Romanian Reports in Physics, Vol. 64, No. 4, P. 907–914, 2012

THEORETICAL PHYSICS

FRACTIONAL-ORDER TWO-ELECTRIC PENDULUM

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Received March 2, 2012

Abstract. In this paper we study the fractional Lagrangian of the two-electric pendulum. We obtained the fractional Euler-Lagrangian equation of the system and then we studied the obtained Euler-Lagrangian equation analytically, and numerically. The numerical method used here is based on Grünwald-Letnikov definition of left and right fractional derivatives.

Key words: Riemann-Liouville derivatives, two-electric pendulum, Grünwald-Letnikov derivative.

1. INTRODUCTION

As it is well known the Newtonian mechanics is mathematically fairly straightforward, and can be applied to a wide variety of classical problems. It is not a unique formulation of mechanics, however other formulations are possible such as the two common alternative formulations of classical mechanics: Lagrangian mechanics and Hamiltonian mechanics [1, 2]. It is important to note that all of these formulations of mechanics equivalent. In principle, any of them could be used to solve any problem in classical mechanics. Their importance is that to set up the equations of motion for certain physical systems is more convenient and useful than the use of Newtonian mechanics. Unlike Newtonian mechanics, neither Lagrangian nor Hamiltonian mechanics requires the concept of force; instead, these systems are expressed in terms of energy.

Lagrangian and Hamiltonian mechanics were used to solve wide range of physical problems in classical mechanics. One of these mechanical physical systems is two electric pendulum [3].

Fractional calculus is a branch of calculus that deals with half-integer power derivatives and integrals. The mathematical tools from fractional calculus have been used widely to study many phenomena in engineering, physics, as well as in other sciences [4–13]. The formulation of the fractional Euler-Lagrange problem has been recently drew the attention of many authors in their works [14–19]. Particularly, finding the numerical solutions of the equations involving the left and right derivatives is still an open problem in the field of the fractional dynamics [see for Ref. 19 and the references therein].

Numerical analysis of fractional differential equations has been used by many authors to solve wide ranges of differential equations [20–25]. In his recent work, Podlubny [26, 27] shows how we can numerically investigate differential equations using the so-called matrix approach method.

Having these things in mind, in this paper, we pay attention to study numerically the fractional Euler-Lagrange equation of a mechanical physical system called two electric pendulum.

This work is organized as follows:

In Sect. 2 we discussed briefly the basic definitions of the fractional derivatives. In Sect. 3 we study the fractional two electric pendulum model. In Sect. 4 numerical analysis of the obtained Euler-Lagrange equation of our model is carried out. Finally, we closed our paper with concluding remarks.

2. BASIC DEFINITIONS

In this section we discussed the definitions of the fractional derivatives (left and right Riemann-Liouville fractional derivatives). These definitions are used in the Hamiltonian formulation and the solution of examples leading to the equations of motion of the fractional order. The left Riemann-Liouville fractional integral (LRLFI) is defined as follows [8, 9]:

$$ {}_aI_t^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)\mathrm{d}\tau \tag{1}$$

The right Riemann-Liouville fractional integral (RRLFI) has the form

$$ {}_tI_b^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}x(\tau)\mathrm{d}\tau\,. \tag{2}$$

The left Riemann-Liouville fractional derivative (LRLFD) reads

$$ {}_aD_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha-n+1}}\mathrm{d}\tau\,. \tag{3}$$

The right Riemann-Liouville fractional derivative (RRLFD) reads

$$ {}_{x}D_{b}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n} \int_{x}^{b} \frac{f(\tau)}{(\tau - x)^{\alpha-n+1}} \mathrm{d}\tau . \quad (4) $$

Here α is the order of the derivative such that $n-1 \leq \alpha \leq n$ and is not equal to zero. If α is an integer, these derivatives are defined in the usual sense, i.e.,

$$ {}_{a}D_{x}^{\alpha} f(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\alpha} f(x) $$

$$ {}_{x}D_{b}^{\alpha} f(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\alpha} f(x) ; \ \alpha = 1, 2, \ldots . \quad (5) $$

3. THE MODEL

The model of the two electric pendulum consists of two planar pendula, both of length l and mass m suspended a distance d apart on a horizontal line so that they swing in the same plane. The kinetic energy is given by:

$$ T = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2) , \quad (6) $$

where q_1 and q_2 denotes the corresponding coordinates.

Potential energy is the sum of two terms; one gravitational, the other is an electrostatic, and they are written successively as:

$$ U_G = \frac{1}{2} \frac{mg}{l} (q_1^2 + q_2^2) , \quad (7) $$

where g is the gravity constant.

$$ U_E = \frac{e^2}{d + q_2 - q_1} , \quad (8) $$

where e is the electron charge. As a result, the entire Lagrangian function is:

$$ L = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2} \frac{mg}{l} (q_1^2 + q_2^2) - \frac{e^2}{d + q_2 - q_1} . \quad (9) $$

The fractional form of the above equation has the following form:

$$ L^F = \frac{1}{2} m[{}_{a}D_{t}^{\alpha} q_1]^2 + \frac{1}{2} m[{}_{a}D_{t}^{\alpha} q_2]^2 - \frac{1}{2} \frac{mg}{l} (q_1^2 + q_2^2) - \frac{e^2}{d + q_2 - q_1} . \quad (10) $$

Now, to obtain Euler-Lagrange equations for the generalized coordinates (q_1, q_2) we use:

$$\frac{\partial L}{\partial q_i} + {}_tD_b^{\alpha}\frac{\partial L}{\partial\, {}_aD_t^{\alpha} q_i} + {}_aD_t^{\beta}\frac{\partial L}{\partial\, {}_tD_b^{\beta} q_i} = 0 . \tag{11}$$

Thus, for $q_i = q_1$, we get:

$${}_tD_b^{\alpha}\, {}_aD_t^{\alpha} q_1 - \frac{g}{l} q_1 - \frac{1}{m}\frac{e^2}{(d + q_2 - q_1)^2} = 0 , \tag{12a}$$

while, for $q_i = q_2$, we get:

$${}_tD_b^{\alpha}\, {}_aD_t^{\alpha} q_2 - \frac{g}{l} q_2 + \frac{1}{m}\frac{e^2}{(d + q_2 - q_1)^2} = 0 . \tag{12b}$$

As $\alpha \to 1$, we obtain the following two classical Euler-Lagrange equations

$$\ddot{q}_1 + \frac{g}{l} q_1 + \frac{e^2}{m(d + q_2 - q_1)^2} = 0 , \tag{13a}$$

$$\ddot{q}_2 + \frac{g}{l} q_2 - \frac{e^2}{m(d + q_2 - q_1)^2} = 0 . \tag{13b}$$

Our aim is to obtain a numerical solution for Eqs. (11a, and 11b) for arbitrary values of $m, l, d,$ and for arbitrary boundary conditions for different values of α.

4. NUMERICAL RESULTS OF FRACTIONAL LAGRANGIAN OF TWO-ELECTRIC PENDULUM

For numerical solution of the linear fractional-order equations (12a) and (12b) we can use the decomposition to its canonical form with substitutions $q_1 \equiv x_1$ and $q_2 \equiv x_2$. We obtain the set of equation in the form:

$$\begin{aligned}
&{}_aD_t^{\alpha} x_1 = x_3 ,\\
&{}_tD_b^{\alpha} x_3 = -\frac{g}{l} x_1 - \frac{1}{m}\frac{e^2}{(d + x_2 - x_1)^2} ,\\
&{}_aD_t^{\alpha} x_2 = x_4 ,\\
&{}_tD_b^{\alpha} x_4 = -\frac{g}{l} x_2 + \frac{1}{m}\frac{e^2}{(d + x_2 - x_1)^2} ,
\end{aligned} \tag{14}$$

where we can set four initial conditions: $x_1(0) \equiv q_1(0),\ x_2(0) \equiv q_2(0)$ and $x_3(0) \equiv {}_aD_t^\alpha q_1(0),\ x_4(0) \equiv {}_aD_t^\alpha q_2(0)$. Instead left and right side Riemann-Liouville fractional derivatives (3) and (4) in the set of equations (14) can be used the left and right Grünwald-Letnikov derivatives, which are equivalent to the Riemann-Liouville fractional derivatives for a wide class of the functions [8]. The Grünwald-Letnikov derivatives can be defined by using upper and lower triangular strip matrices (Podlubny's matrix approach) or we can directly apply the formula derived from the Grünwald-Letnikov definitions, backward and forward, respectively, for discrete time step *kh, k* = 1, 2, 3, … Le us consider the second approach, which works very well for linear as well as for nonlinear fractional differential equations [28]. Time interval [*a, b*] is discretized by (N + 1) equal grid points, where $N = (b-a)/h$. Thus, we obtain the following formula for discrete equivalents of left and right fractional derivatives:

$$ {}_aD_t^\alpha x_k = h^{-\alpha}\sum_{i=0}^{k} c_i x_{k-i}, \quad k = 0,\dots,N, \tag{15a}$$

$$ {}_tD_b^\alpha x_k = h^{-\alpha}\sum_{i=0}^{N-k} c_i x_{k+i}, \quad k = N,\dots,0, \tag{15b}$$

respectively, where $x_k \approx x(t_k)$ and $t_k = kh$. The binomial coefficients c_i, i = 1, 2, 3,…, can be calculated according to relation

$$ c_i = \left(1 - \frac{1+\alpha}{i}\right)c_{i-1}, \tag{16}$$

for $c_0 = 1$. Then, general numerical solution of the fractional linear differential equation with left side derivative in the form

$$ {}_aD_t^\alpha x(t) = f(x(t),t) \tag{17}$$

can be expressed for discrete time $t_k = kh$ in the following form:

$$ x(t_k) = f(x(t_k),t_k)h^\alpha - \sum_{i=m}^{k} c_i x(t_{k-i}), \tag{18}$$

where m = 0 if we do not use a short memory principle, otherwise it can be related to memory length. Similarly we can derive a solution for an equation with right side fractional derivative.

Let us consider the different value of order α for simulation time 2 sec and time step h = 0.0005. The parameters set up are the following: m = 1 kg, l = 1 m, d = 1m.

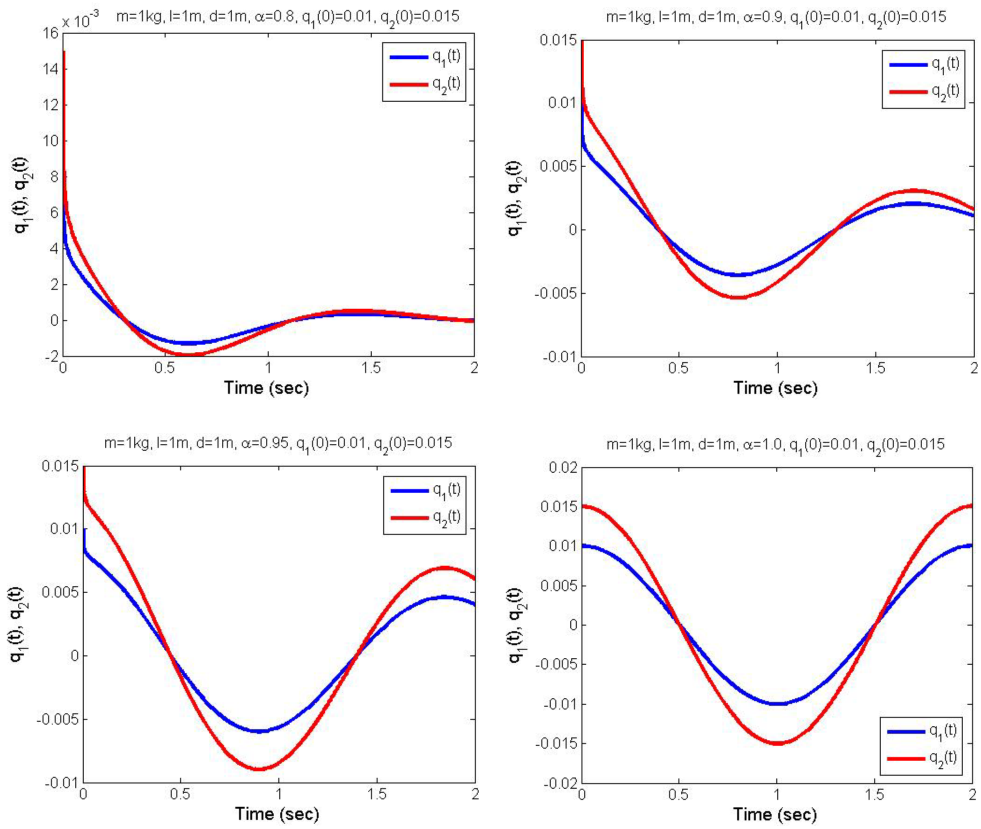

Fig. 1 – Simulation results for various parameters and orders.

In Fig. 1 are depicted the simulation results of equation (14) for parameters m=1kg, l=1m, d=1m, and various order α, where derivative interval is a=0 and b=2, initial conditions $q_1(0)$=0.01, $q_2(0)$=0.015, α^{th} derivatives of $q_1(0)$ and $q_2(0)$ are zeros, for total simulation time 2 sec and computational time step h = 0.0005.

5. CONCLUSIONS

The fractional Euler-Lagrange equations have the particularity that they contain both the left and the right derivatives. These kind of equations started recently to be analyzed both from theoretical and numerically point of views. In this manuscript the numerical solutions of the fractional Euler-Lagrange equations corresponding to the fractional electric model are presented for various values of α.

Acknowledgments. The work of Ivo Petras was supported in part by grants VEGA: 1/0729/12, 1/0497/11, 1/0746/11.

REFERENCES

1. J. B. Marion, S. T. Thornton, *Classical Dynamics of Particles and Systems*, Third Edition, Harcourt Brace Jovanovich, 1988.
2. H. Goldstein, C. P. Poole, J. L. Safko, *Classical Mechanics*, Third Edition, Addison Wesley, 1980.
3. *** www.math.uchicago.edu/~may/VIGRE/VIGRE2009/.../Tolish.pdf
4. R.L. Bagley, P.J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol., **27**, 201–210 (1983).
5. F. Mainardi, *On the Initial Value Problem for the Fractional Diffusion-wave Equations*, in: S. Rionero and T. Ruggeri (eds.); *Waves and Stability in Continuous Media*, Word Scientific, Singapore, 1994, pp. 246–251.
6. F. Mainardi, *The Fundamental Solutions for the Fractional Diffusion-wave Equation*, Appl. Math. Lett., **9**, *6*, 23–28 (1996).
7. A.A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204, North-Holland, Amsterdam 2006.
8. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
9. S. G. Samko, A.A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.
10. D. Baleanu, S. I. Muslih, E. M. Rabei, Golmankhaneh, Alireza K, Golmankhaneh, Ali K, *On fractional Hamiltonian systems possessing first-class constraints within Caputo derivatives*, Rom. Rep. Phys., **63**, *1*, 3 (2011).
11. D. Baleanu, S. I. Vacaru, *Fedosov quantization of fractional Lagrange spaces*, Int. J. Theor. Phys., **50**, *1*, 233 (2011).
12. S. I. Muslih, S. Madhat, D. Baleanu,E.M. Rabei, *Lagrangian formulation of Maxwell's field in fractional D dimensional space-time*, Rom. J. Phys., **55**, 7–8, 659 (2010).
13. R. R. Nigmatullin, D. Baleanu, *Is it possible to derive Newtonian equations of motion with memory?*, Int. J. Theor. Phys., **49**, *4*, 701 (2010).
14. F. Riewe, *Nonconservative Lagrangian and Hamiltonian mechanics*, Phys. Rev., E **53**, 1890 (1996).
15. O. P. Agrawal, *Formulation of Euler–Lagrange equations for fractional variational problems*, J. Math. Anal., Appl., **272**, 368 (2002).
16. D. Baleanu, J. J. Trujillo, *On exact solutions of a class of fractional Euler-Lagrange equations*, Nonlinear Dynamics, **52**, *4*,331 (2007).
17. M. Klimek, *Lagrangian and Hamiltonian fractional sequential mechanics*, Czech. J. Phys., **52**, 1247 (2002).
18. D. Baleanu, T. Avkar, *Lagrangians with linear velocities within Riemann-Liouville fractional derivatives*, Nuovo Cim., B **119**, 73 (2004).
19. D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, Series on Complexity, Nonlinearity and Chaos, World Scientific, 2012.
20. P. Kumar, O.P. Agrawal, *An approximate method for numerical solution of fractional differential equations*, Signal Proc., **86**, *10*, 234 (2006).
21. R. Scherer, S. L. Kalla, Y. Tang, J. Huang, *The Grünwald-Letnikov method for fractional differential equations*, Comput. Math. Appl., **62**, *3*, 432 (2011).
22. D. Baleanu, I. Petras, J. H. Asad, M. P. Velasco, *Fractional Pais–Uhlenbeck Oscillator*, Int. J. Theor. Phys., 2012; doi: 10.1007/s10773-011-1000-y.

23. K. Diethelm, N. J. Ford, *Numerical solution of the Bagley-Torvik equation*, BIT, **42**, *3*, 490 (2002).
24. S.S. Ray, R. K. Bera, *Analytical solution of the Bagley Torvik equation by Adomian decomposition method*, Appl. Math. Comput., **168**, *1*, 398 (2005).
25. K. Diethelm, N.J. Ford, A.D. Freed, Yu. Luchko, *Algorithms for the fractional calculus: A selection of numerical methods*, Comput. Methods Appl. Mech. Engr., **194**, 743 (2005).
26. I. Podlubny, *Matrix approach to discrete fractional calculus*, Fract. Calcul. Appl. Anal., **3**, *4*, 359 (2010).
27. I. Podlubny, A.V. Chechkin, T. Skovranek, Y. Q. Chen, B. Vinagre, *Matrix approach to discrete fractional calculus II: partial fractional differential equations*, J. Comput. Phys., **228**, *8*, 3137 (2009).
28. I. Petras, *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*, Series: Nonlinear Physical Science, Springer, HEP, London, 2011.