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[5134] First Galley Proofs

SOME INEQUALITIES FOR POWERS OF POSITIVE DEFINITE MATRICES

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Abstract. We give several matrix versions of the inequalities $a^b + b^a > 1$ and $a^a > e^{-e^{-1}}$ for positive scalars a and b. For instance, for all positive definite matrices A,B, any Hermitian matrix X, and any unitarily invariant norm,

$$\left|\left|\left|A^b X + X B^a\right|\right|\right| \geqslant |||X|||,$$

where a and b are the smallest eigenvalues of A and B, respectively.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. For a matrix $A \in \mathbb{M}_n(\mathbb{C})$, let $\lambda_1(A),\ldots,\lambda_n(A)$ be the eigenvalues of A repeated according to multiplicity. The singular values of A, denoted by $s_1(A),\ldots,s_n(A)$, are the eigenvalues of the matrix $|A| = (A^*A)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. A Hermitian matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be positive semidefinite if $x^*Ax \geqslant 0$ for all $x \in \mathbb{C}^n$ and it is called positive definite if $x^*Ax > 0$ for all $x \in \mathbb{C}^n$ with $x \neq 0$. The direct sum of matrices $A_1,\ldots,A_m \in \mathbb{M}_n(\mathbb{C})$ is the matrix $\oplus_{i=1}^m A_i = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_m \end{bmatrix}$. For two matrices $A_1,A_2 \in \mathbb{M}_n(\mathbb{C})$, we write $A \oplus B$ instead of $\oplus_{i=1}^2 A_i$.

The usual matrix norm $\|\cdot\|$, the Schatten p-norm ($p \geqslant 1$), and the Ky Fan k-norms $\|\cdot\|_{(k)}$ $(k = 1,\ldots,n)$ are the norms defined on $\mathbb{M}_n(\mathbb{C})$ by $\|A\| = \sup\{\|Ax\| : x \in \mathbb{C}, \|x\| = 1\}$, $\|A\|_p = \sum_{j=1}^n s_j^p(A)$, and $\|A\|_{(k)} = \sum_{j=1}^k s_j(A)$, $k = 1,\ldots,n$. It is known that (see, e.g., [1, p. 76]) for every $A \in \mathbb{M}_n(\mathbb{C})$ we have

$$\|A\| = s_1(A) \tag{1.1}$$

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and for each $k = 1, \ldots, n,$ we have

$$\|A\|_{(k)} = \max \left| \sum_{j=1}^{k} y_j^* A x_j \right|, \tag{1.2}$$

where the maximum is taken over all choices of orthonormal k-tuples $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$. In fact, replacing each y_j by $z_j y_j$ for some suitable complex number z_j of modulus 1 for which $\bar{z}_j y_j^* A x_j = \left| y_j^* A x_j \right|$, implies that the k-tuple $z_1 y_1, \ldots, z_k y_k$ is still orthonormal, and so an identity equivalent the identity (1.2) can be seen as follows:

$$\|A\|_{(k)} = \max \sum_{j=1}^{k} \left| y_j^* A x_j \right|, \tag{1.3}$$

where the maximum is taken over all choices of orthonormal k-tuples $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$.

A unitarily invariant norm $|||\cdot|||$ is a norm defined on $\mathbb{M}_n(\mathbb{C})$ that satisfies the invariance property $|||UAV||| = |||A|||$ for every $A \in \mathbb{M}_n(\mathbb{C})$ and every unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. It is known that

$$|||A \oplus A||| \geqslant |||B \oplus B||| \quad \text{for every unitarily invariant norm}$$

if and only if

$$|||A||| \geqslant |||B||| \quad \text{for every unitarily invariant norm.}$$

Also,

$$|||A \oplus B||| = |||B \oplus A||| = \left|\left|\left| \begin{bmatrix} 0 & B \\ A^* & 0 \end{bmatrix} \right|\right|\right|$$

for every unitarily invariant norm. Typical examples of unitarily invariant norms are the usual matrix norm, the Schatten p-norms, and the Ky Fan k-norms. For further properties and examples of unitarily invariant norms, the reader is referred to [1], [9], or [10].

An elementary inequality (see [8, p. 281]) for positive scalars a, b, asserts that

$$a^b + b^a > 1. \tag{1.4}$$

It can be easily seen that the inequality (1.4) can be written as: If a and b are positive real numbers such that $a > b \geqslant 0$, then

$$a^b + b^a \geqslant 1 \tag{1.5}$$

with equality if and only if $b = 0$.

In Section 2 of this paper, we give new inequalities for singular value powers of matrices that present generalizations of the inequality (1.5). In Section 3, we extend our generalizations of the inequality (1.4) for several matrices and we give singular value inequalities of convex functions. In Section 4, we derive new singular value inequalities for the direct sums of matrices.

2. Matrix versions of the inequality (1.5)

In this section we derive inequalities for matrices that present generalizations of the inequality (1.5). First we need the following lemma.

LEMMA 2.1. *Let* $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ *such that* A *and* B *are positive definite. Then*

$$s_j\left(X^*A^{s_n(B)}X+Y^*B^{s_n(A)}Y\right) \geqslant \min\left(s_j^2(X), s_n^2(Y)\right) \tag{2.1}$$

for $j=1,\ldots,n$.

Proof. Since B is positive definite, then $Y^*B^{s_n(A)}Y \geqslant s_n\left(Y^*B^{s_n(A)}Y\right)I_n$, and since A is positive definite we have

$$\begin{aligned} s_j\left(X^*A^{s_n(B)}X+Y^*B^{s_n(A)}Y\right) &\geqslant s_j\left(X^*A^{s_n(B)}X+s_n\left(Y^*B^{s_n(A)}Y\right)I_n\right)\\ &= s_j\left(X^*A^{s_n(B)}X\right)+s_n\left(Y^*B^{s_n(A)}Y\right)\\ &\geqslant s_n^{s_n(B)}(A)\,s_j^2(X)+s_n^{s_n(A)}(B)\,s_n^2(Y)\\ &\geqslant \min\left(s_j^2(X), s_n^2(Y)\right)\left(s_n^{s_n(B)}(A)+s_n^{s_n(A)}(B)\right)\\ &\geqslant \min\left(s_j^2(X), s_n^2(Y)\right)\\ &\qquad\text{(by the inequality (1.4))}\end{aligned}$$

for $j=1,\ldots,n$. □

Applications of Lemma 2.1 can be seen in the following two results.

COROLLARY 2.2. *Let* $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ *such that* A *and* B *are positive definite. Then*

$$X^*A^{s_n(B)}X+Y^*B^{s_n(A)}Y \geqslant \min\left(s_j^2(X), s_n^2(Y)\right)I_n.$$

Proof. Since $X^*A^{s_n(B)}X+Y^*B^{s_n(A)}Y$ is positive semidefinite, we have

$$\begin{aligned} X^*A^{s_n(B)}X+Y^*B^{s_n(A)}Y &\geqslant s_n\left(X^*A^{s_n(B)}X+Y^*B^{s_n(A)}Y\right)I_n\\ &\geqslant \min\left(s_j^2(X), s_n^2(Y)\right)I_n\\ &\qquad\text{(by The inequality (2.1)),}\end{aligned}$$

as required. □

The following result presents a natural generalization of the inequality (1.4).

COROLLARY 2.3. *Let* $A,B \in \mathbb{M}_n(\mathbb{C})$ *such that* A *and* B *are positive definite. Then*

$$A^{s_n(B)}+B^{s_n(A)} > I_n. \tag{2.2}$$

REMARK 2.4. In view of the proof of Lemma 2.1, a matrix version of the inequality (1.5) can be stated as follows: If $A,B \in \mathbb{M}_n(\mathbb{C})$ such that A is positive definite and B is positive semidefinite, then

$$A^{s_n(B)} + B^{s_n(A)} \geqslant I$$

with equality if and only if $B = 0$.

Now, we need the following Fan Dominance Theorem [1, p. 93].

LEMMA 2.5. *Let $A,B \in \mathbb{M}_n(\mathbb{C})$. If $\|A\|_{(k)} \leqslant \|B\|_{(k)}$ for $k = 1,\ldots,n$, then $|||A||| \leqslant |||B|||$ for every unitarily invariant norm.*

The following result is our first main result. It presents a natural generalization of the inequality (1.4) in the setting of unitarily invariant norms.

THEOREM 2.6. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A,B are positive definite and X is Hermitian. Then*

$$\left|\left|\left|A^{s_n(B)}X + XB^{s_n(A)}\right|\right|\right| \geqslant |||X||| \tag{2.3}$$

for every unitarily invariant norm with equality if and only if $X = 0$.

Proof. Since X is Hermitian, then there is an orthonormal basis $\{e_j\}$ of $\mathbb{C}^n$ consists of eigenvectors corresponding to the eigenvalues $\{\lambda_j(X)\}$ arranged in a way such that $|\lambda_1(X)| \geqslant \cdots \geqslant |\lambda_n(X)|$. Since $s_j(X) = \left|\lambda_j(X)\right|$ for $j = 1,\ldots,n$, then

$$\begin{aligned}
\left\|A^{s_n(B)}X + XB^{s_n(A)}\right\|_{(k)} &\geqslant \sum_{j=1}^{k} \left|e_j^*\left(A^{s_n(B)}X + XB^{s_n(A)}\right)e_j\right| \\
&\qquad \text{(by the identity (1.3))} \\
&= \sum_{j=1}^{k} \left|\left(e_j^*A^{s_n(B)}Xe_j + e_j^*XB^{s_n(A)}e_j\right)\right| \\
&= \sum_{j=1}^{k} \left|\left(e_j^*A^{s_n(B)}Xe_j + (Xe_j)^*B^{s_n(A)}e_j\right)\right| \\
&= \sum_{j=1}^{k} \left|\lambda_j(X)e_j^*\left(A^{s_n(B)} + B^{s_n(A)}\right)e_j\right| \\
&= \sum_{j=1}^{k} \left|\lambda_j(X)\right| \left|e_j^*\left(A^{s_n(B)} + B^{s_n(A)}\right)e_j\right| \\
&\geqslant \sum_{j=1}^{k} s_j(X) \ \text{(by Corollary (2.3))} \\
&= \|X\|_{(k)}
\end{aligned} \tag{2.4}$$

for $k = 1,\ldots,n$. Now, the result follows from the inequality (2.4) and Lemma 2.5.

For the equality case, suppose that equality holds in the inequality (2.3). Then

$$s_j^p(X)e_j^*\left(A^{s_n(B)}+B^{s_n(A)}\right)e_j = s_j^p(X) \tag{2.5}$$

for $j=1,\dots,n$. Corollary (2.3) implies that $e_j^*\left(A^{s_n(B)}X+XB^{s_n(A)}\right)e_j > 1$ for $j=1,\dots,n$. So, the identity (2.5) implies that $s_j(X)=0$ for $j=1,\dots,n$. This means that $X=0$. The converse is trivial, and the proof is complete. □

REMARK 2.7. In the setting of the Schatten p-norms, a particular case of Theorem 2.6 is the following: If $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A,B are positive definite and X is Hermitian, then

$$\left|\left|A^{s_n(B)}X+XB^{s_n(A)}\right|\right|_p \geqslant ||X||_p \tag{2.6}$$

for $p \geqslant 1$ with equality if and only if $X=0$.
In fact, the inequality (2.6) can be derived from Corollary 2.3 and Theorem 8 in [7], where Theorem 8 in [7] must be understood for Hermitian operators X.

Applications of Theorem 2.6 can be seen in the following three results.

COROLLARY 2.8. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. Then*

$$\left|\left|\left|\left(A^{s_n(B)}X+XB^{s_n(A)}\right)\oplus\left(XA^{s_n(B)}+B^{s_n(A)}X\right)\right|\right|\right| \geqslant |||X\oplus X||| \tag{2.7}$$

for every unitarily invariant norm with equality if and only if $X=0$.

Proof. Let $\tilde{A}=\begin{bmatrix} A & 0 \\ 0 & A\end{bmatrix}$, $\tilde{B}=\begin{bmatrix} B & 0 \\ 0 & B\end{bmatrix}$, and $\tilde{X}=\begin{bmatrix} 0 & X \\ X^* & 0\end{bmatrix}$. Then $\tilde{A}$, $\tilde{B}$ are positive definite and $\tilde{X}$ is Hermitian. It follows, from Theorem 2.6, that

$$\left|\left|\left|\tilde{A}^{s_n(\tilde{B})}\tilde{X}+\tilde{X}\tilde{B}^{s_n(\tilde{A})}\right|\right|\right| \geqslant |||\tilde{X}||| \tag{2.8}$$

for every unitarily invariant norm. Since

$$\begin{aligned}
\left|\left|\left|\tilde{A}^{s_n(\tilde{B})}\tilde{X}+\tilde{X}\tilde{B}^{s_n(\tilde{A})}\right|\right|\right| &= \left|\left|\left|\begin{bmatrix} 0 & A^{s_n(B)}X+XB^{s_n(A)} \\ A^{s_n(B)}X^*+X^*B^{s_n(A)} & 0\end{bmatrix}\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} A^{s_n(B)}X^*+X^*B^{s_n(A)} & 0 \\ 0 & A^{s_n(B)}X+XB^{s_n(A)}\end{bmatrix}\right|\right|\right| \\
&= \left|\left|\left|\left(A^{s_n(B)}X^*+X^*B^{s_n(A)}\right)\oplus\left(A^{s_n(B)}X+XB^{s_n(A)}\right)\right|\right|\right| \\
&= \left|\left|\left|\left(A^{s_n(B)}X+XB^{s_n(A)}\right)\oplus\left(XA^{s_n(B)}+B^{s_n(A)}X\right)\right|\right|\right| \qquad (2.9)
\end{aligned}$$

and

$$\begin{aligned}
|||\tilde{X}||| &= \left|\left|\left|\begin{bmatrix} 0 & X \\ X^* & 0\end{bmatrix}\right|\right|\right| \\
&= |||X\oplus X|||, \qquad (2.10)
\end{aligned}$$

then the inequality (2.7) follows from the inequality (2.8) and the identities (2.9), (2.10). Equality holds in the inequality (2.7) if and only if equality holds in the inequality (2.8) and by the equality condition of Theorem 2.6, the last assertion is equivalent to saying $\tilde{X}=0$, that is $X=0$. □

COROLLARY 2.9. *Let $A,X\in\mathbb{M}_n(\mathbb{C})$ such that A is positive definite. Then*

$$\left|\left|\left|A^{s_n(A)}X+XA^{s_n(A)}\right|\right|\right|\geqslant |||X||| \tag{2.11}$$

for every unitarily invariant norm with equality if and only if $X=0$.

Proof. In Corollary 2.8, replacing B by A, we have

$$\left|\left|\left|\left(A^{s_n(A)}X+XA^{s_n(A)}\right)\oplus\left(A^{s_n(A)}X+XA^{s_n(A)}\right)\right|\right|\right|\geqslant |||X\oplus X|||$$

for every unitarily invariant norm. So,

$$\left|\left|\left|A^{s_n(A)}X+XA^{s_n(A)}\right|\right|\right|\geqslant |||X|||$$

for every unitarily invariant norm. □

COROLLARY 2.10. *Let $A,B,X\in\mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. Then*

$$|||A^{\alpha}X+XB^{\alpha}|||\geqslant |||X||| \tag{2.12}$$

for every unitarily invariant norm with equality if and only if $X=0$, where $\alpha=$ $\min\{s_n(A),s_n(B)\}$.

Proof. Let $\mathscr{A}=\begin{bmatrix}A&0\\0&B\end{bmatrix}$ and $\tilde{X}=\begin{bmatrix}0&X\\X^*&0\end{bmatrix}$. Then $\mathscr{A}$ is positive definite and $\tilde{X}$ is Hermitian. It follows, from Corollary 2.10, that

$$\begin{aligned}\left|\left|\left|\mathscr{A}^{s_n(\mathscr{A})}\tilde{X}+\tilde{X}\mathscr{A}^{s_n(\mathscr{A})}\right|\right|\right| &\geqslant |||\tilde{X}|||\\ &=|||X\oplus X|||.\end{aligned} \tag{2.13}$$

Since $s_n(\mathscr{A})=\min\left(s_n(A),s_n(B)\right)=\alpha$, then

$$\begin{aligned}|||A^{\alpha}X+XB^{\alpha}||| &=\left|\left|\left|\mathscr{A}^{s_n(\mathscr{A})}\tilde{X}+\tilde{X}\mathscr{A}^{s_n(\mathscr{A})}\right|\right|\right|\\ &=\left|\left|\left|\begin{bmatrix}0&A^{\alpha}X+XB^{\alpha}\\B^{\alpha}X^*+X^*A^{\alpha}&0\end{bmatrix}\right|\right|\right|\\ &=|||(B^{\alpha}X^*+X^*A^{\alpha})\oplus(A^{\alpha}X+XB^{\alpha})|||\\ &=|||(A^{\alpha}X+XB^{\alpha})\oplus(A^{\alpha}X+XB^{\alpha})|||.\end{aligned} \tag{2.14}$$

Now, the result follows from the inequalities (2.13) and (2.14). □

REMARK 2.11. It should be mentioned here that optimal inequalities with sharp constants related to the inequalities (2.11) and (2.12) will be given at the end of this section.

In order to give another type of inequalities related to the inequality (1.4), we need the following lemma.

LEMMA 2.12. *Let a be a positive real number. Then $a^a \geqslant e^{-e^{-1}}$ with equality if and only if $a = e^{-1}$.*

Proof. Let $f(x) = x^x, x \in (0,\infty)$. Then the minimum value of f occurs only at $x = e^{-1}$. Thus, $a^a = f(a) \geqslant f(e^{-1}) = e^{-e^{-1}}$ with equality if and only if $a = e^{-1}$. □

Based on Lemma 2.12, we have the following result. Its proofs is similar to that of Lemma 2.1.

LEMMA 2.13. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. Then*

$$s_j\left(X^*A^{s_n(A)}X^* + Y^*B^{s_n(B)}Y^*\right) \geqslant 2e^{-e^{-1}} \min\left(s_j^2(X), s_n^2(Y)\right) \tag{2.15}$$

for $j = 1,...,n$.

The following two Corollaries follow from Lemma 2.13 by using proofs similar to those of Corollaries 2.2, 2.3.

COROLLARY 2.14. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. Then*

$$X^*A^{s_n(A)}X^* + Y^*B^{s_n(B)}Y^* \geqslant 2e^{-e^{-1}} \min\left(s_j^2(X), s_n^2(Y)\right) I_n.$$

COROLLARY 2.15. *Let $A,B \in \mathbb{M}_n(\mathbb{C})$ be positive definite. Then*

$$A^{s_n(A)} + B^{s_n(B)} \geqslant 2e^{-e^{-1}} I_n.$$

The following is our second main result in this section. It follows by a proof similar to that of Theorem 2.6.

THEOREM 2.16. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A,B are positive definite and X is Hermitian. Then*

$$\left|\left|\left|A^{s_n(A)}X + XB^{s_n(B)}\right|\right|\right| \geqslant 2e^{-e^{-1}} |||X||| \tag{2.16}$$

for every unitarily invariant norm.

Applications of Theorem 2.16 can be seen in the following three results.

COROLLARY 2.17. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. Then*

$$\left|\left|\left|\left(A^{s_n(A)}X + XB^{s_n(B)}\right) \oplus \left(B^{s_n(B)}X + XA^{s_n(A)}\right)\right|\right|\right| \geqslant 2e^{-e^{-1}} |||X \oplus X||| \tag{2.17}$$

for every unitarily invariant norm.

COROLLARY 2.18. *Let $A,X \in \mathbb{M}_n(\mathbb{C})$ such that A is positive definite. Then*

$$\left|\left|\left|A^{s_n(A)}X + XA^{s_n(A)}\right|\right|\right| \geqslant 2e^{-e^{-1}} |||X||| \tag{2.18}$$

for every unitarily invariant norm.

COROLLARY 2.19. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive definite. Then*

$$|||A^{\alpha}X + XB^{\alpha}||| \geqslant 2e^{-e^{-1}} |||X||| \tag{2.19}$$

for every unitarily invariant norm, where $\alpha = \min\{s_n(A), s_n(B)\}$.

REMARK 2.20. It can be seen that the inequalities (2.18) and (2.19) are optimal with sharp constants. Since $e^{-e^{-1}} > \frac{1}{2}$, then the inequalities (2.18) and (2.19) are better than the inequalities (2.11) and (2.12).

3. Extensions for several matrices

This section is devoted to generalize our results in Section 2. First, we start by the following generalization of the inequality (1.4).

LEMMA 3.1. *Let $a_1,\ldots,a_m$ be positive real numbers. Then* $\sum_{i=1}^{m} a_i^{a_{m+1-i}} > \frac{m}{2}$.

Proof. We have two cases for m:
Case 1. If m is even, then

$$\begin{aligned}\sum_{i=1}^{m} a_j^{a_{m+1-j}} &= \sum_{i=1}^{m/2} \left(a_j^{a_{m+1-j}} + a_{m+1-j}^{a_j}\right)\\ &> \sum_{i=1}^{m/2} 1 \quad \text{(by the inequality (1.4))}\\ &= \frac{m}{2}.\end{aligned}$$

Case 2. If m is odd, then

$$\begin{aligned}\sum_{i=1}^{m} a_j^{a_{m+1-j}} &= a_{\frac{m+1}{2}}^{a_{\frac{m+1}{2}}} + \sum_{i=1}^{\frac{m-1}{2}} \left(a_i^{a_{m+1-i}} + a_{m+1-j}^{a_j}\right) \\ &> e^{-e^{-1}} + \sum_{i=1}^{\frac{m-1}{2}} 1 \ \text{(by Lemma 2.12)} \\ &= e^{-e^{-1}} + \frac{m-1}{2} \\ &> \frac{m}{2} \left(\text{since } e^{-e^{-1}} > \frac{1}{2}\right),\end{aligned}$$

this completes the proof of the lemma. □

Based on Lemma 3.1, we have the following generalizations of Lemma 2.1, Corollaries 2.2, and 2.3. The proofs will follow by arguments similar to those used in Section 2.

LEMMA 3.2. *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1,...m$, such that each A_i is positive definite. Then*

$$s_j\left(\sum_{i=1}^{m} X_i^* A_i^{s_n(A_{m+1-i})} X_i\right) \geqslant \frac{mc_j}{2}, \tag{3.1}$$

for $j = 1,...,n$, where $c_j = \min\{s_j^2(X_1), s_n^2(X_2), ..., s_n^2(X_m)\}$.

Applications of Lemma 3.2 can be seen in the following three results.

COROLLARY 3.3. *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1,...m$, such that each A_i is positive definite. Then*

$$\sum_{i=1}^{m} X_i^* A_i^{s_n(A_{m+1-i})} X_i \geqslant \frac{mc_n}{2} I_n, \tag{3.2}$$

where $c_j = \min\{s_j^2(X_1), s_n^2(X_2), ..., s_n^2(X_m)\}$.

COROLLARY 3.4. *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1,...m$, such that each A_i is positive definite. Then*

$$\sum_{i=1}^{m} A_i^{s_n(A_{m+1-i})} > \frac{m}{2} I_n, \tag{3.3}$$

where $c_j = \min\{s_j^2(X_1), s_n^2(X_2), ..., s_n^2(X_m)\}$.

We close this section by the following conjecture.

CONJECTURE 3.5. Let $a_1, ..., a_m$ be positive real numbers and let σ be a permutation of the set $\{1, ..., m\}$. Then

$$\sum_{i=1}^{m} a_i^{a_{\sigma(i)}} > \frac{m}{2}. \tag{3.4}$$

In particular,

$$\left(\sum_{i=1}^{m-1} a_i^{a_{i+1}}\right) + a_m^{a_1} > \frac{m}{2}.$$

If the inequality (3.4) is true, then other matrix type inequalities related to the inequalities (3.1), (3.2), and (3.3) can be obtained.

In the rest of this section we apply our results that we obtained in this section to some known results for convex functions. First, we need the following lemma [2]. Other related results can be found in [3] and [4]. Also, all convex functions here are assumed to be continuous.

LEMMA 3.6. *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1, \ldots m$, such that each A_i is Hermitian and $\sum_{i=1}^m X_i^* X_i = I$. If f is a monotone convex function, then*

$$s_j\left(\sum_{i=1}^m X_i^* f(A_i) X_i\right) \geqslant s_j\left(f\left(\sum_{i=1}^m X_i^* A_i X_i\right)\right)$$

for $j = 1, \ldots, n$.

Our third main result in this section can be stated as follows.

THEOREM 3.7. *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1, \ldots m$, such that each A_i is positive definite and $\sum_{i=1}^m X_i^* X_i = I_n$. If f is a monotone convex function on $[0, \infty)$, then*

$$s_j\left(\sum_{i=1}^m X_i^* f\left(A_i^{s_n(A_{m+1-i})}\right) X_i\right) \geqslant f\left(\frac{mc_j}{2}\right) \tag{3.5}$$

for $j = 1, \ldots, n$, where $c_j = \min\{s_j^2(X_1), s_n^2(X_2), \ldots, s_n^2(X_m)\}$.

Proof.

$$\begin{aligned} s_j\left(\sum_{i=1}^m X_i^* f\left(A_i^{s_n(A_{m+1-i})}\right) X_i\right) &\geqslant s_j\left(f\left(\sum_{i=1}^m X_i^* A_i^{s_n(A_{m+1-i})} X_i\right)\right) \text{ (by Lemma 3.6)} \\ &= f\left(s_j\left(\sum_{i=1}^m X_i^* A_i^{s_n(A_{m+1-i})} X_i\right)\right) \\ &\geqslant f\left(\frac{mc_j}{2}\right) \text{ (by Theorem 3.2),} \end{aligned}$$

this proves the inequality (3.5). □

Applications of Theorem 3.7 can be seen in the following two results.

COROLLARY 3.8. *Let* $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1,\dots m$, *such that each* A_i *is positive definite and* $\sum_{i=1}^{m} X_i^* X_i = I_n$. *If* f *is a monotone convex function on* $[0,\infty)$, *then*

$$\sum_{i=1}^{m} X_i^* f\left(A_i^{s_n(A_{m+1-i})}\right) X_i \geqslant f\left(\frac{mc_n}{2}\right) I_n, \tag{3.6}$$

where $c_j = \min\{s_j^2(X_1), s_n^2(X_2), \dots, s_n^2(X_m)\}$.

COROLLARY 3.9. *Let* $A_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1,\dots m$, *such that each* A_i *is positive definite. If* f *is a monotone convex function on* $[0,\infty)$, *then*

$$\sum_{i=1}^{m} f\left(A_i^{s_n(A_{m+1-i})}\right) \geqslant f\left(\frac{m}{2}\right) I_n. \tag{3.7}$$

REMARK 3.10. A result of J.-C. Bourin [2] asserts the following: Let $A, X \in \mathbb{M}_n(\mathbb{C})$ such that A is Hermitian and X is contractive. If f is a monotone convex function such that $f(0) \leqslant 0$, then

$$s_j\left(X^* f(A) X\right) \geqslant s_j\left(f\left(X^* A X\right)\right) \tag{3.8}$$

for $j = 1,\dots,n$. Thus, a result related to the inequality (3.8) can be stated as follows: Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1,\dots m$, such that each A_i is positive definite and $\sum_{i=1}^{m} X_i^* X_i \leqslant I_n$. If f is a monotone convex function on $[0,\infty)$ such that $f(0) \leqslant 0$, then

$$s_j\left(\sum_{i=1}^{m} X_i^* f\left(A_i^{s_n(A_{m+1-i})}\right) X_i\right) \geqslant f\left(\frac{mc_j}{2}\right) \tag{3.9}$$

for $j = 1,\dots,n$, where $c_j = \min\{s_j^2(X_1), s_n^2(X_2), \dots, s_n^2(X_m)\}$. In fact, the inequality (3.9) follows by applying the inequality (3.8) to the partitioned matrices $A = \oplus_{i=1}^{m} A_i$ and $X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ X_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ X_m & 0 & \cdots & 0 \end{bmatrix}$ and observing that X is contractive.

Applications of Theorem 3.7 can be seen in the following two results.

COROLLARY 3.11. *Let* $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1,\dots m$, *such that each* A_i *is positive definite and* $\sum_{i=1}^{m} X_i^* X_i = I_n$. *Then*

$$s_j\left(\sum_{i=1}^{m} X_i^* \left(e^{A_i^{s_n(A_{m+1-i})}} - I_n\right) X_i\right) \geqslant s_j\left(e^{\frac{mc_j}{2}} - 1\right)$$

for $j = 1,\dots,n$, *where* $c_j = \min\{s_j^2(X_1), s_n^2(X_2), \dots, s_n^2(X_m)\}$.

Proof. The result follows from Theorem 3.7 by letting $f(t) = e^t - 1$. □

COROLLARY 3.12. *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1, \ldots m$, such that each A_i is positive definite and $\sum_{i=1}^{m} X_i^* X_i = I_n$. Then*

$$\sum_{i=1}^{m} X_i^* e^{A_i^{s_n(A_{m+1-i})}} X_i \geqslant \left(e^{\frac{mc_j}{2}} - 1\right) I_n + \sum_{i=1}^{m} X_i^* X_i,$$

where $c_j = \min\{s_j^2(X_1), s_n^2(X_2), \ldots, s_n^2(X_m)\}$.

4. Singular values and direct sums

In this section, we give singular value inequalities related to the inequality (2.2) that involve direct sums of matrices. In order to do that, we need the following lemma [1, p. 62] that constitute the Wely's inequalities.

LEMMA 4.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite. Then*

$$s_j(A+B) \leqslant s_k(A) + s_{j-k+1}(B) \tag{4.1}$$

for $j, k = 1, \ldots, n$ with $k \leqslant j$, and

$$s_j(A+B) \geqslant s_k(A) + s_{j-k+n}(B) \tag{4.2}$$

for $j, k = 1, \ldots, n$ with $k \geqslant j$.

The following result presents a variation of Lemma 4.1 for several matrices.

LEMMA 4.2. *Let $A_1, \ldots, A_m \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then*

(1)

$$s_j\left(\sum_{i=1}^{m} A_i\right) \leqslant s_{k_{m-1}}(A_m) + \sum_{i=1}^{m} s_{k_{i-1}-k_i+1}(A_i) \tag{4.3}$$

for $j = 1, \ldots, n$, where $k_i \leqslant k_{i-1}$, $i = 1, \ldots, m-1$ with $k_0 = j$.

(2)

$$s_j\left(\sum_{i=1}^{m} A_i\right) \geqslant s_{k_{m-1}}(A_m) + \sum_{i=1}^{m} s_{k_{i-1}-k_i+n}(A_i) \tag{4.4}$$

for $j = 1, \ldots, n$, where $k_i \geqslant k_{i-1}$, $i = 1, \ldots, m-1$ with $k_0 = j$.

Proof.
(1)

$$\begin{aligned}
s_j\left(\sum_{i=1}^{m} A_i\right) &\leqslant s_{k_1}\left(\sum_{i=2}^{m} A_i\right) + s_{k_0-k_1+1}(A_1) \\
&\qquad \text{(by the inequality (4.1))} \\
&\leqslant s_{k_2}\left(\sum_{i=3}^{m} A_i\right) + s_{k_1-k_2+1}(A_2) + s_{k_0-k_1+1}(A_1) \\
&\leqslant s_{k_{m-2}}(A_m + A_{m-1}) + s_{k_{m-3}-k_{m-2}+1}(A_{m-2}) + \cdots + s_{k_0-k_1+1}(A_1) \\
&\leqslant s_{k_{m-1}}(A_m) + \sum_{i=1}^{m-1} s_{k_{i-1}-k_i+1}(A_i),
\end{aligned}$$

this proves the inequality (4.3).
(2)

$$\begin{aligned}
s_j\left(\sum_{i=1}^{m} A_i\right) &\geqslant s_{k_1}\left(\sum_{i=2}^{m} A_i\right) + s_{k_0-k_1+n}(A_1) \\
&\qquad \text{(by the inequality (4.2))} \\
&\geqslant s_{k_2}\left(\sum_{i=3}^{m} A_i\right) + s_{k_1-k_2+n}(A_2) + s_{k_0-k_1+n}(A_1) \\
&\geqslant s_{k_{m-2}}(A_m + A_{m-1}) + s_{k_{m-3}-k_{m-2}+n}(A_{m-2}) + \cdots + s_{k_0-k_1+n}(A_1) \\
&\geqslant s_{k_{m-1}}(A_m) + \sum_{i=1}^{m-1} s_{k_{i-1}-k_i+n}(A_i),
\end{aligned}$$

as required. □

It is shown in [6] that if $X, Y \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(X \oplus Y) \geqslant \frac{1}{2} s_j(X+Y) \tag{4.5}$$

for $j = 1, \dots, n$. A natural generalization of the inequality (4.5) has been recently given in [5] as follows.

LEMMA 4.3. *Let* $X_1, \dots, X_m \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$s_j\left(\oplus_{i=1}^{m} X_i\right) \geqslant \frac{1}{m} s_j\left(\sum_{i=1}^{m} X_i\right) \tag{4.6}$$

for $j = 1, \dots, n$.

Based on Lemmas 4.2 and 4.3, we have the following result. It is our main result in this section.

THEOREM 4.4. *Let $A_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1,\dots,m$ such that each A_i is positive definite and for $j \in \{1,\dots,n\}$ let $k_0,\dots,k_{m-1}$ be positive integers satisfying $k_0 = j$, $k_i \geqslant k_{i-1}$, $i = 1,\dots,m-1$. Then*

$$s_j\left(\oplus_{i=1}^m X_i^* A_i^{s_n(A_{m+1-i})} X_i\right) \geqslant \frac{\alpha_j}{2} \tag{4.7}$$

for $j = 1,\dots,n$, where $\alpha_j = \min\{s_{k_{m-1}}^2(X_m), s_{k_{i-1}-k_i+n}^2(X_i) : i = 1,\dots,m-1\}$.

Proof. Let $j \in \{1,\dots,n\}$. Then

$$\begin{aligned}
& s_j\left(\oplus_{i=1}^m X_i^* A_i^{s_n(A_{m+1-i})} X_i\right) \\
\geqslant\ & \frac{1}{m} s_j\left(\sum_{i=1}^m X_i^* A_i^{s_n(A_{m+1-i})} X_i\right) \quad \text{(by Lemma 4.3)} \\
\geqslant\ & \frac{1}{m} s_j\left(\sum_{i=1}^m s_n\left(A_i^{s_n(A_{m+1-i})}\right) X_i^* X_i\right) \\
=\ & \frac{1}{m} s_j\left(\sum_{i=1}^m s_n^{s_n(A_{m+1-i})}(A_i) X_i^* X_i\right) \\
\geqslant\ & \frac{1}{m}\left(s_n^{s_n(A_1)}(A_m)\, s_{k_{m-1}}^2(X_m) + \sum_{i=1}^{m-1} s_n^{s_n(A_{m+1-i})}(A_i)\, s_{k_{i-1}-k_i+n}^2(X_i)\right) \\
& \hspace{6cm} \text{(by the inequality (4.4))} \\
\geqslant\ & \frac{\alpha_j}{m}\left(s_n^{s_n(A_1)}(A_m) + \sum_{i=1}^{m-1} s_n^{s_n(A_{m+1-i})}(A_i)\right) \\
=\ & \frac{\alpha_j}{m} \sum_{i=1}^m s_n^{s_n(A_{m+1-i})}(A_i) \\
\geqslant\ & \frac{\alpha_j}{2} \quad \text{(by Lemma 3.1)},
\end{aligned}$$

as required. □

An application of Theorem 4.4 is the following.

COROLLARY 4.5. *Let $A_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1,\dots,m$ such that each A_i is positive definite. Then*

$$s_j\left(\oplus_{i=1}^m A_i^{s_n(A_{m+1-i})}\right) \geqslant \frac{1}{2} \tag{4.8}$$

for $j = 1,\dots,n$.

REMARK 4.6. The inequality (4.8) is not true for $j > n$. This can be demonstrated by the following example. Let $A_1 = [1]$ and $A_2 = \left[\frac{1}{4}\right]$ be 1×1 matrices. Then $n = 1$

and $m = 1$. So, by taking $j = 2$, we have

$$
\begin{aligned}
s_2\left(A_1^{s_n(A_2)} \oplus A_2^{s_n(A_1)}\right) &= s_2\left(\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\right) \\
&= \frac{1}{4} \\
&\ngeq \frac{1}{2}.
\end{aligned}
$$

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