Advances in Fuzzy Mathematics.
ISSN No. 973-533X Volume 6, Number 2 (2011), pp. 253-268

http://www.ripublication.com/afm.htm

A Particular Class of Ergodic Finite Fuzzy Markov Chains

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Abstract

In this paper, depending on max-min composition, we study the Ergodicity of a particular class of finite fuzzy Markov chains where the first row of the transition matrices consists of arbitrary values (between zero and 1) while the other rows' entries are one in one place and zero elsewhere. Under certain conditions, we show that a fuzzy Markov chain in this class is Ergodic.

Keywords: A Fuzzy Set, A Fuzzy Relation, Max-Min Composition, Fuzzy transition matrix, Ergodic Fuzzy Markov Chain.

Introduction

Fuzzy set theory is a branch that deals precisely with imprecision and ambiguity, and first introduced by Lotfi Zadeh in his well-known paper entitled "Fuzzy Sets" in 1965[15].

Fuzzy Markov chains have been discussed in the literature and many authors published articles in this area [1-5], [7-9], [12], and [14]. In [10] and [11] Sanchez first introduced the concept of greatest eigen fuzzy set. In [1] and [2] Avrachenkov and Sanchez used the concept of greatest eigen fuzzy set to find the stationary solution of fuzzy Markov chains. In [5] Garcia (et al.) have performed a simulation study on fuzzy Markov chains from which they have shown that most of fuzzy Markov chains are not Ergodic. In [12] Sujatha (et al.) studied the limit behavior of cyclic non-homogeneous fuzzy Markov chains.

In this paper, we consider finite fuzzy Markov chains where the first row of the transition matrices consists of arbitrary values (between zero and 1) while the other rows' entries are one in one place and zero elsewhere. For such a class of fuzzy Markov chains, we first study the limit behavior of 2×2 and 3×3 cases and investigate conditions that guarantee ergodicity. Next for $n \times n$, with $n \geq 4$ case, we state and prove a theorem about the ergodicity.

Finite Fuzzy Markov Chains [1] and [2]

Let $S = \{1, \dots, n\}$ be a finite state space.

Definition 1: A (finite) fuzzy set or a fuzzy distribution, on S, is defined by a mapping x from S to $[0,1]$, represented by a vector $x = (x_1, \dots, x_n)$, with x_i denoting $x(i)$, $0 \leq x_i \leq 1, i \in S$. The set of all fuzzy sets on S is denoted by $\mathcal{F}(S)$.

Definition 2: A fuzzy relation P is defined as a fuzzy set on the Cartesian product $S \times S$. P is represented by a matrix $\{p_{ij}\}_{i,j=1}^{n}$, with p_{ij} denoting $P(i,j)$, $0 \leq p_{ij} \leq 1, i, j \in S$.

Definition 3: At each time instant $t, t = 0,1, \dots$, the state of the system is described by the fuzzy set (or distribution) $x^{(t)} \in \mathcal{F}(S)$. The transition law of the fuzzy Markov chain given by the fuzzy relation P as follows, at time instant $t, t = 1,2, \dots$

$$x_j^{(t+1)} = \max_{i \in S}\left\{x_i^{(t)} \wedge p_{ij}\right\}, j \in S.$$

We refer to $x^{(0)}$ as the initial fuzzy set (or the initial distribution).

It is natural to define the powers of the fuzzy transition matrix. Namely,

$$p_{ij}^{(t)} = \max_{k \in S}\left\{p_{ik} \wedge p_{kj}^{(t-1)}\right\}, p_{ij}^{(1)} = p_{ij}\, , p_{ij}^{(0)} = \delta_{ij},$$

where δ_{ij} is a Kronecker delta.

Note that the fuzzy state $x_k^{(t)}$ at time instant $t, t = 1,2, \dots$ can be calculated by the formula

$$x_k^{(t)} = \max_{l \in S}\left\{x_l^{(0)} \wedge p_{lk}^{(t)}\right\}, k = 1, \dots, n.$$

Theorem 4([6] and [13]): The powers of the fuzzy transition matrix $\{p_{ij}\}_{i,j=1}^{n}$ either converge to idempotent $\left\{p_{ij}^{(\tau)}\right\}_{i,j=1}^{n}$, where τ is a finite number, or oscillate with a finite period υ starting from some finite power.

Definition 5: Let the powers of fuzzy transition matrix converge in τ steps to a non periodic solution, then the associated fuzzy Markov chain is called nonperiodic (or aperiodic) and $P^* = P^\tau$ is called a limiting fuzzy transition matrix.

Definition 6: The fuzzy Markov chain is called ergodic if it is aperiodic and the limiting transition matrix has identical rows.

Motivation

Theorem 4 above is general and does not give us information which fuzzy Markov chains have the ergodic behavior. Moreover, Garcia (et al.) have performed a simulation study on fuzzy Markov chains from which they have shown that most of fuzzy Markov chains are not ergodic [5]. Besides, in [1] Avrachenkov and Sanchez introduced an open problem about the general conditions that guarantee the ergodicity of fuzzy Markov chains. These results motivated us to study the ergodicity of fuzzy Markov chains.

Let $\bar{P} = [\bar{p}_{ij}]$ be an $n \times n$ fuzzy transition matrix corresponding to finite fuzzy Markov chains. Suppose that $\bar{p}_{ij} = 0$ or 1 for $i = 2,3, \dots, n$, $j = 1, \dots, n$ and in each of these rows –all rows except the first row– exactly one entry is 1. We want to study the ergodicity of such fuzzy Markov chains.

$\mathbf{2} \times \mathbf{2}$ and $\mathbf{3} \times \mathbf{3}$ Cases

2×2 Case:

$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{11} \leq \bar{p}_{12} \leq 1$.

$\bar{P}^2 = \begin{bmatrix} \bar{p}_{12} & \bar{p}_{11} \\ \bar{p}_{11} & \bar{p}_{12} \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{12} & \bar{p}_{11} \end{bmatrix}, \bar{P}^4 = \begin{bmatrix} \bar{p}_{12} & \bar{p}_{11} \\ \bar{p}_{11} & \bar{p}_{12} \end{bmatrix}$.

Therefore,

$$\bar{P}^n = \begin{cases} \bar{P}^2, for\ n\ even \\ \bar{P}^3, for\ n\ odd \end{cases}.$$

$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1$.

$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} \end{bmatrix}$. So $\bar{P}^n = \bar{P}^2$ for $n = 2,3,4, \dots$.

According to Theorem 4, $\tau = 2$.

$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 0 & 1 \end{bmatrix}$ with $0 \leq \bar{p}_{11} \leq \bar{p}_{12} \leq 1$.

$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 0 & 1 \end{bmatrix} = \bar{P}$, so $\bar{P}^n = \bar{P}$ for $n = 1,2,3, \dots$.

According to Theorem 4, $\tau = 1$.

$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 0 & 1 \end{bmatrix}$ with $0 \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1$.

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 0 & 1 \end{bmatrix} = \bar{P}, \text{ so } \bar{P}^n = \bar{P} \text{ for } n = 1,2,3, \dots .$$

According to Theorem 4, $\tau = 1$.

We conclude that case 2 above is the only ergodic one, from which we have $\bar{p}_{11} \geq \bar{p}_{12}$ and $\bar{p}_{22} \neq 1$.

3 × 3 Case: Assume the following two conditions :

$\bar{p}_{11}$ is the maximum entry in the first row.

$\bar{p}_{22} \neq 1$ and $\bar{p}_{33} \neq 1$.

We have the following cases:

$$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ with } 0 \leq \bar{p}_{13} \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1.$$

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}. \text{ So } \bar{P}^n = \bar{P}^2 \text{ for } n = 2,3,4, \dots .$$

According to Theorem 4, $\tau = 2$.

$$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ with } 0 \leq \bar{p}_{12} \leq \bar{p}_{13} \leq \bar{p}_{11} \leq 1.$$

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}. \text{ So } \bar{P}^n = \bar{P}^2 \text{ for } n = 2,3,4, \dots .$$

According to Theorem 4, $\tau = 2$.

$$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ with } 0 \leq \bar{p}_{13} \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1.$$

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \bar{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}.$$

So $\bar{P}^n = \bar{P}^3$ for $n = 3,4,5, \dots$. According to Theorem 4, $\tau = 3$.

$$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ with } 0 \leq \bar{p}_{12} \leq \bar{p}_{13} \leq \bar{p}_{11} \leq 1.$$

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \bar{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \end{bmatrix}, \bar{P}^5 =$$

$$\begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \end{bmatrix}.$$

So $\bar{P}^n = \bar{P}^4$ for $n = 4,5,6, \dots$. According to Theorem 4, $\tau = 4$.

$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{13} \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1$.

$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \bar{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \end{bmatrix}, \bar{P}^5 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \end{bmatrix}.$

So $\bar{P}^n = \bar{P}^4$ for $n = 4,5,6, \dots$. According to Theorem 4, $\tau = 4$.

$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{12} \leq \bar{p}_{13} \leq \bar{p}_{11} \leq 1$.

$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \bar{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}.$

So $\bar{P}^n = \bar{P}^3$ for $n = 3,4,5, \dots$. According to Theorem 4, $\tau = 3$.

$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{13} \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1$.

$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \bar{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Therefore,

$$\bar{P}^n = \begin{cases} \bar{P}^2, for\ n\ even \\ \bar{P}^3, for\ n\ odd \end{cases}.$$

$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{12} \leq \bar{p}_{13} \leq \bar{p}_{11} \leq 1$.

$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \bar{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Therefore,

$$\bar{P}^n = \begin{cases} \bar{P}^2, for\ n\ even \\ \bar{P}^3, for\ n\ odd \end{cases}.$$

We conclude that cases 1-6 are ergodic, cases 7 and 8 are not, since in both of them $\bar{p}_{23} = \bar{p}_{32} = 1$ and so in $\bar{P}^2$ we have $\bar{p}_{22}^{(2)} = \bar{p}_{33}^{(2)} = 1$ and so $\bar{p}_{22}^{(2n)} = \bar{p}_{33}^{(2n)} = 1$ for $n = 1,2,3, \dots$.

Main Result

Notation: Let e_{nk} be a $1 \times n$ row vector whose all entries are 0 except the k^{th} entry which is 1. That is,

$$e_{nk} = \underset{\substack{\uparrow \\ k^{th}\text{ entry}}}{[0} \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad \cdots \quad 0].$$

Lemma 1: Let $\bar{P} = \left[\bar{p}_{ij}\right]$ be an $n \times n$ fuzzy matrix such that $0 \le \bar{p}_{ij} \le 1$ for all i, j.Then, by the max-min composition $e_{nk}\bar{P}$ is the k^{th} row of $\bar{P}$.

Proof. Follows directly.

Theorem1: For $n \ge 4$ let $\bar{P} = \left[\bar{p}_{ij}\right]$ be an $n \times n$ fuzzy transition matrix, such that $\bar{p}_{ij} = 0$ or 1 for $i = 2, \dots n$, $j = 1, \dots, n$ and in each row except possibly the first one, exactly one entry is 1. Assuming the following conditions hold:

$\bar{p}_{11}$ is the maximum among the entries in the first row.

$\bar{p}_{ii} \ne 1$ for $i = 2, \dots, n$.

If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \ne j$, $i = 2, \dots, n, j = 1, \dots, n$.

$\bar{p}_{i_1 1} = \bar{p}_{i_2 1} = \cdots = \bar{p}_{i_k 1} = 1$ where $k \in \{n-3, n-2, n-1\}$ and $i_1, i_2, \dots, i_k \in \{2,3, \dots, n\}$.

Then, by max-min composition $\bar{P}$ is ergodic.

Proof. If $k = n - 3$ then $\bar{p}_{i_1 1} = \bar{p}_{i_2 1} = \cdots = \bar{p}_{i_{n-3} 1} = 1$, $i_1, i_2, \dots, i_{n-3} \in \{2,3, \dots, n\}$, and $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{2,3, \dots, n\} - \{i_1, i_2, \dots, i_{n-3}\}$ for $j_1, j_2 \in \{2,3, \dots, n\}$.

Either $i_{n-1} < i_{n-2}$ or $i_{n-1} > i_{n-2}$ we may assume that $i_{n-1} < i_{n-2}$.

Let $R_i^{(m)}$ denote the i^{th} row in $\bar{P}^m$ (the m^{th} power of $\bar{P}$), then $R_i^{(m+1)} = R_i^{(1)}\bar{P}^k$. During the proof $R_1^{(m+1)}$ will be computed by $R_1^{(m+1)} = R_1^{(m)}\bar{P}$ and $R_i^{(m+1)} = R_i^{(1)}\bar{P}^m$ for $i = 2, \dots, n$.

Now we consider two cases:

Case 1: $j_1 = j_2$ then $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_1} = 1$ for $i_{n-1}, i_{n-2} \in \{2,3, \dots, n\} - \{i_1, i_2, \dots, i_{n-3}\}$, and $j_1 = i_k$ for some $k \in \{1,2, \dots, n-3\}$ otherwise (i.e. $j_1 = i_{n-1}$ or $j_1 = i_{n-2}$) we have $\bar{p}_{j_1 j_1} = 1$ which contradicts condition 2.

$$\bar{P} = \begin{array}{c} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array}$$

Consider, $\bar{P}^2$ then $R_{i_1}^{(2)} = R_{i_2}^{(2)} = \cdots = R_{i_{n-3}}^{(2)} = R_1^{(1)}$, $R_{i_{n-1}}^{(2)} = R_{i_{n-2}}^{(2)} = R_{j_1}^{(1)} = R_{i_k}^{(1)}$ by Lemma 1.

$R_1^{(2)} = \left[\bar{p}_{1j}^{(2)}\right]$, for $j \neq j_1$ $\bar{p}_{1j}^{(2)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(2)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}, \bar{p}_{1i_{n-2}}\}$ by condition 1. Therefore,

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array}$$

Consider, $\bar{P}^3$ then $R_{i_1}^{(3)} = R_{i_2}^{(3)} = \cdots = R_{i_{n-3}}^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(3)} = R_{i_{n-2}}^{(3)} = R_{j_1}^{(2)} = R_{i_k}^{(2)} = R_1^{(1)}$ by Lemma 1. $R_1^{(3)} = \left[\bar{p}_{1j}^{(3)}\right]$, for $j \neq j_1$ $\bar{p}_{1j}^{(3)} = \bar{p}_{1j}$, and $\bar{p}_{1j_1}^{(3)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_1}^{(2)}$ by condition 1. So, $R_1^{(3)} = R_1^{(2)}$.

If $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ then $R_1^{(2)} = R_1^{(1)}$, so in $\bar{P}^3$ we have $R_1^{(3)} = R_2^{(3)} = \cdots = R_n^{(3)} = R_1^{(1)}$. It is obvious that $\bar{P}^4 = \bar{P}^3$. Hence, $\bar{P}^m = \bar{P}^3$ for $m = 3,4,5, \ldots$. Therefore, $\bar{P}$ is ergodic.

If $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$ then

$$\bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array}$$

Consider, $\bar{P}^4$ then $R_{i_1}^{(4)} = R_{i_2}^{(4)} = \cdots = R_{i_{n-3}}^{(4)} = R_1^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(4)} = R_{i_{n-2}}^{(4)} = R_{j_1}^{(3)} = R_{i_k}^{(3)} = R_1^{(2)}$ by Lemma 1. $R_1^{(4)} = \left[\bar{p}_{1j}^{(4)}\right]$, for $j \neq j_1$, $\bar{p}_{1j}^{(4)} = \bar{p}_{1j}$, and $\bar{p}_{1j_1}^{(4)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$, by condition1. So $R_1^{(4)} = R_1^{(2)}$, and so in $\bar{P}^4$ we have, $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(2)}$. It is obvious that $\bar{P}^5 = \bar{P}^4$. Hence, $\bar{P}^m = \bar{P}^4$ for $m = 4,5,6, \dots$. Therefore, $\bar{P}$ is ergodic.

Similarly if $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-2}}$ then $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(2)}$. Hence, $\bar{P}^m = \bar{P}^4$ for $m = 4,5,6, \dots$. Therefore, $\bar{P}$ is ergodic.

Case 2: $j_1 \neq j_2$ then either $j_1 < j_2$ or $j_1 > j_2$ we may assume that $j_1 < j_2$. So $\bar{p}_{i_{n-1}j_1} = \bar{p}_{i_{n-2}j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{2,3, \dots, n\} - \{i_1, i_2, \dots, i_{n-3}\}$, and $j_1, j_2 \in \{2,3, \dots, n\}$.

Now we have the following subcases:
$j_1 = i_k, j_2 = i_m$,where $k, m \in \{1,2, \dots, n-3\}$.
$j_2 = i_{n-1}, j_1 = i_k$, where $k \in \{1,2, \dots, n-3\}$.
$j_1 = i_{n-2}, j_2 = i_k$, where $k \in \{1,2, \dots, n-3\}$.

Note that the cases $j_1 = i_{n-1}, j_2 = i_{n-2}$ and $j_1 = i_{n-2}, j_2 = i_{n-1}$ are not taken into account since they contradict conditions 2 and 3 respectively. Again we keep in the mind that $i_{n-1} < i_{n-2}$, and continue with this assumption throughout the proof.

We first deal with the subcase (1):

$$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array}$$

Consider, $\bar{P}^2$ then $R_{i_1}^{(2)} = R_{i_2}^{(2)} = \cdots = R_{i_{n-3}}^{(2)} = R_1^{(1)}$, $R_{i_{n-1}}^{(2)} = R_{j_1}^{(1)} = R_{i_k}^{(1)}$, $R_{i_{n-2}}^{(2)} = R_{j_2}^{(1)} = R_{i_m}^{(1)}$ by Lemma 1. $R_1^{(2)} = \left[\bar{p}_{1j}^{(2)}\right]$, for $j \neq j_1, j_2$, $\bar{p}_{1j}^{(2)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(2)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\}, \bar{p}_{1j_2}^{(2)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\}$, by condition 1.

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array}$$

Consider, $\bar{P}^3$ then
$R_{i_1}^{(3)} = R_{i_2}^{(3)} = \cdots = R_{i_{n-3}}^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(3)} = R_{j_1}^{(2)} = R_{i_k}^{(2)} = R_1^{(1)}$, $R_{i_{n-2}}^{(3)} = R_{j_2}^{(2)} = R_{i_m}^{(2)} = R_1^{(1)}$ by Lemma 1. $R_1^{(3)} = \left[\bar{p}_{1j}^{(3)}\right]$, for $j \neq j_1, j_2$, $\bar{p}_{1j}^{(3)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(3)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = \bar{p}_{1j_1}^{(2)}$,
$\bar{p}_{1j_2}^{(3)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_2}^{(2)}$, by condition 1. So $R_1^{(3)} = R_1^{(2)}$. We have the following subcases:

$\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1j_2}$.

$\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

$\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1j_2}$.

$\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

For the subcase i, $R_1^{(3)} = R_2^{(3)} = \cdots = R_n^{(3)} = R_1^{(1)}$and it is obvious that $\bar{P}^4 = \bar{P}^3$. Hence, $\bar{P}^m = \bar{P}^3$ for $m = 3,4,5, \dots$. Therefore, $\bar{P}$ is ergodic.

For the subcases ii and iii we need to find $\bar{P}^4$ from which we have –as previously shown in Case 1- $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(2)}$ and it is obvious that $\bar{P}^5 = \bar{P}^4$. Hence, $\bar{P}^m = \bar{P}^4$ for $m = 4,5,6, \dots$. Therefore, $\bar{P}$ is ergodic.

For the subcase iv:

$$\bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{matrix}$$

Consider $\bar{P}^4$, then $R_{i_1}^{(4)} = R_{i_2}^{(4)} = \cdots = R_{i_{n-3}}^{(4)} = R_1^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(4)} = R_{j_1}^{(3)} = R_{i_k}^{(3)} = R_1^{(2)}$, $R_{i_{n-2}}^{(4)} = R_{j_2}^{(3)} = R_{i_m}^{(3)} = R_1^{(2)}$ by Lemma 1. $R_1^{(4)} = \left[\bar{p}_{1j}^{(4)}\right]$, for $j \neq j_1, j_2$, $\bar{p}_{1j}^{(4)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(4)} = max\{\bar{p}_{1i_{n-1}}, \bar{p}_{1j_1}\} = \bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$, $\bar{p}_{1j_2}^{(4)} = max\{\bar{p}_{1i_{n-2}}, \bar{p}_{1j_2}\} = \bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-2}}$. In $\bar{P}^4$ we have $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(2)}$ and it is obvious that $\bar{P}^5 = \bar{P}^4$. Hence, $\bar{P}^m = \bar{P}^4$ for $m = 4,5,6, \dots$. Therefore, $\bar{P}$ is ergodic.

Next we deal with the subcase (2) in which $j_2 = i_{n-1}, j_1 = i_k$, where $k \in \{1,2, \dots, n-3\}$.

$$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{matrix}$$

Consider, $\bar{P}^2$ then $R_{i_1}^{(2)} = R_{i_2}^{(2)} = \cdots = R_{i_{n-3}}^{(2)} = R_1^{(1)}$, $R_{i_{n-1}}^{(2)} = R_{j_1}^{(1)} = R_{i_k}^{(1)}$, $R_{i_{n-2}}^{(2)} = R_{j_2}^{(1)} = R_{i_{n-1}}^{(1)}$by Lemma 1. $R_1^{(2)} = \left[\bar{p}_{1j}^{(2)}\right]$, for $j \neq j_1, j_2$ $\bar{p}_{1j}^{(2)} = \bar{p}_{1j}$, and condition 1 implies that$\bar{p}_{1j_1}^{(2)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = max\{\bar{p}_{1j_1}, \bar{p}_{1j_2}\}$, $\bar{p}_{1j_2}^{(2)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\} = max\{\bar{p}_{1i_{n-1}}, \bar{p}_{1i_{n-2}}\}$.

$$\bar{P}^2 = \begin{array}{cl} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Consider, $\bar{P}^3$ then $R_{i_1}^{(3)} = R_{i_2}^{(3)} = \cdots = R_{i_{n-3}}^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(3)} = R_{j_1}^{(2)} = R_{i_k}^{(2)} = R_1^{(1)}$, $R_{i_{n-2}}^{(3)} = R_{j_2}^{(2)} = R_{i_{n-1}}^{(2)} = R_{j_1}^{(1)} = R_{i_k}^{(1)}$by Lemma 1. $R_1^{(3)} = \left[\bar{p}_{1j}^{(3)}\right]$, for $j \neq j_1, j_2$ $\bar{p}_{1j}^{(3)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(3)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = \bar{p}_{1j_1}^{(2)}$, $\bar{p}_{1j_2}^{(3)} = max\{\bar{p}_{1i_{n-2}}, \bar{p}_{1j_2}\} = \bar{p}_{1j_2}^{(2)}$. So, $R_1^{(3)} = R_1^{(2)}$.

$$\bar{P}^3 = \begin{array}{cl} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Consider, $\bar{P}^4$ then $R_{i_1}^{(4)} = R_{i_2}^{(4)} = \cdots = R_{i_{n-3}}^{(4)} = R_1^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(4)} = R_{j_1}^{(3)} =$

$R_{i_k}^{(3)} = R_1^{(2)}$, $R_{i_{n-2}}^{(4)} = R_{j_2}^{(3)} = R_{i_{n-1}}^{(3)} = R_{j_1}^{(2)} = R_{i_k}^{(2)} = R_1^{(1)}$by Lemma 1. $R_1^{(4)} = \left[\bar{p}_{1j}^{(4)}\right]$, for $j \neq j_1, j_2$ $\bar{p}_{1j}^{(4)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(4)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = \bar{p}_{1j_1}^{(2)}$, $\bar{p}_{1j_2}^{(4)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_2}^{(2)}$, by condition 1.

So, $R_1^{(4)} = R_1^{(2)}$. As before, we have the following subcases:

$\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1j_2}$.

$\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

$\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1j_2}$.

$\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

For the subcase i, $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(1)}$and it is obvious that $\bar{P}^5 = \bar{P}^4$. Hence, $\bar{P}^n = \bar{P}^4$ for $n = 4,5,6, \ldots$. Therefore, $\bar{P}$ is ergodic.

For the subcases ii and iii we need to find $\bar{P}^5$ from which we have $R_1^{(5)} = R_2^{(5)} = \cdots = R_n^{(5)} = R_1^{(3)} = R_1^{(2)}$ and it is obvious that $\bar{P}^6 = \bar{P}^5$. Hence, $\bar{P}^m = \bar{P}^5$ for $m = 5,6,7, \ldots$. Therefore, $\bar{P}$ is ergodic.

For the subcase iv:

$$\bar{P}^4 = \begin{array}{cc} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Consider, $\bar{P}^5$ then $R_{i_1}^{(5)} = R_{i_2}^{(5)} = \cdots = R_{i_{n-3}}^{(5)} = R_1^{(4)} = R_1^{(2)}$, $R_{i_{n-1}}^{(5)} = R_{j_1}^{(4)} = R_{i_k}^{(4)} = R_1^{(2)}$, $R_{i_{n-2}}^{(5)} = R_{j_2}^{(4)} = R_{i_{n-1}}^{(4)} = R_{j_1}^{(3)} = R_{i_k}^{(3)} = R_1^{(2)}$by Lemma 1. $R_1^{(5)} = \left[\bar{p}_{1j}^{(5)}\right]$, for $j \neq j_1, j_2$ $\bar{p}_{1j}^{(5)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(5)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = \bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}, \bar{p}_{1j_2}^{(5)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

So, $R_1^{(5)} = R_1^{(2)}$. In $\bar{P}^5$, $R_1^{(5)} = R_2^{(5)} = \cdots = R_n^{(5)} = R_1^{(3)} = R_1^{(2)}$ and it is obvious that $\bar{P}^6 = \bar{P}^5$.

Hence, $\bar{P}^m = \bar{P}^5$ for $m = 5,6,7, \ldots$. Therefore, $\bar{P}$ is ergodic.

For the subcase (3) in which we have $j_1 = i_{n-2}, j_2 = i_k$, where $k \in \{1,2, \ldots, n -$

3}, we deal with it similar to the subcase(2) above.

We have proved the result when $k = n - 3$ so $\bar{p}_{i_1 1} = \bar{p}_{i_2 1} = \cdots = \bar{p}_{i_{n-3} 1} = 1$, $i_1, i_2, \dots, i_{n-3} \in \{2,3, \dots, n\}$, and $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{2,3, \dots, n\} - \{i_1, i_2, \dots, i_{n-3}\}$, $j_1, j_2 \in \{2,3, \dots, n\}$.

Similarly we can prove the theorem when $k = n - 2$ so $\bar{p}_{i_1 1} = \bar{p}_{i_2 1} = \cdots = \bar{p}_{i_{n-2} 1} = 1$, $i_1, i_2, \dots, i_{n-2} \in \{2,3, \dots, n\}$, and $\bar{p}_{i_{n-1} j_1} = 1$ for $i_{n-1} \in \{2,3, \dots, n\} - \{i_1, i_2, \dots, i_{n-2}\}$, $j_1 \in \{2,3, \dots, n\}$.

Finally, for the case $k = n - 1$, we have $\bar{p}_{21} = \bar{p}_{31} = \cdots = \bar{p}_{n1} = 1$ and by considering $\bar{P}^2$ we get $R_1^{(2)} = R_2^{(2)} = \cdots = R_n^{(2)} = R_1^{(1)}$. It is obvious that $\bar{P}^m = \bar{P}^2$ for $m = 2,3,4, \dots$. Therefore, $\bar{P}$ is ergodic, and this completes the proof.

Examples and the Conditions of Theorem 1:

If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ with $\bar{p}_{12} \geq \bar{p}_{11} \geq \bar{p}_{13} \geq \bar{p}_{14}$.

Then by max-min composition we have

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix},$$

$$\bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}, \bar{P}^4 = \begin{bmatrix} \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}.$$

Therefore, $\bar{P}^{2n} = \bar{P}^2$ for $n = 1,2,3, \dots$ and $\bar{P}^{2n+1} = \bar{P}^3$ for $n = 1,2,3, \dots$. Hence, $\bar{P}$ is not ergodic. Here conditions 2,3 and 4 are satisfied but condition 1 is not satisfied.

If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ with $\bar{p}_{11} \geq \bar{p}_{12} \geq \bar{p}_{13} \geq \bar{p}_{14}$, (note $\bar{p}_{22} = 1$).

Then by max-min composition we have

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}, \text{ so}$$

$\bar{P}^n = \bar{P}^2$ for $n = 2,3,4, \dots$. Therefore, $\bar{P}$ is not ergodic. Here conditions 1,3 and 4 are satisfied but condition 2 is not satisfied.

If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ with $\bar{p}_{11} \geq \bar{p}_{12} \geq \bar{p}_{13} \geq \bar{p}_{14}$,(note that $\bar{p}_{23} = 1$and $\bar{p}_{32} = 1$). Then by max-min composition we have

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \end{bmatrix},$$

$$\bar{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \end{bmatrix}, \bar{P}^5 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \end{bmatrix}.$$

So, $\bar{P}^{2n} = \bar{P}^4$ for $n = 2,3,4,\ldots$ and $\bar{P}^{2n+1} = \bar{P}^3$ for $n = 1,2,3,\ldots$. Therefore, $\bar{P}$ is not ergodic. Here conditions 1,2 and 4 are satisfied but condition 3 is not satisfied.

If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ with $\bar{p}_{11} \geq \bar{p}_{12} \geq \bar{p}_{13} \geq \bar{p}_{14} \geq \bar{p}_{15}$,(note only $\bar{p}_{21} = 1$). Then by max-min composition we have

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\bar{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \bar{P}^5 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\bar{P}^6 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \bar{P}^7 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\bar{P}^8 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \bar{P}^9 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, $\bar{P}^{3n+1} = \bar{P}^4$, $\bar{P}^{3n+2} = \bar{P}^5$, and $\bar{P}^{3n+3} = \bar{P}^6$ for $n = 1,2,3,\ldots$. Therefore, $\bar{P}$ is not ergodic. Here conditions 1,2 and 3 are satisfied but condition 4 is not satisfied.

If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ with $\bar{p}_{11} \geq \bar{p}_{12} \geq \bar{p}_{13} \geq \bar{p}_{14} \geq \bar{p}_{15}$,(note only $\bar{p}_{21} = 1$). Then by max-min composition we have

$$\bar{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \bar{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{P}^5 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \end{bmatrix},$$

$\bar{P}^5 = \bar{P}^6$. Hence, $\bar{P}^m = \bar{P}^5$ for $m = 5,6,7, \ldots$. Therefore, $\bar{P}$ is ergodic.

We see from 5 above that even though condition 4 of Theorem 1 does not hold the result is satisfied.

Conclusion

Although, we put strong conditions to guarantee the ergodicity, Theorem 1 introduces a wide class of ergodic finite fuzzy Markov chains. We do believe that these conditions can be reduced.

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