UNBOUNDED NORM TOPOLOGY IN BANACH LATTICES

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ABSTRACT. A net (x_{α}) in a Banach lattice X is said to un-converge to a vector x if $|||x_{\alpha} - x| \wedge u|| \to 0$ for every $u \in X_+$. In this paper, we investigate un-topology, i.e., the topology that corresponds to un-convergence. We show that un-topology agrees with the norm topology iff X has a strong unit. Un-topology is metrizable iff X has a quasi-interior point. Suppose that X is order continuous, then un-topology is locally convex iff X is atomic. An order continuous Banach lattice X is a KB-space iff its closed unit ball B_X is un-complete. For a Banach lattice X, B_X is un-compact iff X is an atomic KB-space. We also study un-compact operators and the relationship between un-convergence and weak*-convergence.

1. INTRODUCTION AND PRELIMINARIES

For a net (x_{α}) in a vector lattice X, we write $x_{\alpha} \xrightarrow{o} x$ if (x_{α}) converges to x in order. That is, there is a net (u_{γ}) , possibly over a different index set, such that $u_{\gamma} \downarrow 0$ and for every γ there exists α_0 such that $|x_{\alpha} - x| \leq u_{\gamma}$ whenever $\alpha \geq \alpha_0$. We write $x_{\alpha} \xrightarrow{uo} x$ and say that (x_{α}) uo-converges to x if $|x_{\alpha} - x| \land u \xrightarrow{o} 0$ for every $u \in X_+$; "uo" stands for "unbounded order". For a net (x_{α}) in a normed lattice X, we write $x_{\alpha} \xrightarrow{\parallel \cdot \parallel} x$ if (x_{α}) converges to x in norm. We write $x_{\alpha} \xrightarrow{un} x$

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and say that (x_{α}) **un-converges** to x if $|x_{\alpha} - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for every $u \in X_+$; "un" stands for "unbounded norm".

A variant of uo-convergence was originally introduced in [Nak48], while the term "uo-convergence" was first coined in [DeM64]. Relationships between uo, weak, and weak* convergences were investigated in [Wic77, GX14, Gao14]. Relationships between uo-convergence and almost everywhere convergence were investigated and applied in [GX14, EM16, GTX]. We refer the reader to [GTX] for a further review of properties of uo-convergence. Un-convergence was introduced in [Tro04] and further investigated in [DOT]. For unexplained terminology on vector and Banach lattices we refer the reader to [AA02, AB06]. All vector lattices are assumed to be Archimedean.

Let us start by briefly going over some of the known properties of these modes of convergence; we refer the reader to [GTX, DOT] for details. Both uo-convergence and un-convergence respect linear and lattice operations; limits are unique. In particular, $x_{\alpha} \xrightarrow{\text{uo}} x$ iff $|x_{\alpha} - x| \xrightarrow{\text{uo}} 0$; similarly, $x_{\alpha} \xrightarrow{\text{un}} x$ iff $|x_{\alpha} - x| \xrightarrow{\text{un}} 0$. For order bounded nets, uo-convergence agrees with order convergence while unconvergence agrees with norm convergence. It follows that order intervals are uo- and un-closed. For sequences in $L_p(\mu)$, where $1 \leq p < \infty$ and μ is a finite measure, it is easy to see that uo-convergence agrees with convergence almost everywhere, see, e.g., [DeM64, Example 2]. Under the same assumptions, un-convergence agrees with convergence in measure, see [Tro04, Example 23]. We write L_p for $L_p[0, 1]$.

Suppose that X is a vector lattice. By [GTX, Corollary 3.6], every disjoint sequence in X is uo-null. Recall that a sublattice Y of X is **regular** if the inclusion map preserves suprema and infima of arbitrary subsets. It was shown in [GTX, Theorem 3.2] that uo-convergence is stable under passing to and from regular sublattices. That is, if (y_{α}) is a net in a regular sublattice Y of X then $y_{\alpha} \xrightarrow{\text{uo}} 0$ in Y iff $y_{\alpha} \xrightarrow{\text{uo}} 0$ in X (in fact, this property characterizes regular sublattices).

It is clear that if X is an order continuous normed lattice then uoconvergence implies un-convergence. Let X be a Banach lattice and (x_n) a un-null sequence in X. Then (x_n) has a uo-null subsequence by Proposition 4.1 of [DOT]. A disjoint sequence need not be un-null. For example, the standard unit sequence (e_n) in ℓ_{∞} is not un-null. However, a un-null sequence has an asymptotically disjoint subsequence. More precisely, we have the following.

Theorem 1.1. ([DOT, Theorem 3.2]) Let (x_{α}) be a un-null net. There is an increasing sequence of indices (α_k) and a disjoint sequence (d_k) such that $x_{\alpha_k} - d_k \xrightarrow{\|\cdot\|} 0$.

While uo-convergence need not be given by a topology, it was observed in [DOT] that un-convergence is topological. For every $\varepsilon > 0$ and non-zero $u \in X_+$, put

$$V_{\varepsilon,u} = \left\{ x \in X : \| |x| \wedge u \| < \varepsilon \right\}$$

The collection of all sets of this form is a base of zero neighborhoods for a topology, and the convergence in this topology agrees with unconvergence. We will refer to this topology as *un-topology*.

Every time a new linear topology is discovered, one is expected to ask several natural questions: is this topology metrizable? Is it locallyconvex? Complete? Can one characterize (relatively) compact sets? Is this topology stronger or weaker than other known topologies? In this paper, we study these and similar questions for un-topology. In other words, our motivation for this paper is to investigate topological properties of un-topology.

Throughout this paper, X will be assumed to be a Banach lattice, unless specified otherwise. We write B_X for the closed unit ball of X. It was observed in [DOT] that $x_{\alpha} \xrightarrow{\text{un}} x$ implies $||x|| \leq \liminf ||x_{\alpha}||$. This yields that B_X is un-closed.

The following facts will be used throughout the paper.

- **Lemma 1.2.** (i) If (x_{α}) is an increasing net in a vector lattice X and $x_{\alpha} \xrightarrow{uo} x$ then $x_{\alpha} \uparrow x$;
 - (ii) If (x_{α}) is an increasing net in a normed lattice X and $x_{\alpha} \xrightarrow{\text{un}} x$ then $x_{\alpha} \uparrow x$ and $x_{\alpha} \xrightarrow{\parallel \cdot \parallel} x$.

Proof. Without loss of generality, $x_{\alpha} \ge 0$ for all α ; otherwise, pick any index α_0 and consider the net $(x_{\alpha} - x_{\alpha_0})_{\alpha \ge \alpha_0}$, which converges to

 $x - x_{\alpha_0}$. Since lattice operations are uo- and un-continuous, we have $x \ge 0$.

(i) Take any $z \in X_+$. It follows from uo-continuity of lattice operations that $x_{\alpha} \wedge z \xrightarrow{u_0} x \wedge z$. Since the net $(x_{\alpha} \wedge z)$ is order bounded and increasing, this yields $x_{\alpha} \wedge z \xrightarrow{o} x \wedge z$ and, therefore $x_{\alpha} \wedge z \uparrow x \wedge z$. It follows that $x_{\alpha} \wedge z \leq x$ for every α and every $z \in X_+$. Applying this with $z = x_{\alpha}$ we get $x_{\alpha} \leq x$. Thus, the net (x_{α}) is order bounded and, therefore, $x_{\alpha} \xrightarrow{o} x$, hence $x_{\alpha} \uparrow x$.

(ii) The proof is similar and uses the fact that every monotone norm convergent net converges in order to the same limit. We note that $x_{\alpha} \wedge z \xrightarrow{\|\cdot\|} x \wedge z$ and, therefore, $x_{\alpha} \wedge z \uparrow x \wedge z$ for every $z \in X_+$. It follows that the net (x_{α}) is order bounded, which yields $x_{\alpha} \xrightarrow{\|\cdot\|} x$ and, therefore, $x_{\alpha} \uparrow x$.

Recall that [DOT, Question 2.14] asks whether $x_{\alpha} \xrightarrow{\text{un}} 0$ implies that there exists an increasing sequence of indices (α_k) such that $x_{\alpha_k} \xrightarrow{\text{un}} 0$. The following counterexample was kindly provided to us by E. Emelyanov.

Example 1.3. Let Ω be an uncountable set; let X be the closed sublattice of $\ell_{\infty}(\Omega)$ consisting of all the functions with countable support. For $\omega \in \Omega$, we write e_{ω} for the characteristic function of $\{\omega\}$.

Let Λ be the set of all countable subsets of Ω , ordered by inclusion. For each $\alpha \in \Lambda$, pick any $\omega \notin \alpha$ and put $x_{\alpha} = e_{\omega}$. We claim that $x_{\alpha} \xrightarrow{\text{un}} 0$. Indeed, let $u \in X_+$; let α_0 be the support of u. Then $x_{\alpha} \wedge u = 0$ whenever $\alpha \ge \alpha_0$.

On the other hand, let (ω_k) be any sequence in Ω ; we claim that the sequence (e_{ω_k}) is not un-null. Indeed, put $\beta = \{\omega_k : k \in \mathbb{N}\}$ and let u be the characteristic function of β . Then $e_{\omega_k} \wedge u = e_{\omega_k}$ for every k; hence it does not converge in norm to zero.

In particular, if (α_k) is an increasing sequence of indices in Λ then (x_{α_k}) is not un-null.

Let $e \in X_+$. Recall that the band B_e generated by e is norm closed and contains the principal ideal I_e ; hence $I_e \subseteq \overline{I_e} \subseteq B_e$. Recall also that

- e is a **strong unit** when $I_e = X$; equivalently, for every $x \ge 0$ there exists $n \in \mathbb{N}$ such that $x \le ne$;
- e is a **quasi-interior point** if $\overline{I_e} = X$; equivalently, $x \wedge ne \xrightarrow{\|\cdot\|} x$ for every $x \in X_+$;
- e is a **weak unit** if $B_e = X$; equivalently, $x \wedge ne \uparrow x$ for every $x \in X_+$.

In particular, strong unit \Rightarrow quasi-interior point \Rightarrow weak unit.

2. Strong units

It is easy to see that each $V_{\varepsilon,u}$ is solid. It is also absorbing, that is, for every $x \in X$ there exists $\lambda > 0$ such that $\lambda x \in V_{\varepsilon,u}$. The following lemma is a dichotomy: it says that $V_{\varepsilon,u}$ is either "very small" or "very large".

Lemma 2.1. Let $\varepsilon > 0$, and $0 \neq u \in X_+$. Then $V_{\varepsilon,u}$ is either contained in [-u, u] or contains a non-trivial ideal.

Proof. Suppose that $V_{\varepsilon,u}$ is not contained in [-u, u]. Then there exists $x \in V_{\varepsilon,u}$ such that $x \notin [-u, u]$. Replacing x with |x|, we may assume that x > 0. Let $y = (x - u)^+$; then y > 0. It is an easy exercise to show that $(\lambda y) \wedge u \leq x \wedge u$ for every $\lambda \geq 0$; it follows that $\lambda y \in V_{\varepsilon,u}$. Since $V_{\varepsilon,u}$ is solid, it contains the principal ideal I_y .

Lemma 2.2. If $V_{\varepsilon,u}$ is contained in [-u, u] then u is a strong unit.

Proof. Let $x \in X_+$. There exists $\lambda > 0$ such that $\lambda x \in V_{\varepsilon,u}$, hence $\lambda x \in [-u, u]$. It follows that u is a strong unit.

Recall that if e is a positive vector in X then the principal ideal I_e equipped with the norm

$$||x||_e = \inf\{\lambda > 0 : |x| \le \lambda e\}$$

is lattice isometric to C(K) for some compact Hausdorff space K, with e corresponding to the constant one function 1; see, e.g., Theorems 3.4 and 3.6 in [AA02]. If e is a strong unit in X then $I_e = X$; it is easy to see that in this case $\|\cdot\|_e$ is equivalent to the original norm; it follows that X is lattice and norm isomorphic to C(K).

It is easy to see that if $x_{\alpha} \xrightarrow{\|\cdot\|} x$ then $x_{\alpha} \xrightarrow{\mathrm{un}} x$, so norm topology generally is stronger than un-topology.

Theorem 2.3. Let X be a Banach lattice. The following are equivalent.

- (i) Un-topology agrees with norm topology;
- (ii) X has a strong unit.

Proof. Suppose that un-topology and norm topology agree. It follows that $V_{\varepsilon,u}$ is contained in B_X for some $\varepsilon > 0$ and u > 0. By Lemma 2.1, we conclude that $V_{\varepsilon,u}$ is contained in [-u, u]; hence u is a strong unit by Lemma 2.2.

Suppose now that X has a strong unit. Then X is lattice and norm isomorphic to C(K) for some compact Hausdorff space K. Without loss of generality, X = C(K). It follows from $x_{\alpha} \xrightarrow{\text{un}} 0$ that $|x_{\alpha}| \wedge 1 \xrightarrow{\|\cdot\|} 0$. Since the norm in C(K) is the sup-norm, it is easy to see that $x_{\alpha} \xrightarrow{\|\cdot\|} 0$.

3. QUASI-INTERIOR POINTS AND METRIZABILITY

Given a net (x_{α}) in a vector lattice with a weak unit e, then $x_{\alpha} \xrightarrow{u_{0}} x$ iff $|x_{\alpha} - x| \wedge e \xrightarrow{\circ} 0$; see, e.g., [GTX, Corollary 3.5] (this was proved in [Kap97] in the special case when the lattice is order complete). That is, it suffices to test uo-convergence on a weak unit. Lemma 2.11 in [DOT] provides a similar statement for un-convergence and quasiinterior points. We now prove that this property actually characterizes quasi-interior points.

Theorem 3.1. Let $e \in X_+$. The following are equivalent.

- (i) *e* is a quasi-interior point;
- (ii) For every net (x_{α}) in X_+ , if $x_{\alpha} \wedge e \xrightarrow{\parallel \cdot \parallel} 0$ then $x_{\alpha} \xrightarrow{\operatorname{un}} 0$;
- (iii) For every sequence (x_n) in X_+ , if $x_n \wedge e \xrightarrow{\parallel \cdot \parallel} 0$ then $x_n \xrightarrow{\operatorname{un}} 0$.

Proof. The implication $(i) \Rightarrow (ii)$ was proved in [DOT, Lemma 2.11]. $(ii) \Rightarrow (iii)$ is trivial. This leaves $(iii) \Rightarrow (i)$.

Suppose (iii). Fix $x \in X_+$. We need to show that $x \wedge ne \xrightarrow{\|\cdot\|} x$ or, equivalently $(x - ne)^+ \xrightarrow{\|\cdot\|} 0$ as a sequence of n. Put $u = x \vee e$. The

ideal I_u is lattice isomorphic (as a vector lattice) to C(K) for some compact space K, with u corresponding to 1. Since $x, e \in I_u$, we may consider x and e as elements of C(K). Note that $x \vee e = 1$ implies that x and e never vanish simultaneously.

For each $n \in \mathbb{N}$, we define

$$F_n = \{t \in K : x(t) \ge ne(t)\}$$
 and $O_n = \{t \in K : x(t) > ne(t)\}.$

Clearly, $O_n \subseteq F_n$, O_n is open, and F_n is closed.

Claim 1: $F_{n+1} \subseteq O_n$. Indeed, let $t \in F_{n+1}$. Then $x(t) \ge (n+1)e(t)$. If e(t) > 0 then x(t) > ne(t), so that $t \in O_n$. If e(t) = 0 then x(t) > 0, hence $t \in O_n$.

By Urysohn's Lemma, we find $z_n \in C(K)$ such that $0 \leq z_n \leq x, z_n$ agrees with x on F_{n+1} and vanishes outside of O_n . We can also view z_n as an element of X.

Claim 2: $n(z_n \wedge e) \leq x$. Let $t \in K$. If $t \in O_n$ then $n(z_n \wedge e)(t) \leq ne(t) < x(t)$. If $t \notin O_n$ then $z_n(t) = 0$, so that the inequality is satisfied trivially.

Claim 3: $(x - (n+1)e)^+ \leq z_n$. Again, let $t \in K$. If $t \in F_{n+1}$ then $(x - (n+1)e)^+ \leq x(t) = z_n(t)$. If $t \notin F_{n+1}$ then x(t) < (n+1)e(t), so that $(x - (n+1)e)^+(t) = 0$ and the inequality is satisfied trivially.

Now, Claim 2 yields $0 \leq z_n \wedge e \leq \frac{1}{n}x \xrightarrow{\|\cdot\|} 0$, so that $z_n \wedge e \xrightarrow{\|\cdot\|} 0$. By assumption, this yields $z_n \xrightarrow{\mathrm{un}} 0$. Since $0 \leq z_n \leq x$ for every n, the sequence (z_n) is order bounded and, therefore, $z_n \xrightarrow{\|\cdot\|} 0$. Now Claim 3 yields $(x - (n+1)e)^+ \xrightarrow{\|\cdot\|} 0$, which concludes the proof. \Box

Theorem 3.2. Un-topology is metrizable iff X has a quasi-interior point. If e is a quasi-interior point then $d(x, y) = |||x - y| \wedge e||$ is a metric for un-topology.

Proof. Suppose that $e \in X_+$ is a quasi-interior point and put $d(x, y) = ||x - y| \wedge e||$ for $x, y \in X$. It can be easily verified that this defines a metric on X. Indeed, d(x, x) = 0 and d(x, y) = d(y, x) for every $x, y \in X$. If d(x, y) = 0 then $|x - y| \wedge e = 0$, hence |x - y| = 0 because e is a weak unit, so that x = y. The triangle inequality follows from the fact that

$$|x - z| \land e \leqslant |x - y| \land e + |y - z| \land e.$$

Note also that $x_{\alpha} \xrightarrow{\text{un}} x$ iff $d(x_{\alpha}, x) \to 0$ for every net (x_{α}) in X.

Conversely, suppose that un-topology is metrizable; let d be a metric for it. For each n, let $B_{\frac{1}{n}}$ be the ball of radius $\frac{1}{n}$ centred at zero for the metric, that is,

$$B_{\frac{1}{n}} = \left\{ x \in X : d(x,0) \leq \frac{1}{n} \right\}.$$

Since $B_{\frac{1}{n}}$ is a neighborhood of zero for the un-topology, it contains V_{ε_n,u_n} for some $\varepsilon_n > 0$ and $u_n > 0$. Let $M_n = 2^n ||u_n|| + 1$; then the series $e = \sum_{n=1}^{\infty} \frac{u_n}{M_n}$ converges. Note that $M_n > 1$ and $u_n \leq M_n e$ for every n. We claim that e is a quasi-interior point.

It suffices that Theorem 3.1(ii) is satisfied. Suppose that $x_{\alpha} \wedge e \xrightarrow{\|\cdot\|} 0$ for some net (x_{α}) in X_+ . Fix *n*. It follows from

$$x_{\alpha} \wedge u_n \leqslant (M_n x_{\alpha}) \wedge (M_n e) = M_n (x_{\alpha} \wedge e) \xrightarrow{\|\cdot\|} 0$$

that $x_{\alpha} \wedge u_n \xrightarrow{\|\cdot\|} 0$. Then there exists α_0 such that $\|x_{\alpha} \wedge u_n\| < \varepsilon_n$ whenever $\alpha \ge \alpha_0$. Consequently, x_{α} is in V_{ε_n, u_n} and, therefore, in $B_{\frac{1}{n}}$. It follows that $x_{\alpha} \to 0$ in the metric, hence $x_{\alpha} \xrightarrow{\mathrm{un}} 0$.

Note that a linear Hausdorff topological space is metrizable iff it is first countable, i.e., has a countable base of neighborhoods of zero, see, e.g., [KN63, pp. 49]. Therefore, Theorem 3.2 implies, in particular, that un-topology is first countable iff X has a quasi-interior point. This should be compared with Corollary 2.13 and Question 2.14 in [DOT] (we now know from Example 1.3 that Question 2.14 has a negative answer).

Proposition 3.3. Un-topology is stronger than or equal to a metric topology iff X has a weak unit.

Proof. Suppose that un-topology is stronger than or equal to a topology given by a metric. Construct e as in the second part of the proof of Theorem 3.2. We claim that e is a weak unit. Suppose that $x \wedge e = 0$. It follows that $x \wedge u_n = 0$ for every n and, therefore, $x \in V_{\varepsilon_n, u_n}$, hence $x \in B_{\frac{1}{n}}$. It follows that x = 0.

Conversely, let $e \in X_+$ be a weak unit. For $x, y \in X$, define $d(x, y) = ||x - y| \wedge e||$. As in the first part of the proof of Theorem 3.2, this is a metric and $x_{\alpha} \xrightarrow{\text{un}} x$ implies $d(x_{\alpha}, x) \to 0$.

When is every un-null sequence norm bounded? If X has a strong unit then, by Theorem 2.3, un-topology agrees with norm topology, hence every un-null sequence is norm null and, in particular, norm bounded. This justifies the following question: If every un-null sequence in X is norm bounded (or even norm null), does this imply that X has a strong unit? The following example shows that, in general, the answer in negative.

Example 3.4. Let X be as in Example 1.3. Clearly, X does not have a strong unit; it does not even have a weak unit. Yet, every un-null sequence in X is norm null. Indeed, suppose that $x_n \xrightarrow{\text{un}} 0$. Let u be the characteristic function of $\bigcup_{n=1}^{\infty} \text{supp } x_n$. By assumption, $|x_n| \wedge u \xrightarrow{\|\cdot\|} 0$. It follows that for every $\varepsilon \in (0, 1)$ there exists n_0 such that for every $n \ge n_0$ we have $\||x_n| \wedge u\|| < \varepsilon$. It follows that $\|x_n\| < \varepsilon$.

However, we will see that the answer is affirmative under certain additional assumptions.

Recall that every disjoint sequence is uo-null. Thus, if dim $X = \infty$, one can take any non-zero disjoint sequence, scale it to make it norm unbounded, and thus produce a uo-null sequence which is not norm bounded. However, this trick does not work for un-topology because a disjoint sequence need not be un-null. Moreover, we have the following.

Proposition 3.5. The following are equivalent.

- (i) X is order continuous;
- (ii) Every disjoint sequence in X is un-null;
- (iii) Every disjoint net in X is un-null.

Proof. (i) \Rightarrow (ii) because every disjoint sequence is uo-null and, therefore, un-null. To show that (ii) \Rightarrow (i), note that every order bounded disjoint sequence is norm null and apply [AB06, Theorem 4.14].

(iii) \Rightarrow (ii) is trivial. To show that (ii) \Rightarrow (iii), suppose that there exists a disjoint net (x_{α}) which is not un-null. Then there exist $\varepsilon > 0$ and $u \in X_+$ such that for every α there exists $\beta > \alpha$ with $|||x_{\beta}| \wedge u|| > \varepsilon$. Inductively, we find an increasing sequence (α_k) of indices such that $|||x_{\alpha_k}| \wedge u|| > \varepsilon$. Hence, the sequence (x_{α_k}) is disjoint but not unnull. **Corollary 3.6.** If X is order continuous and every un-null sequence in X is norm bounded then dim $X < \infty$ (and, therefore, X has a strong unit).

Proof. Suppose dim $X = \infty$. Then there exists a non-zero disjoint sequence in X. Scaling it if necessary, we may assume that it is not norm bounded. Yet it is un-null. A contradiction.

Note that Example 2.7 in [DOT] is an example of a disjoint but non un-null sequence in an infinite-dimensional Banach lattice which is not order continuous and lacks a strong unit.

Proposition 3.7. If X has a quasi-interior point and every un-null sequence is norm bounded then X has a strong unit.

Proof. By Theorem 3.2, the un-topology on X is metrizable. Fix such a metric. As before, for each n, let $B_{\frac{1}{n}}$ be the ball of radius $\frac{1}{n}$ centred at zero for the metric. For each n, $B_{\frac{1}{n}}$ contains V_{ε_n,u_n} for some $\varepsilon_n > 0$ and $u_n > 0$. If $V_{\varepsilon_n,u_n} \subseteq [-u_n, u_n]$ for some n then u_n is a strong unit by Lemma 2.2. Otherwise, by Lemma 2.1, each V_{ε_n,u_n} contains a non-trivial ideal. Pick any x_n in this ideal with $||x_n|| = n$. Then the sequence (x_n) is norm unbounded; yet $x_n \in B_{\frac{1}{n}}$ for every n, so that $x_n \xrightarrow{\mathrm{un}} 0$; a contradiction.

4. UN-CONVERGENCE IN A SUBLATTICE

Recall that if (y_{α}) is a net in a regular sublattice Y of a vector lattice X then $y_{\alpha} \xrightarrow{u_{0}} 0$ in Y iff $y_{\alpha} \xrightarrow{u_{0}} 0$ in X. The situation is very different for un-convergence. Let Y be a sublattice of a normed lattice X and (y_{α}) a net in Y. If $y_{\alpha} \xrightarrow{u_{0}} 0$ in X then, clearly, $y_{\alpha} \xrightarrow{u_{0}} 0$ in Y. However, the following examples show that the converse fails even for closed ideals or bands.

Example 4.1. The sequence of the standard unit vectors (e_n) is unnull in c_0 but not in ℓ_{∞} , even though c_0 is a closed ideal in ℓ_{∞} .

Example 4.2. Let X = C[-1, 1] and Y be the set of all $f \in X$ which vanish on [-1, 0]. It is easy to see that Y is a band (though it is not a projection band). Let (f_n) be a sequence in Y_+ such that $||f_n|| = 1$

and $\operatorname{supp} f_n \subseteq [\frac{1}{n+1}, \frac{1}{n}]$. Since X has a strong unit, the un-topology on X agrees with the norm topology, hence (f_n) is not un-null in X. However, it is easy to see that (f_n) is un-null in Y.

Nevertheless, there are some good news. Recall that a sublattice Y of a vector lattice X is **majorizing** if for every $x \in X_+$ there exists $y \in Y_+$ with $x \leq y$.

Theorem 4.3. Let Y be a sublattice of a normed lattice X and (y_{α}) a net in Y such that $y_{\alpha} \xrightarrow{\text{un}} 0$ in Y. Each of the following conditions implies that $y_{\alpha} \xrightarrow{\text{un}} 0$ in X.

- (i) Y is majorizing in X;
- (ii) Y is norm dense in X;
- (iii) Y is a projection band in X.

Proof. Without loss of generality, $y_{\alpha} \ge 0$ for every α . (i) is straightforward. To prove (ii), take $u \in X_+$ and fix $\varepsilon > 0$. Find $v \in Y_+$ with $||u - v|| < \varepsilon$. By assumption, $y_{\alpha} \wedge v \xrightarrow{||\cdot||} 0$. We can find α_0 such that $||y_{\alpha} \wedge v|| < \varepsilon$ whenever $\alpha \ge \alpha_0$. It follows from $u \le v + |u - v|$ that $y_{\alpha} \wedge u \le y_{\alpha} \wedge v + |u - v|$, so that

$$||y_{\alpha} \wedge u|| \leq ||y_{\alpha} \wedge v|| + ||u - v|| < 2\varepsilon.$$

It follows that $y_{\alpha} \wedge u \xrightarrow{\|\cdot\|} 0$. Hence $y_{\alpha} \xrightarrow{\operatorname{un}} 0$ in X.

To prove (iii), let $u \in X_+$. Then u = v + w for some positive $v \in Y$ and $w \in Y^d$. It follows from $y_{\alpha} \perp w$ that $y_{\alpha} \wedge u = y_{\alpha} \wedge v \xrightarrow{\|\cdot\|} 0$. \Box

Recall that every (Archimedean) vector lattice X is majorizing in its order (or Dedekind) completion X^{δ} ; see , e.g., [AB06, p. 101].

Corollary 4.4. If X is a normed lattice and $x_{\alpha} \xrightarrow{\text{un}} x$ in X then $x_{\alpha} \xrightarrow{\text{un}} x$ in the order completion X^{δ} of X.

Corollary 4.5. If X is a KB-space and $x_{\alpha} \xrightarrow{\text{un}} 0$ in X then $x_{\alpha} \xrightarrow{\text{un}} 0$ in X^{**} .

Proof. By [AB06, Theorem 4.60], X is a projection band in X^{**} . The conclusion now follows from Theorem 4.3(iii).

Example 4.1 shows that the assumption that X is a KB-space cannot be removed.

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Corollary 4.6. Let Y be a sublattice of an order continuous Banach lattice X. If $y_{\alpha} \xrightarrow{\text{un}} 0$ in Y then $y_{\alpha} \xrightarrow{\text{un}} 0$ in X.

Proof. Suppose that $y_{\alpha} \xrightarrow{\text{un}} 0$ in Y. By Theorem 4.3(i), $y_{\alpha} \xrightarrow{\text{un}} 0$ in the ideal I(Y) generated by Y in X. By Theorem 4.3(ii), $y_{\alpha} \xrightarrow{\text{un}} 0$ in the closure $\overline{I(Y)}$ of the ideal. Since X is order continuous, $\overline{I(Y)}$ is a projection band in X. It now follows from Theorem 4.3(ii) that $y_{\alpha} \xrightarrow{\text{un}} 0$ in X. \Box

Question 4.7. Let B be a band in X. Suppose that every net in B which is un-null in B is also un-null in X. Does this imply that B is a projection band?

Proposition 4.8. Every band in a normed lattice is un-closed.

Proof. Let *B* be a band and (x_{α}) a net in *B* such that $x_{\alpha} \xrightarrow{\text{un}} x$. Fix $z \in B^d$. Then $|x_{\alpha}| \wedge z = 0$ for every α . Since lattice operations are un-continuous, we have $|x| \wedge z = 0$. It follows that $x \in B^{dd} = B$. \Box

Remark 4.9. Let *B* be a projection band a normed lattice *X*. We write P_B for the corresponding band projection. It follows easily from $0 \leq P_B \leq I$ that if $x_{\alpha} \xrightarrow{\text{un}} x$ in *X* then $P_B x_{\alpha} \xrightarrow{\text{un}} P_B x$ both in *X* and in *B*.

Dense band decompositions. Let X be a Banach lattice. By a *dense band decomposition* of X we mean a family \mathcal{B} of pairwise disjoint projection bands in X such that the linear span of all of the bands in \mathcal{B} is norm dense in X.

Lemma 4.10. Let \mathcal{B} be a family of pairwise disjoint projection bands in a Banach lattice X. \mathcal{B} is a dense band decomposition of X iff for every $x \in X$ and every $\varepsilon > 0$ there exist B_1, \ldots, B_n in \mathcal{B} such that $\|x - \sum_{i=1}^n P_{B_i} x\| < \varepsilon.$

Proof. Suppose that \mathcal{B} is a dense band decomposition of X. Let $x \in X$ and $\varepsilon > 0$. By assumption, we can find distinct bands B_1, \ldots, B_n and vectors $x_1 \in B_1, \ldots, x_n \in B_n$ such that $||x - \sum_{i=1}^n x_i|| < \varepsilon$. Put $Q = I - \sum_{i=1}^n P_{B_i}$. Then Q is also a band projection, hence it is a

lattice homomorphism and $0 \leq Q \leq I$. Note also that $Qx_i = 0$ for i = 1, ..., n. We have

$$|x - \sum_{i=1}^{n} x_i| \ge Q|x - \sum_{i=1}^{n} x_i| = |Qx - \sum_{i=1}^{n} Qx_i| = |x - \sum_{i=1}^{n} P_{B_i}x|.$$

It follows that $||x - \sum_{i=1}^{n} P_{B_i}x|| < \varepsilon$. The converse implication is trivial

The converse implication is trivial.

Our definition of a disjoint band decomposition is partially motivated by following fact.

Theorem 4.11. ([LT79, Proposition 1.a.9]) Every order continuous Banach lattice admits a dense band decomposition \mathcal{B} such that each band in \mathcal{B} has a weak unit.

It is easy to see that if X is an order continuous Banach lattice and \mathcal{B} is a pairwise disjoint collection of bands such that $x = \sup\{P_B x : B \in \mathcal{B}\}$ for every $x \in X_+$ then \mathcal{B} is a dense band decomposition.

Theorem 4.12. Suppose that \mathcal{B} is a dense band decomposition of a Banach lattice X. Then $x_{\alpha} \xrightarrow{\text{un}} x$ in X iff $P_B x_{\alpha} \xrightarrow{\text{un}} P_B x$ in B for each $B \in \mathcal{B}$.

Proof. Without loss of generality, x = 0 and $x_{\alpha} \ge 0$ for every α . The forward implication follows immediately from Remark 4.9. To prove the converse, suppose that $P_B x_{\alpha} \xrightarrow{\text{un}} 0$ in B for each $B \in \mathcal{B}$. Let $u \in X_+$; it suffices to show that $x_{\alpha} \wedge u \xrightarrow{\|\cdot\|} 0$. Fix $\varepsilon > 0$. Find $B_1, \ldots, B_n \in \mathcal{B}$ such that $\|u - \sum_{i=1}^n P_{B_i} u\| < \varepsilon$. Since $P_{B_i} x_{\alpha} \xrightarrow{\text{un}} 0$ in B_i as $i = 1, \ldots, n$, we can find α_0 such that $\|P_{B_i} x_{\alpha} \wedge P_{B_i} u\| < \frac{\varepsilon}{n}$ for every $\alpha \ge \alpha_0$ and every $i = 1, \ldots, n$. It follows from $x_{\alpha} \wedge P_{B_i} u \in B_i$ that $x_{\alpha} \wedge P_{B_i} u = P_{B_i} x_{\alpha} \wedge P_{B_i} u$. Therefore,

$$\|x_{\alpha} \wedge u\| \leq \left\|x_{\alpha} \wedge \sum_{i=1}^{n} P_{B_{i}}u\right\| + \left\|u - \sum_{i=1}^{n} P_{B_{i}}u\right\| \leq \left\|\sum_{i=1}^{n} x_{\alpha} \wedge P_{B_{i}}u\right\| + \varepsilon$$
$$= \left\|\sum_{i=1}^{n} P_{B_{i}}x_{\alpha} \wedge P_{B_{i}}u\right\| + \varepsilon \leq n \cdot \frac{\varepsilon}{n} + \varepsilon \leq 2\varepsilon.$$

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Remark 4.13. Recall that a positive non-zero vector a in a vector lattice X is an **atom** if the principal ideal I_a generated by a coincides with span a. In this case, I_a is a projection band, and the corresponding band projection P_a has form $f_a \otimes a$ for some positive functional f_a , that is, $P_a x = f_a(x)a$. We say that X is **non-atomic** if it has no atoms. We say that X is **atomic** if X is the band generated by all the atoms. In the latter case, $x = \sup\{f_a(x)a : a \text{ is an atom}\}$ for every $x \in X_+$. See, e.g., [Sch74, p. 143].

It follows that if X is an order continuous atomic Banach lattice, the family $\{I_a : a \text{ is an atom}\}$ is a dense band decomposition of X. Applying Theorem 4.12, we conclude that in such spaces un-convergence is exactly the "coordinate-wise" convergence:

Corollary 4.14. Let X be an atomic order continuous Banach lattice. Then $x_{\alpha} \xrightarrow{\text{un}} x$ iff $f_a(x_{\alpha}) \to f_a(x)$ for every atom a.

Remark 4.15. The order continuity assumption cannot be removed. Indeed, ℓ_{∞} is atomic, the sequence (e_n) converges to zero coordinatewise, yet it is not un-null.

The following results extends [DOT, Proposition 6.2].

Proposition 4.16. The following are equivalent:

- (i) $x_{\alpha} \xrightarrow{w} 0$ implies $x_{\alpha} \xrightarrow{un} 0$ for every net (x_{α}) in X;
- (ii) $x_n \xrightarrow{w} 0$ implies $x_n \xrightarrow{un} 0$ for every sequence (x_n) in X;
- (iii) X is atomic and order continuous.

Proof. (i) \Rightarrow (ii) is trivial. The implication (ii) \Rightarrow (iii) is a part of [DOT, Proposition 6.2]. The implication (iii) \Rightarrow (i) follows from Corollary 4.14.

5. AL-representations and local convexity

In this section, we will show that un-topology on an order continuous Banach lattice X is locally convex iff X is atomic. Our main tool is the relationship between un-convergence in X and in an AL-representation of X. It was observed in [Tro04, Example 23] that for a net (x_{α}) in $L_p(\mu)$ where μ is a finite measure and $1 \leq p < \infty$, one has $x_{\alpha} \xrightarrow{\text{un}} 0$ iff $x_{\alpha} \xrightarrow{\mu} 0$ (i.e., the net converges to zero in measure). Note that this does not extend to σ -finite measures. Indeed, let $X = L_p(\mathbb{R})$ and let x_n be the characteristic function of [n, n + 1]. Then $x_n \xrightarrow{\text{un}} 0$ but (x_n) does not converge to zero in measure. On the other hand, let (x_{α}) be a net in $L_p(\mu)$ where μ is a σ -finite measure, let (Ω_n) be a countable partition of Ω into sets of finite measure; it follows from Theorem 4.12 that $x_{\alpha} \xrightarrow{\text{un}} 0$ iff the restriction of x_{α} to Ω_n converges to zero in measure for every n.

Suppose that X is an order continuous Banach lattice with a weak unit e. By [LT79, Theorem 1.b.14], X can be represented as an ideal of $L_1(\mu)$ for some probability measure μ . More precisely, there is a lattice isomorphism from X onto a norm-dense ideal of $L_1(\mu)$; with a slight abuse of notation we will view X itself as an ideal of $L_1(\mu)$. Moreover, this representation may be chosen so that e corresponds to 1, $L_{\infty}(\mu)$ is a norm-dense ideal in X, and both inclusions in $L_{\infty}(\mu) \subseteq X \subseteq L_1(\mu)$ are continuous. We call $L_1(\mu)$ an **AL-representation** for X and e. Let (x_n) be a sequence in X. It was shown in [GTX, Remark 4.6] that $x_n \xrightarrow{\text{uo}} 0$ in X iff $x_n \xrightarrow{\text{a.e.}} 0$ in $L_1(\mu)$. It was shown in [DOT, Theorem 4.6] that $x_n \xrightarrow{\text{un}} 0$ in X iff $x_n \xrightarrow{\mu} 0$ in $L_1(\mu)$. Since un-topology and the topology of convergence in measure are both metrizable on Xbecause X has a weak unit, it follows that these two topologies coincide on X. In particular, $x_{\alpha} \xrightarrow{\text{un}} 0$ in X iff $x_{\alpha} \xrightarrow{\mu} 0$ in $L_1(\mu)$ for every net (x_{α}) in X. This may also be deduced from Amemiya's Theorem (see, e.g., Theorem 2.4.8 in [MN91]) as follows:

 $x_{\alpha} \xrightarrow{\mathrm{un}} 0 \text{ in } X \Leftrightarrow \|x_{\alpha} \wedge e\|_{X} \to 0 \stackrel{\mathrm{Amemiya}}{\Leftrightarrow} \|x_{\alpha} \wedge 1\|_{L_{1}} \to 0 \Leftrightarrow x_{\alpha} \xrightarrow{\mu} 0 \text{ in } L_{1}(\mu)$ for every net (x_{α}) in X_{+} .

Proposition 5.1. Let X be a non-atomic order continuous Banach lattice and W a neighborhood of zero for un-topology. If W is convex then W = X.

Proof. Fix $e \in X_+$; we will show that $e \in W$. We know that $V_{\varepsilon,u} \subseteq W$ for some $\varepsilon > 0$ and u > 0. Consider the principal band B_e . Since X is order continuous, B_e is a projection band in X; let P_e be the

corresponding band projection. Furthermore, B_e is a non-atomic order continuous Banach lattice with a weak unit. Let $L_1(\Omega, \mathcal{F}, \mu)$ be an AL-representation for B_e with e = 1. Note that the measure μ is non-atomic because if a measurable set A were an atom for μ then its characteristic function χ_A would be an atom in X. Fix $n \in \mathbb{N}$. Using the non-atomicity of μ , we find a measurable partition $A_{n,1}, \ldots, A_{n,n}$ of Ω with $\mu(A_{n,i}) = \frac{1}{n}$ as $i = 1, \ldots, n$; see, e.g., Exercise 2 in [Hal70, p. 174]. Since $L_{\infty}(\mu) \subseteq B_e \subseteq L_1(\mu)$, we may view the characteristic functions $\chi_{A_{n,i}}$ as elements of B_e . Consider the vectors $(n\chi_{A_{n,i}}) \wedge u$ as $i = 1, \ldots, n$; they belong to B_e , so that we may view them as functions in $L_1(\mu)$. Let g_n be the function in this list whose norm in X is maximal; if there are more than one, pick any one. Repeating this construction for every $n \in \mathbb{N}$, we produce a sequence (g_n) in $[0, u] \cap B_e$. It follows that $g_n \leq P_e u$ for every n. Since $P_e u$ may be viewed as an element of $L_1(\mu)$ and the measure of the support of g_n tends to zero, it follows that $||g_n||_{L_1} \to 0$. Amemiya's Theorem yields $||g_n||_X \to 0$. Fix n such that $||g_n||_X < \varepsilon$. It follows from the definition of g_n that $\|(n\chi_{A_{n,i}}) \wedge u\|_{\chi} < \varepsilon$ as $i = 1, \ldots, n$, so that $n\chi_{A_{n,i}}$ is in $V_{\varepsilon,u}$ and, therefore, in W. Since W is convex and

$$e = 1 = \frac{1}{n} \sum_{i=1}^{n} n \chi_{A_{n,i}},$$

we have $e \in W$. Therefore, $X_+ \subseteq W$. Furthermore, it follows from $n\chi_{A_{n,i}} \in V_{\varepsilon,u}$ that $-n\chi_{A_{n,i}} \in V_{\varepsilon,u}$ for all $i = 1, \ldots, n$ and, therefore, $-e \in W$. This yields $X_- \subseteq W$. Finally, for every $x \in X$ we have $x = \frac{1}{2}(2x^+ + 2(-x^-))$, so that $x \in W$.

Theorem 5.2. Let X be an order continuous Banach lattice. Untopology on X is locally convex iff X is atomic.

Proof. Suppose that X is atomic. By Corollary 4.14, un-topology is determined by the family of seminorms $x \mapsto |f_a(x)|$ where a is an atom of X; hence the topology is locally convex.

Suppose that un-topology is locally convex but X is not atomic. It follows that there is $e \in X_+$ such that B_e is non-atomic. By Theorem 4.3, un-topology on B_e agrees with the relative topology induced

on B_e by un-topology on X; in particular, it is locally convex. On the other hand, Proposition 5.1 asserts that this topology on B_e has no proper convex neighborhoods; a contradiction.

Un-continuous functionals. Theorem 5.2 allows us to describe uncontinuous linear functionals. For a functional $\varphi \in X^*$, we say that φ is **un-continuous** if it is continuous with respect to the un-topology on X or, equivalently, if $x_{\alpha} \xrightarrow{\text{un}} 0$ implies $\varphi(x_{\alpha}) \to 0$.

Proposition 5.3. The set of all un-continuous functionals in X^* is an ideal.

Proof. It is straightforward to verify that this set is a linear subspace. Suppose that φ in X^* is un-continuous; we will show that $|\varphi|$ is also un-continuous. Fix $\delta > 0$. One can find $\varepsilon > 0$ and u > 0 such that $|\varphi(x)| < \delta$ whenever $x \in V_{\varepsilon,u}$. Fix $x \in V_{\varepsilon,u}$. Since $V_{\varepsilon,u}$ is solid, $|y| \leq |x|$ implies $y \in V_{\varepsilon,u}$ and, therefore, $|\varphi(y)| < \delta$. By the Riesz-Kantorovich formula, we get

$$\left| |\varphi|(x) \right| \leq |\varphi| \left(|x| \right) = \sup \left\{ \left| \varphi(y) \right| : |y| \leq |x| \right\} \leq \delta.$$

It follows that $|\varphi|$ is un-continuous. Hence, the set of all un-continuous functionals in X^* forms a sublattice. It is easy to see that if $\varphi \in X^*_+$ is un-continuous and $0 \leq \psi \leq \varphi$ then ψ is also un-continuous; this completes the proof.

Recall that if a is an atom then f_a stands for the corresponding "coordinate functional".

Corollary 5.4. Suppose that X is an order continuous Banach lattice and $\varphi \in X^*$ is un-continuous.

- (i) If X is atomic then $\varphi = \lambda_1 f_{a_1} + \dots + \lambda_n f_{a_n}$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and a_1, \dots, a_n are atoms;
- (ii) If X is non-atomic then $\varphi = 0$.

Proof. By Proposition 5.3, we may assume that $\varphi \ge 0$; otherwise we consider φ^+ and φ^- .

Suppose X is atomic; let A be a maximal disjoint family of atoms. We claim that the set $F := \{a \in A : \varphi(a) \neq 0\}$ is finite. Indeed, otherwise, take a sequence (a_n) of distinct atoms in F and put $x_n = \frac{1}{\varphi(a_n)}a_n$. Then $x_n \xrightarrow{\text{un}} 0$ by Corollary 4.14, yet $\varphi(x_n) = 1$; a contradiction. This proves the claim.

Since X is order continuous, it follows from Remark 4.13 that X has a disjoint band decomposition $X = B_F \oplus B_{A\setminus F}$. Since $\varphi(a) = 0$ for all $a \in A \setminus F$, φ vanishes on the ideal $I_{A\setminus F}$ and, therefore, on $B_{A\setminus F}$ because φ is order continuous. On the other hand, since F is finite, $B_F = \operatorname{span} F$ and, therefore, is finite-dimensional. It follows that φ is a linear combination of $\{f_a : a \in F\}$.

Suppose now that X is non-atomic. Let $W = \varphi^{-1}(-1, 1)$. Then W is a convex neighborhood of zero for the un-topology. By Proposition 5.1, W = X. This easily implies $\varphi = 0$.

Case (i) of the preceding corollary essentially says that every uncontinuous functional on an atomic order continuous space has finite support.

Example 5.5. Let $X = \ell_2$. By Corollary 5.4, the set of all uncontinuous functionals in X^* may be identified with c_{00} , the linear subspace of all sequences with finite support. Clearly, it is neither norm closed nor order closed; it is not even σ -order closed in X^* .

Example 5.6. Let $X = C_0(\Omega)$ where Ω is a locally compact Hausdorff topological space. It was observed in [Tro04, Example 20] that the un-topology in X agrees with the topology of uniform convergence on compact subsets of Ω .

Let $\varphi \in X_{+}^{*}$. By the Riesz Representation Theorem, there exists a regular Borel measure μ such that $\varphi(f) = \int f d\mu$ for every $f \in X$; see, e.g., [Con99, Theorem III.5.7]. An argument similar to the proof of [Con99, Proposition IV.4.1] shows that φ is un-continuous iff μ has compact support.

6. UN-COMPLETENESS

Throughout this section, X is assumed to be an order continuous Banach lattice. Since un-topology is linear, one can talk about un-Cauchy nets. That is, a net (x_{α}) is un-Cauchy if for every un-neighborhood U of zero there exists α_0 such that $x_{\alpha} - x_{\beta} \in U$ whenever $\alpha, \beta \ge \alpha_0$. We investigate whether X itself or some "nice" subset of X is un-complete. First, we observe that the entire space is un-complete only when X is finite-dimensional.

Lemma 6.1. Let (x_n) be a positive disjoint sequence in an order continuous Banach lattice X such that (x_n) is not norm null. Put $s_n = \sum_{i=1}^n x_i$. Then (s_n) is un-Cauchy but not un-convergent.

Proof. The sequence (s_n) is monotone increasing and does not converge in norm; hence it is not un-convergent by Lemma 1.2(ii). To show that (s_n) is un-Cauchy, fix any $\varepsilon > 0$ and a non-zero $u \in X_+$. Since x_i 's are disjoint, we have $s_n \wedge u = \sum_{i=1}^n (x_i \wedge u)$. The sequence $(s_n \wedge u)$ is increasing and order bounded, hence is norm Cauchy by Nakano's Theorem; see [AB06, Theorem 4.9]. We can find n_0 such that $||s_m \wedge u - s_n \wedge u|| < \varepsilon$ whenever $m \ge n \ge n_0$. Observe that

$$s_m \wedge u - s_n \wedge u = \sum_{i=n+1}^m (x_i \wedge u) = (s_m - s_n) \wedge u = |s_m - s_n| \wedge u.$$

It follows that $|||s_m - s_n| \wedge u|| < \varepsilon$, so that $s_m - s_n \in V_{\varepsilon,u}$.

Proposition 6.2. Let X be an order continuous Banach lattice. X is un-complete iff X is finite-dimensional.

Proof. If X is finite-dimensional then it has a strong unit, so that untopology agrees with norm topology and is, therefore, un-complete. Suppose now that dim $X = \infty$. Then X contains a disjoint normalized positive sequence. By Lemma 6.1, X is not un-complete.

Example 6.3. Let $X = L_p$ with $1 . Pick <math>0 \leq x \in L_1 \setminus L_p$ and put $x_n = x \land (n1)$. It is easy to see that (x_n) is un-Cauchy in L_p , yet it does not un-converge in L_p .

Even when the entire space is not un-complete, the closed unit ball B_X may still be un-complete; that is, complete in the topology induced by un-topology on X. Since B_X is un-closed, it is un-complete iff every norm bounded un-Cauchy net in X is un-convergent. The following theorem should be compared with [GX14, Theorem 4.7], where a similar statement was proved for uo-convergence. **Theorem 6.4.** Let X be an order continuous Banach lattice. Then B_X is un-complete iff X is a KB-space.

Proof. Suppose X is not KB. Then X contains a lattice copy of c_0 . Let (x_n) be the sequence in X corresponding to the unit basis of c_0 . Let $s_n = \sum_{i=1}^n x_i$. Clearly, (s_n) is norm bounded. However, by Lemma 6.1, (s_n) is un-Cauchy but not un-convergent.

Suppose now that X is a KB-space. First, we consider the case when X has a weak unit. In this case, un-topology on X and, therefore, on B_X , is metrizable by Theorem 3.2. Hence, it suffices to prove that B_X is sequentially un-complete. Let (x_n) be a sequence in B_X which is un-Cauchy in X. Let $L_1(\mu)$ be an AL-representation for X. It follows that (x_n) is Cauchy with respect to convergence in measure in $L_1(\mu)$. By [Fol99, Theorem 2.30], there is a subsequence (x_{n_k}) which converges a.e. It follows that (x_{n_k}) is uo-Cauchy in X by [GTX, Remark 4.6]. Then [GX14, Theorem 4.7] yields that $x_{n_k} \xrightarrow{\text{uo}} x$ for some $x \in X$. It follows that $x_{n_k} \xrightarrow{\text{un}} x$.

Now consider the general case. Let X be a KB-space and (x_{α}) a net in B_X such that (x_{α}) is un-Cauchy in X; we need to prove that the net is un-convergent. We may assume without loss of generality that $x_{\alpha} \ge 0$ for every α ; otherwise, consider (x_{α}^+) and (x_{α}^-) , which are also un-Cauchy because $|x_{\alpha}^+ - x_{\beta}^+| \le |x_{\alpha} - x_{\beta}|$ and $|x_{\alpha}^- - x_{\beta}^-| \le |x_{\alpha} - x_{\beta}|$. By Theorem 4.11, there exists a dense band decomposition \mathcal{B} of X such that each B in \mathcal{B} has a weak unit. Put

$$\mathcal{C} = \{B_1 \oplus \cdots \oplus B_n : B_1, \dots, B_n \in \mathcal{B}\}.$$

Note that C is a family of bands with weak units. Furthermore, C is a directed set when ordered by inclusion, so the family of band projections $(P_C)_{C \in C}$ may be viewed as a net.

For every $C \in \mathcal{C}$, the net $(P_C x_\alpha)$ is un-Cauchy by Remark 4.9. Since C has a weak unit, the first part of the proof yields that $(P_C x_\alpha)$ unconverges to some positive vector x_C in C. This produces a net $(x_C)_{C \in \mathcal{C}}$. It is easy to verify that $x_C = x_{B_1} + \cdots + x_{B_n}$ whenever $C = B_1 \oplus \cdots \oplus B_n$ for some $B_1, \ldots, B_n \in \mathcal{B}$. It follows that the net $(x_C)_{C \in \mathcal{C}}$ is increasing. On the other hand, $||x_C|| \leq \liminf_{\alpha} ||P_C x_\alpha|| \leq 1$, so that this net is norm bounded. Since X is a KB-space, the net $(x_C)_{C \in \mathcal{C}}$ converges in norm to some $x \in X$.

Fix $B \in \mathcal{B}$. On one hand, norm continuity of P_B yields $\lim_{C \in \mathcal{C}} P_B x_C = P_B x$. On the other hand, for every $C \in \mathcal{C}$ with $B \subseteq C$ we have $P_B x_C = x_B$, so that $\lim_{C \in \mathcal{C}} P_B x_C = x_B$. It follows that $P_B x = x_B$, so that $P_B x_{\alpha} \xrightarrow{\text{un}} P_B x$ for every $B \in \mathcal{B}$. Now Theorem 4.12 yields $x_{\alpha} \xrightarrow{\text{un}} x$.

The assumption that X is order continuous cannot be removed: for example, ℓ_{∞} is not a KB-space, yet its closed unit ball is un-complete (because the un and the norm topologies on ℓ_{∞} agree).

Example 6.5. The following examples show that in general B_X in Theorem 6.4 cannot be replaced with an arbitrary convex closed bounded set. Let $X = \ell_1$; let C be the set of all vectors in B_X whose coordinates sum up to zero. Clearly, C is convex, closed, and bounded. Let $x_n = \frac{1}{2}(e_1 - e_n)$. Then (x_n) is a sequence in C which un-converges to $\frac{1}{2}e_1$ which is not in C. Thus, C is not un-closed in X; in particular, C is not un-complete.

It is easy to construct a similar example in $X = L_1$; take $C = \{x \in B_X : \int x = 0\}$ and put $x_n = \chi_{[0,\frac{1}{2}]} - \frac{n}{2}\chi_{[\frac{1}{2},\frac{1}{2}+\frac{1}{n}]}, n \ge 2$.

Proposition 6.6. Suppose that X^* is order continuous and C is a norm closed convex norm bounded subset of X. Then C is un-closed.

Proof. Suppose that $x_{\alpha} \xrightarrow{\text{un}} x$ for a net (x_{α}) in C and a vector x in X. Since (x_{α}) is norm bounded and X^* is order continuous, [DOT, Theorem 6.4] guarantees that (x_{α}) converges to x weakly. Since C is convex and closed, it is weakly closed, hence $x \in C$.

Corollary 6.7. Let X be a reflexive Banach lattice and C a closed convex norm bounded subset of X. Then C is un-complete.

Proof. Since X is reflexive, X is a KB-space and X^* is order continuous. Let (x_{α}) be a un-Cauchy net in C. Theorem 6.4 yields that $x_{\alpha} \xrightarrow{\text{un}} x$ for some $x \in X$, while Proposition 6.6 implies that $x \in C$.

7. UN-COMPACT SETS

The main result of this section is Theorem 7.5, which asserts that B_X is (sequentially) un-compact iff X is an atomic KB-space. We start with some auxiliary results. The following theorem shows that, under certain assumptions, un-compactness is a "local" property.

Theorem 7.1. Let X be a KB-space, \mathcal{B} a dense band decomposition of X, and A a un-closed norm bounded subset of X. Then A is uncompact iff $P_B(A)$ is un-compact in B for every $B \in \mathcal{B}$.

Proof. If A is un-compact then $P_B(A)$ is un-compact in B for every $B \in \mathcal{B}$ because P_B is un-continuous by Remark 4.9. To prove the converse, suppose that $P_B(A)$ is un-compact in B for every $B \in \mathcal{B}$. Let $H = \prod_{B \in \mathcal{B}} B$, the formal product of all the bands in \mathcal{B} . That is, H consists of families $(x_B)_{B \in \mathcal{B}}$ indexed by \mathcal{B} , where $x_B \in B$. We equip H with the topology of coordinate-wise un-convergence; this is the product of un-topologies on the bands that make up H. This makes H a topological vector space. Define $\Phi: X \to H$ via $\Phi(x) = (P_B x)_{B \in \mathcal{B}}$. Clearly, Φ is linear. Since \mathcal{B} is a dense band decomposition, Φ is one-to-one. By Theorem 4.12, Φ is a homeomorphism from X equipped with un-topology onto its range in H.

Let K be the subset of H defined by $K = \prod_{B \in \mathcal{B}} P_B(A)$. By Tikhonov's Theorem, K is compact in H. It is easy to see that $\Phi(A) \subseteq K$.

We claim that $\Phi(A)$ is closed in H. Indeed, suppose that $\Phi(x_{\alpha}) \to h$ in H for some net (x_{α}) in A. In particular, the net $(\Phi(x_{\alpha}))$ is Cauchy in H. Since Φ is a homeomorphism, the net (x_{α}) is un-Cauchy in A. Since (x_{α}) is bounded and X is a KB-space, (x_{α}) un-converges to some $x \in X$ by Theorem 6.4. Since A is un-closed, we have $x \in A$. It follows that $h = \Phi(x)$, so that $h \in \Phi(A)$.

Being a closed subset of a compact set, $\Phi(A)$ is itself compact. Since Φ is a homeomorphism, we conclude that A is un-compact. \Box

Next, we discuss relationships between the sequential and the general variants of un-closedness and un-compactness. Recall that for a set A in a topological space, we write \overline{A} for the closure of A; we write \overline{A}^{σ} for the sequential closure of A, i.e., $a \in \overline{A}^{\sigma}$ iff a is the limit of a

sequence in A. We say that A is **sequentially closed** if $\overline{A}^{\sigma} = A$. It is well known that for a metrizable topology, we always have $\overline{A}^{\sigma} = \overline{A}$.

For a set A in a Banach lattice, we write \overline{A}^{un} and $\overline{A}^{\sigma\text{-un}}$ for the un-closure and the sequential un-closure of A, respectively. Obviously, $\overline{A}^{\sigma\text{-un}} \subseteq \overline{A}^{\text{un}}$.

Example 7.2. In general, $\overline{A}^{\text{un}} \neq \overline{A}^{\sigma\text{-un}}$. Indeed, in the notation of Example 1.3, let $A = \{e_{\omega} : \omega \in \Omega\}$. It follows from Example 1.3 that zero is in \overline{A}^{un} but not in $\overline{A}^{\sigma\text{-un}}$.

Proposition 7.3. Let A be a subset of a Banach lattice X. If X has a quasi-interior point or X is order continuous then $\overline{A}^{un} = \overline{A}^{\sigma-un}$.

Proof. If X has a quasi-interior point then its un-topology is metrizable by Theorem 3.2, hence $\overline{A}^{un} = \overline{A}^{\sigma-un}$.

Suppose that X is order continuous. Suppose that $x \in \overline{A}^{\mathrm{un}}$; we need to show that $x \in \overline{A}^{\sigma-\mathrm{un}}$. Without loss of generality, x = 0. This means that A contains a un-null net (x_{α}) . By Theorem 1.1, there exists an increasing sequence of indices (α_k) and a disjoint sequence (d_k) such that $x_{\alpha_k} - d_k \xrightarrow{\|\cdot\|} 0$. It follows that $x_{\alpha_k} - d_k \xrightarrow{\mathrm{un}} 0$. Since (d_k) is disjoint, it is uo-null and, since X is order continuous, un-null. It follows that $x_{\alpha_k} \xrightarrow{\mathrm{un}} 0$ and, therefore, $0 \in \overline{A}^{\sigma-\mathrm{un}}$.

Recall that a topological space is said to be *sequentially compact* if every sequence has a convergent subsequence. In a Hausdorff topological vector space which is metrizable (or, equivalently, first countable), sequential compactness is equivalent to compactness, see, e.g., [Roy88, Theorem 7.21]. We do not know whether un-compactness and sequential un-compactness are equivalent in general, yet we have the following partial result.

Proposition 7.4. Let A be a subset of a Banach lattice X.

- (i) If X has a quasi-interior point, then A is sequentially uncompact iff A is un-compact.
- (ii) Suppose that X is order continuous. If A is un-compact then A is sequentially un-compact.
- (iii) Suppose that X is a KB-space. If A is norm bounded and sequentially un-compact then A is un-compact.

Proof. (i) follows immediately from Theorem 3.2.

(ii) Let (x_n) be a sequence in A. Find $e \in X_+$ such that (x_n) is contained in B_e (e.g., take $e = \sum_{n=1}^{\infty} \frac{x_n}{2^n ||x_n||+1}$). Since B_e is un-closed, the set $A \cap B_e$ is un-compact in B_e . Since e is a quasi-interior point for B_e , the un-topology on B_e is metrizable, hence $A \cap B_e$ is sequentially un-compact. It follows that there is a subsequence (x_{n_k}) which unconverges in B_e to some $x \in A \cap B_e$. By Theorem 4.3(iii), $x_{n_k} \xrightarrow{\text{un}} x$ in X.

(iii) Clearly, A is sequentially un-closed and, therefore, un-closed by Proposition 7.3. Let \mathcal{B} be as in Theorem 4.11. For each $B \in \mathcal{B}$, the band projection P_B is un-continuous by Remark 4.9, so that $P_B(A)$ is sequentially un-compact in B. Since B has a weak unit, the untopology on B is metrizable, so that $P_B(A)$ is un-compact in B. The conclusion now follows from Theorem 7.1.

Theorem 7.5. For a Banach lattice X, TFAE:

- (i) B_X is un-compact;
- (ii) B_X is sequentially un-compact;
- (iii) X is an atomic KB-space.

Proof. First, observe that both (i) and (ii) imply that X is order continuous and atomic. Indeed, since order intervals are bounded and un-closed, they are (sequentially) un-compact. But on order intervals, the un-topology agrees with the norm topology, hence order intervals are norm compact. This implies that X is atomic and order continuous; see, e.g., [Wnuk99, Theorem 6.1].

Suppose (i). Since X is order continuous, Proposition 7.4(ii) yields (ii).

Suppose (ii). We already know that X is atomic. To show that X is a KB-space, let (x_n) be an increasing norm bounded sequence in X_+ . By assumption, it has a un-convergent subsequence (x_{n_k}) . By Lemma 1.2(ii), (x_{n_k}) converges in norm, hence (x_n) converges in norm. This yields (iii).

Suppose (iii). Let A be a maximal disjoint family of atoms in X. Then $\{B_a : a \in A\}$ is a dense band decomposition of X. For every $a \in A$, $P_a(B_X)$ is a closed bounded subset of the one-dimensional band

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 B_a , hence $P_a(B_X)$ is norm and un-compact in B_a . Theorem 7.1 now implies that B_X is un-compact, which yields (i).

Example 7.6. Let $X = c_0$ and $x_n = e_1 + \cdots + e_n$. Then (x_n) is a sequence in B_X with no un-convergent subsequences.

Proposition 7.7. Let A be a subset of an order continuous Banach lattice X. If A is relatively un-compact then A is relatively sequentially un-compact.

Proof. Let (x_n) be a sequence in A. Find $e \in X_+$ such that (x_n) is contained in B_e . Since \overline{A}^{un} is un-compact, the set $\overline{A}^{un} \cap B_e$ is uncompact in B_e and, therefore, sequentially un-compact in B_e because the un-topology on B_e is metrizable. Hence, there is a subsequence (x_{n_k}) which un-converges in B_e and, therefore, in X. \Box

8. UN-CONVERGENCE AND WEAK*-CONVERGENCE

When does un-convergence imply weak*-convergence? It is easy to see that, in general, un-convergence does not imply weak*convergence. Indeed, let X be an infinite-dimensional Banach lattice with order continuous dual. Pick any unbounded disjoint sequence (f_n) in X*. Being unbounded, (f_n) cannot be weak*-null. Yet it is un-null by Proposition 3.5. However, if we restrict ourselves to norm bounded nets, the situation is more interesting. The following result is analogous to [Gao14, Theorem 2.1]. Recall that for a net (f_{α}) in X*, we write $f_{\alpha} \xrightarrow{|\sigma|(X^*,X)} 0$ if $|f_{\alpha}|(x) \to 0$ for every $x \in X_+$.

Theorem 8.1. Let X be a Banach lattice such that X^* is order continuous. The following are equivalent:

- (i) X is order continuous;
- (ii) for any norm bounded net (f_{α}) in X^* , if $f_{\alpha} \xrightarrow{\text{un}} 0$, then $f_{\alpha} \xrightarrow{\text{w}^*} 0$;
- (iii) for any norm bounded net (f_{α}) in X^* , if $f_{\alpha} \xrightarrow{\text{un}} 0$, then $f_{\alpha} \xrightarrow{|\sigma|(X^*,X)} 0$;
- (iv) for any norm bounded sequence (f_n) in X^* , if $f_n \xrightarrow{\text{un}} 0$, then $f_n \xrightarrow{\text{w}^*} 0$;

(v) for any norm bounded sequence (f_n) in X^* , if $f_n \xrightarrow{\text{un}} 0$, then $f_n \xrightarrow{|\sigma|(X^*,X)} 0$.

The proof is similar to that of [Gao14, Theorem 2.1] except that in the proof of $(iv) \Rightarrow (i)$ we use Proposition 3.5. Note that without the assumption that X^* is order continuous, we still get the following implications:

$$(i) \Rightarrow [(ii) \Leftrightarrow (iii)] \Rightarrow [(iv) \Leftrightarrow (v)].$$

When does weak*-convergence imply un-convergence? Recall that for norm bounded nets, weak*-convergence implies uo-convergence in X^* iff X is atomic and order continuous by [Gao14, Theorem 3.4]. Furthermore, Proposition 4.16 immediately yields the following.

Corollary 8.2. If $f_n \xrightarrow{w^*} 0$ implies $f_n \xrightarrow{un} 0$ for every sequence in X^* then X^* is atomic and order continuous.

The following example shows that the converse is false in general.

Example 8.3. Let X = c, the space of all convergent sequences. By [AB06a, Theorem 16.14], X^* may be identified with $\ell_1 \oplus \mathbb{R}$ with the duality given by

$$\langle (f,r), x \rangle = r \cdot \lim_{n} x_n + \sum_{n=1}^{\infty} f_n x_n,$$

where $x \in c, f \in \ell_1$, and $r \in \mathbb{R}$. It is easy to see that X^* is atomic and order continuous. Consider the sequence $((e_n, 0))$ in X^* , where e_n is the *n*-th standard unit vector in ℓ_1 . It is easy to see that $(e_n, 0) \xrightarrow{w^*} (0, 1)$ in X^* . On the other hand, this sequence is disjoint and, therefore, un-null. Take $f_n = (e_n, -1)$; it follows that (f_n) is weak*-null but not un-null. Note that in this example, X^* is order continuous while X is not.

Nevertheless, we will show that the converse implication is true under the additional assumption that X is order continuous.

Theorem 8.4. The following are equivalent:

- (i) For every net (f_{α}) in X^* , if $f_{\alpha} \xrightarrow{w^*} 0$ then $f_{\alpha} \xrightarrow{un} 0$;
- (ii) X^* is atomic and both X and X^* are order continuous.

Proof. (i) \Rightarrow (ii) By Corollary 8.2, X^* is atomic and order continuous. Suppose X is not order continuous. By [MN91, Corollary 2.4.3] there exists a disjoint norm-bounded sequence (f_n) in X^* which is not weak^{*}-null. One can then find a subsequence (f_{n_k}) , a vector $x_0 \in X$ and a positive real ε so that $|f_{n_k}(x_0)| > \varepsilon$ for every k. By the Alaoglu-Bourbaki Theorem, there is a subnet (g_α) of (f_{n_k}) such that $g_\alpha \xrightarrow{\mathrm{w}^*} g$ for some $g \in X^*$. Since (f_{n_k}) is disjoint and X^* is order continuous, we have $f_{n_k} \xrightarrow{\mathrm{un}} 0$ and, therefore, $g_\alpha \xrightarrow{\mathrm{un}} 0$. By assumption, this yields g = 0, so that $g_\alpha \xrightarrow{\mathrm{w}^*} 0$. This contradicts $|g_\alpha(x_0)| > \varepsilon$ for every α .

(ii) \Rightarrow (i) Let $f_{\alpha} \xrightarrow{w^*} 0$ in X. Let A be a maximal disjoint collection of atoms in X^{*}; for each atom $a \in A$ let P_a and φ_a be the corresponding band projection and the coordinate functional, respectively; P_a and φ_a are defined on X^{*}. By [MN91, Corollary 2.4.7], P_a and, therefore, φ_a , is weak^{*}-continuous. It follows that $\varphi_a(f_{\alpha}) \to 0$ in α . Corollary 4.14 yields that $f_{\alpha} \xrightarrow{\text{un}} 0$.

Proposition 8.5. Suppose that X^* is atomic. The following are equivalent.

- (i) For every net (f_{α}) in X^* , if $f_{\alpha} \xrightarrow{|\sigma|(X^*,X)} 0$ then $f_{\alpha} \xrightarrow{\text{un}} 0$;
- (ii) For every sequence (f_n) in X^* , if $f_n \xrightarrow{|\sigma|(X^*,X)} 0$ then $f_n \xrightarrow{\mathrm{un}} 0$;
- (iii) X^* is order continuous.

Proof. (i) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (iii) The proof is similar to that of Proposition 4.16. To show that X^* is order continuous, suppose that (f_n) is an order bounded positive disjoint sequence in X^*_+ . It follows that $f_n \xrightarrow{|\sigma|(X^*,X)} 0$ and, by assumption, $f_n \xrightarrow{\text{un}} 0$. Since the sequence is order bounded, this yields $f_n \xrightarrow{\parallel \cdot \parallel} 0$. Therefore, X^* is order continuous.

(iii) \Rightarrow (i) By [MN91, Proposition 2.4.5], band projections on X^* are $|\sigma|(X^*, X)$ -continuous. The proof is now analogous to the implication (ii) \Rightarrow (i) in Theorem 8.4.

Simultaneous weak^{*} and un-convergence. Section 4 of [Gao14] contains several results that assert that if a sequence or a net in X^* converges in both weak^{*} and uo-topology then it also converges in some other topology. Several of these results remain valid if uo-convergence

is replaced with un-convergence. In particular, this works for Proposition 4.1 in [Gao14]. Propositions 4.3, 4.4, and 4.6 in [Gao14] remain valid under the additional assumption that X^* is order continuous (note that the dual positive Schur property already implies that X^* is order continuous by [Wnuk13, Proposition 2.1]). The proofs are analogous to the corresponding proofs in [Gao14]. Alternatively, the un-versions of these may be deduced from the uo-versions using the following two facts: first, every un-convergent sequence has a uo-convergent subsequence and, second, a sequence (x_n) converges to x in a topology τ iff every subsequence (x_{n_k}) has a further subsequence $(x_{n_{k_i}})$ such that $x_{n_{k_i}} \xrightarrow{\tau} x$.

9. UN-COMPACT OPERATORS

Throughout this section, let E be a Banach space, X a Banach lattice, and $T \in L(E, X)$. We say that T is *(sequentially) un-compact* if TB_E is relatively (sequentially) un-compact in E. Equivalently, for every bounded net (x_{α}) (respectively, every bounded sequence (x_n)) its image has a subnet (respectively, subsequence), which is un-convergent.

Clearly, if T is compact then it is un-compact and sequentially uncompact. Theorems 3.2 and 7.5 and Proposition 7.7 yield the following.

Proposition 9.1. Let $T \in L(E, X)$.

- (i) If X has a quasi-interior point then T is un-compact iff it is sequentially un-compact;
- (ii) If X is order continuous and T is un-compact then T is sequentially un-compact;
- (iii) If X is an atomic KB-space then T is un-compact and sequentially un-compact.

Proposition 9.2. The set of all un-compact operators is a linear subspace of L(E, X). The set of all sequentially un-compact operators in L(E, X) is a closed subspace.

Proof. Linearity is straightforward. To prove closedness, suppose that (T_m) is a sequence of sequentially un-compact operators in L(E, X) and $T_m \xrightarrow{\parallel \cdot \parallel} T$. We will show that T is sequentially un-compact.

Let (x_n) be a sequence in B_E . For every m, the sequence $(T_m x_n)_n$ has a un-convergent subsequence. By a standard diagonal argument, we can find a common subsequence for all these sequences. Passing to this subsequence, we may assume without loss of generality that for every m we have $T_m x_n \xrightarrow{\text{un}} y_m$ for some y_m . Note that

$$||y_m - y_k|| \leq \liminf_n ||T_m x_n - T_k x_n|| \leq ||T_m - T_k|| \to 0,$$

so that the sequence (y_m) is Cauchy and, therefore, $y_m \xrightarrow{\|\cdot\|} y$ for some $y \in X$.

Fix $u \in X_+$ and $\varepsilon > 0$. Find m_0 such that $||T_{m_0} - T|| < \varepsilon$ and $||y_{m_0} - y|| < \varepsilon$. Find n_0 such that $|||T_{m_0}x_n - y_{m_0}| \wedge u|| < \varepsilon$ whenever $n \ge n_0$. It follows from

$$|Tx_n - y| \wedge u \leqslant |Tx_n - T_{m_0}x_n| + |T_{m_0}x_n - y_{m_0}| \wedge u + |y_{m_0} - y|$$

that $|||Tx_n - y| \wedge u|| < 3\varepsilon$, so that $Tx_n \xrightarrow{\mathrm{un}} y$.

We do not know whether the set of all un-compact operators is closed.

It is easy to see that if we multiply a (sequentially) un-compact operator by another bounded operator on the right, the product is again (sequentially) un-compact. The following example shows that this fails when we multiply on the left.

Example 9.3. The class of all (sequentially) un-compact operators is not a left ideal. Let $T: \ell_1 \to L_1$ be defined via $Te_n = r_n^+$, where (e_n) is the standard unit basis of ℓ_1 and (r_n) is the Rademacher sequence in L_1 . Note that T is neither un-compact nor sequentially un-compact because the sequence (Te_n) has no un-convergent subsequences. On the other hand, $T = TI_{\ell_1}$, where I_{ℓ_1} is the identity operator on ℓ_1 . Observe that I_{ℓ_1} is un-compact by Proposition 9.1(iii).

Proposition 9.4. In the diagram $E \xrightarrow{T} X \xrightarrow{S} Y$, suppose that T is (sequentially) un-compact and S is a lattice homomorphism. If the ideal generated by Range S is dense in Y then ST is (sequentially) un-compact.

Proof. We will prove the statement for the sequential case; the other case is analogous. Let (h_n) be a norm bounded sequence in E. By

assumption, there is a subsequence (h_{n_k}) such that $Th_{n_k} \xrightarrow{\mathrm{un}} x$ for some $x \in X$. Let $Z = \operatorname{Range} S$; it is a sublattice of Y. Fix $u \in Z_+$. Then u = Sv for some $v \in X_+$, and $|Th_{n_k} - x| \wedge v \xrightarrow{\|\cdot\|} 0$. Applying S, we get $|STh_{n_k} - Sy| \wedge u \xrightarrow{\|\cdot\|} 0$. Therefore, $STh_{n_k} \xrightarrow{\mathrm{un}} Sx$ in Z. It follows from Theorem 4.3(i) and (ii) that $STh_{n_k} \xrightarrow{\mathrm{un}} Sx$ in Y. \Box

Example 9.5. The set of all sequentially un-compact operators is not order closed. Let T be as in Example 9.3. Let $T_n = TP_n$, where P_n is the *n*-th basis projection on ℓ_1 , i.e., $T_n h = \sum_{i=1}^n h_i r_i^+$ for $h \in \ell_1$. It is easy to see that each T_n is finite rank and, therefore, sequentially un-compact. Note that $T_n \uparrow T$, yet T is not sequentially un-compact.

Proposition 9.6. Suppose that for every sequence (T_n) of sequentially un-compact operators in $L(c_0, X)$, $T_n \uparrow T$ implies that T is sequentially un-compact. Then X is a KB-space.

Proof. Suppose not. Then there is a lattice isomorphism $T: c_0 \to X$. Put $x_n = Te_n$, where (e_n) is the standard unit basis of c_0 . Put $T_n = TP_n$, where P_n is the *n*-th basis projection on c_0 , i.e., $T_n h = \sum_{i=1}^n h_i x_i$ for $h \in c_0$. It follows that $T_n h \xrightarrow{\|\cdot\|} Th$, so that $T_n h \uparrow Th$ for every $h \ge 0$ and, therefore, $T_n \uparrow T$. For each n, T_n has finite rank and, therefore, is sequentially un-compact.

We claim that, nevertheless, T is not sequentially un-compact. Put $w_n = e_1 + \cdots + e_n$ in c_0 . Note that (w_n) is norm bounded and $Tw_n = x_1 + \cdots + x_n$. Since T is an isomorphism, (Tw_n) is not norm-convergent. Since (Tw_n) is increasing, it is not un-convergent by Lemma 1.2(ii). Similarly, no subsequence of (Tw_n) is un-convergent.

We do not know whether the converse is true.

Next, we study whether un-compactness is inherited under domination. The following example shows that, in general, the answer is negative.

Example 9.7. Let T be as in Example 9.3. Let $S: \ell_1 \to L_1$ be defined via $Se_n = 1$. Then S is a rank-one operator; hence it is compact and un-compact. Clearly, $0 \leq T \leq S$. Yet T is not un-compact.

Proposition 9.8. Suppose that $S,T: E \to X$, $0 \leq S \leq T$, X is a KB-space and T is a lattice homomorphism. If T is (sequentially) un-compact then so is S.

Proof. We will prove the sequential case; the other case is similar. Let (h_n) be a bounded sequence in E. Passing to a subsequence, we may assume that (Th_n) is un-convergent. In particular, it is un-Cauchy. Fix $u \in X_+$. Note that

$$|Sh_n - Sh_m| \wedge u \leqslant \left(S|h_n - h_m|\right) \wedge u \leqslant \left(T|h_n - h_m|\right) \wedge u = |Th_n - Th_m| \wedge u \xrightarrow{\|\cdot\|} 0$$

as $n, m \to \infty$. It follows that (Sh_n) is un-Cauchy and, therefore, unconverges by Theorem 6.4.

We would like to mention that the class of un-compact operators is different from several other known classes of operators. We already mentioned that every compact operator is un-compact. The converse is false as the identity operator on any infinite-dimensional atomic KBspace is un-compact but not compact.

Recall that an operator between Banach lattices is AM-compact if it maps order intervals to relatively compact sets.

Proposition 9.9. Every order bounded un-compact operator is AMcompact.

Proof. Let $T: X \to Y$ be an order bounded un-compact operator between Banach lattices. Fix an order interval [a, b] in X. Since T is un-compact, $T[a, b] \subseteq C$ for some un-compact set C. Since T is order bounded, $T[a, b] \subseteq [c, d]$ for some $c, d \in Y$. Note that [c, d] is un-closed, hence $C \cap [c, d]$ is un-compact and, being order bounded, is compact. It follows that T[a, b] is relatively compact. \Box

Note that the converse is false: the identity operator on c_0 is AMcompact but not un-compact.

The identity operator on ℓ_1 is un-compact, yet it is neither L-weakly compact nor M-weakly compact.

Finally, we note that if T is sequentially un-compact and semicompact then T is compact. Indeed, let (h_n) be a bounded sequence in E. There is a subsequence (h_{n_k}) such that $Th_{n_k} \xrightarrow{\text{un}} x$ for some $x \in X$. Since T is semi-compact, the sequence (Th_{n_k}) is almost order bounded and, therefore, $Th_{n_k} \xrightarrow{\parallel \cdot \parallel} x$ by [DOT, Lemma 2.9].

Finally, we discuss when weakly compact operators are un-compact.

Lemma 9.10. If $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{un} y$ then x = y.

Proof. Without loss of generality, y = 0. By Theorem 1.1, there exist a subsequence (x_{n_k}) and a disjoint sequence (d_k) such that $x_{n_k} - d_k \xrightarrow{\|\cdot\|} 0$. It follows that $x_{n_k} - d_k \xrightarrow{w} 0$, so that $d_k \xrightarrow{w} x$. Now [AB06, Theorem 4.34] yields x = 0.

Theorem 9.11. A Banach lattice X is atomic and order continuous iff T is sequentially un-compact for every Banach space E and every weakly compact operator $T: E \to X$.

Proof. The forward implication follows immediately from Proposition 4.16. To prove the converse, let (x_n) be a weakly null sequence in X. By Proposition 4.16, it suffices to show that $x_n \xrightarrow{\text{un}} 0$. Define $T: \ell_1 \to X$ via $Te_n = x_n$. By [AB06, Theorem 5.26], T is weakly compact. By assumption, T is sequentially un-compact. It follows that (Te_n) has a un-convergent subsequence, i.e., $x_{n_k} \xrightarrow{\text{un}} x$ for some $x \in X$ and a subsequence (x_{n_k}) . Lemma 9.10 yields x = 0. By the same argument, every subsequence of (x_n) has a further subsequence which is un-null; since un-convergence is topological, it follows that $x_n \xrightarrow{\text{un}} 0$.

Corollary 9.12. Every operator from a reflexive Banach space to an atomic order continuous Banach lattice is sequentially un-compact.

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References

- [AA02] Y. Abramovich and C.D. Aliprantis, An invitation to operator theory, Vol. 50. Providence, RI: American Mathematical Society, 2002.
- [AB06] C.D. Aliprantis and O. Burkinshaw, *Positive operators*, 2nd edition, Springer 2006.
- [AB06a] C.D. Aliprantis, K. C. Border, *Infinite Dimensional Analysis: A hitchhiker's guide*, 3rd edition, Springer, Berlin, 2006.
- [CW98] Z.L. Chen and A.W. Wickstead, Relative weak compactness of solid hulls in Banach lattices, *Indag. Math.*, 9(2), 1998, 187–196.
- [Con99] J.B. Conway, A course in functional analysis, 2nd edition, Springer-Verlag, New York, 1990
- [DeM64] R. DeMarr, Partially ordered linear spaces and locally convex linear topological spaces, *Illinois J. Math.* 8, 1964, 601-606.
- [DOT] Y. Deng, M. O'Brien, and V.G. Troitsky, Unbounded norm convergence in Banach lattices, *Positivity*, to appear. doi:10.1007/s11117-016-0446-9.
- [EM16] E.Yu. Emelyanov, M.A.A. Marabeh, Two measure-free versions of the Brezis-Lieb Lemma, Vladikavkaz Math. J., 18(1), 2016, 21–25.
- [Fol99] G.B. Folland, Real analysis: Modern techniques and their applications, 2nd edition, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1999.
- [GTX] N. Gao, V.G. Troitsky, and F. Xanthos, Uo-convergence and its applications to Cesàro means in Banach lattices, *Israel J. Math.*, to appear. arXiv:1509.07914 [math.FA].
- [GX14] N. Gao and F. Xanthos, Unbounded order convergence and application to martingales without probability, J. Math. Anal. Appl., 415 (2014), 931–947.
- [Gao14] N. Gao, Unbounded order convergence in dual spaces, J. Math. Anal. Appl., 419, 2014, 347–354.
- [GLX] N. Gao, D.H. Leung, and F. Xanthos, The dual representation problem of risk measures. Preprint. arXiv:1610.08806 [q-fin.MF]
- [Hal70] P.R. Halmos, *Measure Theory*, Springer-Verlag, New York, 1970.
- [Kap97] S. Kaplan, On Unbounded Order Convergence, Real Anal. Exchange 23(1), 1997, 175–184.
- [KN63] J.L. Kelley and I. Namioka, *Linear topological spaces*. Van Nostrand Co., Inc., Princeton, N.J. 1963
- [LT79] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. II, Springer-Verlag, Berlin, 1979.
- [MN91] P. Meyer-Nieberg, Banach lattices, Springer-Verlag, Berlin, 1991.

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- [Nak48] H. Nakano, Ergodic theorems in semi-ordered linear spaces, Ann. of Math. (2), 49, 1948, 538–556.
- [Roy88] H.L. Royden, *Real analysis*, 3rd ed., Macmillan Publishing Company, New York, 1988.
- [Sch74] H.H. Schaefer, Banach lattices and positive operators, Springer-Verlag, Berlin, 1974.
- [Tro04] V.G. Troitsky, Measures of non-compactness of operators on Banach lattices, *Positivity*, 8(2), 2004, 165–178.
- [Wic77] A.W. Wickstead, Weak and unbounded order convergence in Banach lattices, J. Austral. Math. Soc. Ser. A, 24(3), 1977, 312–319.
- [Wnuk99] W. Wnuk, Banach lattices with order continuous norms, Polish scientific publishers PWN, Warszawa, 1999.
- [Wnuk13] W. Wnuk, On the dual positive Schur property in Banach lattices, *Positivity*, 17(2013), 759–773.
- [Zab] O. Zabeti, Unbounded absolute weak convergence in Banach lattices, preprint. arXiv:1608.02151 [math.FA].

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