Advances in Fuzzy Mathematics.
ISSN No. 973-533X Volume 6, Number 2 (2011), pp. 269-281

http://www.ripublication.com/afm.htm

Further Particular Classes of Ergodic Finite Fuzzy Markov Chains

[1]Saed F. Mallak, [2]Mohammad Mara'Beh and [3]Abdelhalim Zaiqan

[1]*Department of Applied Mathematics, Palestine Technical University-Kadoorie*
E-mail: s.mallak@ptuk.edu.ps, saedmallak@yahoo.com
[2]*Master Program of Applied Mathematics, The Arab American University-Jenin*
E-mail: m.maraabeh@gmail.com
[3]*Department of Mathematics, The Arab American University–Jenin*
E-mail: azaiqan@aauj.edu

Abstract

In this paper, depending on max-min composition, and continuing the work in [10], we study the Ergodicity of a particular class of finite fuzzy Markov chains where the last row of the fuzzy transition matrices consists of arbitrary values (between zero and 1) while the other rows' entries are one in one place and zero elsewhere. Under certain conditions, we show that a fuzzy Markov chain in this class is Ergodic. We do the same when the last row is replaced by any row.

Keywords: A Fuzzy Set, A Fuzzy Relation, Max-Min Composition, Fuzzy transition matrix, Ergodic Fuzzy Markov Chain.

Introduction

Fuzzy set theory is a branch that deals precisely with imprecision and ambiguity, and first introduced by Lotfi Zadeh in his well-known paper entitled "Fuzzy Sets" in 1965[16].

Fuzzy Markov chains have been discussed in the literature and many authors published articles in this area [1-5], [7-9], [13], and [15]. In [11] and [12] Sanchez first introduced the concept of greatest eigen fuzzy set. In [1] and [2] Avrachenkov and Sanchez used the concept of greatest eigen fuzzy set to find the stationary solution of fuzzy Markov chains. In [5] Garcia (et al.) have performed a simulation study on fuzzy Markov chains from which they have shown that most of fuzzy Markov chains are not Ergodic. In [13] Sujatha (et al.) studied the limit behavior of cyclic non-homogeneous fuzzy Markov chains.

In [10], we considered finite fuzzy Markov chains where the first row of the transition matrices consists of arbitrary values (between zero and 1) while the other rows' entries are one in one place and zero elsewhere. For such a class of fuzzy Markov chains, we first studied the limit behavior of 2×2 and 3×3 cases and investigated conditions that guarantee ergodicity. After that for $n \times n$, with $n \geq 4$ case, we stated and proved a theorem about the ergodicity of such fuzzy Markov chains. In this paper we replace the first row by the last row and an arbitrary row respectively. We state and prove propositions related with ergodicity of such fuzzy Markov chains.

Fuzzy Markov Chains [1] and [2]

Let $S = \{1, \ldots, n\}$ be a finite state space.

Definition 1: A (finite) fuzzy set or a fuzzy distribution, on S, is defined by a mapping x from S to [0,1], represented by a vector $x = (x_1, \ldots, x_n)$, with x_i denoting $x(i)$, $0 \leq x_i \leq 1, i \in S$. The set of all fuzzy sets on S is denoted by $\mathcal{F}(S)$.

Definition 2: A fuzzy relation P is defined as a fuzzy set on the Cartesian product $S \times S$. P is represented by a matrix $\{p_{ij}\}_{i,j=1}^{n}$, with p_{ij} denoting $P(i,j)$, $0 \leq p_{ij} \leq 1, i, j \in S$.

Definition 3: At each time instant $t, t = 0,1, \ldots$, the state of the system is described by the fuzzy set (or distribution) $x^{(t)} \in \mathcal{F}(S)$. The transition law of the fuzzy Markov chain given by the fuzzy relation P as follows, at time instant $t, t = 1,2, \ldots$

$$x_j^{(t+1)} = \max_{i \in S}\left\{x_i^{(t)} \wedge p_{ij}\right\}, j \in S.$$

We refer to $x^{(0)}$ as the initial fuzzy set (or the initial distribution).

It is natural to define the powers of the fuzzy transition matrix. Namely,

$$p_{ij}^{(t)} = \max_{k \in S}\left\{p_{ik} \wedge p_{kj}^{(t-1)}\right\}, p_{ij}^{(1)} = p_{ij}, p_{ij}^{(0)} = \delta_{ij},$$

where δ_{ij} is a Kronecker delta.

Note that the fuzzy state $x_k^{(t)}$ at time instant $t, t = 1,2, \ldots$ can be calculated by the formula

$$x_k^{(t)} = \max_{l \in S}\left\{x_l^{(0)} \wedge p_{lk}^{(t)}\right\}, k = 1, \ldots, n.$$

Theorem 4([6] and [14]): The powers of the fuzzy transition matrix $\{p_{ij}\}_{i,j=1}^{n}$ either converge to idempotent $\left\{p_{ij}^{(\tau)}\right\}_{i,j=1}^{n}$, where τ is a finite number, or oscillate with a finite period υ starting from some finite power.

Definition 5: Let the powers of fuzzy transition matrix converge in τ steps to a non periodic solution, then the associated fuzzy Markov chain is called nonperiodic (or aperiodic) and $P^* = P^\tau$ is called a limiting fuzzy transition matrix.

Definition 6: The fuzzy Markov chain is called ergodic if it is aperiodic and the limiting transition matrix has identical rows.

A Quick Review

As we mentioned in [10], theorem 4 above is general and does not give us information which fuzzy Markov chains are ergodic . Also, J. C. F. Garcia et al. have performed a simulation study on fuzzy Markov chains from which they have shown that most of fuzzy Markov chains are not ergodic [5]. Besides, in [1] Avrachenkov and Sanchez introduced an open problem about the general conditions that guarantee the ergodicity of fuzzy Markov chains. These results motivates to study the ergodicity of fuzzy Markov chains.

Let $\overline{P} = [\overline{p}_{ij}]$ be an $n \times n$ fuzzy transition matrix. Suppose that $\overline{p}_{ij} = 0$ or 1 for $i \in \{1, \dots, n\} - \{k\}$, $j = 1, \dots, n$ and in each of these rows –all rows except the k^{th} one– exactly one entry is 1, where $1 \leq k \leq n$. In [10] we studied the ergodicity of such fuzzy Markov chains when $k = 1$. In this paper we study the ergodicity when $k = n$ and when k is arbitrary, respectively.

2×2 and 3×3 Cases

We consider $k = 2$ and $k = 3$ for 2×2 and 3×3 fuzzy Markov chains respectively, from which, the following are ergodic:

1. $\overline{P} = \begin{bmatrix} 0 & 1 \\ \overline{p}_{21} & \overline{p}_{22} \end{bmatrix}$ with $0 \leq \overline{p}_{21} \leq \overline{p}_{22} \leq 1$.
2. $\overline{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \overline{p}_{31} & \overline{p}_{32} & \overline{p}_{33} \end{bmatrix}$ with $0 \leq \overline{p}_{31} \leq \overline{p}_{32} \leq \overline{p}_{33} \leq 1$.
3. $\overline{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \overline{p}_{31} & \overline{p}_{32} & \overline{p}_{33} \end{bmatrix}$ with $0 \leq \overline{p}_{32} \leq \overline{p}_{31} \leq \overline{p}_{33} \leq 1$.
4. $\overline{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \overline{p}_{31} & \overline{p}_{32} & \overline{p}_{33} \end{bmatrix}$ with $0 \leq \overline{p}_{31} \leq \overline{p}_{32} \leq \overline{p}_{33} \leq 1$.
5. $\overline{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \overline{p}_{31} & \overline{p}_{32} & \overline{p}_{33} \end{bmatrix}$ with $0 \leq \overline{p}_{32} \leq \overline{p}_{31} \leq \overline{p}_{33} \leq 1$.

6. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \bar{p}_{31} & \bar{p}_{32} & \bar{p}_{33} \end{bmatrix}$ with $0 \leq \bar{p}_{31} \leq \bar{p}_{32} \leq \bar{p}_{33} \leq 1$.
7. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \bar{p}_{31} & \bar{p}_{32} & \bar{p}_{33} \end{bmatrix}$ with $0 \leq \bar{p}_{32} \leq \bar{p}_{31} \leq \bar{p}_{33} \leq 1$.

We conclude from cases 2-7 above that in addition to $\bar{p}_{ij} = 0$ or 1 for $i = 1,2$, $j = 1,2,3$ and in each of the first and second rows exactly one entry is 1 the following conditions were satisfied:

1. $\bar{p}_{33}$ is the maximum among the entries in the last row.
2. $\bar{p}_{ii} \neq 1$ for $i = 1,2$.
3. If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \neq j$, $i = 1,2$, $j = 1,2,3$.

Next we similarly consider $k = 2$ for 3×3 fuzzy transition matrices from which we have the following are ergodic:

1. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{21} \leq \bar{p}_{23} \leq \bar{p}_{22} \leq 1$.
2. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{23} \leq \bar{p}_{21} \leq \bar{p}_{22} \leq 1$.
3. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{21} \leq \bar{p}_{23} \leq \bar{p}_{22} \leq 1$.
4. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{23} \leq \bar{p}_{21} \leq \bar{p}_{22} \leq 1$.
5. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{21} \leq \bar{p}_{23} \leq \bar{p}_{22} \leq 1$.
6. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{23} \leq \bar{p}_{21} \leq \bar{p}_{22} \leq 1$.

We conclude from above that in addition to $\bar{p}_{ij} = 0$ or 1 for $i = 1,3$, $j = 1,2,3$, and in each of the first and third rows exactly one entry is 1 the following conditions were satisfied:

1. $\bar{p}_{22}$ is the maximum among the entries in the second row.
2. $\bar{p}_{ii} \neq 1$ for $i = 1,3$.
3. If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \neq j$, $i = 1,3$, $j = 1,2,3$.

The general Case

Proposition 1: For $n \geq 4$ let $\bar{P} = [\bar{p}_{ij}]$ be an $n \times n$ fuzzy transition matrix, such that $\bar{p}_{ij} = 0$ or 1 for $i = 1, \ldots n-1$, $j = 1, \ldots, n$ and in each row except possibly the last one, exactly one entry is 1. Assuming the following conditions hold:

1. $\bar{p}_{nn}$ is the maximum among the entries in the last row.
2. $\bar{p}_{ii} \neq 1$ for $i = 1, \ldots, n-1$.
3. If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \neq j$, $i = 1, \ldots, n-1, j = 1, \ldots, n$.
4. $\bar{p}_{i_1 n} = \bar{p}_{i_2 n} = \cdots = \bar{p}_{i_k n} = 1$ where $k \in \{n-3, n-2, n-1\}$ and $i_1, i_2, \ldots, i_k \in \{1,2, \ldots, n-1\}$.

Then, by max-min composition $\bar{P}$ is ergodic.

Proof. If $k = n-3$ then $\bar{p}_{i_1 n} = \bar{p}_{i_2 n} = \cdots = \bar{p}_{i_{n-3} n} = 1$, $i_1, i_2, \ldots, i_{n-3} \in \{1,2, \ldots, n-1\}$, and $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{1,2, \ldots, n-1\} - \{i_1, i_2, \ldots, i_{n-3}\}$ for $j_1, j_2 \in \{1,2, \ldots, n-1\}$.

Either $i_{n-1} < i_{n-2}$ or $i_{n-1} > i_{n-2}$ we may assume that $i_{n-1} < i_{n-2}$.

Case 1: If $j_1 = j_2$ then

$$\bar{P} = \left[\begin{array}{cccccc}
0 & \cdots & 0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & \ddots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 \\
\bar{p}_{n1} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{nn-1} & \bar{p}_{nn}
\end{array}\right]
\begin{array}{l}
\\ \\ \\
\leftarrow row\ i_{n-1} \\
\\ \\ \\
\leftarrow row\ i_{n-2} \\
\\ \\ \\ \\
\end{array}$$

$$\text{Let } E_n = \begin{array}{r} \\ \\ \\ \\ row\ n-k+1 \rightarrow \\ \\ \\ \end{array}
\left[\begin{array}{c} e_{nn} \\ e_{n(n-1)} \\ \vdots \\ \vdots \\ e_{nk} \\ \vdots \\ e_{n1} \end{array}\right]
= \begin{array}{c} \begin{array}{ccccccc} & & & column\ n-k+1 & & & \\ & & & \downarrow & & & \end{array} \\ \left[\begin{array}{ccccccc} e_{nn}{}^T & e_{n(n-1)}{}^T & \cdots & e_{nk}{}^T & \cdots & e_{n2}{}^T & e_{n1}{}^T \end{array}\right] \end{array}$$, then E_n is an $n \times n$ permutation matrix and $E_n E_n = I_n$. Consider $\bar{T} = E_n \bar{P} E_n$.

$$E_n\bar{P} = \begin{array}{l} \begin{bmatrix} \bar{p}_{n1} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{n(n-1)} & \bar{p}_{nn} \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ n-i_{n-2}+1 \\ \\ \\ \\ \leftarrow row\ n-i_{n-1}+1 \\ \\ \\ \\ \end{array}$$

$$\bar{T} = E_n\bar{P}E_n = \begin{array}{c} \begin{array}{cccccc} & & & column\ n-j_1+1 & & \\ & & & \downarrow & & \end{array} \\ \begin{bmatrix} \bar{p}_{nn} & \bar{p}_{n(n-1)} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{n_1} \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \\ \leftarrow row\ n-i_{n-2}+1 \\ \\ \\ \\ \leftarrow row\ n-i_{n-1}+1 \\ \\ \\ \\ \end{array}$$

Therefore, $\bar{T}$ satisfies the conditions of Theorem 1[10] and there is $k \in \mathbb{N}$ such that $\bar{T}^m = \bar{T}^K$, for $m = k, k+1, k+2, \ldots$, where the rows are identical in $\bar{T}^K$. Therefore, $\bar{T}^m = \bar{T}^K \Longrightarrow (E_n\bar{P}E_n)^m = \bar{T}^K$

$$\underbrace{(E_n\bar{P}E_n)(E_n\bar{P}E_n)\cdots\cdots(E_n\bar{P}E_n)}_{m-times} = \bar{T}^K$$

$E_n\bar{P}I_n\bar{P}\cdots\cdots I_n\bar{P}E_n = \bar{T}^K \Longrightarrow E_n\bar{P}^mE_n = \bar{T}^K \Longrightarrow \bar{P}^m = E_n\bar{T}^KE_n$, $m = k, k+1, \cdots$. Since E_n is a permutation matrix we conclude that the rows are identical in $E_n\bar{T}^KE_n$. Hence, $\bar{P}$ is ergodic.

Case 2: $j_1 \neq j_2$then either $j_1 < j_2$ or $j_1 > j_2$ we may assume that $j_1 < j_2$.

$$\bar{P} = \begin{array}{cl} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cdots & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \bar{p}_{n1} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{nj_2} & \cdots & \bar{p}_{n(n-1)} & \bar{p}_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \\ \end{array} \end{array}$$

We use the same E_n as in Case 1and consider the composition $\bar{T} = E_n\bar{P}E_n$

$$E_n\bar{P} = \begin{array}{cl} \begin{bmatrix} \bar{p}_{n1} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{nj_2} & \cdots & \bar{p}_{n(n-1)} & \bar{p}_{nn} \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ n-i_{n-2}+1 \\ \\ \\ \\ \leftarrow row\ n-i_{n-1}+1 \\ \\ \\ \\ \end{array} \end{array}$$

$$\bar{T} = E_n\bar{P}E_n = \begin{array}{cl} \begin{array}{cccccccc} & & & \substack{column\ n-j_2+1 \\ \downarrow} & & \substack{column\ n-j_1+1 \\ \downarrow} & & \end{array} & \\ \begin{bmatrix} \bar{p}_{nn} & \bar{p}_{n(n-1)} & \cdots & \bar{p}_{nj_2} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{n1} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ n-i_{n-2}+1 \\ \\ \\ \\ \leftarrow row\ n-i_{n-1}+1 \\ \\ \\ \\ \end{array} \end{array}$$

Therefore $\bar{T}$ satisfies the conditions of Theorem 1 [10] and as in case 1 above we conclude that $\bar{P}$ is ergodic.

Similar argument applies for $k = n-2$ and $k = n-1$.

Proposition 2: For $n \geq 4$ let $\bar{P} = [\bar{p}_{ij}]$ be an $n \times n$ fuzzy transition matrix such that, $\bar{p}_{ij} = 0$ or 1 for $i \in \{1, \dots, n\} - \{k\}, j = 1, \dots, n$, where $1 < k < n$, and in each row except possibly the k^{th} one, exactly one entry is 1.Assuming the following conditions hold:

1. $\bar{p}_{kk}$ is the maximum among the entries in the k^{th} row.
2. $\bar{p}_{ii} \neq 1$ for , $i \in \{1, \dots, n\} - \{k\}$.
3. If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \neq j$, $i \in \{1, \dots, n\} - \{k\}, j = 1, \dots, n$.
4. $\bar{p}_{i_1 k} = \bar{p}_{i_2 k} = \cdots = \bar{p}_{i_l k} = 1$ where $l \in \{n-3, n-2, n-1\}$ and $i_1, i_2, \dots, i_l \in \{1, \dots, n\} - \{k\}$.

Then, by max-min composition $\bar{P}$ is ergodic.

Proof. If $l = n - 3$ then $\bar{p}_{i_1 k} = \bar{p}_{i_2 k} = \cdots = \bar{p}_{i_{n-3} k} = 1$, $i_1, i_2, \dots, i_{n-3} \in \{1, \dots, n\} - \{k\}$, and $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{1, \dots, n\} - \{i_1, i_2, \dots, i_{n-3}, k\}$ for $j_1, j_2 \in \{1, \dots, n\} - \{k\}$. Either $i_{n-1} < i_{n-2}$ or $i_{n-1} > i_{n-2}$ we may assume that $i_{n-1} < i_{n-2}$.

Case 1: If $j_1 = j_2$ then

$$\bar{P} = \begin{array}{cl} \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{k1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

$$\text{Let } E_n = \begin{array}{cl} \begin{bmatrix} e_{nk} \\ e_{n2} \\ e_{n3} \\ \vdots \\ \vdots \\ e_{n(k-1)} \\ e_{n1} \\ e_{n(k+1)} \\ e_{n(k+2)} \\ \vdots \\ e_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \\ \\ \leftarrow k^{th} row \\ \\ \\ \\ \\ \end{array} \end{array} = \begin{array}{c} \begin{array}{cccccccccc} & & & & & k^{th}column \downarrow & & & & \end{array} \\ \begin{bmatrix} e_{nk}{}^T & e_{n2}{}^T & e_{n3}{}^T & \cdots & e_{n(k-1)}{}^T & e_{n1}{}^T & e_{n(k+1)}{}^T & e_{n(k+2)}{}^T & \cdots & e_{nn}{}^T \end{bmatrix} \end{array},$$

then E_n is an $n \times n$ permutation matrix and $E_n E_n = I_n$. Consider the composition $\bar{T} = E_n \bar{P} E_n$.

$$E_n\bar{P} = \left[\begin{matrix} \bar{p}_{k1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \end{matrix}\right] \begin{matrix} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{matrix}$$

$$\bar{T} = E_n\bar{P}E_n = \left[\begin{matrix} \bar{p}_{kk} & \bar{p}_{k2} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{k1} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \end{matrix}\right] \begin{matrix} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \leftarrow k^{th}\ row \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{matrix}$$

Therefore $\overline{T}$ satisfies the conditions of Theorem 1 [10] and as in Case 1 of proposition 1 above we conclude that $\overline{P}$ is ergodic.

Case 2: $j_1 \neq j_2$ then either $j_1 < j_2$ or $j_1 > j_2$ we may assume that $j_1 < j_2$.
We use the same E_n as in Case 1 and consider the composition $\overline{T} = E_n\overline{P}E_n$

$$\bar{P} = \left[\begin{matrix} 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{k1} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \end{matrix}\right] \begin{matrix} \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{matrix}$$

$$E_n\bar{P} = \begin{array}{cl} \left[\begin{array}{ccccccccccc} \bar{p}_{k1} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \leftarrow k^{th}\ row \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

$$\bar{T} = E_n\bar{P}E_n = \begin{array}{cl} \begin{array}{c} k^{th} column \\ \downarrow \end{array} & \\ \left[\begin{array}{ccccccccccc} \bar{p}_{kk} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{k1} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\ 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \leftarrow k^{th}\ row \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Therefore $\overline{\mathrm{T}}$ satisfies the conditions of Theorem 1 [10] and as in Case 1 of proposition 1 above we conclude that $\overline{\mathrm{P}}$ is ergodic.

Similar argument applies for $l = n - 2$ and $l = n - 1$.

We may prove proposition 2 using another permutation matrix E_n as follows:

If $l = n - 3$ then $\bar{p}_{i_1k} = \bar{p}_{i_2k} = \cdots = \bar{p}_{i_{n-3}k} = 1$, $i_1, i_2, \dots, i_{n-3} \in \{1, \dots, n\} - \{k\}$, and $\bar{p}_{i_{n-1}j_1} = \bar{p}_{i_{n-2}j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{1, \dots, n\} - \{i_1, i_2, \dots, i_{n-3}, k\}$ for $j_1, j_2 \in \{1, \dots, n\} - \{k\}$. Either $i_{n-1} < i_{n-2}$ or $i_{n-1} > i_{n-2}$ we may assume that $i_{n-1} < i_{n-2}$.

Case 1: If $j_1 = j_2$ then

$$\bar{P} = \begin{array}{cl} \left[\begin{array}{ccccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{k1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Let

$$E_n = \left[\begin{matrix} e_{nk} \\ e_{n(n-1)} \\ e_{n(n-2)} \\ \vdots \\ \vdots \\ e_{n(n-k+2)} \\ e_{n1} \\ e_{n(n-k)} \\ \vdots \\ e_{n2} \\ e_{nn} \end{matrix}\right] \begin{matrix} \\ \\ \\ \\ \\ \\ \leftarrow k^{th}\, row \\ \\ \\ \\ \\ \end{matrix} = \left[e_{nk}{}^{T} \quad e_{n(n-1)}{}^{T} \quad e_{n(n-2)}{}^{T} \quad \cdots \quad e_{n(n-k+2)}{}^{T} \quad \overset{\substack{k^{th} column \\ \downarrow}}{e_{n1}{}^{T}} \quad e_{n(n-k)}{}^{T} \quad \cdots \quad e_{n2}{}^{T} \quad e_{nn}{}^{T} \right],$$

E_n then E_n is an $n \times n$ permutation matrix and $E_n E_n = I_n$. Consider the composition $\overline{T} = E_n \overline{P} E_n$.

$$E_n\overline{P} = \left[\begin{matrix}
\bar{p}_{k1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{matrix}\right]
\begin{matrix} \\ \\ \\ \\ \\ \leftarrow row\ n-i_{n-2}+1 \\ \\ \\ \leftarrow k^{th}\, row \\ \\ \\ \leftarrow row\ n-i_{n-1}+1 \\ \\ \\ \end{matrix}$$

$$\overline{T} = \begin{array}{c}
\qquad\qquad\qquad\qquad \text{column } n-j_1+1 \\
\qquad\qquad\qquad\qquad \downarrow \\
\left[\begin{matrix}
\bar{p}_{kk} & \bar{p}_{k(n-1)} & \bar{p}_{k(n-2)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(n-k+2)} & \bar{p}_{k1} & \bar{p}_{k(n-k)} & \cdots & \bar{p}_{k2} & \bar{p}_{kn} \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0
\end{matrix}\right]
\begin{matrix} \\ \\ \\ \\ \leftarrow row\ n-i_{n-2}+1 \\ \\ \\ \leftarrow k^{th}\, row \\ \\ \\ \leftarrow row\ n-i_{n-1}+1 \\ \\ \\ \\ \end{matrix}
\end{array}$$

Therefore $\overline{T}$ satisfies the conditions of Theorem 1 [10] and as in Case 1 of proposition 1 above we conclude that $\overline{P}$ is ergodic.

Case 2: $j_1 \neq j_2$ then either $j_1 < j_2$ or $j_1 > j_2$ we may assume that $j_1 < j_2$.

We use the same E_n as in case 1 and consider the composition $\overline{T} = E_n \overline{P} E_n$.

$$\bar{P} = \left[\begin{array}{ccccccccccc}
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\bar{p}_{k1} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\
\end{array}$$

$$E_n\bar{P} = \left[\begin{array}{ccccccccccc}
\bar{p}_{k1} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ \leftarrow row\ n-i_{n-2}+1 \\ \\ \\ \leftarrow k^{th}\, row \\ \\ \\ \leftarrow row\ n-i_{n-1}+1 \\ \\ \\ \\
\end{array}$$

$$\begin{array}{rc}
 & \begin{array}{ccccccccccccccccc}
 & & & & column\ n-j_2+1 \downarrow & & & k^{th} column \downarrow & & & column\ n-j_1+1 \downarrow & & & & & &
\end{array} \\
\bar{T} = & \left[\begin{array}{ccccccccccccccc}
\bar{p}_{kk} & \bar{p}_{k(n-1)} & \bar{p}_{k(n-2)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{k(n-k+2)} & \bar{p}_{k1} & \bar{p}_{k(n-k)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k2} & \bar{p}_{kn} \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ row\ n-i_{n-2}+1 \\ \\ \\ \leftarrow k^{th}\, row \\ \\ \\ row\ n-i_{n-1}+1 \\ \\ \\ \\
\end{array}
\end{array}$$

Therefore $\overline{\mathrm{T}}$ satisfies the conditions of Theorem 1 [10] and as in Case 1 of proposition 1 above we conclude that $\overline{\mathrm{P}}$ is ergodic.

Similar argument applies for $\mathrm{l} = \mathrm{n} - 2$ and $\mathrm{l} = \mathrm{n} - 1$.

Conclusion

In this paper we continued studying the ergodocity of a particular class of finite fuzzy Markov chains. As we mentioned in [10] we do believe that the introduced conditions can be reduced.

References

[1] K. E. Avrachenkov and E. Sanchez, Fuzzy Markov Chains: Specificities and Properties, Proc. IPMU, July 3-7 Madrid, Spain, 2000, pp. 1851-1856.

[2] K. E. Avrachenkov and E. Sanchez, Fuzzy Markov Chains and Decision-making, Fuzzy Optimization and Decision Making 1, 143-159 (2002).

[3] R. E. Bellman and L. A. Zadeh, Decision-Making in a Fuzzy Environment, Management Science 17 (1970), pp. 141-164.

[4] J. Buckley, E. Eslami, Fuzzy Markov Chains: Uncertain Probabilities, Mathware and Soft Computing, 9, (2002), 33-41.

[5] J. C. F. Garcia et al., A Simulation Study on Fuzzy Markov Chains, D.-S. Huang et al. (Eds.): ICIC 2008, CCIS 15, pp. 109-117, 2008, Springer-Verlag, Berlin Heidelberg 2008.

[6] Y. Give'on, Lattice matrices, Information and Control, 7 (1964), 477-484.

[7] R. M. Kleyle and A. DeKorvin, Constructing One-Step and Limiting Fuzzy Transition Probabilities for Finite Markov Chains, J. Intelligent and Fuzzy Systems, 6 (1998), pp. 223-235.

[8] R. Kruse, R. Buck-Emden and R. Cordes, Processor Power Considerations- An Application of Fuzzy Markov Chains, Fuzzy Sets and Systems, 21 (1987), pp. 289-299.

[9] M. Kurano, M. Yasuda, J. Nakagami and Y. Yoshida, Markov-type Fuzzy Decision Processes with a Discounted Reward on a Closed Interval, European J. Oper. Res., 92 (1996), pp. 649-662.

[10] Saed F. Mallak, Mohammad Mara'Beh and Abdelhalim Zaiqan, A Particular Class of Ergodic Finite Fuzzy Markov Chains (submitted).

[11] E. Sanchez, Resolution of Eigen Fuzzy Sets Equations, Fuzzy Sets and Systems, 1, 69-74 (1978).

[12] E. Sanchez, Eigen Fuzzy Sets and Fuzzy Relations, J. Math. Anal. Appl. 81, 399-42 (1981)

[13] R. Sujatha et al., Long Term Behavior of Cyclic Non-Homogeneous Fuzzy Markov Chain, Int. J. Contemp. Math. Sciences, Vol. 5,(2010), no. 22, 1077-1089.

[14] M. G. Thomason, Convergence of Powers of a Fuzzy Matrix, J. Math. Anal. Appl. 57,476-480, (1977).

[15] Y. Yoshida, Markov Chains with a Transition Possibility Measure and Fuzzy Dynamic Programming, Fuzzy Sets and Systems, 66 (1994), pp. 39-57.

[16] L. A. Zadeh, Fuzzy Sets, Information and Control, (1965), 8 :338-353.