beta) \exp(-in\gamma). \tag{19}$$

Let us choose $Q_{0,0,0} = 1$, and $Q_{0,m,n} = 0$, unless both m and n vanish. This implies that

$$F(\beta, \gamma) = \frac{i}{8\pi^2 \sin \alpha}. \tag{20}$$

Substituting Eq. (20) into Eq. (16) yields the potential $V_{l,m,n}$. That is,

$$V_{l,m,n} = \int_0^{2\pi} d\beta \int_0^{2\pi} d\gamma \frac{i}{8\pi^2 \sin \alpha} v_{l,m,n}(\beta, \gamma). \tag{21}$$

As in Sec. 2, we can compute the capacitance $C_{l,m,n}$ by assuming that a charge of one micro-Coulomb enters the origin and leaves the lattice site (l, m, n) . Thus,

$$C_{l,m,n} = \frac{1}{\frac{i}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{d\beta d\gamma}{\sin \alpha} [1 - \exp(i|l|\alpha) + im\beta + in\gamma]}. \tag{22}$$

Again, this expression is symmetric under any permutation of the indices. A manifestly real form of Eq. (22) is

$$C_{l,m,n} = \frac{1}{\frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{d\beta d\gamma}{\sinh |\alpha|} [1 - \exp(-|l||\alpha|) \cos m\beta \cos n\gamma]}. \tag{23}$$

As $l \rightarrow \infty$, $C_{\infty,m,n} \rightarrow \frac{1}{\frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{d\beta d\gamma}{\sinh |\alpha|}} \rightarrow 0$.

Using Eq. (23), we can calculate $C_{l,m,n}$ by means of Mathematica. Table 2 shows some calculated values.

Table 2. Numerical values of $C_{l,m,n}$ in units of C for an infinite SC grid.

l, m, n	$C_{l,m,n}$	l, m, n	$C_{l,m,n}$	l, m, n	$C_{l,m,n}$
0	∞	410	2.144	531	2.08958
100	3.000003	411	2.138018	532	2.084667
110	2.531139	420	2.12867	533	2.077909
111	2.3906	421	2.124356	540	2.080503
200	2.382751	422	2.113825	541	2.079348
210	2.306284	430	2.111192	542	2.075559
211	2.264847	431	2.108277	543	2.070342
220	2.225432	432	2.100721	544	2.064235
221	2.206804	433	2.09079	550	2.070179
222	2.173162	440	2.094776	551	2.069768
300	2.220392	441	2.092865	552	2.066637
310	2.200632	442	2.087565	553	2.062804
311	2.186289	443	2.0803	554	2.057948
320	2.167735	444	2.072238	555	2.05287
321	2.159146	500	2.11299	600	2.088777
322	2.14053	510	2.109767	610	2.087086
330	2.136601	511	2.106833	633	2.069498
331	2.131646	520	2.101692	644	2.056729
332	2.119735	521	2.099336	655	2.047817
333	2.105161	522	2.093075	700	2.071745
400	2.15107	530	2.091324	531	2.08958

This method can be generalized in a straightforward way to four and more dimensions. In a $(d + 1)$ -dimensional hypercubic lattice, the capacitance between the origin and the lattice site (m_1, m_2, \dots, m_d) is

$$C_{m_1, \dots, m_d} = \frac{1}{\frac{i}{(2\pi)^d} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{d\beta_1 \dots d\beta_d}{\sin \alpha} [1 - \exp(i|m_1|\alpha) + im_2\beta_2 + \dots + im_d\beta_d]}, \quad (24)$$

where $\cos \alpha + \cos \beta_1 + \dots + \cos \beta_d = d$.

As a further generalization, consider the SC network with different capacitances in the three directions. For example, say, 1 micro-Farad along the x -direction, $\frac{1}{p}$ micro-Farad along the y -direction and $\frac{1}{q}$ micro-Farad along the z -direction. In this case, the charge entering the lattice site (l, m, n) is given by

$$Q_{l,m,n} = 2(1+p+q)V_{l,m,n} - V_{l+1,m,n} - pV_{l,m+1,n} - pV_{l,m-1,n} - qV_{l,m,n+1} - qV_{l,m,n-1}. \quad (25)$$

Therefore, the capacitance $C_{l,m,n}$ is still given by Eq. (24), but with

$$\alpha = \cos^{-1}(1 + p + q - p \cos \beta - q \cos \gamma). \quad (26)$$

Now, with $p = q = 1$ we recover the symmetric SC lattice, while for $p = 1$ and $q = 0$, we give the square lattice discussed in Sec. 2. Finally, for $p \neq 1$ and $q = 0$,

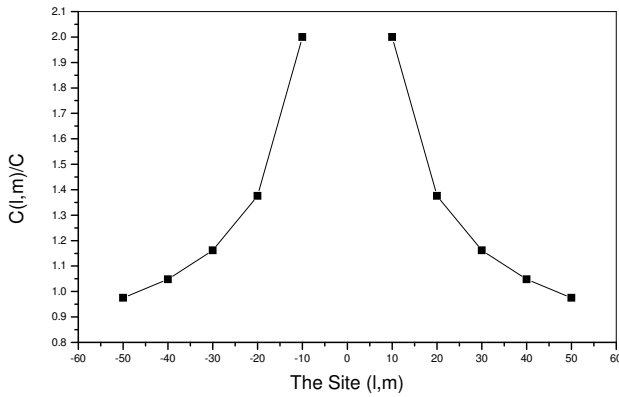


Fig. 1. The capacitance $C_{l,m}$ in terms of l and m for an infinite square grid along the [10] direction.

we have the “rectangular” lattice (i.e. a square lattice with unequal capacitances in the two coordinate directions).

4. Results and Discussion

This work is divided into two main parts. In part one, the capacitance between the site $(0, 0)$ and the site (l, m) in an infinite square grid consisting of identical capacitors is calculated using the superposition of charge distribution. The capacitance $C_{l,m}$ is expressed in an integral form which is evaluated analytically and numerically (Table 1). In part two, the capacitance between the site $(0, 0, 0)$ and the site (l, m, n) in an infinite SC grid consisting of identical capacitors is also calculated using the superposition of charge distribution. The capacitance $C_{l,m,n}$ is expressed in an integral form as the infinite square grid which is evaluated analytically and numerically (Table 2).

In Figs. 1 and 2, the capacitance is plotted against the site (l, m) . Figure 1 shows the capacitance of the infinite square grid as a function of l and m along the [10] direction and Fig. 2 shows the capacitance of the infinite square grid as a function of l and m along the [11] direction. Inversion symmetry is present in both figures around the origin.

In Figs. 3 and 4, the capacitance is plotted against the site (l, m, n) . Figure 3 shows the capacitance of an infinite SC grid as a function of l, m , and n along the [100] direction and Fig. 4 shows the capacitance of the infinite SC grid as a function of l, m , and n along the [111] direction. One can notice from these two figures that there are inversion symmetries about the origin.

The asymptotic form of Eqs. (14) and (23) corresponding to the identical capacitors of infinite square lattice and infinite simple cubic lattice, respectively, leads to zero as l goes to infinity. (See Figs. 1 and 3.)

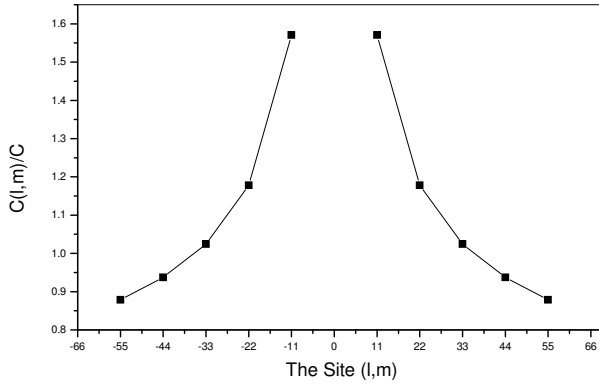


Fig. 2. The capacitance $C_{l,m}$ in terms of l and m for an infinite square grid along the $[11]$ direction.

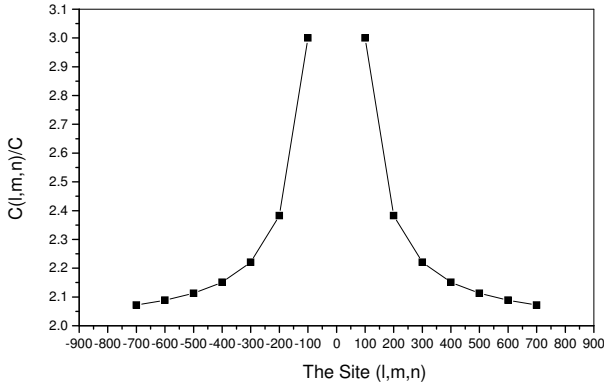


Fig. 3. The capacitance $C_{l,m,n}$ in terms of $l, m,$ and n for an infinite SC grid along the $[100]$ direction.

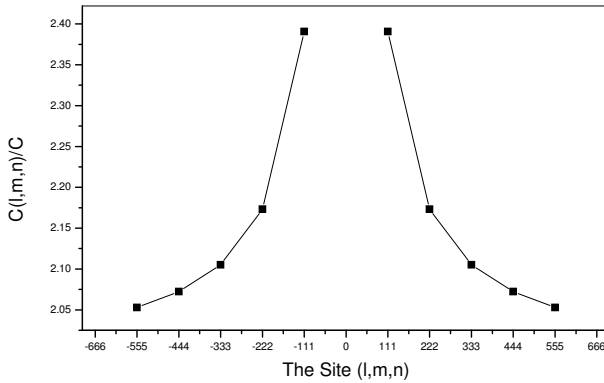


Fig. 4. The capacitance $C_{l,m,n}$ in terms of $l, m,$ and n for an infinite SC grid along the $[111]$ direction.

An investigation of infinite complicated lattices and of lattices with missing capacitor (bond) is in progress. The analysis of complex infinite networks of identical capacitors (i.e. triangular, honeycomb, and Kagome networks) has been demonstrated, in a recent paper²⁸ using basic concepts of physics.

References

1. P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks*, in The Carus Mathematical Monograph, Ser. 22 (The Mathematical Association of America, USA, 1984), p. 83.
2. L. Lovász, Random walks on graphs: A survey, in *Combinatorics, Paul Erdős is Eighty*, Vol. 2, eds. D. Miklós, V. T. Sós and T. Szónyi (János Bolyai Mathematical Society, Budapest, 1996), pp. 353–398, <http://research.microsoft.com~Lovász>.
3. B. Van der Pol, The finite-difference analogy of the periodic wave equation and the potential equation, in *Probability and Related Topics in Physical Sciences*, Lectures in Applied Mathematics, Vol. 1, ed. M. Kac (Interscience Publ., London, 1959), pp. 237–257.
4. S. Redner, *A Guide to First-Passage Processes* (Cambridge Press, Cambridge, UK, 2001).
5. S. Katsura and T. Horiguchi, *J. Math. Phys.* **12**(2) (1971) 230–231.
6. G. Kirchhoff, *Ann. Phys. und Chemie* **72** (1847) 497–508.
7. B. Van der Pol and H. Bremmer, *Operational Calculus Based on the Two-Sided Laplace Integral* (Cambridge University Press, England, 1955).
8. F. J. Bartis, *Am. J. Phys.* **35** (1967) 354–362.
9. G. Venezian, *Am. J. Phys.* **62** (1994) 1000–1004.
10. D. Atkinson and F. J. Van Steenwijk, *Am. J. Phys.* **67** (1999) 486–492.
11. M. Jeng, *Am. J. Phys.* **68**(1) (2000) 37–40.
12. J. Cserti, *Am. J. Phys.* **68** (2000) 896–906.
13. J. Cserti, G. David and A. Piroth, *Am. J. Phys.* **70** (2002) 153–159.
14. J. H. Asad, Ph.D Thesis, University of Jordan, 2004, unpublished.
15. J. H. Asad, A. Sakaji, R. S. Hijjawi and J. M. Khalifeh, *Int. J. Theo. Phys.* **43**(11) (2004) 2223–2235.
16. J. H. Asad, A. Sakaji, R. S. Hijjawi and J. M. Khalifeh, On the resistance of an infinite square network of identical resistors — Theoretical and experimental comparison, 2005, to be published.
17. J. H. Asad, R. S. Hijjawi, A. J. Sakaji and J. M. Khalifeh, *Int. J. Theo. Phys.* **44**(4) (2005) 471–483.
18. R. S. Hijjawi, J. H. Asad, A. J. Sakaji, M. Al-Sabayleh and J. M. Khalifeh, *Eur. Phys. J. — Appl. Phys.* **41** (2008) 111–114.
19. F. Y. Wu, *J. Phys. A: Math. Gen.* **37** (2004) 6653.
20. S. J. Van Enk, *Am. J. Phys.* **68**(9) (2000) 854–856.
21. J. H. Asad, A. J. Sakaji, R. S. Hijjawi and J. M. Khalifeh, *Int. J. Mod. Phys. B.* **19**(24) (2005) 3713–3721.
22. J. H. Asad, A. J. Sakaji, R. S. Hijjawi and J. M. Khalifeh, *Eur. Phys. J. — Appl. Phys.* **32**(3) (2005) 149–154.
23. J. H. Asad, A. J. Sakaji, R. S. Hijjawi and J. M. Khalifeh, *Eur. Phys. J. — Appl. Phys.* **40**(3) (2007) 257–264.
24. R. S. Hijjawi, J. H. Asad, A. J. Sakaji and J. M. Khalifeh, *Int. J. Mod. Phys. B* **21**(2) (2007) 199–209.

25. E. M. Purcell, *Electricity and Magnetism*, 1st edn., in Berkeley Physics Course Vol. 2 (McGraw-Hill, New York, 1965), p. 424.
26. E. M. Purcell, *Electricity and Magnetism*, 2nd edn., in Berkeley Physics Course Vol. 2 (McGraw-Hill, New York, 1985), pp. 167–168.
27. A. H. Zemanian, *IEEE Trans. Circuits Syst.* **35** (1988) 1346–1358; A. H. Zemanian, Transfiniteness for graphs, *Electrical Networks and Random Walks* (Birkhauser, Boston, 1966).
28. J. H. Asad, *Asian J. Appl. Sci.* **2**(3) (2009) 296–299.