$u\tau$-CONVERGENCE IN LOCALLY SOLID VECTOR LATTICES

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Abstract. Let x_α be a net in a locally solid vector lattice (X,τ); we say that x_α is unbounded τ-convergent to a vector $x\in X$ if $|x_\alpha - x|\wedge w \xrightarrow{\tau} 0$ for all $w\in X_+$. In this paper, we study general properties of unbounded τ-convergence (shortly, $u\tau$-convergence). $u\tau$-Convergence generalizes unbounded norm convergence and unbounded absolute weak convergence in normed lattices that have been investigated recently. Besides, we introduce $u\tau$-topology and study briefly metrizabililty and completeness of this topology.

1. Introduction and preliminaries

The subject of "unbounded convergence" has attracted many researchers [25, 23, 11, 13, 9, 8, 27, 15, 5, 17, 16, 12, 22]. It is well-investigated in vector lattices and normed lattices [11, 14, 13, 27]. In the present paper, we study unbounded convergence in locally solid vector lattices. Results in this article extend previous works [8, 13, 15, 27].

For a net x_α in a vector lattice X, we write $x_\alpha \xrightarrow{o} x$, if x_α *converges* to x *in order*. This means that there is a net y_β, possibly over a different index set, such that $y_\beta \downarrow 0$ and, for every β, there exists α_β satisfying $|x_\alpha - x|\leqslant y_\beta$ whenever $\alpha \geqslant \alpha_\beta$. A net x_α is *unbounded order convergent* to a vector $x\in X$ if $|x_\alpha - x|\wedge u \xrightarrow{o} 0$ for every $u\in X_+$. We write $x_\alpha \xrightarrow{uo} x$ and say that x_α *uo-converges* to x. Clearly, order convergence implies uo-convergence and they coincide for order bounded nets. For a measure space (Ω,Σ,μ) and for a sequence f_n in $L_p(\mu)$ $(0\le p\le\infty)$, $f_n \xrightarrow{uo} 0$ iff $f_n\to 0$ almost everywhere (cf. [13, Rem. 3.4]). It is well known that almost everywhere convergence is not topological in general [18]. Therefore, the uo-convergence might not be topological. Quite recently, it has been shown that order convergence is never topological in infinite dimensional vector lattices [7].

For a net x_α in a normed lattice $(X,\|\cdot\|)$, we write $x_\alpha \xrightarrow{\|\cdot\|} x$ if x_α converges to x in norm. We say that x_α *unbounded norm converges* to $x\in X$ (or x_α

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un-converges to x) if $|x_\alpha - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for every $u \in X_+$. We write $x_\alpha \xrightarrow{un} x$. Clearly, norm convergence implies *un*-convergence. The *un*-convergence is topological, and the corresponding topology (which is known as *un-topology*) was investigated in [15]. A net x_α is *unbounded absolute weak convergent* to $x \in X$ (or x_α *uaw-converges* to x) if $|x_\alpha - x| \wedge u \xrightarrow{w} 0$ for all $u \in X_+$, where "w" refers the weak convergence. We write $x_\alpha \xrightarrow{uaw} x$. Absolute weak convergence implies *uaw*-convergence. The notions of *uaw*-convergence and *uaw*-topology were introduced in [27].

If X is a vector lattice, and τ is a linear topology on X that has a base at zero consisting of solid sets, then the pair (X, τ) is called a *locally solid vector lattice*. It should be noted that all topologies considered throughout this article are assumed to be Hausdorff. It follows from [2, Thm. 2.28] that a linear topology τ on a vector lattice X is locally solid iff it is generated by a family $\{\rho_j\}_{j\in J}$ of Riesz pseudonorms. Moreover, if a family of Riesz pseudonorms generates a locally solid topology τ on a vector lattice X, then $x_\alpha \xrightarrow{\tau} x$ in X iff $\rho_j(x_\alpha - x) \underset{\alpha}{\to} 0$ in $\mathbb{R}$ for each $j \in J$. Since X is Hausdorff, then the family $\{\rho_j\}_{j\in J}$ of Riesz pseudonorms is separating; i.e., if $\rho_j(x) = 0$ for all $j \in J$, then $x = 0$. In this article, unless otherwise, the pair (X, τ) refers to as a locally solid vector lattice.

A subset A in a topological vector space (X, τ) is called *topologically bounded* (or simply *τ-bounded*) if, for every τ-neighborhood V of zero, there exists some $\lambda > 0$ such that $A \subseteq \lambda V$. If ρ is a Riesz pseudonorm on a vector lattice X and $x \in X$, then $\frac{1}{n}\rho(x) \le \rho(\frac{1}{n}x)$ for all $n \in \mathbb{N}$. Indeed, if $n \in \mathbb{N}$ then $\rho(x) = \rho(n\frac{1}{n}x) \le n\rho(\frac{1}{n}x)$. The following standard fact is included for the sake of completeness.

Proposition 1. *Let (X, τ) be a locally solid vector lattice with a family of a Riesz pseudonorms $\{\rho_j\}_{j\in J}$ that generates the topology τ. If a subset A of X is τ-bounded then $\rho_j(A)$ is bounded in $\mathbb{R}$ for any $j \in J$.*

Proof. Let $A \subseteq X$ be τ-bounded and $j \in J$. Put $V := \{x \in X : \rho_j(x) < 1\}$. Clearly, V is a neighborhood of zero in X. Since A is τ-bounded, there is $\lambda > 0$ satisfying $A \subseteq \lambda V$. Thus $\rho_j(\frac{1}{\lambda}a) \le 1$ for all $a \in A$. There exists $n \in \mathbb{N}$ with $n > \lambda$. Now, $\frac{1}{n}\rho_j(a) \le \rho_j(\frac{1}{n}a) \le \rho_j(\frac{1}{\lambda}a) \le 1$ for all $a \in A$. Hence, $\sup_{a\in A} \rho_j(a) \le n < \infty$. □

Next, we discuss the converse of the proposition above.

Let $\{\rho_j\}_{j\in J}$ be a family of Riesz pseudonorms for a locally solid vector lattice (X, τ). For $j \in J$, let $\tilde{\rho}_j := \frac{\rho_j}{1+\rho_j}$. Then $\tilde{\rho}_j$ is a Riesz pseudonorm on X. Moreover, the family $(\tilde{\rho}_j)_{j\in J}$ generates the topology τ on X. Clearly, $\tilde{\rho}_j(A) \le 1$ for any subset A of X, but still we might have a subset that is not τ-bounded.

Recall that a locally solid vector lattice (X, τ) is said to have the *Lebesgue property* if $x_\alpha \downarrow 0$ in X implies $x_\alpha \xrightarrow{\tau} 0$; or equivalently $x_\alpha \xrightarrow{o} 0$ implies

$x_\alpha \xrightarrow{\tau} 0$; and (X,τ) is said to have the *σ-Lebesgue property* if $x_n \downarrow 0$ in X implies $x_n \xrightarrow{\tau} 0$. Finally, (X,τ) is said to have the *Levi property* if $0 \le x_\alpha \uparrow$ and the net x_α is τ-bounded, then x_α has the supremum in X; and (X,τ) is said to have the *σ-Levi property* if $0 \le x_n \uparrow$ and x_n is τ-bounded, then x_n has supremum in X, see [2, Def. 3.16].

Let X be a vector lattice, and take $0 \neq u \in X_+$. Then a net x_α in X is said to be *u-uniformly convergent* to a vector $x \in X$ if, for each $\varepsilon > 0$, there exists some α_ε such that $|x_\alpha - x| \le \varepsilon u$ holds for all $\alpha \geqslant \alpha_\varepsilon$; and x_α is said to be *u-uniformly Cauchy* if, for each $\varepsilon > 0$, there exists some α_ε such that, for all $\alpha, \alpha' \geqslant \alpha_\varepsilon$, we have $|x_\alpha - x_{\alpha'}| \le \varepsilon u$. A vector lattice X is said to be *u-uniformly complete* if every u-uniformly Cauchy sequence in X is u-uniformly convergent; and X is said to be *uniformly complete* if X is u-uniformly complete for each $0 \neq u \in X_+$.

Let X be a vector lattice. An element $0 \neq e \in X_+$ is called a *strong unit* if $I_e = X$ (equivalently, for every $x \geqslant 0$, there exists $n \in \mathbb{N}$ such that $x \leqslant ne$), and $0 \neq e \in X_+$ is called a *weak unit* if $B_e = X$ (equivalently, $x \wedge ne \uparrow x$ for every $x \in X_+$). Here B_e denotes the band generated by e. If (X,τ) is a topological vector lattice, then $0 \neq e \in X_+$ is called a *quasi-interior point*, if the principal ideal I_e is τ-dense in X [20, Def. II.6.1]. It is known that

$$\text{strong unit} \Rightarrow \text{quasi-interior point} \Rightarrow \text{weak unit}.$$

Recall that a Banach lattice X is called an *AM-space* if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X$ with $x \wedge y = 0$.

Let (X,τ) be a sequentially complete locally solid vector lattice. Then it follows from the proof of [4, Cor. 2.59] that it is uniformly complete. So, for each $0 \neq u \in X_+$, let I_u be the ideal generated by u and $\|\cdot\|_u$ be the norm on I_u given by

$$\|x\|_u = \inf\{r > 0 : |x| \le ru\} \quad (x \in X).$$

Then, by [4, Thm. 2.58], the pair $(I_u, \|.\|_u)$ is a Banach lattice. Now Theorem 3.4 in [1] implies that $(I_u, \|\cdot\|_u)$ is an AM-space with a strong unit u, and then, by [1, Thm. 3.6], it is lattice isometric (uniquely, up to a homeomorphism) to $C(K)$ for some compact Hausdorff space K in such a way, that the strong unit u is identified with the constant function $\mathbb{1}$ on K.

For unexplained terminologies and notions we refer to [2, 3].

2. Unbounded τ-convergence

Suppose (X,τ) is a locally solid vector lattice. Let x_α be a net in X. We say that x_α is unbounded τ-convergent to $x \in X$ if, for any $w \in X_+$, we have $|x_\alpha - x| \wedge w \xrightarrow{\tau} 0$. In this case, we write $x_\alpha \xrightarrow{u\tau} x$ and say that x_α *$u\tau$-converges to x*. Obviously, if $x_\alpha \xrightarrow{\tau} x$ then $x_\alpha \xrightarrow{u\tau} x$. The converse holds if the net x_α is order bounded. Note also that $u\tau$-convergence respects linear and lattice operations. It is clear that $u\tau$-convergence is a generalization of un-convergence [8, 15] and, of uaw-convergence [27].

Let $\mathcal{N}_\tau$ be a neighborhood base at zero consisting of solid sets for (X, τ). For each $0 \neq w \in X_+$ and $V \in \mathcal{N}_\tau$, let

$$U_{V,w} := \{x \in X : |x| \wedge w \in V\}.$$

It can be easily shown that the collection

$$\mathcal{N}_{u\tau} := \{U_{V,w} : V \in \mathcal{N}_\tau, 0 \neq w \in X_+\}$$

forms a neighborhood base at zero for a locally solid topology; we call it $u\tau$-topology, where u refers to as *unbounded*. Moreover, $x_\alpha \xrightarrow{u\tau} 0$ iff $x_\alpha \to 0$ with respect to $u\tau$-topology. Indeed, suppose $x_\alpha \xrightarrow{u\tau} 0$. Given a neighborhood $U_{V,w} \in \mathcal{N}_{u\tau}$. Then there are $0 \neq w \in X_+$ and $V \in \mathcal{N}_\tau$ such that

$$U_{V,w} = \{x \in X : |x| \wedge w \in V\}.$$

Now, $x_\alpha \xrightarrow{u\tau} 0$ implies $|x_\alpha| \wedge w \xrightarrow{\tau} 0$. So, there is α_0 such that, for all $\alpha \geq \alpha_0$, we have $|x_\alpha| \wedge w \in V$. That is $x_\alpha \in U_{V,w}$ for all $\alpha \geq \alpha_0$. Thus, $x_\alpha \to 0$ in the $u\tau$-topology.

Conversely, assume $x_\alpha \to 0$ in the $u\tau$-topology. Given $0 \neq w \in X_+$ and $V \in \mathcal{N}_\tau$. Then, $U_{V,w}$ is a zero neighborhood in the $u\tau$-topology. So, there is α' such that $x_\alpha \in U_{V,w}$ for all $\alpha \geq \alpha'$. That is, $|x_\alpha| \wedge w \in V$ for all $\alpha \geq \alpha'$. Thus, $|x_\alpha| \wedge w \xrightarrow{\tau} 0$ or $x_\alpha \xrightarrow{u\tau} 0$. The locally solid $u\tau$-topology will be referred to as *unbounded τ-topology.*

The neighborhood base at zero for the $u\tau$-topology on X has an equivalent representation in terms of a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms that generates the topology τ. For $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$, let $V_{\varepsilon,w,j} := \{x \in X : \rho_j(|x| \wedge w) < \varepsilon\}$. Clearly, the collection $\{V_{\varepsilon,w,j} : \varepsilon > 0, 0 \neq w \in X_+, j \in J\}$ generates the $u\tau$-topology.

It is known that the topology of any linear topological space can be derived from a unique translation-invariant uniformity, i.e., any linear topological space is uniformisable (cf. [21, Thm. 1.4]). It follows from [10, Thm. 8.1.20] that any linear topological space is completely regular. In particular, the unbounded τ-convergence is completely regular.

Since $x_\alpha \xrightarrow{\tau} 0$ implies $x_\alpha \xrightarrow{u\tau} 0$, then the τ-topology in general is finer than $u\tau$-topology. The next result should be compared with [15, Lm. 2.1].

Lemma 1. *Let (X, τ) be a sequentially complete locally solid vector lattice, where τ is generated by a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms. Let $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$. Then either $V_{\varepsilon,w,j}$ is contained in $[-w, w]$, or it contains a non-trivial ideal.*

Proof. Suppose that $V_{\varepsilon,w,j}$ is not contained in $[-w, w]$. Then there exists $x \in V_{\varepsilon,w,j}$ such that $x \notin [-w, w]$. Replacing x with $|x|$, we may assume $x > 0$. Since $x \notin [-w, w]$, then $y = (x - w)^+ > 0$. Now, letting $z = x \vee w$, we have that the ideal I_z generated by z, is lattice and norm isomorphic to

$C(K)$ for some compact and Hausdorff space K, where z corresponds to the constant function $\mathbb{1}$. Also x, y, and w in I_z correspond to $x(t)$, $y(t)$, and $w(t)$ in $C(K)$ respectively.

Our aim is to show that for all $\alpha \geq 0$ and $t \in K$, we have

$$(\alpha y)(t) \wedge w(t) \leq x(t) \wedge w(t).$$

For this, note that $y(t) = (x-w)^+(t) = (x-w)(t) \vee 0$.

Let $t \in K$ be arbitrary.

- Case (1): If $(x-w)(t) > 0$, then $x(t) \wedge w(t) = w(t) \geq (\alpha y)(t) \wedge w(t)$ for all $\alpha \geq 0$, as desired.
- Case (2): If $(x-w)(t) < 0$, then $(\alpha y)(t) \wedge w(t) \leq (\alpha y)(t) = \alpha(x-w)(t) \vee 0 = 0 \leq x(t) \wedge w(t)$, as desired.

Hence, for all $\alpha \geq 0$ and $t \in K$, we have $(\alpha w)(t) \wedge w(t) \leq x(t) \wedge w(t)$ and so $(\alpha y) \wedge w \leq x \wedge w$ for all $\alpha \geq 0$. Note, that $\alpha y, w, x \in X_+$. Thus $\rho_j(|\alpha y| \wedge w) \leq \rho_j(|x| \wedge w) < \varepsilon$, so $\alpha y \in V_{\varepsilon,w,j}$ and, since $V_{\varepsilon,w,j}$ is solid, then $I_z \subseteq V_{\varepsilon,w,j}$. □

Note that the sequential completeness in Lemma 1 can be removed, as we see in the following corollary.

Theorem 1. *Let (X,τ) be a locally solid vector lattice, where τ is generated by a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms. Let $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$. Then either $V_{\varepsilon,w,j}$ is contained in $[-w,w]$ or $V_{\varepsilon,w,j}$ contains a non-trivial ideal.*

Proof. Given $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$. Let $(\hat{X},\hat{\tau})$ be the topological completion of (X,τ). In particular, $(\hat{X},\hat{\tau})$ is sequentially complete. Let $\hat{V}_{\varepsilon,w,j} = \{\hat{x} \in \hat{X} \colon \hat{\rho}_j(|\hat{x}| \wedge w) < \varepsilon\}$. Then $V_{\varepsilon,w,j} = X \cap \hat{V}_{\varepsilon,w,j}$. By Lemma 1, either $\hat{V}_{\varepsilon,w,j}$ is a subset of $[-w,w]_{\hat{X}}$ in $\hat{X}$ or $\hat{V}_{\varepsilon,w,j}$ contains a non-trivial ideal of $\hat{X}$. If $\hat{V}_{\varepsilon,w,j} \subseteq [-w,w]_{\hat{X}}$, then

$$V_{\varepsilon,w,j} = X \cap \hat{V}_{\varepsilon,w,j} \subseteq X \cap [-w,w]_{\hat{X}} = [-w,w] \subseteq X.$$

If $\hat{V}_{\varepsilon,w,j}$ contains a non-trivial ideal, then $\hat{V}_{\varepsilon,w,j} \nsubseteq [-w,w]_{\hat{X}}$. So, there is $\hat{x} \in \hat{V}_{\varepsilon,w,j}$ with $\hat{x} \notin [-w,w]_{\hat{X}}$. Since $[-w,w]_{\hat{X}}$ is $\hat{\tau}$-closed, then there is a solid neighborhood $N_{\hat{x}}$ of $\hat{x}$ in $\hat{X}$ such that $N_{\hat{x}} \cap [-w,w]_{\hat{X}} = \emptyset$. Hence, $N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j} \cap [-w,w]_{\hat{X}} = \emptyset$, and $N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j}$ is open in $\hat{X}$ with $\hat{x} \in N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j}$. By τ-density of X in $\hat{X}$, we may take $x \in X \cap N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j}$. Since $|x| \in X \cap N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j}$, we may also assume that $x \in X_+$.

Let $y := (x-w)^+$, then $y > 0$ and $y \in X_+$. By the same argument in Lemma 1, we get $(\alpha y) \wedge w \leq x \wedge w$ for all $\alpha \in \mathbb{R}_+$. Since $x \in \hat{V}_{\varepsilon,w,j}$, then $\alpha y \in \hat{V}_{\varepsilon,w,j}$ for all $\alpha \in \mathbb{R}_+$. But $\alpha y \in X_+$ for all $\alpha \in \mathbb{R}_+$ and, since $V_{\varepsilon,w,j} = X \cap \hat{V}_{\varepsilon,w,j}$, we get $\alpha y \in V_{\varepsilon,w,j}$ for all $\alpha \in \mathbb{R}_+$. Since $V_{\varepsilon,w,j}$ is solid, we conclude that the principal ideal I_y taken in X is a subset of $V_{\varepsilon,w,j}$. □

Lemma 2. *Let (X,τ) be a locally solid vector lattice, where τ is generated by a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms. If $V_{\varepsilon,w,j}$ is contained in $[-w,w]$, then w is a strong unit.*

Proof. Suppose $V_{\varepsilon,w,j}\subseteq[-w,w]$. Since $V_{\varepsilon,w,j}$ is absorbing, for any $x\in X_+$, there exist $\alpha>0$ such that $\alpha x\in V_{\varepsilon,w,j}$, and so $\alpha x\in[-w,w]$, or $x\le\frac{1}{\alpha}w$. Thus w is a strong unit, as desired. □

Proposition 2. *Let $e\in X_+$. Then e is a quasi-interior point in (X,τ) iff e is a quasi-interior point in the topological completion $(\hat{X},\hat{\tau})$.*

Proof. The backward implication is trivial.
For the forward implication let $\hat{x}\in\hat{X}_+$. Our aim is to show that $\hat{x}-\hat{x}\wedge ne\xrightarrow{\tau}0$ in $\hat{X}$ as $n\to\infty$. By [2, Thm. 2.40], $\hat{X}_+=\overline{X_+}^{\hat{\tau}}$. So, there is a net x_α in X_+ such that $x_\alpha\xrightarrow{\hat{\tau}}\hat{x}$ in $\hat{X}$. Let $j\in J$ and $\varepsilon>0$. Since $\hat{\rho}_j(x_\alpha-\hat{x})\to0$, then there is α_ε satisfying

$$\hat{\rho}_j(x_{\alpha_\varepsilon}-\hat{x})<\varepsilon. \tag{2.1}$$

Since e is a quasi-interior point in X and $x_{\alpha_\varepsilon}\in X_+$, then $x_{\alpha_\varepsilon}-x_{\alpha_\varepsilon}\wedge ne\xrightarrow{\tau}0$ in X as $n\to\infty$. Thus, there is $n_\varepsilon\in\mathbb{N}$ such that

$$\hat{\rho}_j(x_{\alpha_\varepsilon}-ne\wedge x_{\alpha_\varepsilon})=\rho_j(x_{\alpha_\varepsilon}-ne\wedge x_{\alpha_\varepsilon})<\varepsilon\qquad(\forall n\geqslant n_\varepsilon). \tag{2.2}$$

Now, $0\le\hat{x}-\hat{x}\wedge ne=\hat{x}-x_{\alpha_\varepsilon}+x_{\alpha_\varepsilon}-ne\wedge x_{\alpha_\varepsilon}+ne\wedge x_{\alpha_\varepsilon}-\hat{x}\wedge ne$. So $\hat{\rho}_j(\hat{x}-\hat{x}\wedge ne)\le\hat{\rho}_j(\hat{x}-x_{\alpha_\varepsilon})+\hat{\rho}_j(x_{\alpha_\varepsilon}-ne\wedge x_{\alpha_\varepsilon})+\hat{\rho}_j(ne\wedge x_{\alpha_\varepsilon}-\hat{x}\wedge ne)$. For $n\geqslant n_\varepsilon$, we have, by (2.1), (2.2), and [3, Thm. 1.9(2)], that

$$\hat{\rho}_j(\hat{x}-\hat{x}\wedge ne)\le\varepsilon+\varepsilon+\hat{\rho}_j(x_{\alpha_\varepsilon}-\hat{x})\le3\varepsilon.$$

Therefore, e is a quasi-interior point in $\hat{X}$. □

The technique used in the proof of [15, Thm. 3.1] can be used in the following theorem as well, and so we omit its proof.

Theorem 2. *Let (X,τ) be a sequentially complete locally solid vector lattice, where τ is generated by a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms. Let $e\in X_+$. The following are equivalent:*

1. *e is a quasi-interior point;*
2. *for every net x_α in X_+, if $x_\alpha\wedge e\xrightarrow{\tau}0$ then $x_\alpha\xrightarrow{u\tau}0$;*
3. *for every sequence x_n in X_+, if $x_n\wedge e\xrightarrow{\tau}0$ then $x_n\xrightarrow{u\tau}0$.*

3. Unbounded τ-convergence in sublattices

Let Y be a sublattice of a locally solid vector lattice (X,τ). If y_α is a net in Y such that $y_\alpha\xrightarrow{u\tau}0$ in X, then clearly, $y_\alpha\xrightarrow{u\tau}0$ in Y. The converse does not hold in general. For example, the sequence e_n of standard unit vectors is *un*-null in c_0, but not in ℓ_∞. In this section, we study when the $u\tau$-convergence passes from a sublattice to the whole space.

Recall that a sublattice Y of a vector lattice X is *majorizing* if, for every $x \in X_+$, there exists $y \in Y_+$ with $x \leqslant y$. The following theorem extends [15, Thm. 4.3] to locally solid vector lattices.

Theorem 3. *Let (X, τ) be a locally solid vector lattice and Y be a sublattice of X. If y_α is a net in Y and $y_\alpha \xrightarrow{u\tau} 0$ in Y, then $y_\alpha \xrightarrow{u\tau} 0$ in X in each of the following cases*:

(1) *Y is majorizing in X;*
(2) *Y is τ-dense in X;*
(3) *Y is a projection band in X.*

Proof. (1) Trivial.

(2) Let $u \in X_+$. Fix $\varepsilon > 0$ and take $j \in J$. Since Y is τ-dense in X, then there is $v \in Y_+$ such that $\rho_j(u-v) < \varepsilon$. But $y_\alpha \xrightarrow{u\tau} 0$ in Y and so, in particular, $\rho_j(|y_\alpha| \wedge v) \to 0$. So there is α_0 such that $\rho_j(|y_\alpha| \wedge v) < \varepsilon$ for all $\alpha \geqslant \alpha_0$. It follows from $u \leq v + |u - v|$, that $|y_\alpha| \wedge u \leq |y_\alpha| \wedge v + |u-v|$, and so $\rho_j(|y_\alpha| \wedge u) < \rho_j(|y_\alpha| \wedge v) + \rho_j(u-v) < 2\varepsilon$. Thus, $\rho_j(|y_\alpha| \wedge u) \to 0$ in $\mathbb{R}$. Since $j \in J$ was chosen arbitrary, we conclude that $y_\alpha \xrightarrow{u\tau} 0$ in X.

(3) Let $u \in X_+$. Then $u = v + w$, where $v \in Y_+$ and $w \in Y^d_+$. Now $|y_\alpha| \wedge u = |y_\alpha| \wedge v + |y_\alpha| \wedge w = |y_\alpha| \wedge v$, since $y_\alpha \in Y$. Then $|y_\alpha| \wedge u = |y_\alpha| \wedge v \xrightarrow{\tau} 0$ in X.

□

Corollary 1. *If (X, τ) is a locally solid vector lattice and $x_\alpha \xrightarrow{u\tau} 0$ in X, then $x_\alpha \xrightarrow{u\tau} 0$ in the Dedekind completion X^δ of X.*

Corollary 2. *If (X, τ) is a locally solid vector lattice and $x_\alpha \xrightarrow{u\tau} 0$ in X, then $x_\alpha \xrightarrow{u\tau} 0$ in the topological completion $\hat{X}$ of X.*

The next result generalizes Corollary 4.6 in [15] and Proposition 16 in [27].

Theorem 4. *Let (X, τ) be a topologically complete locally solid vector lattice that possesses the Lebesgue property, and Y be a sublattice of X. If $y_\alpha \xrightarrow{u\tau} 0$ in Y, then $y_\alpha \xrightarrow{u\tau} 0$ in X.*

Proof. Suppose $y_\alpha \xrightarrow{u\tau} 0$ in Y. By Theorem 3(1), $y_\alpha \xrightarrow{u\tau} 0$ in the ideal $I(Y)$ generated by Y in X. By Theorem 3(2), $y_\alpha \xrightarrow{u\tau} 0$ in the closure $\overline{\{I(Y)\}}^\tau$ of $I(Y)$. It follows from [2, Thm. 3.7] that $\overline{\{I(Y)\}}^\tau$ is a band in X. Now, [2, Thm. 3.24] assures that X is Dedekind complete, and so $\overline{\{I(Y)\}}^\tau$ is a projection band in X. Then $y_\alpha \xrightarrow{u\tau} 0$ in X, in view of Theorem 3(3). □

Suppose that (X, τ) is a locally solid vector lattice possessing the Lebesgue property. Then, in view of [2, Thms. 3.23 and 3.26], its topological completion $(\hat{X}, \hat{\tau})$ possesses the Lebesgue property as well. Hence, by [2, Thm.

3.24], $\hat{X}$ is Dedekind complete. Since $X \subseteq \hat{X}$, there holds $X^\delta \subseteq (\hat{X})^\delta = \hat{X}$. So, $X \subseteq X^\delta \subseteq \hat{X}$. Now, Theorem 4 assures that, given a net z_α in X^δ, if $z_\alpha \xrightarrow{u\tau} 0$ in X^δ then $z_\alpha \xrightarrow{u\tau} 0$ in $\hat{X}$.

4. Unbounded relatively uniformly convergence

In this section we discuss unbounded relatively uniformly convergence. Recall that a net x_α in a vector lattice X is said to be *relatively uniformly convergent* to $x \in X$ if, there is $u \in X_+$ such that for any $n \in \mathbb{N}$, there exists α_n satisfying $|x_\alpha - x| \leq \frac{1}{n}u$ for $\alpha \geqslant \alpha_n$. In this case we write $x_\alpha \xrightarrow{ru} x$ and the vector $u \in X_+$ is called *regulator*, see [24, Def. III.11.1].

If $x_\alpha \xrightarrow{ru} 0$ in a locally solid vector lattice (X, τ), then $x_\alpha \xrightarrow{\tau} 0$. Indeed, let V be a solid neighborhood at zero. Since $x_\alpha \xrightarrow{ru} 0$, then there is $u \in X_+$ such that, for a given $\varepsilon > 0$, there is α_ε satisfying $|x_\alpha| \leq \varepsilon u$ for all $\alpha \geq \alpha_\varepsilon$. Since V is absorbing, there is $c \geq 1$ such that $\frac{1}{c}u \in V$. There is some α_0 such that $|x_\alpha| \leq \frac{1}{c}u$ for all $\alpha \geq \alpha_0$. Since V is solid and $|x_\alpha| \leq \frac{1}{c}u$ for all $\alpha \geq \alpha_0$, then $x_\alpha \in V$ for all $\alpha \geq \alpha_0$. That is $x_\alpha \xrightarrow{\tau} 0$.

The following result might be considered as an ru-version of Theorem 1 in [7].

Theorem 5. *Let X be a vector lattice. Then the following conditions are equivalent.*

(1) *There exists a linear topology τ on X such that, for any net x_α in X: $x_\alpha \xrightarrow{ru} 0$ iff $x_\alpha \xrightarrow{\tau} 0$.*

(2) *There exists a norm $\|\cdot\|$ on X such that, for any net x_α in X: $x_\alpha \xrightarrow{ru} 0$ iff $\|x_\alpha\| \to 0$.*

(3) *X has a strong order unit.*

Proof. (1) $\Rightarrow$ (3) It follows from [7, Lem. 1].

(3) $\Rightarrow$ (2) Let $e \in X$ be a strong order unit. Then $x_\alpha \xrightarrow{ru} 0$ iff $\|x_\alpha\|_e \to 0$, where $\|x\|_e := \inf\{r : |x| \leqslant re\}$.

(2) $\Rightarrow$ (1) It is trivial. □

Let X be a vector lattice. A net x_α in X is said to be *unbounded relatively uniformly convergent* to $x \in X$ if $|x_\alpha - x| \wedge w \xrightarrow{ru} 0$ for all $w \in X_+$. In this case, we write $x_\alpha \xrightarrow{uru} x$. Clearly, if $x_\alpha \xrightarrow{uru} 0$ in a locally solid vector lattice (X, τ), then $x_\alpha \xrightarrow{u\tau} 0$.

In general, uru-convergence is also not topological. Indeed, consider the vector lattice $L_1[0,1]$. It satisfies the diagonal property for order convergence by [19, Thm. 71.8]. Now, by combining Theorems 16.3, 16.9, and 68.8 in [19] we get that for any sequence f_n in $L_1[0,1]$ $f_n \xrightarrow{o} 0$ iff $f_n \xrightarrow{ru} 0$. In particular, $f_n \xrightarrow{uo} 0$ iff $f_n \xrightarrow{uru} 0$. But the uo-convergence in $L_1[0,1]$ is equivalent to $a.e.$-convergence which is not topological, see [18].

However, in some vector lattices the uru-convergence could be topological. For example, if X is a vector lattice with a strong unit e, It follows from Theorem 5, that ru-convergence is equivalent to the norm convergence $\|\cdot\|_e$, where $\|x\|_e := \inf\{\lambda > 0 : |x| \leq \lambda e\}$, $x \in X$. Thus uru-convergence in X is topological.

Consider vector lattice c_{00} of eventually zero sequences. It is well known that in c_{00}: $x_\alpha \xrightarrow{ru} 0$ iff $x_\alpha \xrightarrow{o} 0$. For the sake of completeness we include a proof of this fact. Clearly, $x_\alpha \xrightarrow{ru} 0 \Rightarrow x_\alpha \xrightarrow{o} 0$. For the converse, suppose $x_\alpha \xrightarrow{o} 0$ in c_{00}. Then there is a net $y_\beta \downarrow 0$ in c_{00} such that, for any β, there is α_β satisfying $|x_\alpha| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. Let e_n denote the sequence of standard unit vectors in c_{00}. Fix β_0. Then $y_{\beta_0} = c_1^{\beta_0} e_{k_1} + \cdots + c_n^{\beta_0} e_{k_n}$, $c_i^{\beta_0} \in \mathbb{R}, i = 1, \ldots, n$. Since y_β is decreasing, then $y_\beta \leq y_{\beta_0}$ for all $\beta \geq \beta_0$. So, $y_\beta = c_1^\beta e_{k_1} + \cdots + c_n^\beta e_{k_n}$ for all $\beta \geq \beta_0, c_i^\beta \in \mathbb{R}, i = 1, \ldots, n$. Since $y_\beta \downarrow 0$ then $\lim_\beta c_i^\beta = 0$ for all $i = 1, \ldots, n$. Let $u = e_{k_1} + \cdots + e_{k_n}$. Given $\varepsilon > 0$. Then, there is $\beta_\varepsilon \geq \beta_0$ such that $c_i^\beta < \varepsilon$ for all $\beta \geq \beta_\varepsilon$ for $i = 1, \ldots, n$. Consider y_{β_ε} then there is α_ε such that $|x_\alpha| \leq y_{\beta_\varepsilon}$ for all $\alpha \geq \beta_\varepsilon$. But $y_{\beta_\varepsilon} = c_1^{\beta_\varepsilon} e_{k_1} + \cdots + c_n^{\beta_\varepsilon} e_{k_n} \leq \varepsilon u$. So, $|x_\alpha| \leq \varepsilon u$ for all $\alpha \geq \alpha_\varepsilon$. That is $x_\alpha \xrightarrow{ru} 0$. Thus, the uru-convergence in c_{00} coincides with the uo-convergence which is pointwise convergence and, therefore, is topological.

Proposition 3. *Let X be Lebesgue and complete metrizable locally solid vector lattice. then $x_\alpha \xrightarrow{ru} 0$ iff $x_\alpha \xrightarrow{o} 0$.*

Proof. The necessity is obvious. For the sufficiency assume that $x_\alpha \xrightarrow{o} 0$. Then there exists $y_\beta \downarrow 0$ such that for any β there is α_β with $|x_\alpha| \leqslant y_\beta$ as $\alpha \geqslant \alpha_\beta$. Since $d(y_\beta, 0) \to 0$, there exists an increasing sequence $(\beta_k)_k$ of indeces with $d(ky_{\beta_k}, 0) \leqslant \frac{1}{2^k}$. Let $s_n = \sum_{k=1}^n ky_{\beta_k}$. We show the sequence s_n is Cauchy. For $n > m$,

$$d(s_n, s_m) = d(s_n - s_m, 0) = d\Big(\sum_{k=m+1}^{n} ky_{\beta_k}, 0\Big) \leq \sum_{k=m+1}^{n} d\big(ky_{\beta_k}, 0\big)$$

$$\leq \sum_{k=m+1}^{n} \frac{1}{2^k} \to 0, \text{ as } n, m \to \infty.$$

Since X is complete, then the sequence s_n converges to some $u \in X_+$. That is, $u := \sum\limits_{k=1}^{\infty} ky_{\beta_k}$. Then

$$k|x_\alpha| \leqslant ky_{\beta_k} \leqslant u \quad (\forall \alpha \geqslant \alpha_{\beta_k})$$

which means that $x_\alpha \xrightarrow{ru} 0$. ☐

Let $X = \mathbb{R}^\Omega$ be the vector lattice of all real-valued functions on a set Ω.

Proposition 4. *In the vector lattice $X = \mathbb{R}^\Omega$, the following conditions are equivalent*:

(1) *for any net f_α in X: $f_\alpha \xrightarrow{o} 0$ iff $f_\alpha \xrightarrow{ru} 0$;*

(2) *Ω is countable.*

Proof. (1) $\Rightarrow$ (2) Suppose $f_\alpha \xrightarrow{o} 0 \Leftrightarrow f_\alpha \xrightarrow{ru} 0$ for any sequence f_α in $X = \mathbb{R}^\Omega$. Our aim is to show that Ω is countable. Assume, in contrary, that Ω is uncountable. Let $\mathcal{F}(\Omega)$ be the collection of all finite subsets of Ω. For each $\alpha \in \mathcal{F}(\Omega)$, put $f_\alpha = \mathcal{X}_\alpha$. Clearly, $f_\alpha \uparrow \mathbb{1}$, where $\mathbb{1}$ denotes the constant function one on Ω. Then $\mathbb{1} - f_\alpha \downarrow 0$ or $\mathbb{1} - f_\alpha \xrightarrow{o} 0$ in $\mathbb{R}^\Omega$. So, there is $0 \le g \in \mathbb{R}^\Omega$ such that, for any $\varepsilon > 0$, there exists α_ε satisfying $\mathbb{1} - f_\alpha \le \varepsilon g$ for all $\alpha \geqslant \alpha_\varepsilon$. Let $n \in \mathbb{N}$. Then there is a finite set $\alpha_n \subseteq \Omega$ such that $\mathbb{1} - f_{\alpha_n} \le \frac{1}{n} g$. Consequently, $g(x) \geqslant n$ for all $x \in \Omega \setminus \alpha_n$. Let $S = \cup_{n=1}^\infty \alpha_n$. Then S is countable and $\Omega \setminus S \neq \emptyset$. Moreover, for each $x \in \Omega \setminus S$, we have $g(x) \geqslant n$ for all $n \in \mathbb{N}$, which is impossible.

(2) $\Rightarrow$ (1) Suppose that Ω is countable. So, we may assume that $X = s$, the space of all sequences. Since, from $x_\alpha \xrightarrow{ru} 0$ always follows that $x_\alpha \xrightarrow{o} 0$, it is enough to show that if $x_\alpha \xrightarrow{o} 0$ then $x_\alpha \xrightarrow{ru} 0$. To see this, let $(x_\alpha^n)_n = x_\alpha \xrightarrow{o} 0$. Then, the net x_α is eventually bounded, say $|x_\alpha| \leqslant u = (u_n)_n \in s$. Take $w := (nu_n)_n \in s$. We show that $x_\alpha \xrightarrow{ru} 0$ with the regulator w. Let $k \in \mathbb{N}$. Since $x_\alpha \xrightarrow{o} 0$, then for each $n \in \mathbb{N}$, $x_\alpha^n \to 0$ in $\mathbb{R}$. Hence, there is α_k such that $k|x_\alpha^1| < u_1$, $k|x_\alpha^2| < u_2$, $\cdots$, $k|x_\alpha^{k-1}| < u_{k-1}$ for all $\alpha \geqslant \alpha_k$. Note that for $n \geqslant k$, $k|x_\alpha^n| < u_n$. Therefore, $k|x_\alpha| < w$ for all $\alpha \geqslant \alpha_k$. □

It follows from Proposition 4 that, for countable Ω, the *uru*-convergence in $\mathbb{R}^\Omega$ coincides with the *uo*-convergence (which is pointwise) and therefore is topological. We do not know, whether or not the countability of Ω is necessary for the property that *uru*-convergence is topological in $\mathbb{R}^\Omega$.

5. Topological orthogonal systems and metrizabililty

A collection $\{e_\gamma\}_{\gamma\in\Gamma}$ of positive vectors in a vector lattice X is called an *orthogonal system* if $e_\gamma \wedge e_{\gamma'} = 0$ for all $\gamma \neq \gamma'$. If, moreover, $x \wedge e_\gamma = 0$ for all $\gamma \in \Gamma$ implies $x = 0$, then $\{e_\gamma\}_{\gamma\in\Gamma}$ is called a *maximal orthogonal system*. It follows from Zorn's Lemma that every vector lattice containing at least one non-zero element has a maximal orthogonal system. Motivated by Definition III.5.1 in [20], we introduce the following notion.

Definition 1. *Let (X, τ) be a topological vector lattice. An orthogonal system $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ of non-zero elements in X_+ is said to be a* topological orthogonal system *if the ideal I_Q generated by Q is τ-dense in X.*

Lemma 3. *If $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ is a topological orthogonal system in a topological vector lattice (X, τ), then Q is a maximal orthogonal system in X.*

Proof. Assume $x \wedge e_\gamma = 0$ for all $\gamma \in \Gamma$. By the assumption, there is a net x_α in the ideal I_Q such that $x_\alpha \xrightarrow{\tau} x$. Without lost of generality, we may assume $0 \leq x_\alpha \leq x$ for all α. Since $x_\alpha \in I_Q$, then there are $0 < \mu_\alpha \in \mathbb{R}$ and $\gamma_1, \gamma_2, \dots, \gamma_n$, such that $0 \leq x_\alpha \leq \mu_\alpha(e_{\gamma_1} + e_{\gamma_2} + \cdots + e_{\gamma_n})$. So $0 \leq x_\alpha = x_\alpha \wedge x \leq \mu_\alpha(e_{\gamma_1} + e_{\gamma_2} + \cdots + e_{\gamma_n}) \wedge x = \mu_\alpha e_{\gamma_1} \wedge x + \cdots + \mu_\alpha e_{\gamma_n} \wedge x$ $= 0$. Hence $x_\alpha = 0$ for all α, and so $x = 0$. □

We recall the following construction from [20, p.169]. Let X be a vector lattice and $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ be a maximal orthogonal system of X. Let $\mathcal{F}(\Gamma)$ denote the collection of all finite subsets of Γ ordered by inclusion. For each $(n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma)$ and $x \in X_+$, define

$$x_{n,H} := \sum_{\gamma\in H} x \wedge n e_\gamma.$$

Clearly $\{x_{n,H} : (n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma)\}$ is directed upward, and

$$x_{n,H} \leq x \quad \text{for all} \quad (n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma). \tag{5.1}$$

Moreover, Proposition II.1.9 in [20] implies $x_{n,H} \uparrow x$.

Theorem 6. *Let $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ be an orthogonal system of a locally solid vector lattice (X, τ), then Q is a topological orthogonal system iff we have $x_{n,H} \xrightarrow{\tau} x$ over $(n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma)$ for each $x \in X_+$.*

Proof. For the backward implication take $x \in X_+$. Since

$$x_{n,H} = \sum_{\gamma\in H} x \wedge n e_\gamma \leq n \sum_{\gamma\in H} e_\gamma,$$

then $x_{n,H} \in I_Q$ for each $(n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma)$. Also, we have, by assumption, $x_{n,H} \xrightarrow{\tau} x$. Thus, $x \in \overline{I_Q}^\tau$, i.e., Q is a topological orthogonal system of X.

For the forward implication, note that Q is a maximal orthogonal system, by Lemma 3. Let $x \in X_+$, and $j \in J$. Given $\varepsilon > 0$. Let $V_{\varepsilon,x,j} := \{z \in X : \rho_j(z - x) < \varepsilon\}$. Then $V_{\varepsilon,x,j}$ is a neighborhood of x in the τ-topology. Since I_Q is dense in X with respect to the τ-topology, there is $x_\varepsilon \in I_Q$ with $0 \leq x_\varepsilon \leq x$ such that $\rho_j(x_\varepsilon - x) < \varepsilon$. Now, $x_\varepsilon \in I_Q$ implies that there are $H_\varepsilon \in \mathcal{F}(\Gamma)$ and $n_\varepsilon \in \mathbb{N}$ such that

$$x_\varepsilon \leq n_\varepsilon \sum_{\gamma\in H_\varepsilon} e_\gamma. \tag{5.2}$$

Let

$$w := x \wedge \sum_{\gamma\in H_\varepsilon} n_\varepsilon e_\gamma. \tag{5.3}$$

It follows from $0 \leq w \leq \sum\limits_{\gamma\in H_\varepsilon} n_\varepsilon e_\gamma$ and the Riesz decomposition property, that, for each $\gamma \in H_\varepsilon$, there exists y_γ with

$$0 \leq y_\gamma \leq n_\varepsilon e_\gamma \tag{5.4}$$

such that

$$w = \sum_{\gamma \in H_\varepsilon} y_\gamma. \tag{5.5}$$

From (5.3) and (5.5), we have

$$y_\gamma \leq x \quad (\forall \gamma \in H_\varepsilon). \tag{5.6}$$

Also, (5.4) and (5.6) imply that $y_\gamma \leq n_\varepsilon e_\gamma \wedge x$. Now

$$w = \sum_{\gamma \in H_\varepsilon} y_\gamma \leq \sum_{\gamma \in H_\varepsilon} x \wedge n_\varepsilon e_\gamma = x_{n_\varepsilon, H_\varepsilon}. \tag{5.7}$$

But, from (5.2) and (5.3), we get

$$0 \leq x_\varepsilon \leq w. \tag{5.8}$$

Thus, it follows from (5.7), (5.8), and (5.1), that $0 \leq x_\varepsilon \leq x_{n_\varepsilon, H_\varepsilon} \leq x$. Hence, $0 \leq x - x_{n_\varepsilon, H_\varepsilon} \leq x - x_\varepsilon$ and so $\rho_j(x - x_{n,H}) \leq \rho_j(x - x_{n_\varepsilon, H_\varepsilon}) \leq \rho_j(x - x_\varepsilon)$ for each $(n, H) \geq (n_\varepsilon, H_\varepsilon)$. Therefore $x_{n,H} \xrightarrow{\tau} x$. □

The following corollary can be proven easily.

Corollary 3. *Let (X, τ) be a locally solid vector lattice. The following statements are equivalent:*

(1) *$e \in X_+$ is a quasi-interior point;*
(2) *for each $x \in X_+$, $x - x \wedge ne \xrightarrow{\tau} 0$ as $n \to \infty$.*

Corollary 4. *Let (X, τ) be a locally solid vector lattice possessing the σ-Lebesgue property. Then every weak unit in X is a quasi-interior point.*

Proof. Let $x \in X^+$, and let e be a weak unit. Then $x \wedge ne \uparrow x$. So, by the σ-Lebesgue property, we get $x - x \wedge ne \xrightarrow{\tau} 0$ as $n \to \infty$. □

Theorem 7. *Let (X, τ) be a locally solid vector lattice, and $Q = \{e_\gamma\}_{\gamma \in \Gamma}$ be a topological orthogonal system of (X, τ). Then $x_\alpha \xrightarrow{u\tau} 0$ iff $|x_\alpha| \wedge e_\gamma \xrightarrow{\tau} 0$ for every $\gamma \in \Gamma$.*

Proof. The forward implication is trivial. For the backward implication, assume $|x_\alpha| \wedge e_\gamma \xrightarrow{\tau} 0$ for every $\gamma \in \Gamma$. Let $u \in X_+$, $j \in J$. Fix $\varepsilon > 0$. We have

$$\begin{aligned}|x_\alpha|\wedge u &= |x_\alpha|\wedge(u-u_{n,H}+u_{n,H})\\ &\leq |x_\alpha|\wedge(u-u_{n,H})+|x_\alpha|\wedge u_{n,H}\\ &\leq (u-u_{n,H})+|x_\alpha|\wedge\sum_{\gamma\in H}u\wedge ne_\gamma\\ &\leq (u-u_{n,H})+|x_\alpha|\wedge\sum_{\gamma\in H}ne_\gamma\\ &\leq (u-u_{n,H})+n\big(|x_\alpha|\wedge\sum_{\gamma\in H}e_\gamma\big)\\ &= (u-u_{n,H})+n\sum_{\gamma\in H}|x_\alpha|\wedge e_\gamma.\end{aligned}$$

Now, Theorem 6 assures that $u_{n,H}\xrightarrow{\tau}u$, and so, there exists $(n_\varepsilon,H_\varepsilon)\in\mathbb{N}\times\mathscr{F}(\Gamma)$ such that

$$\rho_j(u-u_{n_\varepsilon,H_\varepsilon})<\varepsilon. \tag{5.9}$$

Thus, $|x_\alpha|\wedge u\leq u-u_{n_\varepsilon,H_\varepsilon}+\sum\limits_{\gamma\in H_\varepsilon}n_\varepsilon(e_\gamma\wedge|x_\alpha|)$. But, by the assumption, $e_\gamma\wedge|x_\alpha|\xrightarrow{\tau}0$ for all $\gamma\in\Gamma$, and so $n_\varepsilon(e_\gamma\wedge|x_\alpha|)\xrightarrow{\tau}0$. Hence, there is $\alpha_{\varepsilon,H_\varepsilon}$ such that

$$\rho_j\big(n_\varepsilon(e_\gamma\wedge|x_\alpha|)\big)<\frac{\varepsilon}{|H_\varepsilon|}\qquad(\forall\alpha\geq\alpha_{\varepsilon,H_\varepsilon},\ \forall\gamma\in H_\varepsilon). \tag{5.10}$$

Here $|H_\varepsilon|$ denotes the cardinality of H_ε. For $\alpha\geq\alpha_{\varepsilon,H_\varepsilon}$, we have

$$\begin{aligned}\rho_j(|x_\alpha|\wedge u)&\leq\rho_j(u-u_{n_\varepsilon,H_\varepsilon})+\rho_j\big(n_\varepsilon\sum_{\gamma\in H_\varepsilon}|x_\alpha|\wedge e_\gamma\big)\\ &\leq\varepsilon+\sum_{\gamma\in H_\varepsilon}\rho_j\big(n_\varepsilon(e_\gamma\wedge|x_\alpha|)\big)<\varepsilon+\sum_{\gamma\in H_\varepsilon}\frac{\varepsilon}{|H_\varepsilon|}=2\varepsilon,\end{aligned}$$

where the second inequality follows from (5.9) and the third one from (5.10). Therefore, $\rho_j(|x_\alpha|\wedge u)\to 0$, and so $x_\alpha\xrightarrow{u\tau}0$. □

The following corollary is immediate.

Corollary 5. *Let (X,τ) be a locally solid vector lattice, and $e\in X_+$ be a quasi-interior point. Then $x_\alpha\xrightarrow{u\tau}0$ iff $|x_\alpha|\wedge e\xrightarrow{\tau}0$.*

Recall that a topological vector space is metrizable iff it has a countable neighborhood base at zero, [2, Thm. 2.1]. In particular, a locally solid vector lattice (X,τ) is metrizable iff its topology τ is generated by a countable family $(\rho_k)_{k\in\mathbb{N}}$ of Riesz pseudonorms. The following result gives a sufficient condition for the metrizabililty of $u\tau$-topology.

Proposition 5. *Let (X,τ) be a complete metrizable locally solid vector lattice. If X has a countable topological orthogonal system, then the $u\tau$-topology is metrizable.*

Proof. First note that, since (X,τ) is metrizable, τ is generated by a countable family $(\rho_k)_{k\in\mathbb{N}}$ of Riesz pseudonorms.

Now suppose $(e_n)_{n\in\mathbb{N}}$ to be a topological orthogonal system. For each $n\in\mathbb{N}$, put $d_n(x,y) := \sum\limits_{k=1}^{\infty}\frac{1}{2^k}\frac{\rho_k(|x-y|\wedge e_n)}{1+\rho_k(|x-y|\wedge e_n)}$. Note that each d_n is a semi-metric, and $d_n(x,y)\leq 1$ for all $x,y\in X$. If $d_n(x,y)=0$, then $\rho_k(|x-y|\wedge e_n)=0$ for all $k\in\mathbb{N}$, so $(|x-y|\wedge e_n)=0$. For $x,y\in X$, let $d(x,y) := \sum\limits_{n=1}^{\infty}\frac{1}{2^n}d_n(x,y)$. Clearly, $d(x,y)$ is nonnegative and satisfies the triangle inequality, and $d(x,y)=d(y,x)$ for all $x,y\in X$. Now $d(x,y)=0$ iff $d_n(x,y)=0$ for all $n\in\mathbb{N}$ iff $\rho_k(|x-y|\wedge e_n)=0$ for all $k\in\mathbb{N}$ iff $(|x-y|\wedge e_n)=0$ for all $n\in\mathbb{N}$ iff $|x-y|=0$ iff $x=y$. Thus (X,d) is a metric space. Finally, it is easy to see from Theorem 7 that d generates the $u\tau$-topology. □

Recall that a topological space X is called *submetrizable* if its topology is finer that some metric topology on X.

Proposition 6. *Let (X,τ) be a metrizable locally solid vector lattice. If X has a weak unit, then the $u\tau$-topology is submetrizable.*

Proof. Note that, since (X,τ) is metrizable, then τ is generated by a countable family $(\rho_k)_{k\in\mathbb{N}}$ of Riesz pseudonorms.

Suppose that $e\in X_+$ is a weak unit. Put $d(x,y) := \sum\limits_{k=1}^{\infty}\frac{1}{2^k}\frac{\rho_k(|x-y|\wedge e)}{1+\rho_k(|x-y|\wedge e)}$. Note that $d(x,y)=0$ iff $\rho_k(|x-y|\wedge e)=0$ for all $k\in\mathbb{N}$ iff $|x-y|\wedge e=0$ and, since e is a weak unit, $x=y$. It can easily be shown that d satisfies the triangle inequality. Assume $x_\alpha\xrightarrow{u\tau}x$. Then, for all $u\in X_+$, $\rho_k(|x-y|\wedge u)\to 0$ for all $k\in\mathbb{N}$. In particular, $\rho_k(|x-y|\wedge e)\to 0$ for all $k\in\mathbb{N}$. Then in a similar argument to [24, p.200], it can be shown that $x_\alpha\xrightarrow{d}x$. Therefore, the $u\tau$-topology is finer than the metric topology generated by d, and hence $u\tau$-topology is submetrizable. □

We do not know whether the converse of propositions 5, and 6 is true or not.

6. Unbounded τ-Completeness

A subset A of a locally solid vector lattice (X,τ) is said to be (*sequentially*) *$u\tau$-complete* if, it is (sequentially) complete in the $u\tau$-topology. In this section, we relate sequential $u\tau$-completeness of subsets of X with the Lebesgue and Levi properties. First, we remind the following theorem.

Theorem 8. [26, Thm. 1] *If (X,τ) is a locally solid vector lattice, then the following statements are equivalent*:

1. (X,τ) *has the Lebesgue and Levi properties*;
2. X *is τ-complete, and c_0 is not lattice embeddable in (X,τ).*

Recall that two locally solid vector lattices (X_1,τ_1) and (X_2,τ_2) are said to be *isomorphic*, if there exists a lattice isomorphism from X_1 onto X_2 that is also a homeomorphism; in other words, if there exists a mapping from X_1 onto X_2 that preserves the algebraic, the lattice, and the topological structures. A locally solid vector lattice (X_1,τ_1) is said to be *lattice embeddable* into another locally solid vector lattice (X_2,τ_2) if there exists a sublattice Y_2 of X_2 such that (X_1,τ_1) and (Y_2,τ_2) are isomorphic.

Note that (X,τ) can have the Lebesgue and Levi properties and simultaneously contains c_0 as a sublattice, but not as a lattice embeddable copy. The following example illustrates this.

Example 1. *Let s denote the vector lattice of all sequences in $\mathbb{R}$ with coordinatewise ordering. Clearly, c_0 is a sublattice of s. Define the following separating family of Riesz pseudonorms*

$$\mathcal{R} := \{\rho_j : \rho_j((x_n)_{n\in\mathbb{N}}) := |x_j|\}$$

for each $j\in\mathbb{N}$ and $(x_n)_n\in s$. Then $\mathcal{R}$ generates a locally solid topology τ on s. It can be easily shown that (s,τ) has the Lebesgue and Levi properties. Although c_0 is a sublattice of s, but $(c_0,\|\cdot\|_\infty)$ is not lattice embeddable in (s,τ). To see this, consider the sequence e_n of the standard unit vectors in c_0. Then the sequence e_n is not norm null in $(c_0,\|\cdot\|_\infty)$, whereas $e_n\xrightarrow{\tau}0$ in (s,τ).

Proposition 7. *Let (X,τ) be a complete locally solid vector lattice. If every τ-bounded subset of X is sequentially $u\tau$-complete, then X has the Lebesgue and Levi properties.*

Proof. Suppose X does not possess the Lebesgue or Levi properties. Then, by Theorem 8, c_0 is lattice embeddable in (X,τ). Let $s_n=\sum_{k=1}^n e_k$, where e_k's denote the standard unit vectors in c_0. Clearly, the sequence s_n is norm-bounded in c_0 and so it is τ-bounded in (X,τ). Note that $\|e_k\|_\infty = 1 \nrightarrow 0$, and so e_k is not τ-null. It follows from [15, Lm. 6.1] that s_n is un-Cauchy in c_0, but is not un-convergent in c_0. That is s_n is $u\tau$-Cauchy which is not $u\tau$-convergent, a contradiction. □

Using the proof of the previous result and [26, Thm. 1′], one can easily prove the following result.

Proposition 8. *Let X be a Dedekind complete vector lattice equipped with a sequentially complete topology τ. If every τ-bounded subset of X is sequentially $u\tau$-complete, then X has the σ-Lebesgue and σ-Levi properties.*

Clearly, every finite dimensional locally solid vector lattice (X, τ) is $u\tau$-complete. On the contrary of [15, Prop. 6.2], we provide an example of a τ-complete locally solid vector lattice (X, τ) possessing the Lebesgue property such that it is $u\tau$-complete and $\dim X = \infty$.

Example 2. *Let $X = s$ and $\mathcal{R} = (\rho_j)_{j\in\mathbb{N}}$ such that $\rho_j((x_n)) := |x_j|$, where $(x_n)_{n\in\mathbb{N}} \in s$. It is easy to see that $(X, \mathcal{R})$ is τ-complete and has the Lebesgue property. Now, we show that $(X, \mathcal{R})$ is $u\tau$-complete. Suppose x^α is $u\tau$-Cauchy net. Then, for each $u \in X_+$, we have $|x^\alpha - x^\beta| \wedge u \xrightarrow{\tau} 0$. Now, $u = u_n$ and, $x^\alpha = x^\alpha_n$. Let $j \in \mathbb{N}$, then $\rho_j(|x^\alpha - x^\beta| \wedge u) \to 0$ in $\mathbb{R}$ over α, β iff $|x^\alpha_j - x^\beta_j| \wedge u_j \to 0$ in $\mathbb{R}$ iff $|x^\alpha_j - x^\beta_j| \to 0$ in $\mathbb{R}$ over α, β.*
Thus, $(x^\alpha_j)_\alpha$ is Cauchy in $\mathbb{R}$ and so there is $x_j \in \mathbb{R}$ such that $x^\alpha_j \to x_j$ in $\mathbb{R}$ over α. Let $x = (x_j)_{j\in\mathbb{N}} \in s$, then, clearly, $x^\alpha \xrightarrow{u\tau} x$.

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