um-TOPOLOGY IN MULTI-NORMED VECTOR LATTICES

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Abstract. Let $\mathcal{M} = \{m_\lambda\}_{\lambda\in\Lambda}$ be a separating family of lattice seminorms on a vector lattice X, then $(X, \mathcal{M})$ is called a multi-normed vector lattice (or MNVL). We write $x_\alpha \xrightarrow{\mathrm{m}} x$ if $m_\lambda(x_\alpha - x) \to 0$ for all $\lambda \in \Lambda$. A net x_α in an MNVL $X = (X, \mathcal{M})$ is said to be unbounded m-convergent (or um-convergent) to x if $|x_\alpha - x| \wedge u \xrightarrow{\mathrm{m}} 0$ for all $u \in X_+$. um-Convergence generalizes un-convergence [7, 15] and uaw-convergence [25], and specializes up-convergence [3] and $u\tau$-convergence [6]. um-Convergence is always topological, whose corresponding topology is called unbounded m-topology (or um-topology). We show that, for an m-complete metrizable MNVL $(X, \mathcal{M})$, the um-topology is metrizable iff X has a countable topological orthogonal system. In terms of um-completeness, we present a characterization of MNVLs possessing both Lebesgue's and Levi's properties. Then, we characterize MNVLs possessing simultaneously the σ-Lebesgue and σ-Levi properties in terms of sequential um-completeness. Finally, we prove that any m-bounded and um-closed set is um-compact iff the space is atomic and has Lebesgue's and Levi's properties.

1. Introduction and preliminaries

Unbounded convergences have attracted many researchers (see for instance [13, 9, 10, 8, 7, 25, 15, 3, 19, 17, 16, 11, 12, 21, 6]. Unbounded convergences are well-investigated in vector and normed lattices (cf. [7, 10, 15, 22, 24]). In the present paper, we also extend several previous results from [7, 10, 15, 22, 24, 25] to multi-normed setting. This work is a continuation of [6], in which unbounded topological convergence was studied in locally solid vector lattices.

For a net x_α in a vector lattice X, we write $x_\alpha \xrightarrow{\mathrm{o}} x$ if x_α *converges to x in order*. That is, there is a net y_β, possibly over a different index set, such that $y_\beta \downarrow 0$ and, for every β, there exists α_β satisfying $|x_\alpha - x| \leqslant y_\beta$ whenever $\alpha \geqslant \alpha_\beta$. A net x_α in a vector lattice X is *unbounded order convergent* (*uo-convergent*) to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{\mathrm{o}} 0$ for every $u \in X_+$.

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We write $x_\alpha \xrightarrow{\mathrm{uo}} x$ in this case. Clearly, order convergence implies *uo*-convergence and they coincide for order bounded nets. For a measure space (Ω, Σ, μ) and a sequence f_n in $L_p(\mu)$ $(0 \leq p \leq \infty)$, $f_n \xrightarrow{\mathrm{uo}} 0$ iff $f_n \to 0$ almost everywhere [10, Rem.3.4]. It is known that almost everywhere convergence is not topological. Therefore, *uo*-convergence might not be topological in general. It was also shown recently that order convergence is never topological in infinite dimensional vector lattices [5].

Let $(X, \|\cdot\|)$ be a normed lattice. For a net x_α in X, we write $x_\alpha \xrightarrow{\|\cdot\|} x$ if x_α converges to x in norm. We say that x_α *unbounded norm converges* to x (x_α *un-converges* to x or $x_\alpha \xrightarrow{\mathrm{un}} x$) if $|x_\alpha - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for every $u \in X_+$. Clearly, norm convergence implies *un*-convergence. The *un*-convergence is topological, and the corresponding topology (which is known as *un-topology*) was investigated in [15]. A net x_α *uaw-converges* to x if $|x_\alpha - x| \wedge u \xrightarrow{w} 0$ for all $u \in X_+$, where "w" stands for the weak convergence. Absolute weak convergence implies *uaw*-convergence. *uaw*-Convergence and *uaw*-topology were introduced and investigated in [25].

All topologies considered throughout this article are assumed to be Hausdorff. If a linear topology τ on a vector lattice X has a base at zero consisting of solid sets, then the pair (X, τ) is called a *locally solid vector lattice*. Furthermore, if τ has base at zero consisting of convex-solid sets, then (X, τ) is called a *locally convex-solid vector lattice*. It is known that a linear topology τ on X is locally convex-solid iff there exists a family $\mathcal{M} = \{m_\lambda\}_{\lambda \in \Lambda}$ of lattice seminorms that generates τ (cf. [1, Thm.2.25]). Moreover, for such $\mathcal{M}$, $x_\alpha \xrightarrow{\tau} x$ iff $m_\lambda(x_\alpha - x) \underset{\alpha}{\to} 0$ in $\mathbb{R}$ for each $m_\lambda \in \mathcal{M}$. Since τ is Hausdorff then the family $\mathcal{M}$ is separating.

A subset A in a topological vector space (X, τ) is called τ-*bounded* if, for every τ-neighborhood V of zero, there exists $\lambda > 0$ such that $A \subseteq \lambda V$. In the case when the topology τ is generated by a family $\{m_\lambda\}_{\lambda \in \Lambda}$ of seminorms, a subset A of X is τ-bounded iff $\sup_{a \in A} m_\lambda(a) < \infty$ for all $\lambda \in \Lambda$.

Recall that a locally solid vector lattice (X, τ) is said to have the *Lebesgue property* if $x_\alpha \downarrow 0$ in X implies $x_\alpha \xrightarrow{\tau} 0$ or, equivalently, if $x_\alpha \xrightarrow{\mathrm{o}} 0$ implies $x_\alpha \xrightarrow{\tau} 0$; (X, τ) is said to have the σ-*Lebesgue property* if $x_n \downarrow 0$ in X implies $x_n \xrightarrow{\tau} 0$; and (X, τ) is said to have the *pre-Lebesgue* property if $0 \leq x_n \uparrow \leq x$ implies only that x_n is τ-Cauchy. Finally, (X, τ) is said to have the *Levi property* if, when $0 \leq x_\alpha \uparrow$ and x_α is τ-bounded, then $x_\alpha \uparrow x$ for some $x \in X$; (X, τ) is said to have the σ-*Levi property* if x_n has supremum in X provided by $0 \leq x_n \uparrow$ and by the τ-boundedness of x_n, see [1, Def. 3.16].

2. Multi-Normed Vector Lattices

Let (X, τ) be a locally convex-solid vector lattice with an upward directed family $\mathcal{M} = \{m_\lambda\}_{\lambda \in \Lambda}$ of lattice seminorms generating τ. Throughout this article, the pair $(X, \mathcal{M})$ will be referred to as a *multi-normed vector lattice* (*MNVL*). Also, τ-convergence, τ-Cauchy, τ-complete, etc. will be denoted by m-convergence, m-Cauchy, m-complete, etc.

Let X be a vector space, E be a vector lattice, and $p : X \to E_+$ be a vector norm (i.e. $p(x) = 0 \Leftrightarrow x = 0$, $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$, $x \in X$, and $p(x+y) \leq p(x)+p(y)$ for all $x, y \in X$), then (X, p, E) is called a *lattice-normed space*, abbreviated as *LNS*, see [18]. If X is a vector lattice, and the vector norm p is monotone (i.e. $|x| \leq |y| \Rightarrow p(x) \leq p(y)$), then the triple (X, p, E) is called a lattice-normed vector lattice, abbreviated as *LNVL* (cf. [3, 4]).

Given an LNS (X, p, E). Recall that a net x_α in X is said to be *p-convergent* to x (see [3]) if $p(x_\alpha - x) \xrightarrow{o} 0$ in E. In this case, we write $x_\alpha \xrightarrow{p} x$. A subset A of X is called *p-bounded* if there exists $e \in E$ such that $p(a) \leq e$ for all $a \in A$.

Proposition 1. *Every MNVL induces an LNVL. Moreover, for arbitrary nets, p-convergence in the induced LNVL implies m-convergence, and they coincide in the case of p-bounded nets.*

Proof. Let $(X, \mathcal{M})$ be an MNVL, then there is a separating family $\{m_\lambda\}_{\lambda \in \Lambda}$ of lattice seminorms on X. Let $E = \mathbb{R}^\Lambda$ be the vector lattice of all real-valued functions on Λ, and define $p : x \mapsto p_x$ from X into E_+ such that $p_x[\lambda] := m_\lambda(x)$.

It is clear that p is a monotone vector norm on X. Therefore (X, p, E) is an LNVL. Let x_α be a net in X. If $x_\alpha \xrightarrow{p} 0$, then $p_{x_\alpha} \xrightarrow{o} 0$ in $\mathbb{R}^\Lambda$, and so $p_{x_\alpha}[\lambda] \to 0$ or $m_\lambda(x_\alpha) \to 0$ for all $\lambda \in \Lambda$. Hence $x_\alpha \xrightarrow{m} 0$.

Finally, assume a net x_α to be p-bounded. If $x_\alpha \xrightarrow{m} 0$, then $m_\lambda(x_\alpha) \to 0$ or $p_{x_\alpha}[\lambda] \to 0$ for each $\lambda \in \Lambda$. Since x_α is p-bounded, then $p_{x_\alpha} \xrightarrow{o} 0$ in $\mathbb{R}^\Lambda$. That is $x_\alpha \xrightarrow{p} 0$. □

Let X be a vector lattice. An element $0 \neq e \in X_+$ is called a *strong unit* if the ideal I_e generated by e is X or, equivalently, for every $x \geqslant 0$, there exists $n \in \mathbb{N}$ such that $x \leqslant ne$; a *weak unit* if the band B_e generated by e is X or, equivalently, $x \wedge ne \uparrow x$ for every $x \in X_+$. If (X, τ) is a topological vector lattice, then $0 \neq e \in X_+$ is called a *quasi-interior point* if the principal ideal I_e is τ-dense in X (see Definition 6.1 in [20]). It is known that

strong unit $\Rightarrow$ quasi-interior point $\Rightarrow$ weak unit.

The following proposition characterizes quasi-interior points, and should be compared with [2, Thm.4.85].

Proposition 2. *Let $(X, \mathcal{M})$ be an MNVL, then the following statements are equivalent:*

(1) *$e \in X_+$ is a quasi-interior point;*

(2) *for all* $x \in X_+$, $x - x \wedge ne \xrightarrow{\text{m}} 0$ *as* $n \to \infty$;
(3) e *is strictly positive on* X^*, *i.e.*, $0 < f \in X^*$ *implies* $f(e) > 0$, *where* X^* *denotes the topological dual of* X.

Proof. (1)⇒(2) Suppose that e is a quasi-interior point of X, then $\overline{I_e}^m = X$. Let $x \in X_+$. Then $x \in \overline{I_e}^m$, so there exists a net x_α in I_e that m-converges to x. But $x_\alpha \xrightarrow{\text{m}} x$ implies $|x_\alpha| \xrightarrow{\text{m}} |x| = x$. Moreover, $x_\alpha \wedge x \xrightarrow{\text{m}} x \wedge x = x$, and $x_\alpha \wedge x \leq x_\alpha$ implies that $x_\alpha \wedge x \in I$, because I_e is an ideal. So we can assume also that $x_\alpha \leq x$. Hence, for any $x \in X_+$, there is a net $0 \leq x_\alpha \in I_e$ and $x_\alpha \leq x$. Then $0 \leq x_\alpha \wedge ne \leq x \wedge ne \leq x$ for all $n \in \mathbb{N}$. Now, take $\lambda \in \Lambda$, and let $\varepsilon > 0$, then there is α_ε such that $m_\lambda(x - x_{\alpha_\varepsilon}) < \varepsilon$. But $0 \leq x_{\alpha_\varepsilon} \in I_e$, so $0 \leq x_{\alpha_\varepsilon} \leq k_\varepsilon e$ for some $k_\varepsilon \in \mathbb{N}$. Since $0 \leq x_{\alpha_\varepsilon} = x_{\alpha_\varepsilon} \wedge k_\varepsilon e \leq x \wedge k_\varepsilon e \leq x$, then $m_\lambda(x - x \wedge ne) \leq m_\lambda(x - x \wedge k_\varepsilon e) \leq m_\lambda(x - x_\alpha \wedge k_\varepsilon e) = m_\lambda(x - x_{\alpha_\varepsilon}) < \varepsilon$ for all $n \geq k_\varepsilon$. Hence $m_\lambda(x - x \wedge ne) \to 0$ as $n \to \infty$. Since $\lambda \in \Lambda$ was chosen arbitrary, we get $x - x \wedge ne \xrightarrow{\text{m}} 0$.

The proofs of the implications (2)⇒(3), and (3)⇒(1) are similar to the proofs of the corresponding implications of Theorem 4.85 in [2].

□

3. *um*-Topology

In this section we introduce the *um*-topology in a analogous manner to the *un*-topology [15] and *uaw*-topology [25]. First we define the *um*-convergence.

Definition 1. *Let* $(X, \mathcal{M})$ *be an MNVL, then a net* x_α *is said to be* unbounded m-convergent *to* x, *if* $|x_\alpha - x| \wedge u \xrightarrow{\text{m}} 0$ *for all* $u \in X_+$. *In this case, we say* x_α *um*-converges *to* x *and write* $x_\alpha \xrightarrow{\text{um}} x$.

Clearly, that *um*-convergence is a generalization of *un*-convergence. The following result generalizes [15, Cor.4.5].

Proposition 3. *If* $(X, \mathcal{M})$ *is an MNVL possessing the Lebesgue and Levi properties, and* $x_\alpha \xrightarrow{\text{um}} 0$ *in* X, *then* $x_\alpha \xrightarrow{\text{um}} 0$ *in* X^{**}.

Proof. It follows from Theorem 6.63 of [1] that $(X, \mathcal{M})$ is m-complete and X is a band in X^{**}. Now, [1, Thm.2.22] shows that X^{**} is Dedekind complete, and so X is a projection band in X^{**}. The conclusion follows now from [6, Thm.3(3)]. □

In a similar way as in [7, Section 7], one can show that $\mathcal{N}_0$, the collection of all sets of the form

$$V_{\varepsilon,u,\lambda} = \{x \in X : m_\lambda(|x| \wedge u) < \varepsilon\} \qquad (\varepsilon > 0, 0 \neq u \in X_+, \lambda \in \Lambda)$$

forms a neighborhood base at zero for some Hausdorff locally solid topology τ such that, for any net x_α in X: $x_\alpha \xrightarrow{\text{um}} 0$ iff $x_\alpha \xrightarrow{\tau} 0$. Thus, the *um*-convergence is topological, and we will refer to its topology as the *um-topology*.

Clearly, if $x_\alpha \xrightarrow{\mathrm{m}} 0$, then $x_\alpha \xrightarrow{\mathrm{um}} 0$, and so the m-topology, in general, is finer than um-topology. On the contrary to Theorem 2.3 in [15], the following example provides an MNVL which has a strong unit, yet the m-topology and um-topology do not agree.

Example 1. *Let $X = C[0,1]$. Let $\mathcal{A} := \{[a,b] \subseteq [0,1] : a < b\}$. For $[a,b] \in \mathcal{A}$ and $f \in X$, let $m_{[a,b]}(f) := \frac{1}{b-a}\int_a^b |f(t)|dt$. Then $\mathcal{M} = \{m_{[a,b]} : [a,b] \in \mathcal{A}\}$ is a separating family of lattice seminorms on X. Thus, $(X, \mathcal{M})$ is an MNVL. For each $2 \le n \in \mathbb{N}$, let*

$$f_n = \begin{cases} n & \text{if } x \in [0, \frac{1}{n}], \\ n^2(1-n)x + n^2 & \text{if } x \in [\frac{1}{n}, \frac{1}{n-1}], \\ 0 & \text{if } x \in [\frac{1}{n-1}, 1]. \end{cases}$$

So we have

$$f_n \wedge \mathbb{1} = \begin{cases} 1 & \text{if } x \in [0, \frac{n+1}{n^2}], \\ n^2(1-n)x + n^2 & \text{if } x \in [\frac{n+1}{n^2}, \frac{1}{n-1}], \\ 0 & \text{if } x \in [\frac{1}{n-1}, 1]. \end{cases}$$

Now, let $0 < b \le 1$, then there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0-1} < b$. So, for $n \ge n_0$, we have $\frac{1}{n-1} < b$, and so we get $m_{[0,b]}(f_n) = \frac{1}{b}(1 + \frac{1}{n-1}) \to \frac{1}{b} \neq 0$ as $n \to \infty$. Thus, $f_n \overset{\mathrm{m}}{\not\to} 0$. On the other hand, if $[a,b] \in \mathcal{A}$ then there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0-1} < b$ so, for $n \ge (n_0 - 1)$, we have $m_{[a,b]}(f_n \wedge \mathbb{1}) = \frac{1}{b-a}(\frac{n+1}{n^2} + \frac{1}{2n^2(n-1)}) \to 0$ as $n \to \infty$. Since $\mathbb{1}$ is a strong unit in X then, by [6, Cor.5], *$f_n \xrightarrow{\mathrm{um}} 0$.*

4. Metrizabililty of *um*-topology

The main result in this section is Proposition 4, which shows that the um-topology is metrizable iff the space has a countable topological orthogonal system.

It is well known (cf. [1, Thm.2.1]) that a topological vector space is metrizable iff it has a countable neighborhood base at zero. Furthermore, an MNVL $(X, \mathcal{M})$ is metrizable iff the m-topology is generated by a countable family of lattice seminorms, see [23, Theorem VII.8.2].

Notice that, in an MNVL $(X, \mathcal{M})$ with countable $\mathcal{M} = \{m_k\}_{k\in\mathbb{N}}$, an equivalent translation-invariant metric $\rho_{\mathcal{M}}$ can be constructed by the formula

$$\rho_{\mathcal{M}}(x,y) = \sum_{k=1}^{\infty} \frac{m_k(x-y)}{2^k(m_k(x-y)+1)} \quad (x, y \in X). \tag{4.1}$$

Since the function $t \to \frac{t}{t+1}$ is increasing on $[0,\infty)$, $|x| \leqslant |y|$ in X implies that $\rho_{\mathcal{M}}(x,0) \leqslant \rho_{\mathcal{M}}(y,0)$.

Recall that a collection $\{e_\gamma\}_{\gamma\in\Gamma}$ of positive vectors in a vector lattice X is called an *orthogonal system* if $e_\gamma \wedge e_{\gamma'} = 0$ for all $\gamma \neq \gamma'$. If, moreover, $x \wedge e_\gamma = 0$ for all $\gamma \in \Gamma$ implies $x = 0$, then $\{e_\gamma\}_{\gamma\in\Gamma}$ is called a *maximal orthogonal system*. It follows from the Zorn's lemma that every vector lattice containing at least one non-zero element has a maximal orthogonal system. Next, we recall the following notion.

Definition 2. [6, Def.1] *Let (X,τ) be a topological vector lattice. An orthogonal system $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ of non-zero elements in X_+ is said to be a* topological orthogonal system, *if the ideal I_Q generated by Q is τ-dense in X.*

A series $\sum_{i=1}^\infty x_i$ in a multi-normed space $(X,\mathcal{M})$ is called *absolutely m-convergent* if $\sum_{i=1}^\infty m_\lambda(x_i) < \infty$ for all $\lambda \in \Lambda$; and the series is *m-convergent*, if the sequence $s_n := \sum_{i=1}^n x_i$ of partial sums is m-convergent. The following lemma can be proven by combining a diagonal argument with the proof of [14, Prop. 3 in Section 3.3] and therefore we omit its proof.

Lemma 1. *A metrizable multi-normed space $(X,\mathcal{M})$ is m-complete iff every absolutely m-convergent series in X is m-convergent.*

The following result extends [15, Thm.3.2].

Proposition 4. *Let $(X,\mathcal{M})$ be a metrizable m-complete MNVL. Then the following conditions are equivalent:*
(i) *X has a countable topological orthogonal system;*
(ii) *the um-topology is metrizable;*
(iii) *X has a quasi interior point.*

Proof. Since $(X,\mathcal{M})$ is metrizable, we may suppose that $\mathcal{M} = \{m_k\}_{k\in\mathbb{N}}$ is countable and directed.

$(i) \Rightarrow (ii)$ It follows directly from [6, Prop.5]. Notice also that a metric d_{um} of the um-topology can be constructed by the following formula:

$$d(x,y) = \sum_{k,n=1}^{\infty} \frac{1}{2^{k+n}} \cdot \frac{m_k(|x-y| \wedge e_n)}{1 + m_k(|x-y| \wedge e_n)}, \tag{4.2}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a countable topological orthogonal system for X.

$(ii) \Rightarrow (iii)$ Assume that the um-topology is generated by a metric d_{um} on X. For each $n \in \mathbb{N}$, let $B_{um}(0,\frac{1}{n}) = \{x \in X : d_{um}(x,0) < \frac{1}{n}\}$. Since the um-topology is metrizable, then, for each $n \in \mathbb{N}$, there are $k_n \in \mathbb{N}, 0 < u_n \in X_+$, and $\varepsilon_n > 0$ such that $V_{\varepsilon_n,u_n,k_n} \subseteq B_{um}(0,\frac{1}{n})$, where

$$V_{\varepsilon,u_n,k} = \{x \in X : m_k(|x| \wedge u_n) < \varepsilon\}.$$

Notice that $\{V_{\varepsilon,u_n,k}\}_{\varepsilon>0,n,k\in\mathbb{N}}$ is a base at zero of the um-topology on X.

Let $B_m(0,1)=\{x\in X: d_m(x,0)<1\}$, where d_m is the metric generating the m-topology. There is a zero neighborhood V in the m-topology such that $V\subseteq B_m(0,1)$. Since V is absorbing, then, for every $n\in\mathbb{N}$, there is $c_n\geq 1$ such that $\frac{1}{c_n}u_n\in V$. Thus $\frac{1}{c_n}u_n\in V\subseteq B_m(0,1)$ for each $n\in\mathbb{N}$. Hence, the sequence $\frac{1}{c_n}u_n$ is d_m-bounded and so it is bounded with respect to the multi-norm $\mathcal{M}=\{m_k\}_{k\in\mathbb{N}}$. Let

$$e:=\sum_{n=1}^{\infty}\frac{u_n}{2^n c_n}. \tag{4.3}$$

Fix $k\in\mathbb{N}$. Since the sequence $\frac{u_n}{c_n}$ is bounded with respect to $\mathcal{M}$, there exists $r_k\in\mathbb{R}_+$ such that $m_k(\frac{u_n}{c_n})\leq r_k<\infty$ for all $n\in\mathbb{N}$. Hence,

$$\sum_{n=1}^{\infty}m_k\left(\frac{u_n}{2^n c_n}\right)=\sum_{n=1}^{\infty}\frac{1}{2^n}m_k\left(\frac{u_n}{c_n}\right)\leq r_k\sum_{n=1}^{\infty}\frac{1}{2^n}<\infty.$$

Thus, the series $\sum_{n=1}^{\infty}\dfrac{u_n}{2^n c_n}$ is absolutely m-convergent. Since X is m-complete, Lemma 1 assures that the series $\sum_{n=1}^{\infty}\dfrac{u_n}{2^n c_n}$ is m-convergent to some $e\in X$.

Now, we use Theorem 2 in [6] to show that e is a quasi-interior point in X. Let x_α be a net in X_+ such that $x_\alpha\wedge e\xrightarrow{\text{m}}0$. Our aim is to show that $x_\alpha\xrightarrow{\text{um}}0$. Since

$$x_\alpha\wedge u_n\leq 2^n c_n x_\alpha\wedge 2^n c_n e=2^n c_n(x_\alpha\wedge e)\xrightarrow{\text{m}}0\quad(\alpha\to\infty),$$

then $x_\alpha\wedge u_n\xrightarrow{\text{m}}0$ for all $n\in\mathbb{N}$. In particular, $m_{k_n}(x_\alpha\wedge u_n)\to 0$. Thus, there exists α_n such that $m_{k_n}(x_\alpha\wedge u_n)<\varepsilon_n$ for all $\alpha\geq\alpha_n$. That is $x_\alpha\in V_{\varepsilon_n,u_n,k_n}$ for all $\alpha\geq\alpha_n$, which implies $x_\alpha\in B_{um}(0,\frac{1}{n})$. Therefore, $x_\alpha\xrightarrow{\text{d}_{\text{um}}}0$ and so $x_\alpha\xrightarrow{\text{um}}0$. Hence, e is a quasi interior point.

$(iii)\Rightarrow(i)$ It is trivial. □

Similar to [15, Prop.3.3], we have the following result.

Proposition 5. *Let $(X,\mathcal{M})$ be an m-complete metrizable MNVL. The um-topology is stronger than a metric topology iff X has a weak unit.*

Proof. The sufficiency follows from [6, Prop.6].

For the necessity, suppose that the um-topology is stronger than the topology generated by a metric d. Let e be as in (4.3) above. Assume $x\wedge e=0$. Since $e\geq\frac{u_n}{2^n c_n}$ for all $n\in\mathbb{N}$, we get $x\wedge\frac{u_n}{2^n c_n}=0$, and hence $x\wedge u_n=0$ for all n. Then $x\in V_{\varepsilon_n,u_n,k_n}$ for all n, and $x\in B(0,\frac{1}{n})=\{x\in X: d(x,0)<\frac{1}{n}\}$ for each $n\in\mathbb{N}$. So $x=0$, which means that e is a weak unit. □

5. um-Completeness

A subset A of an MNVL $(X,\mathcal{M})$ is said to be (*sequentially*) *um-complete* if, it is (sequentially) complete in the um-topology. In this section, we characterize um-complete subsets of X in terms of the Lebesgue and Levi properties. We begin with the following technical lemma.

Lemma 2. *Let $(X,\mathcal{M})$ be an MNVL, and $A \subseteq X$ be m-bounded, then $\overline{A}^{um}$ is m-bounded.*

Proof. Given $\lambda \in \Lambda$, then $M_\lambda = \sup_{a\in A} m_\lambda(a) < \infty$. Let $x \in \overline{A}^{um}$, then there is a net a_α in A such that $a_\alpha \xrightarrow{\text{um}} x$. So $m_\lambda(|a_\alpha - x| \wedge u) \to 0$ for any $u \in X_+$. In particular,

$$m_\lambda(|x|) = m_\lambda(|x| \wedge |x|) = m_\lambda(|x - a_\alpha + a_\alpha| \wedge |x|) \leq$$
$$m_\lambda(|x - a_\alpha| \wedge |x|) + \sup_{a\in A} m_\lambda(a) = m_\lambda(|x - a_\alpha| \wedge |x|) + M_\lambda.$$

Letting $\alpha \to \infty$, we get $m_\lambda(x) = m_\lambda(|x|) \leq M_\lambda < \infty$ for all $x \in \overline{A}^{um}$. □

Theorem 1. *Let $(X,\mathcal{M})$ be an MNVL and let A be an m-bounded and um-closed subset in X. If X has the Lebesgue and Levi properties, then A is um-complete.*

Proof. Suppose that x_α is um-Cauchy in A, then, without lost of generality, we may assume that x_α consists of positive elements.
Case (1): If X has a weak unit e, then e is a quasi-interior point, by the Lebesgue property of X and Proposition 2. Note that, for each $k \in \mathbb{N}$,

$$|x_\alpha \wedge ke - x_\beta \wedge ke| \leq |x_\alpha - x_\beta| \wedge ke,$$

hence the net $(x_\alpha \wedge ke)_\alpha$ is m-Cauchy in X. Now, [1, Thm.6.63] assures that X is m-complete, and so the net $(x_\alpha \wedge ke)_\alpha$ is m-convergent to some $y_k \in X$. Given $\lambda \in \Lambda$. Then

$$\begin{aligned} m_\lambda(y_k) &= m_\lambda(y_k - x_\alpha \wedge ke + x_\alpha \wedge ke) \\ &\leq m_\lambda(y_k - x_\alpha \wedge ke) + m_\lambda(x_\alpha) \\ &\leq m_\lambda(y_k - x_\alpha \wedge ke) + \sup_\alpha m_\lambda(x_\alpha). \end{aligned}$$

Taking limit over α, we get $m_\lambda(y_k) \leq \sup_\alpha m_\lambda(x_\alpha) < \infty$. Hence the sequence y_k is m-bounded in X. Note also that y_k is increasing in X, but X has the Lebesgue and Levi properties, so, by [1, Thm.6.63], y_k m-converges to some $y \in X$.

It remains to show that y is the um-limit of x_α. Given $\lambda \in \Lambda$. Note that, by Birkhoff's inequality,

$$|x_\alpha \wedge ke - x_\beta \wedge ke| \wedge e \leq |x_\alpha - x_\beta| \wedge e.$$

Thus

$$m_\lambda(|x_\alpha \wedge ke - x_\beta \wedge ke| \wedge e) \le m_\lambda(|x_\alpha - x_\beta| \wedge e).$$

Taking limit over β, we get

$$m_\lambda(|x_\alpha \wedge ke - y_k| \wedge e) \le \lim_\beta m_\lambda(|x_\alpha - x_\beta| \wedge e).$$

Now taking limit over k, we have

$$m_\lambda(|x_\alpha - y| \wedge e) \le \lim_\beta m_\lambda(|x_\alpha - x_\beta| \wedge e).$$

Finally, as x_α is *um*-Cauchy, taking limit over α, yields

$$\lim_\alpha m_\lambda(|x_\alpha - y| \wedge e) \le \lim_{\alpha,\beta} m_\lambda(|x_\alpha - x_\beta| \wedge e) = 0.$$

Thus, $x_\alpha \xrightarrow{\text{um}} y$ and, since A is *um*-closed, $y \in A$.

Case (2): If X has no weak unit. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be a maximal orthogonal system in X. Let Δ be the collection of all finite subsets of Γ. For each $\delta \in \Delta$, $\delta = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$, consider the band B_δ generated by $\{e_{\gamma_1}, e_{\gamma_1}, \dots, e_{\gamma_n}\}$. It follows from [1, Thm.3.24] that B_δ is a projection band. Then B_δ is an m-complete MNVL in its own right. Moreover, the m-topology restricted to B_δ possesses the Lebesgue and Levi properties. Note that B_δ has a weak unit, namely $e_{\gamma_1} + e_{\gamma_2} + \cdots + e_{\gamma_n}$. Let P_δ be the band projection corresponding to B_δ.
For $\delta \in \Delta$, since x_α is *um*-Cauchy in X and P_δ is a band projection, then $P_\delta x_\alpha$ is *um*-Cauchy in B_δ. Lemma 2 assures that $\overline{P_\delta(A)}^{um}$ is m-bounded in B_δ. Thus, by Case (1), there is $z_\delta \in B_\delta$ such that

$$P_\delta x_\alpha \xrightarrow{\text{um}} z_\delta \ge 0 \text{ in } B_\delta \quad (\alpha \to \infty).$$

Since B_δ is a projection band, then $P_\delta x_\alpha \xrightarrow{\text{um}} z_\delta \ge 0$ in X (over α). It is easy to see that $0 \le z_\delta \uparrow$, and z_δ is m-bounded. Since X has the Lebesgue and Levi properties, it follows from [1, Thm.6.63], that there is $z \in X_+$ such that $z_\delta \xrightarrow{\text{m}} z$, and so $z_\delta \uparrow z$. It remains to show that $x_\alpha \xrightarrow{\text{um}} z$. The argument is similar to the proof of [13, Thm.4.7], and we leave it as an exercise. Since A is *um*-closed, then $z \in A$ and so A is *um*-complete. □

The following lemma and its proof are analogous to Lemma 1.2 in [15].

Lemma 3. *Let $(X, \mathcal{M})$ be an MNVL. If x_α is an increasing net in X and $x_\alpha \xrightarrow{\text{um}} x$, then $x_\alpha \uparrow x$ and $x_\alpha \xrightarrow{\text{m}} x$.*

Lemma 4. *Let $(X, \mathcal{M})$ be an MNVL possessing the pre-Lebesgue property. Let x_n be a positive disjoint sequence which is not m-null. Put $s_n := \sum_{k=1}^n x_k$. Then the sequence s_n is um-Cauchy, which is not um-convergent.*

Proof. The sequence s_n is monotone increasing and, since x_n is not m-null, s_n is not m-convergent. Hence, by Lemma 3, the sequence s_n is not um-convergent. To show that s_n is um-Cauchy, fix any $\varepsilon > 0$ and take $0 \neq w \in X_+$. Since x_n is a positive disjoint sequence, we have $s_n \wedge w = \sum_{k=1}^{n} w \wedge x_k$. The sequence $s_n \wedge w$ is increasing and order bounded by w, hence it is m-Cauchy, by [1, Thm.3.22]. Let $\lambda \in \Lambda$. We can find n_{ε_λ} such that $m_\lambda(s_m \wedge w - s_n \wedge w) < \varepsilon$ for all $m \geq n \geq n_{\varepsilon_\lambda}$. Observe that

$$\begin{aligned} s_m \wedge w - s_n \wedge w &= \sum_{k=1}^{m} w \wedge x_k - \sum_{k=1}^{n} w \wedge x_k \\ &= \sum_{k=n+1}^{m} w \wedge x_k = w \wedge \sum_{k=n+1}^{m} x_k = w \wedge |s_m - s_n|. \end{aligned}$$

It follows $m_\lambda(|s_m - s_n| \wedge w) < \varepsilon$ for all $m \geq n \geq n_{\varepsilon_\lambda}$. But $\lambda \in \Lambda$ was chosen arbitrary. Hence s_n is um-Cauchy. □

Next theorem generalizes Theorem 6.4 in [15].

Theorem 2. *Let $(X, \mathcal{M})$ be an m-complete MNVL with the pre-Lebesgue property. Then X has the Lebesgue and Levi properties iff every m-bounded um-closed subset of X is um-complete.*

Proof. The necessity follows directly from Theorem 1.

For the sufficiency, first notice that, in an m-complete MNVL, the pre-Lebesgue and Lebesgue properties coincide [1, Thm.3.24].

If X does not have the Levi property then, by [1, Thm.6.63], there is a disjoint sequence $x_n \in X_+$, which is not m-null, such that its sequence of partial sums $s_n = \sum_{j=1}^{n} x_j$ is m-bounded. Let $A = \overline{\{s_n : n \in \mathbb{N}\}}^{um}$. By Lemma 2, we have that A is m-bounded. By Lemma 4, the sequence s_n is um-Cauchy in X and so in A, in contrary with that the sequence $s_{n+1} - s_n = x_{n+1}$ is not m-null. □

Theorem 3. *Let $(X, \mathcal{M})$ be an m-complete metrizable MNVL, and let A be an m-bounded sequentially um-closed subset of X. If X has the σ-Lebesgue and σ-Levi properties then A is sequentially um-complete. Moreover, the converse holds if, in addition, X is Dedekind complete.*

Proof. Suppose $\mathcal{M} = \{m_k\}_{k \in \mathbb{N}}$. Let $0 \leq x_n$ be a um-Cauchy sequence in A. Let $e := \sum_{n=1}^{\infty} \frac{x_n}{2^n}$. For $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} m_k\left(\frac{x_n}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} m_k(x_n) \leq c_k \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty,$$

where $m_k(a) \leq c_k < \infty$ for all $a \in A$. Since $\sum_{n=1}^{\infty} \frac{x_n}{2^n}$ is absolutely m-convergent, then, by Lemma 1, $\sum_{n=1}^{\infty} \frac{x_n}{2^n}$ is m-convergent in X. Note that, $x_n \leq 2^n e$, so $x_n \in B_e$ for all $n \in \mathbb{N}$. Since X has the Levi property, then X

is σ-order complete (see [1, Definition 3.16]). Thus B_e is a projection band. Also e is a weak unit in B_e. Then, by the same argument as in Theorem 1, we get that there is $x \in B_e$ such that $x_n \xrightarrow{\text{um}} x$ in B_e and so $x_n \xrightarrow{\text{um}} x$ in X. Since A is sequentially *um*-closed, we get $x \in A$. Thus A is sequentially *um*-complete.

The converse follows from Proposition 8 in [6]. □

6. *um*-Compact sets

A subset A of an MNVL $(X, \mathcal{M})$ is said to be (*sequentially*) *um-compact* if, it is (sequentially) compact in the *um*-topology. In this section, we characterize *um*-compact subsets of X in terms of the Lebesgue and Levi properties. We begin with the following result which shows that *um*-compactness can be "localized" under certain conditions.

Theorem 4. *Let $(X, \mathcal{M})$ be an MNVL possessing the Lebesgue property. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be a maximal orthogonal system. For each $\gamma \in \Gamma$, let B_γ be the band generated by e_γ, and P_γ be the corresponding band projection onto B_γ. Then $x_\alpha \xrightarrow{\text{um}} 0$ in X iff $P_\gamma x_\alpha \xrightarrow{\text{um}} 0$ in B_γ for all $\gamma \in \Gamma$.*

Proof. For the forward implication, we assume that $x_\alpha \xrightarrow{\text{um}} 0$ in X. Let $b \in (B_\gamma)_+$. Then

$$|P_\gamma x_\alpha| \wedge b = P_\gamma |x_\alpha| \wedge b \le |x_\alpha| \wedge b \xrightarrow{\text{m}} 0,$$

that implies $P_\gamma x_\alpha \xrightarrow{\text{um}} 0$ in B_γ.

For the backward implication, without lost of generality, we may assume that $x_\alpha \ge 0$ for all α. Let $u \in X_+$. Our aim is to show that $x_\alpha \wedge u \xrightarrow{\text{m}} 0$. It is known that $x_\alpha \wedge u = \sum_{\gamma\in\Gamma} P_\gamma(x_\alpha \wedge u)$. Let F be a finite subset of Γ. Then

$$x_\alpha \wedge u = \sum_{\gamma\in F} P_\gamma(x_\alpha \wedge u) + \sum_{\gamma\in\Gamma\setminus F} P_\gamma(x_\alpha \wedge u). \tag{6.1}$$

Note

$$\sum_{\gamma\in F} P_\gamma(x_\alpha \wedge u) = \sum_{\gamma\in F} P_\gamma x_\alpha \wedge P_\gamma u \xrightarrow{\text{m}} 0. \tag{6.2}$$

We have to control the second term in (6.1).

$$\sum_{\gamma\in\Gamma\setminus F} P_\gamma(x_\alpha \wedge u) \le \frac{1}{n}\sum_{\gamma\in F} P_\gamma u + \sum_{\gamma\in\Gamma\setminus F} P_\gamma u, \tag{6.3}$$

where $n \in \mathbb{N}$. Let $\mathscr{F}(\Gamma)$ be the collection of all finite subsets of Γ. Let $\Delta = \mathscr{F}(\Gamma) \times \mathbb{N}$. For each $\delta = (F, n)$, put

$$y_\delta = \frac{1}{n}\sum_{\gamma\in F} P_\gamma u + \sum_{\gamma\in\Gamma\setminus F} P_\gamma u.$$

We show that y_δ is decreasing. Let $\delta_1 \leq \delta_2$ then $\delta_1 = (F_1, n_1), \delta_2 = (F_2, n_2)$. Then $\delta_1 \leq \delta_2$ iff $F_1 \subseteq F_2$ and $n_1 \leq n_2$. But $n_1 \leq n_2$ iff $\frac{1}{n_1} \geq \frac{1}{n_2}$. So,

$$\frac{1}{n_1} \sum_{\gamma \in F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u. \tag{6.4}$$

Note also

$$\frac{1}{n_2} \sum_{\gamma \in F_2} P_\gamma u = \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u + \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u. \tag{6.5}$$

Since $F_1 \subseteq F_2$, then $\Gamma \setminus F_1 \supseteq \Gamma \setminus F_2$ and hence, $\sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u$. Note, that

$$\sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u = \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u. \tag{6.6}$$

Now,

$$\sum_{\gamma \in F_2 \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u. \tag{6.7}$$

Combining (6.6) and (6.7), we get

$$\sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u + \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u. \tag{6.8}$$

Adding (6.4) and (6.8), we get

$$\frac{1}{n_1} \sum_{\gamma \in F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u + \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u.$$

It follows from (6.5), that

$$\frac{1}{n_1} \sum_{\gamma \in F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_2} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u,$$

that is $y_{\delta_1} \geq y_{\delta_2}$. Next, we show $y_\delta \downarrow 0$. Assume $0 \leq x \leq y_\delta$ for all $\delta \in \Delta$. Let $\gamma_0 \in \Gamma$ be arbitrary and fix it. Let

$$F = \{\gamma_0\}, \ n \in \mathbb{N}, \ 0 \leq x \leq \frac{1}{n} P_{\gamma_0} u + \sum_{\gamma \in \Gamma \setminus \{\gamma_0\}} P_\gamma u.$$

We apply P_{γ_0} for the expression above, so $0 \leq P_{\gamma_0} x \leq \frac{1}{n} P_{\gamma_0} u$ for all $n \in \mathbb{N}$, and so $P_{\gamma_0} x = 0$. Since $\gamma_0 \in \Gamma$ was chosen arbitrary, we get $P_{\gamma_0} x = 0$ for all $\gamma \in \Gamma$. Hence, $x = 0$ and so $y_\delta \downarrow 0$. Since $(X, \mathcal{M})$ has the Lebesgue property, we get $y_\delta \xrightarrow{\mathrm{m}} 0$. Therefore, by (6.3),

$$\sum_{\gamma \in \Gamma \setminus F} P_\gamma (x_\alpha \wedge u) \leq y_\delta \xrightarrow{\mathrm{m}} 0. \tag{6.9}$$

Hence (6.1), (6.2), and (6.9) imply $x_\alpha \wedge u \xrightarrow{\mathrm{m}} 0$. $\square$

The following result and its proof are similar to Theorem 7.1 in [15]. Therefore we omit its proof.

Theorem 5. *Let $(X, \mathcal{M})$ be an MNVL possessing the Lebesgue and Levi properties. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be a maximal orthogonal system. Let A be a um-closed m-bounded subset of X. Then A is um-compact iff $P_\gamma(A)$ is um-compact in B_γ for each $\gamma \in \Gamma$, where B_γ is the band generated by e_γ and P_γ is the band projection corresponding to B_γ.*

Theorem 6. *Let $(X, \mathcal{M})$ be an $MNVL$. The following are equivalent*:

(1) *Any m-bounded and um-closed subset A of X is um-compact.*
(2) *X is an atomic vector lattice and $(X, \mathcal{M})$ has the Lebesgue and Levi properties.*

Proof. (1) ⇒ (2). Let $[a, b]$ be an order interval in X. For $x \in [a, b]$, we have $a \le x \le b$ and so $0 \le x-a \le b-a$. Consider the order interval $[0, b-a] \subseteq X_+$. Clearly, $[0, b-a]$ is m-bounded and um-closed in X. By (1), the order interval $[0, b-a]$ is um-compact. Let x_α be a net in $[0, b-a]$. Then there is a subset x_{α_β} such that $x_{\alpha_\beta} \xrightarrow{\text{um}} x$ in $[0, b-a]$. That is $|x_{\alpha_\beta} - x| \wedge u \xrightarrow{\text{m}} 0$ for all $u \in [0, b-a]$. Hence, $|x_{\alpha_\beta} - x| = |x_{\alpha_\beta} - x| \wedge (b-a) \xrightarrow{\text{m}} 0$. So, $x_{\alpha_\beta} \xrightarrow{\text{m}} x$ in $[0, b-a]$. Thus, $[0, b-a]$ is m-compact. Consider the following shift operator $T_a : X \to X$ given by $T_a(x) := x + a$. Clearly, T_a is continuous, and so $T_a([0, b-a]) = [a, b]$ is m-compact.
Since any order interval in X is m-compact, then it follows from [1, Cor.6.57] that X is atomic and has the Lebesgue property. It remains to show that X has the Levi property. Suppose $0 \le x_\alpha \uparrow$ and is m-bounded. Let $A = \overline{\{x_\alpha\}}^{um}$. Then A is um-closed and, by Lemma 2, A is an m-bounded subset of X. Thus, A is um-compact and so, there are a subnet x_{α_β} and $x \in A$ such that $x_{\alpha_\beta} \xrightarrow{\text{um}} x$. Hence, by Lemma 3, $x_{\alpha_\beta} \uparrow x$, and so $x_\alpha \uparrow x$. Hence, X has the Levi property.

(2) ⇒(1). Let A be an m-bounded and um-closed subset of X. We show that A is um-compact. Since X is atomic, there is a maximal orthogonal system $\{e_\gamma\}_{\gamma\in\Gamma}$ of atoms. For each $\gamma \in \Gamma$, let P_γ be the band projection corresponding to e_γ. Clearly, $P_\gamma(A)$ is m-bounded. Now, by the same argument as in the proof of Theorem 7.1 in [15], we get that $P_\gamma(A)$ is um-closed in $\prod_{\gamma\in\Gamma} B_\gamma$, and so it is um-closed in B_γ. But um-closedness implies m-closedness. So $P_\gamma(A)$ is m-bounded and m-closed in B_γ for all $\gamma \in \Gamma$. Since each e_γ is an atom in X, then $B_\gamma = \text{span}\{e_\gamma\}$ is a one-dimensional subspace. It follows from the Heine-Borel theorem that $P_\gamma(A)$ is m-compact in B_γ, and so it is um-compact in B_γ for all $\gamma \in \Gamma$. Therefore, Theorem 5 implies that A is um-compact in X. □

Proposition 6. *Let A be a subset of an m-complete metrizable MNVL $(X, \mathcal{M})$.*

(1) *If X has a countable topological orthogonal system, then A is sequentially um-compact iff A is um-compact.*
(2) *Suppose that A is m-bounded, and X has the Lebesgue property. If A is um-compact, then A is sequentially um-compact.*

Proof. (1). It follows immediately from Proposition 4.
(2). Let x_n be a sequence in A. Find $e \in X_+$ such that x_n is contained in B_e (e.g., take $e = \sum_{n=1}^{\infty} \frac{|x_n|}{2^n}$). Since A is *um*-compact, then $A \cap B_e$ is *um*-compact in B_e. Now, since X is m-complete and has the Lebesgue property, then B_e is also m-complete and has the Lebesgue property. Moreover, e is a quasi-interior point of B_e. Hence, by Proposition 4, the *um*-topology on B_e is metrizable, consequently, $A \cap B_e$ is sequentially *um*-compact in B_e. It follows that there is a subsequence x_{n_k} that *um*-converges in B_e to some $x \in A \cap B_e$. Since B_e is a projection band, then [6, Thm.3(3)] implies $x_{n_k} \xrightarrow{\text{um}} x$ in X. Thus, A is sequentially *um*-compact. □

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