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Inequalities for Contraction Matrices

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ABSTRACT

Let A, B, C, X, and Y be $n\times n$ matrices such that A and B are positive definite contractions. It is shown that if $r \geq s_n(A)$ and $t \geq s_n(B)$, then

$$\|A^{-r}X + XB^{-t}\|_2^2 + \|AX + XB\|_2^2 \leq 4\|AXB^{-1} + A^{-1}XB\|_2^2.$$

Moreover, if $0 < Y \leq X \leq C + Y \leq 2C$, then

$$s_j\left((C+X)^{-1/2}A(C+Y)^{-1/2}\right) \leq \frac{\kappa(C)}{\|C\| + \sqrt{s_{n-j+i}(X)s_{n-j+i}(Y)}}$$

for $i, j = 1, \ldots, n$ with $i \leq j \leq 2i-1$, where $\|T\|_2, \|T\|, s_j(T)$, and $\kappa(T)$ denote the Hilbert-Schmidt norm, the spectral matrix norm, the j th singular value, and the condition number of the $n\times n$ matrix T, respectively.

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1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n\times n$ complex matrices. The singular values $s_1(A), \ldots, s_n(A)$ of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ are the eigenvalues of the matrix $(A^*A)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. A Hermitian matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be positive semidefinite, written as $A \geq 0$, if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$ and it is called positive definite, written as $A > 0$, if $x^*Ax > 0$ for all $x \in \mathbb{C}^n$ with $x \neq 0$. The Hilbert–Schmidt norm (or the Frobenius norm) $\|\cdot\|_2$ is the norm defined on $\mathbb{M}_n(\mathbb{C})$ by $\|A\|_2 = (\sum_{j=1}^n s_j^2(A))^{1/2}, A \in \mathbb{M}_n(\mathbb{C})$. The Hilbert–Schmidt norm is unitarily invariant, that is $\|UAV\|_2 = \|A\|_2$ for all $A \in \mathbb{M}_n(\mathbb{C})$ and all unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. Another property of the Hilbert–Schmidt norm is that $\|A\|_2 = (\sum_{i,j=1}^n |f_j^*Ae_i|^2)^{1/2}$, where $\{e_j\}_{j=1}^n$ and $\{f_j\}_{j=1}^n$ are two orthonormal bases of $\mathbb{C}^n$. The spectral matrix norm, denoted by $\|\cdot\|$, of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is the norm defined by $\|A\| = \sup\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}$ or equivalently $\|A\| = s_1(A)$, For further properties of these norms, the reader is

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referred to [1] or [2]. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called contraction if $\|A\| \leq 1$, or equivalently, $A^*A \leq I_n$, where I_n is the identity matrix in $\mathbb{M}_n(\mathbb{C})$.

In this paper, we introduce new norm and singular value inequalities for matrices. In Section 2, we use a recent refinement of Young's inequality for scalars to introduce matrix inequalities for the Hilbert–Schmidt and the spectral norms. In Section 3, we are interested in singular value inequalities of some powers of matrices. In Section 4, we studied a scalar inequality of Borwein and we give a matrix version of it.

2. Norm inequalities for matrices

The classical Young's inequality for scalars asserts that if a and b be positive real number and $\alpha \in (0,1)$, then

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \tag{2.1}$$

with equality if and only if $a = b$. Improvements of the inequality (2.1) have been given by several mathematicians (see, e.g., [3–9], and [10, 11]). One of these improvements is the following [9]: If a and b be positive real number and $\alpha \in (0,1)$, then

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b - \min(\alpha, 1-\alpha)\left(a^{1/2} - b^{1/2}\right)^2. \tag{2.2}$$

Based on the inequality (2.2), we have the following lemma.

Lemma 2.1. *Let $a, b \in (0,1]$. Then*

$$(a^{-b} + b^{-a}) + (a+b) + \left(c_0\left(1 - a^{-1/2}\right)^2 + d_0\left(1 - b^{-1/2}\right)^2\right) \leq 2\left(\frac{a}{b} + \frac{b}{a}\right) \tag{2.3}$$

with equality if and only if $a = b = 1$, where $c_0 = \min(b, 1-b)$ and $d_0 = \min(a, 1-a)$. In particular,

$$a^{-b} + b^{-a} + (a+b) < 2\left(\frac{a}{b} + \frac{b}{a}\right). \tag{2.4}$$

Proof. Let $a, b \in (0,1]$. Then

$$\begin{aligned} &a^{-b} + b^{-a} \\ &= \frac{a^{1-b}1^{b}}{a} + \frac{b^{1-a}1^{a}}{b} \\ &\leq \frac{(1-b)a + b - c_0\left(a^{1/2} - 1\right)^2}{a} + \frac{(1-a)b + a - d_0\left(b^{1/2} - 1\right)^2}{b} \\ &\text{(by the inequality (2.2))} \end{aligned} \tag{2.5}$$

$$= 2 + \frac{a}{b} + \frac{b}{a} - (a+b) - \left(c_0\left(1 - a^{-1/2}\right)^2 + d_0\left(1 - b^{-1/2}\right)^2\right). \tag{2.6}$$

Since $2ab \le a^2 + b^2$, we have

$$\frac{a}{b} + \frac{b}{a} \ge 2. \tag{2.7}$$

Now, the inequality (2.3) follows from the inequalities (2.6) and (2.7). The equality conditions follow by applying the equality condition of the inequality (2.1) to the inequality (2.5). □

Based on Lemma 2.1, we have the following matrix version of the inequality (2.3) in the setting of the Hilbert–Schmidt norm.

Theorem 2.2. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite contractions. If $r \ge s_n(A)$ and $t \ge s_n(B)$, then*

$$\|A^{-r}X + XB^{-t}\|_2 \le \left\| \begin{array}{c} 2\left(AXB^{-1} + A^{-1}XB\right) - (AX + XB) \\ -\left(c_0\left(I_n - A^{-1/2}\right)^2 X + d_0 X\left(I_n - B^{-1/2}\right)^2\right) \end{array} \right\|_2 \tag{2.8}$$

with equality if and only if $A = B = I_n$, where $c_0 = \min(s_n(B), 1 - s_1(B))$ and $d_0 = \min(s_n(A), 1 - s_1(A))$.

Proof. Since A and B are positive definite, then there exist two orthonormal bases $\{e_j\}_{j=1}^n$ and $\{f_j\}_{j=1}^n$ of $\mathbb{C}^n$ such that $Af_j = s_j(A)f_j$ and $Be_j = s_j(B)e_j$ for $j = 1, \ldots, n$. Since A and B are positive definite contractions, then $s_j(A), s_j(B) \in (0, 1]$ for $j = 1, \ldots, n$. Now,

$$\left\| \begin{array}{c} 2\left(AXB^{-1} + A^{-1}XB\right) - (AX + XB) \\ -\left(c_0\left(I_n - A^{-1/2}\right)^2 X + d_0 X\left(I_n - B^{-1/2}\right)^2\right) \end{array} \right\|_2^2$$

$$= \sum_{i,j=1}^{n} \left| f_j^* \left(\begin{array}{c} 2\left(AXB^{-1} + A^{-1}XB\right) - (AX + XB) \\ -\left(c_0\left(I_n - A^{-1/2}\right)^2 X + d_0 X\left(I_n - B^{-1/2}\right)^2\right) \end{array} \right) e_i \right|^2 \tag{2.9}$$

$$= \sum_{i,j=1}^{n} \left(\begin{array}{c} 2\left(\frac{s_j(A)}{s_i(B)} + \frac{s_i(B)}{s_j(A)}\right) - \left(s_j(A) + s_i(B)\right) \\ -\left(c_0\left(1 - s_j^{-1/2}(A)\right)^2 + d_0\left(1 - s_i^{-1/2}(B)\right)^2\right) \end{array} \right)^2 |f_j^* X e_i|^2.$$

Let $\tilde{c}_j = \min(s_j(B), 1 - s_j(B))$ and $\tilde{d}_i = \min(s_i(A), 1 - s_i(A))$ for $i, j = 1, \ldots, n$. Since $c_0 \le \tilde{c}_j$ and $d_0 \le \tilde{d}_i$ for $i, j = 1, \ldots, n$, it follows that

$$\left\| \begin{array}{c} 2(AXB^{-1} + A^{-1}XB) - (AX + XB) \\ -\left(c_0\left(I_n - A^{-1/2}\right)^2 X + d_0 X\left(I_n - B^{-1/2}\right)^2\right) \end{array} \right\|_2^2$$

$$\geq \sum_{i,j=1}^{n} \left(\begin{array}{c} 2\left(\frac{s_j(A)}{s_i(B)} + \frac{s_i(B)}{s_j(A)}\right) - \left(s_j(A) + s_i(B)\right) \\ -\left(\tilde{c}_j\left(1 - s_j^{-1/2}(A)\right)^2 + \tilde{d}_i\left(1 - s_i^{-1/2}(B)\right)^2\right) \end{array} \right)^2 |f_j^* X e_i|^2$$

$\big($by the inequality (2.9)$\big)$

$$\geq \sum_{i,j=1}^{n} \left(s_j^{-s_i(B)}(A) + s_i^{-s_j(A)}(B)\right)^2 |f_j^* X e_i|^2 (\text{by Lemma 2.1}) \tag{2.10}$$

$$\geq \sum_{i,j=1}^{n} \left(s_j^{-s_n(B)}(A) + s_i^{-s_n(A)}(B)\right)^2 |f_j^* X e_i|^2$$

$$\geq \sum_{i,j=1}^{n} \left(s_j^{-r}(A) + s_i^{-t}(B)\right)^2 |f_j^* X e_i|^2$$

$$= \sum_{i,j=1}^{n} |f_j^* (A^{-r}X + XB^{-t}) e_i|^2$$

$$= \|A^{-r}X + XB^{-t}\|_2^2,$$

For the equality conditions, it is clear that if $A = B = I_n$, then equality holds in the inequality (2.8). So suppose that equality holds in the inequality (2.8). Then, we have equality in the inequality (2.10) and by the equality conditions of the inequality (2.3), we have $s_j(A) = s_j(B) = 1$ for $j = 1, \ldots, n$. So, by the spectral theorem of positive matrices we have $A = B = I_n$. □

An application of Theorem 2.2 can be seen as follows.

Corollary 2.3. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite contractions. If $r \geq s_n(A)$ and $t \geq s_n(B)$, then*

$$\begin{aligned} &\|A^{-r}X + XB^{-t}\|_2^2 + \|AX + XB\|_2^2 + \left\|c_0\left(I_n - A^{-1/2}\right)^2 X + d_0 X\left(I_n - B^{-1/2}\right)^2\right\|_2^2 \\ &\leq 4\|AXB^{-1} + A^{-1}XB\|_2^2, \end{aligned} \tag{2.11}$$

where $c_0 = \min(s_n(B), 1 - s_1(B))$ and $d_0 = \min(s_n(A), 1 - s_1(A))$. In particular,

$$\|A^{-r}X + XB^{-t}\|_2^2 + \|AX + XB\|_2^2 \leq 4\|AXB^{-1} + A^{-1}XB\|_2^2. \tag{2.12}$$

Proof. Let $\{e_j\}_{j=1}^n$ and $\{f_j\}_{j=1}^n$ be the two orthonormal bases of $\mathbb{C}^n$ as given in the proof of Theorem 2.2. Consequently,

$$
\begin{aligned}
&4\|AXB^{-1}+A^{-1}XB\|_2^2-\|AX+XB\|_2^2-\left\|c_0\left(I_n-A^{-1/2}\right)^2X+d_0X\left(I_n-B^{-1/2}\right)^2\right\|_2^2\\
&=\sum_{i,j=1}^{n}\left(\begin{array}{c}4\left(\frac{s_j(A)}{s_i(B)}+\frac{s_i(B)}{s_j(A)}\right)^2-\left(s_j(A)+s_i(B)\right)^2\\ -\left(c_0\left(1-s_j^{-1/2}(A)\right)^2+d_0\left(1-s_i^{-1/2}(B)\right)^2\right)^2\end{array}\right)|f_j^*Xe_i|^2\\
&\geq\sum_{i,j=1}^{n}\left(\begin{array}{c}2\left(\frac{s_j(A)}{s_i(B)}+\frac{s_i(B)}{s_j(A)}\right)-\left(s_j(A)+s_i(B)\right)\\ -\left(c_0\left(1-s_j^{-1/2}(A)\right)^2+d_0\left(1-s_i^{-1/2}(B)\right)^2\right)\end{array}\right)^2|f_j^*Xe_i|^2\\
&=\left\|\begin{array}{c}2(AXB^{-1}+A^{-1}XB)-(AX+XB)\\ -\left(c_0\left(I_n-A^{-1/2}\right)^2X+d_0X\left(I_n-B^{-1/2}\right)^2\right)\end{array}\right\|_2^2,
\end{aligned}
\tag{2.13}
$$

where the identity (2.13) follows from the identity (2.9). Now, the inequality (2.11) follows from the inequalities (2.8) and (2.13). □

3. Singular value inequalities for matrices

An elementary inequality for scalars that is equivalent to the arithmetic-geometric mean inequality for scalars asserts that if a and b are two nonzero positive real numbers, then

$$(a+b)^{-1}\leq\frac{a^{-1}+b^{-1}}{4}\tag{3.1}$$

with equality if and only if $a=b$.

We need the following essential lemma (see, e.g., [12, p. 63]).

Lemma 3.1. *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be positive definite. Then*

$$s_j(A+B)\geq s_j(A)+s_n(B)$$

for $j=1,\ldots,n$.

A matrix version of the inequality (3.1) can be seen as follows.

Lemma 3.2. *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be positive definite. Then*

$$s_j^{-1}(A+B)\leq\frac{1}{4}\left(s_j^{-1}(A)+s_n^{-1}(B)\right)$$

for $j=1,\ldots,n$.

Proof. Since A and B are positive definite, Lemma 3.1 implies that

$$s_j(A+B)\geq s_j(A)+s_n(B)$$

for $j=1,\ldots,n$, and so,

$$\begin{aligned} s_j^{-1}(A+B) &\le \left(s_j(A)+s_n(B)\right)^{-1} \\ &\le \frac{1}{4}\left(s_j^{-1}(A)+s_n^{-1}(B)\right)\left(\text{by the inequality (3.1)}\right) \end{aligned}$$

for $j = 1, \ldots, n$. □

Based on Lemmas 2.1 and 3.2, we have the following result.

Theorem 3.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive definite contractions. If $r \le s_n(B)$ and $t \le s_n(A)$, then*

$$\begin{aligned} s_j^{-1}\left(A^r+B^t\right) \le& \frac{1}{2}\left(\frac{s_k(A)}{s_n(B)}+\frac{s_n(B)}{s_k(A)}\right)-\frac{1}{4}\left(s_k(A)+s_n(B)\right) \\ &-\frac{1}{4}\left(c_n\left(1-s_k^{-1/2}(A)\right)+d_k\left(1-s_n^{-1/2}(B)\right)\right) \end{aligned} \tag{3.2}$$

for $j, k = 1, \ldots, n$ with $k \ge j$, where $c_n = \min(s_n(B), 1-s_n(B))$ and $d_k = \min(s_k(A), 1-s_k(A))$. In particular,

$$s_j^{-1}\left(A^r+B^t\right) \le \frac{1}{2}\left(\frac{s_j(A)}{s_n(B)}+\frac{s_n(B)}{s_j(A)}\right)$$

for $j = 1, \ldots, n$.

Proof. Since A and B are positive definite contractions, then $s_j(A), s_j(B) \in (0,1], j = 1, \ldots, n$ and so $A^{s_n(B)}+B^{s_k(A)} \le A^r+B^t$. Consequently,

$$\begin{aligned} s_j^{-1}\left(A^r+B^t\right) &\le s_j^{-1}\left(A^{s_n(B)}+B^{s_k(A)}\right) \\ &\le \frac{1}{4}\left(s_j^{-1}\left(A^{s_n(B)}\right)+s_n^{-1}\left(B^{s_k(A)}\right)\right)\left(\text{by Lemma 3.2}\right) \\ &= \frac{1}{4}\left(s_j^{-s_n(B)}(A)+s_n^{-s_k(A)}(B)\right) \\ &\le \frac{1}{4}\left(s_k^{-s_n(B)}(A)+s_n^{-s_k(A)}(B)\right)\left(\text{since } k \ge j\right) \\ &\le \frac{1}{2}\left(\frac{s_k(A)}{s_n(B)}+\frac{s_n(B)}{s_k(A)}\right)-\frac{1}{4}\left(s_k(A)+s_n(B)\right) \\ &\quad -\frac{1}{4}\left(c_n\left(1-s_k^{-1/2}(A)\right)+d_k\left(1-s_n^{-1/2}(B)\right)\right) \\ &\quad \left(\text{by the inequality (2.3)}\right) \end{aligned}$$

for $j, k = 1, \ldots, n$ with $k \ge j$. □

An application of Theorem 3.3 can be stated as follows.

Corollary 3.4. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive definite contractions. If $r \leq s_n(B)$ and $t \leq s_n(A)$, then*

$$s_j^{-1}(A^r + B^t) + \frac{1}{4}\left(s_j(A) + s_n(B)\right) \leq \frac{1}{2}\left(\frac{s_j(A)}{s_n(B)} + \frac{s_n(B)}{s_j(A)}\right)$$

for $j = 1, ..., n$. In particular,

$$s_j^{-1}(A^s + B^s) \leq \frac{1}{2}\left(\frac{s_j(A)}{s_n(B)} + \frac{s_n(B)}{s_j(A)}\right)$$

for $j = 1, ..., \mathrm{n}$, where $s = \max\left(s_n(B), s_1(A)\right)$.

4. On Borwein inequality

Borwein inequality [13] (see also, [14, p. 283]) asserts that if a, b, x, and y are positive real numbers such that $a + b = 1, a \leq b, y \leq x \leq 1 + y \leq 2$. Then

$$\frac{a}{1+x} + \frac{b}{1+y} \leq \frac{1}{1+x^a y^b}. \tag{4.1}$$

In this section, we introduce matrix versions of the Borwein inequality (4.1). First, we start with the following lemma.

Lemma 4.1. *Let a, b, c, x, and y be positive real numbers such that $a + b = 1, a \leq b, y \leq x \leq c + y \leq 2c$. Then*

$$\frac{a}{c+x} + \frac{b}{c+y} \leq \frac{1}{c+x^a y^b}. \tag{4.2}$$

Proof. Let $\tilde{x} = \frac{x}{c}$ and $\tilde{y} = \frac{y}{c}$.Then the conditions $y \leq x \leq c + y \leq 2c$ imply that $\tilde{y} < \tilde{x} \leq 1 + \tilde{y} \leq 2$. So,

$$\begin{aligned} \frac{a}{c+x} + \frac{b}{c+y} &= \frac{1}{c}\left(\frac{a}{1+\frac{x}{c}} + \frac{b}{1+\frac{y}{c}}\right) \\ &= \frac{1}{c}\left(\frac{a}{1+\tilde{x}} + \frac{b}{1+\tilde{y}}\right) \\ &\leq \frac{1}{c}\frac{1}{1+\tilde{x}^a\tilde{y}^b} \left(\text{by the inequality (4.1)}\right) \\ &= \frac{1}{c+x^a y^b} \left(\text{since } c^{a+b} = c\right), \end{aligned}$$

as required. □

The following lemma is given in [12, p. 62].

Lemma 4.2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then $s_j(A+B) \le s_i(A) + s_{j-i+1}(B)$ for $i, j = 1, \dots, n$ with $i \le j$.*

Let $\kappa(A)$ be the condition number of a matrix $A \in \mathbb{M}_n(\mathbb{C})$, that is $\kappa(A) = \|A\| \|A^{-1}\|$. In the following result, we give a matrix version of the inequality (4.2) that involves the condition number of a matrix.

Theorem 4.3. *Let $A, B, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ be positive definite such that $\|A\| + \|B\| = 1, A \le B$, and $Y \le X \le C + Y \le 2C$. Then*

$$s_j\left(A^{\frac{1}{2}}(C+X)^{-1}A^{\frac{1}{2}} + B^{\frac{1}{2}}(C+Y)^{-1}B^{\frac{1}{2}}\right) \le \frac{\kappa(C)}{\|C\| + s_{n-j+i}^{\|A\|}(X)s_{n-j+i}^{\|B\|}(Y)} \tag{4.3}$$

for $i, j = 1, \dots, n$ with $i \le j \le 2i-1$. In particular, if $C = I_n$, then

$$s_j\left(A^{\frac{1}{2}}(I_n+X)^{-1}A^{\frac{1}{2}} + B^{\frac{1}{2}}(I_n+Y)^{-1}B^{\frac{1}{2}}\right) \le \frac{1}{1 + s_{n-j+i}^{\|A\|}(X)s_{n-j+i}^{\|B\|}(Y)} \tag{4.4}$$

for $i, j = 1, \dots, n$ with $i \le j \le 2i-1$.

Proof. Since $Y \le X \le Y + C$, then

$$s_{n-j+i}(Y) \le s_{n-j+i}(X)$$

and

$$\begin{aligned} s_{n-j+i}(X) &\le s_{n-j+i}(Y+C) \\ &\le s_{n-j+i}(Y) + \|C\| (\text{by Lemma 4.2}). \end{aligned}$$

So,

$$0 \le s_{n-j+i}(X) - s_{n-j+i}(Y) \le \|C\|. \tag{4.5}$$

Lemma 4.1 together with the inequality (4.5) implies that

$$\frac{\|A\|}{\|C\| + s_{n-j+i}(X)} + \frac{\|B\|}{\|C\| + s_{n-j+i}(Y)} \le \frac{1}{\|C\| + s_{n-j+i}^{\|A\|}(X)s_{n-j+i}^{\|B\|}(Y)}. \tag{4.6}$$

Since $j \le 2i-1$, then $n-j+i \ge n-i+1$, and so

$$\begin{aligned} s_{n-i+1}(C^{-1/2}XC^{-1/2}) &\ge s_{n-j+i}(C^{-1/2}XC^{-1/2}) \\ &= s_{n-j+i}(X^{1/2}C^{-1}X^{1/2}) \\ &\ge s_n(C^{-1})s_{n-j+i}(X) \\ &= \frac{s_{n-j+i}(X)}{\|C\|}. \end{aligned} \tag{4.7}$$

Similarly,

$$s_{n-j+i}(C^{-1/2}YC^{-1/2}) \geq \frac{s_{n-j+i}(Y)}{\|C\|}. \tag{4.8}$$

Now,

$$\begin{aligned}
&s_j\left(A^{1/2}(C+X)^{-1}A^{1/2}+B^{1/2}(C+Y)^{-1}B^{1/2}\right)\\
&= s_j\begin{pmatrix} A^{1/2}C^{-1/2}\left(I_n+C^{-1/2}XC^{-1/2}\right)^{-1}C^{-1/2}A^{1/2} \\ +B^{1/2}C^{-1/2}\left(I_n+C^{-1/2}YC^{-1/2}\right)^{-1}C^{-1/2}B^{1/2}\end{pmatrix}\\
&\leq s_i\left(A^{1/2}C^{-1/2}\left(I_n+C^{-1/2}XC^{-1/2}\right)^{-1}C^{-1/2}A^{1/2}\right)\\
&+ s_{j-i+1}\left(B^{1/2}C^{-1/2}\left(I+C^{-1/2}YC^{-1/2}\right)^{-1}C^{-1/2}B^{1/2}\right)\\
&(\text{by Lemma 4.2})\\
&\leq \|A^{1/2}C^{-1/2}\|\|C^{-1/2}A^{1/2}\|s_i\left(\left(I_n+C^{-1/2}XC^{-1/2}\right)^{-1}\right)\\
&+ \|B^{1/2}C^{-1/2}\|\|C^{-1/2}B^{1/2}\|s_{j-i+1}\left(\left(I_n+C^{-1/2}YC^{-1/2}\right)^{-1}\right)\\
&\leq \frac{\|A\|\|C^{-1}\|}{1+s_{n-i+1}(C^{-1/2}XC^{-1/2})}+\frac{\|B\|\|C^{-1}\|}{1+s_{n-j+i}(C^{-1/2}YC^{-1/2})}\\
&\leq \frac{\|A\|\|C^{-1}\|}{1+\frac{s_{n-j+i}(X)}{\|C\|}}+\frac{\|B\|\|C^{-1}\|}{1+\frac{s_{n-j+i}(Y)}{\|C\|}}\ (\text{by the inequalities (4.7) and (4.8)})\\
&= \|C\|\|C^{-1}\|\left(\frac{\|A\|}{\|C\|+s_{n-j+i}(X)}+\frac{\|B\|}{\|C\|+s_{n-j+i}(Y)}\right).
\end{aligned} \tag{4.9}$$

Now, the result follows from the inequalities (4.6) and (4.9). □

An application of Theorem 4.3 can be seen in the following result. First, we need the following Lemma [15].

Lemma 4.4. *Let $A, C, X, Y \in \mathbb{M}_n(\mathbb{C})$. Then $2s_j(AB^*) \leq s_j(A^*A+B^*B)$ for $j = 1, \ldots, n$.*

Corollary 4.5. *Let $A, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ be positive definite contraction such that $Y \leq X \leq C+Y \leq 2C$. Then*

$$s_j\left((I_n+X)^{-1/2}A(I_n+Y)^{-1/2}\right) \leq \frac{\kappa(C)}{1+\sqrt{s_{n-j+i}(X)s_{n-j+i}(Y)}}$$

for $i, j = 1, \ldots, n$ with $i \leq j \leq 2i-1$.

Proof. First, suppose that A is positive definite and $\|A\| \leq \frac{1}{2}$. Then, there exists $\epsilon \geq 0$ such that $D = A + \epsilon I$ is positive definite and $\|D\| = \frac{1}{2}$. In the inequality (4.3) replacing A and B by D, we have

$$\begin{aligned} & s_j\left(D^{1/2}\left((C+X)^{-1} + (C+Y)^{-1}\right)D^{1/2}\right) \\ & = s_j\left(D^{1/2}(C+X)^{-1}D^{1/2} + D^{1/2}(C+Y)^{-1}D^{1/2}\right) \\ & \leq \frac{\kappa(C)}{\|C\| + s_{n-j+i}^{\|D\|}(X)s_{n-j+i}^{\|D\|}(Y)} \\ & = \frac{\kappa(C)}{\|C\| + \sqrt{s_{n-j+i}(X)s_{n-j+i}(Y)}} \end{aligned} \tag{4.10}$$

for $i,j = 1, \ldots, n$ with $i \leq j \leq 2i-1$. Let $Z = (C+X)^{-1} + (C+Y)^{-1}$. Then, Z is positive definite and we have

$$\begin{aligned} s_j\left(D^{1/2}\left((C+X)^{-1} + (C+Y)^{-1}\right)D^{1/2}\right) & = s_j\left(D^{1/2}ZD^{1/2}\right) \\ & = s_j\left(Z^{1/2}DZ^{1/2}\right) \\ & \geq s_j\left(Z^{1/2}AZ^{1/2}\right) \ (\text{since } D \geq A) \\ & = s_j\left(A^{1/2}ZA^{1/2}\right) \end{aligned} \tag{4.11}$$

for $j = 1, \ldots, n$. Now, let $\tilde{A} = (C+X)^{-1/2}A^{1/2}$ and $\tilde{B} = (C+Y)^{-1/2}A^{1/2}$. Then

$$\begin{aligned} & 2s_j\left((C+X)^{-1/2}A(C+Y)^{-1/2}\right) \\ & = 2s_j(\tilde{A}\tilde{B}) \\ & \leq s_j\left(\tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B}\right)(\text{by Lemma 4.4}) \\ & = s_j\left(A^{1/2}(C+X)^{-1}A^{1/2} + A^{1/2}(C+Y)^{-1}A^{1/2}\right) \\ & = s_j\left(A^{1/2}ZA^{1/2}\right) \end{aligned} \tag{4.12}$$

for $j = 1, \ldots, n$. The inequalities (4.10), (4.11), and (4.12) imply that

$$2s_j\left((I_n + X)^{-1/2}A(I_n + Y)^{-1/2}\right) \leq \frac{\kappa(C)}{1 + \sqrt{s_{n-j+i}(X)s_{n-j+i}(Y)}} \tag{4.13}$$

Now, the general case when $\|A\| \leq 1$ follows from the inequality (4.13) by replacing A by $\frac{A}{2}$. □

We close this section with the following result related to the inequality (4.4).

Theorem 4.6. *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *be positive definite such that* $0 \le X - Y \le I_n$, $Y \le I_n$, *and* $s_j(A) \le s_j(B)$ *for some* $j \in \{1, \ldots, n\}$. *If* $s_j(A) + s_j(B) = 1$, *then*

$$s_j(A)(I_n + X)^{-1} + s_j(B)(I_n + Y)^{-1} \le \frac{1}{1 + s_n^{s_j(A)}(X) s_n^{s_j(B)}(Y)} I_n.$$

Proof.

$$\begin{aligned} & s_j(A)(I + X)^{-1} + s_j(B)(I + Y)^{-1} \\ & \le \Big(s_j(A) s_1\big((I + X)^{-1}\big) + s_j(B) s_1\big((I + Y)^{-1}\big) \Big) I_n \\ & = \left(\frac{s_j(A)}{1 + s_n(X)} + \frac{s_j(B)}{1 + s_n(Y)} \right) I_n \\ & \le \frac{1}{1 + s_n^{s_j(A)}(X) s_n^{s_j(B)}(Y)} I_n \big(\text{by the inequality (4.4)}\big). \end{aligned}$$

□

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