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Finite Fuzzy Markov Chains

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Dedication

With my great love I dedicate this work to my parents, my brothers: Saed, Raed, Baha', Ala', and to all my relatives and friends.

Acknowledgment

First of all, my great continuous thanks to Allah who gave me the ability to finish this work.

I would like to express my deepest appreciation to my supervisor Dr. Abdelhalim Ziqan.

I will never forget my parents, whom without their infinite support and satisfaction, I might not be able to achieve my goal, also my dear brothers and the whole family.

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Contents

Abstract

In this thesis we deal with finite fuzzy Markov chains in two ways. First, we replace the uncertainty in any transition probability of a crisp transition matrix by a fuzzy number. In this case, we were able under certain conditions to prove the uniqueness of the limit of 2×2 matrix powers. Secondly, the transition matrix represents a fuzzy relation on a finite state space. Here, we were able to place some conditions on $n \times n$ fuzzy transition matrices to have the ergodic behavior.

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الملخص

في هذه الأطروحة درسنا سلاسل ماركوف الضبابية المنتهية بطريقتين. أولاً استبدلنا كل احتمال انتقالي غير مؤكد في المصفوفة الانتقالية اللاضبابية بعدد ضبابي، وفي هذه الحالة أثبتنا تحت شروط معينة أن نهاية قوى المصفوفة المربعة من الحجم 2×2 وحيدة. ثانياً المصفوفة الانتقالية تمثل علاقة ضبابية على فضاء منتهي، حيث أثبتنا ضمن شروط معينة أن نهاية قوى المصفوفة من الحجم $n \times n$ تكون موجودة وهي عبارة عن مصفوفة جميع صفوفها متماثلة.

Introduction

Finite Fuzzy Markov chains and their applications have been widely studied in the literature in the last decades [1-3], [6], [8], [9-10], [12], [15-17], [19-22], [24], [27], [30]. In [2] and [3], Avrachenkov and Sanchez used the concept of greatest eigen fuzzy set, which was first defined in [25] and [26] by Sanchez, to find the stationary solution of ergodic fuzzy Markov chains. In [12], Garcia (et al.) have done a simulation study on fuzzy Markov chains from which they have shown the non-ergodicity of a wide set of fuzzy Markov chains. In [27], Sujatha (et al.) studied the limit behavior of cyclic non-homogeneous fuzzy Markov chains. Bellman and Zadeh were the first who considered stochastic systems in a fuzzy environment [4]. In [15], the fuzzy probabilities determine the elements of the transition matrix as fuzzy subsets of $[0, 1]$ where the extension principle to find powers of the transition matrix was used.

Finite fuzzy Markov chains have many advantages over classical Markov chains due to its reality. In real situations, finite fuzzy Markov chains solve the vague in different ways, but due to lack of information, the state of the process may be not completely known. Also, in many cases the transition probabilities of a transition matrix of a Markov chain may be   estimated. In this case, results in “uncertainty” can be modeled using fuzzy numbers. In such situations, finite fuzzy Markov chains are considered to be very important tools.

Finite Fuzzy Markov chains have many applications including the analysis of internet glance behavior, image segmentation, decision-making, calculating effective processor power, multitemporal image analysis, synthetic aperture radar (SAR) images, cascade multitemporal classification, modeling and forecasting credit behavior dynamics of credit card users, multispectral image segmentation, describing both determined and random behavior of complex dynamic plants, individual demand forecasting, and stochastic dynamic programming (SDP) formulations for reservoir operation [1], [8], [9-10], [16], [19-22], [24].

In chapter one of the thesis, we introduce the concept of fuzzy sets, operations on fuzzy sets, fuzzy relations, alpha cuts, convex fuzzy sets, fuzzy numbers and operations on them with concentration on triangular and trapezoidal fuzzy numbers. In the last section of chapter one, we define a new fuzzy number.

General review of Markov chains theory, including classification of chains, main ergodic theorems are discussed in chapter two.

In chapter three, the restricted fuzzy matrix multiplication is defined and is used to find the limit of regular finite transition matrices whose uncertain transition probabilities are modeled by fuzzy numbers. Under certain conditions, the uniqueness of the limit of powers of 2×2 regular fuzzy transition matrices in the case of triangular, trapezoidal, and a special fuzzy number is also proved in chapter three.

In chapter four, a comparison through examples between fuzzy and crisp Markov chains is introduced. Then, we discuss the ergodicity of finite fuzzy Markov chains, and end up with a worth discussion on the ergodic behavior of a particular class of finite fuzzy Markov chains.

Chapter 1

Fuzzy Sets and Fuzzy Numbers

This chapter consists of six sections. In sections 1.1-1.5 we give the definitions of fuzzy sets, operations on fuzzy sets, fuzzy relations, alpha cuts, convex fuzzy sets, triangular and trapezoidal fuzzy numbers [7], [18], [31], [32]. In Section 1.6 we define a new fuzzy number and we study its properties.

1.1 Fuzzy Sets

Fuzzy set theory, dealing precisely with imprecision and ambiguity, was first introduced by Lotfi A. Zadeh in his well-known paper entitled "Fuzzy Sets" in 1965 [31]. In the classical set theory, an element of the universe either belongs or does not belong to the set, and this is represented by the characteristic function, $f_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$. A fuzzy set is a generalization of a set in the usual sense. It allows for each element in the universe of discourse to take a value in the closed interval [0,1].

Definition 1.1.1([7] pages 7 and 8, [31] page 339): If Ω is a nonempty set, then a *fuzzy subset* $\bar{A}$ of Ω is defined by its *membership function*, written $\bar{A}(x)$, which produces values in $[0,1]$ for all x in Ω. So, $\bar{A}(x)$ is a function that maps Ω into $[0,1]$. $\bar{A}(x)$ is the *grade of membership* of x in $\bar{A}$. We have specified a fuzzy set from a set by placing a bar over a letter. The term *crisp* means not *fuzzy*, so a crisp set is a set in the usual sense.

Notation 1.1.2 ([18] page 9, [32] page 12): If $\bar{A}$ is a fuzzy set, we denote the membership function in Ω by $f_{\bar{A}}(x)$ *or* $\mu_{\bar{A}}(x)$, $x \in \Omega$.

Remark 1.1.3 ([31] page 339): The closer the value of $\bar{A}(x)$ to unity, the higher the grade of membership of x in $\bar{A}$.

Remark 1.1.4 ([18] page 9, [32] pages 12 and 13): If $\bar{A}$ is a fuzzy set in X, then there are different ways of denoting $\bar{A}$ as an example:

1. A fuzzy set $\bar{A}$ in X may be viewed as a set of ordered pairs $\bar{A} = \{(x, \bar{A}(x)) | x \in X$.
2. A fuzzy set $\bar{A}$ is represented solely by stating its membership function
3. If X is countable then

$$\bar{A} = \mu_{\bar{A}}(x_1)/x_1 + \mu_{\bar{A}}(x_2)/x_2 + \mu_{\bar{A}}(x_3)/x_3 + \cdots = \sum_i \mu_{\bar{A}}(x_i)/x_i,$$

here " $+$ " denotes the union rather than the arithmetic sum, usually $0/x$ terms are not taken into account in this representation. Also, if X is a continuous set then

$\bar{A} = \int_X \ \mu_{\bar{A}}(x)/x.$

Example 1.1.5 ([32] page 12): $\bar{A}$ =“real numbers close to 10”

$$\bar{A} = \{(x, \mu_{\bar{A}}(x)) | \ \mu_{\bar{A}}(x) = (1 + (x - 10)^2)^{-1}, x \in \mathbb{R}\}.$$

Example 1.1.6 ([32] page 12): $\bar{A}$ =“real numbers considerably larger than 10”

$$\bar{A}(x) = \begin{cases} 0 & , x \leq 10 \\ (1 + (x - 10)^{-2})^{-1} & , x > 10 \end{cases}$$

Example 1.1.7 ([32] page 13): $\bar{A}$ =“integers close to 10”

$$\bar{A} = 0.1/7 + 0.5/8 + 0.8/9 + 1/10 + 0.8/11 + 0.5/12 + 0.1/13.$$

Example 1.1.8 ([32] page 13): $\bar{A}$ =“real numbers close to 10”,

$$\bar{A} = \int_{\mathbb{R}} \frac{1}{1 + (x - 10)^2} / x$$

1.2 Operations on Fuzzy Sets and Fuzzy Relations

In this section, we list several definitions involving fuzzy sets which are obvious extensions of the corresponding definitions for sets in their usual sense.

Definition 1.2.1 ([31] page 340): A fuzzy set is *empty* if and only if its membership function is identically zero on Ω.

Definition 1.2.2 ([31] page 340): Two fuzzy sets $\bar{A}$ and $\bar{B}$ are *equal*, written $\bar{A} = \bar{B}$, if and only if $\bar{A}(x) = \bar{B}(x)$ for all x in Ω.

Definition 1.2.3 ([31] page 340): The *complement* of a fuzzy set $\bar{A}$ is denoted by $\overline{A}^{c}$ and is defined by $\overline{A}^{c}(x) = 1 - \bar{A}(x)$ for all x in Ω.

Definition 1.2.4 ([31] page 340): $\bar{A}$ is *contained* in $\bar{B}$ (or, equivalently, $\bar{A}$ is a *subset* of $\bar{B}$, or $\bar{A}$ is *smaller than or equal* to $\bar{B}$) if and only if $\bar{A}(x) \leq \bar{B}(x)$ for all x in Ω. In symbols $\overline{\mathrm{A}} \subset \overline{\mathrm{B}} \Leftrightarrow \bar{A}(x) \leq \bar{B}(x)$.

Definition 1.2.5 ([31] page 340): The *union* of two fuzzy sets $\bar{A}$ and $\bar{B}$ with respective membership functions $\bar{A}(x)$ and $\bar{B}(x)$ is a fuzzy set $\bar{C}$, written as $\bar{C} = \bar{A} \cup \bar{B}$, whose

membership function is $\bar{C}(x) = max\{\bar{A}(x), \bar{B}(x)\}$, for all $x \in$ Ω.

Definition 1.2.6 ([31] page 341): (Equivalent definition of the union of two fuzzy sets): The union of $\bar{A}$ and $\bar{B}$ is the smallest fuzzy set containing both $\bar{A}$ and $\bar{B}$. More precisely, if $\bar{D}$ is any fuzzy set which contains both $\bar{A}$ and $\bar{B}$, then it also contains $\bar{A} \cup \bar{B}$.

Proof. Let $\bar{C} = \bar{A} \cup \bar{B}$, then $\bar{C}(x) = max\{\bar{A}(x), \bar{B}(x)\}$, so $\bar{C}(x) \geq \bar{A}(x)$ and $\bar{C}(x) \geq \bar{B}(x)$, i.e. $\bar{C}$ contains both $\bar{A}$ and $\bar{B}$. Let $\bar{D}$ be any fuzzy set containing both $\bar{A}$ and $\bar{B}$. We show that $\bar{C} \subset \bar{D}$. We have $\bar{D}(x) \geq \bar{A}(x)$ and $\bar{D}(x) \geq \bar{B}(x)$. So $\bar{D}(x) \geq max\{\bar{A}(x), \bar{B}(x)\} = \bar{C}(x)$. Then, from the definition $\bar{C} \subset \bar{D}$.

Similarly, we give a definition of the intersection of two fuzzy sets in two equivalent ways.

Definition 1.2.7 ([31] page 341): The *intersection* of two fuzzy sets $\bar{A}$ and $\bar{B}$ with respective membership functions $\bar{A}(x)$ and $\bar{B}(x)$ is a fuzzy set $\bar{C}$, written as $\bar{C} = \bar{A} \cap \bar{B}$, whose membership function is $\bar{C}(x) = min\{\bar{A}(x), \bar{B}(x)\}$, for all $x \in \Omega$.

Definition 1.2.8 ([31] page 341): (Equivalent definition of the intersection of two fuzzy sets): The intersection of $\bar{A}$ and $\bar{B}$ is the largest fuzzy set which is contained in both $\bar{A}$ and $\bar{B}$ More precisely, if $\bar{D}$ is any fuzzy set which is contained in both $\bar{A}$ and $\bar{B}$, then $\bar{D}$ is contained in $\bar{A} \cap \bar{B}$.

With the operations of union, intersection, and complementation defined above , it is easy to extend many of the basic identities which hold for sets in the usual sense to fuzzy sets.

$$\left.\begin{array}{ll} 1. & (\bar{A} \cup \bar{B})^c = \overline{A}^c \cap \overline{B}^c. \\ 2. & (\bar{A} \cap \bar{B})^c = \overline{A}^c \cup \overline{B}^c. \end{array}\right\} \text{De Morgan's laws}$$

$$\left.\begin{array}{ll} 3. & \bar{C} \cap (\bar{A} \cup \bar{B}) = (\bar{C} \cap \bar{A}) \cup (\bar{C} \cap \bar{B}). \\ 4. & \bar{C} \cup (\bar{A} \cap \bar{B}) = (\bar{C} \cup \bar{A}) \cap (\bar{C} \cup \bar{B}). \end{array}\right\} \text{Distributive laws}$$

Those and similar identities can readily be established by showing the corresponding relations for the membership functions of $\bar{A}, \bar{B}$ and $\bar{C}$ *([31] page 342).*

Definition 1.2.9 ([31] page 345): A *fuzzy relation* in X is a fuzzy set $\bar{A}$ in the product space $X \times X$.

Definition 1.2.10 ([31] page 346): An n −ary fuzzy relation in X is a fuzzy set $\bar{A}$ in the product space $X \times X \times \ldots \times X$ (n −times). For such relations, the membership function is

of the form $f_{\bar{A}}\,(x_1, \ldots, x_n)$, where $x_i \in X, i = 1, \ldots, n$.

Definition 1.2.11 ([31] page 346): The *composition of two fuzzy relations* $\bar{A}$ and $\bar{B}$ denoted by $\bar{B} \circ \bar{A}$ is defined as a fuzzy relation in X whose membership function is defined by

$$f_{\bar{B}\circ\bar{A}}(x, y) = sup_v \min\{f_{\bar{A}}(x, v), f_{\bar{B}}(v, y)\}.$$

1.3 Alpha Cuts and Convexity

Definition 1.3.1 ([7] page 10): An *α −cut* of the fuzzy set $\bar{A}$ in X, written as $\bar{A}[\alpha]$ or $\bar{A}_\alpha$ is defined as $\{x \in X | \bar{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$, where $\bar{A}(x)$ is the membership function of the fuzzy set $\bar{A}$.

$\bar{A}[0]$ is defined separately as the closure of the union of all the $\bar{A}[\alpha]$, $0 < \alpha \leq 1$ [6].

Remark ([7] page 10) 1.3.2: $\bar{A}[0]$ is called the *support*, and $\bar{A}[1]$ is called the *core* of the fuzzy set $\bar{A}$.

Proposition 1.3.3: If $\alpha \geq \alpha'$ then $\bar{A}[\alpha] \subseteq \bar{A}[\alpha']$, for $0 \leq \alpha, \alpha' \leq 1$.

Proof.

$$\bar{A}[0] = \overline{\bigcup_{0<\alpha\leq 1} \bar{A}[\alpha]}$$

So, $\bar{A}[\alpha] \subseteq \bar{A}[0]$ for $\alpha \geq 0$. For, $\alpha \geq \alpha' > 0$, let $x \in \bar{A}[\alpha]$ then $\bar{A}(x) \geq \alpha$. But $\alpha \geq \alpha'$, so $\bar{A}(x) \geq \alpha'$. Hence, $x \in \bar{A}[\alpha']$. Therefore, $\bar{A}[\alpha] \subseteq \bar{A}[\alpha']$.

Example 1.3.4: Consider the fuzzy set

$$\bar{A} = 0.1/7 + 0.5/8 + 0.8/9 + 1/10 + 0.8/11 + 0.5/12 + 0.1/13.$$

Then,

$\bar{A}[0.1] = \bar{A}_{0.1} = \{7,8,9,10,11,12,13\}$,

$\bar{A}[0.5] = \bar{A}_{0.5} = \{8,9,10,11,12\}$,

$\bar{A}[0.8] = \bar{A}_{0.8} = \{9,10,11\}$,

$\bar{A}[1] = \bar{A}_1 = \{10\}$,

$\bar{A}[0] = \bar{A}_0 = \{7,8,9,10,11,12,13\}$.

Definition 1.3.5: A set $A \subseteq \mathbb{R}^n$ is *convex* if for each x_1 and x_2 in A, the linear combination $\lambda x_1 + (1-\lambda)x_2$ is also in A for $0 \leq \lambda \leq 1$.

Definition 1.3.6 ([31] page 347): A fuzzy set $\bar{A}$ in $\mathbb{R}^n$ is *convex* if and only if all α −cuts of $\bar{A}$, $\bar{A}[\alpha]$, are convex.

Proposition 1.3.7 ([31] page 347): A fuzzy set $\bar{A}$ in $\mathbb{R}^n$ is convex if and only if $f_{\bar{A}}[\lambda x_1 + (1-\lambda)x_2] \geq min\{f_{\bar{A}}(x_1),\ f_{\bar{A}}(x_2)\}$, for all x_1 and x_2 in $\mathbb{R}^n$ and all λ in $[0, 1]$, where $f_{\bar{A}}$ is the membership function of $\bar{A}$ in $\mathbb{R}^n$.

1.4 Fuzzy Numbers

Definition 1.4.1 ([18] page 130): A *fuzzy number* $\bar{A}$ is a fuzzy set satisfying the following

conditions:

1. $\bar{A}$ is a convex fuzzy set.
2. $\bar{A}$ is a normalized fuzzy set, i.e. $\exists x \in \mathbb{R}$, such that $\bar{A}(x) = 1$.
3. The membership function $\bar{A}(x)$ is piecewise continuous.
4. $\bar{A}(x)$ is defined on the real numbers, i.e. the domain of $\bar{A}(x)$ is $\mathbb{R}$ and its co-domain is $[0, 1]$.

The convex condition is that $\bar{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$, where $a_1(\alpha)$ and $a_2(\alpha)$ satisfy

$$(\alpha' < \alpha) \Longrightarrow (\ a_1(\alpha') \leq a_1(\alpha), a_2(\alpha') \geq a_2(\alpha)\ ).$$

Let $\bar{A}$ be a fuzzy set whose membership function given by:

$$\bar{A}(x)\begin{cases} 0, & x \leq 1 \\ \frac{1}{2}(x-1), & 1 \leq x \leq 2 \\ -\frac{1}{2}(x-3), & 2 \leq x \leq 3 \\ 0, & x \geq 3 \end{cases}$$

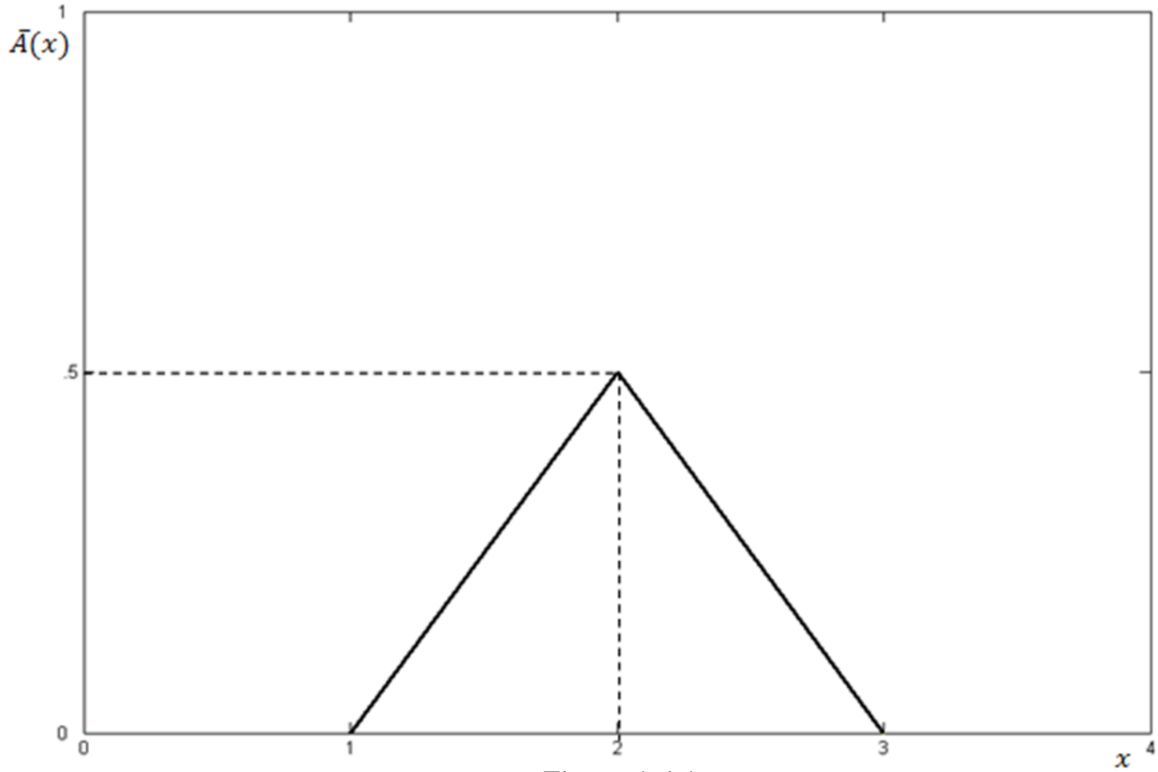

Figure 1.4.1

Fuzzy Set

$\bar{A}$ is not a fuzzy number since it is not normalized.

Now we give the definition of the operations of scalar multiplication, addition, subtraction, multiplication, and division on fuzzy numbers by applying these operations on the α −cut intervals.

Definition 1.4.2 ([7] page 13): For any two fuzzy numbers $\bar{A}$ and $\bar{B}$, let $\bar{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$ and $\bar{B}[\alpha] = [b_1(\alpha), b_2(\alpha)]$, $0 \leq \alpha \leq 1$, be the α −cuts of $\bar{A}$ and $\bar{B}$ respectively. Then,

1. If $\beta \in \mathbb{R}$ then $\beta\bar{A}[\alpha] = [\beta a_1(\alpha), \beta a_2(\alpha)]$ if $\beta > 0$ and $\beta\bar{A}[\alpha] = [\beta a_2(\alpha), \beta a_1(\alpha)]$ if $\beta < 0$.
2. If $\bar{C} = \bar{A} + \bar{B}$,then

   $\bar{C}[\alpha] = \bar{A}[\alpha] + \bar{B}[\alpha] = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)]$
3. If $\bar{C} = \bar{A} - \bar{B}$,then

   $\bar{C}[\alpha] = \bar{A}[\alpha] - \bar{B}[\alpha] = [a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)]$
4. If $\bar{C} = \bar{A} \cdot \bar{B}$, then

   $\bar{C}[\alpha] = \bar{A}[\alpha] \cdot \bar{B}[\alpha] = [m(\alpha), M(\alpha)]$ where

   $m(\alpha) = min\{a_1(\alpha)b_1(\alpha),\ a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}$ and

   $M(\alpha) = max\{a_1(\alpha)b_1(\alpha),\ a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}$.
5. If $\bar{C} = 1/\bar{A}$, then $\bar{C}[\alpha] = [1/a_2(\alpha),\ 1/\,a_1(\alpha)]$, provided that $0 \notin \bar{A}[\alpha], \forall\alpha \in [0,1]$.
6. If $\bar{C} = \bar{A}/\bar{B}$, then $\bar{C} = \bar{A} \cdot (1/\bar{B})$, provided that $0 \notin \bar{B}[\alpha], \forall\alpha \in [0,1]$.

Definition 1.4.3: Let $\bar{A}$ be a fuzzy number with $\bar{A}[\alpha]\ = [a_1(\alpha), a_2(\alpha)]$ then we say that $\bar{A} \geq\ c$ if $a_1(0) \geq\ c$ and $\bar{A} \leq\ d$ if $a_2(0) \leq d$ where $c, d \in \mathbb{R}$.

1.5 Triangular and Trapezoidal Fuzzy Numbers

In this section we introduce two fuzzy numbers namely, *triangular* and *trapezoidal* fuzzy numbers and give their properties.

Definition 1.5.1 ([18] page 137): A *triangular fuzzy number* $\bar{A}$ is a fuzzy number defined by three real numbers a_1, a_2, and a_3, with $a_1 < a_2 < a_3$ and denoted by $\bar{A} = (a_1/a_2/a_3)$ or $\bar{A} = (a_1, a_2, a_3)$, where its membership function is given by

$$\bar{A}(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 0, & x > a_3 \end{cases}$$

If the sides of the triangular fuzzy number are curves other than straight lines, then we call it *triangular shaped fuzzy number* and is denoted by $\bar{A} \approx (a_1/a_2/a_3)$ ([7] page 9).

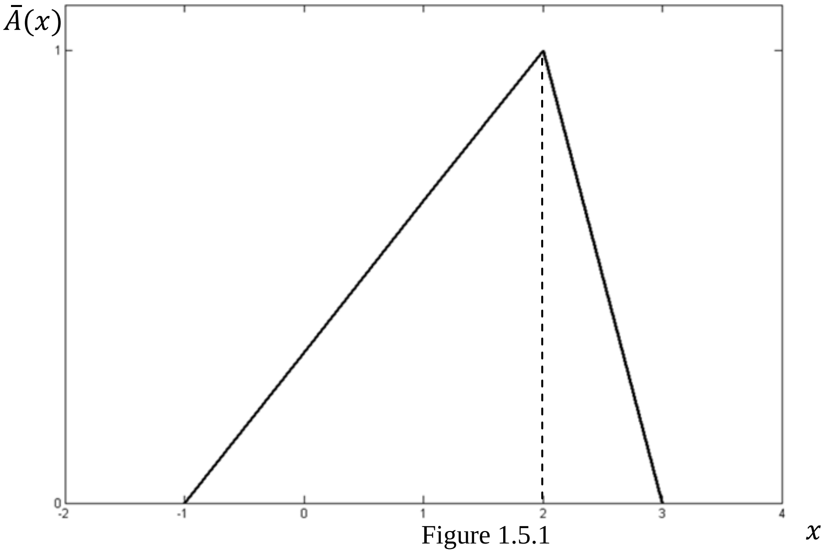

Figure 1.5.1

Triangular Fuzzy Number $(-1/2/3)$

Example 1.5.2 ([18] pages 137 and 138): Let $\bar{A} = (a_1/a_2/a_3)$ be a triangular fuzzy number. If $\bar{A}[\alpha] = [a_1(\alpha), a_3(\alpha)]$, then $a_1(\alpha)$ and $a_3(\alpha)$ can be obtained by solving

$$\frac{a_1(\alpha) - a_1}{a_2 - a_1} = \alpha$$

and,

$$\frac{a_3 - a_3(\alpha)}{a_3 - a_2} = \alpha$$

Therefore, $\bar{A}[\alpha] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3]$.

One can see that:

1. The core of a triangular fuzzy number $\bar{A} = (a_1/a_2/a_3)$ is a_2 and the support is $\bar{A}[0] = [a_1, a_3]$ ([7] page 10).
2. If $\bar{A} \approx (a_1/a_2/a_3)$ and $\bar{A}[\alpha] = [a_1(\alpha), a_3(\alpha)]$, then $a_1(\alpha)$ or $a_3(\alpha)$ are not linear functions in α.

Example 1.5.3 ([18] page 139): If $\bar{A} = (a_1/a_2/a_3)$ and $\bar{B} = (b_1/b_2/b_3)$, be two triangular fuzzy numbers then, the α −cuts of $\bar{A}$ and $\bar{B}$ are:

$$\bar{A}[\alpha] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3],$$

$$\bar{B}[\alpha] = [(b_2 - b_1)\alpha + b_1, -(b_3 - b_2)\alpha + b_3].$$

So by Definition 1.4.2:

1. $\bar{A}[\alpha] + \bar{B}[\alpha] = [(a_2 - a_1 + b_2 - b_1)\alpha + a_1 + b_1, -(a_3 - a_2 + b_3 - b_2)\alpha + a_3 + b_3]$

2. $\bar{A}[\alpha] - \bar{B}[\alpha] = [(a_2 - a_1 + b_3 - b_2)\alpha + a_1 - b_3, -(a_3 - a_2 + b_2 - b_1)\alpha + a_3 - b_1]$

3. $\bar{A}[\alpha] \cdot \bar{B}[\alpha] = [m(\alpha), M(\alpha)]$ where

$$m(\alpha) = min\{a_1(\alpha)b_1(\alpha),\ a_1(\alpha)b_3(\alpha), a_3(\alpha)b_1(\alpha), a_3(\alpha)b_3(\alpha)\}$$

$$M(\alpha) = max\{a_1(\alpha)b_1(\alpha),\ a_1(\alpha)b_3(\alpha), a_3(\alpha)b_1(\alpha), a_3(\alpha)b_3(\alpha)\}$$

and,

$$a_1(\alpha) = (a_2 - a_1)\alpha + a_1, a_3(\alpha) = -(a_3 - a_2)\alpha + a_3,$$

$$b_1(\alpha) = (b_2 - b_1)\alpha + b_1, b_3(\alpha) = -(b_3 - b_2)\alpha + b_3.$$

We can see from Example 1.5.3 that addition and subtraction of two triangular fuzzy numbers is also a triangular fuzzy number. That is,

$$\bar{A} + \bar{B} = (a_1 + b_1/a_2 + b_2/a_3 + b_3)$$

and

$$\bar{A} - \bar{B} = (a_1 - b_3/a_2 - b_2/a_3 - b_1).$$

While $m(\alpha)$ and $M(\alpha)$ as functions of α are not linear in α so multiplication of two triangular fuzzy numbers is a triangular shaped fuzzy number. That is,

$\bar{A} \cdot \bar{B} \approx (m(0)/a_2b_2/M(0))$ where $m(0) = min\{a_1b_1,\ a_1b_3, a_3b_1, a_3b_3\}$,

$M(0) = max\{a_1b_1,\ a_1b_3, a_3b_1, a_3b_3\}$ and $m(1) = M(1) = a_2b_2$.

Definition 1.5.4 ([18] page 145): A *trapezoidal fuzzy number* $\bar{A}$ is a fuzzy number defined by four real numbers a_1, a_2, a_3, and a_4, with $a_1 < a_2 < a_3 < a_4$ and denoted by $\bar{A} = (a_1/a_2, a_3/a_4)$ or $\bar{A} = (a_1, a_2, a_3, a_4)$, where its membership function is given by

$$\bar{A}(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \le x \le a_4 \\ 0, & x > a_4 \end{cases}$$

If the sides of the trapezoidal fuzzy number are curves other than straight lines we call it *trapezoidal shaped fuzzy number* and is denoted by $\bar{A} \approx (a_1, a_2, a_3, a_4)$ ([7] page 9).

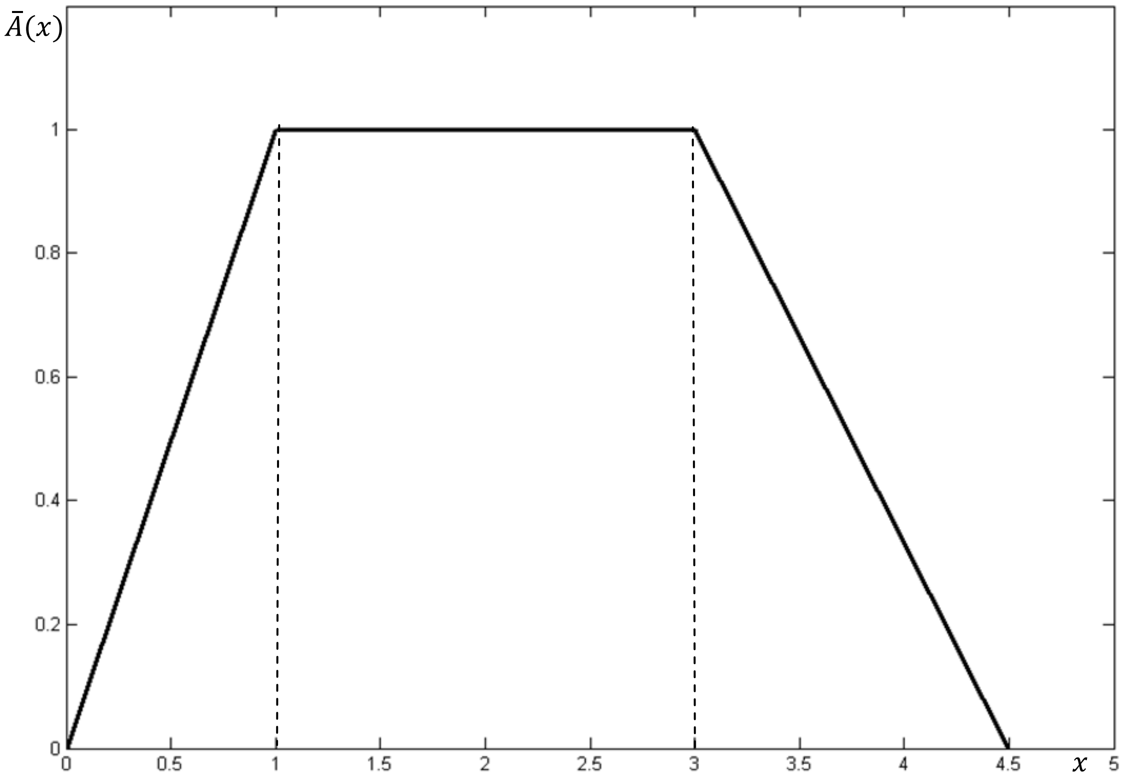

Figure 1.5.2

Trapezoidal Fuzzy Number (0,1,3,4.5)

Example 1.5.5 ([18] page 145): Let $\bar{A} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number. If $\bar{A}[\alpha] = [a_1(\alpha), a_4(\alpha)]$, then $a_1(\alpha)$ and $a_4(\alpha)$ can be obtained by solving

$$\frac{a_1(\alpha) - a_1}{a_2 - a_1} = \alpha$$

and,

$$\frac{a_4 - a_4(\alpha)}{a_4 - a_3} = \alpha$$

Therefore, $\bar{A}[\alpha] = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4]$.

One can see that:

1. The core of a trapezoidal fuzzy number $\bar{A} = (a_1, a_2, a_3, a_4)$ is $[a_2, a_3]$ and the support is $\bar{A}[0] = [a_1, a_4]$ ([7] page 10).
2. If $\bar{A} \approx (a_1, a_2, a_3, a_4)$ and $\bar{A}[\alpha] = [a_1(\alpha), a_4(\alpha)]$, then $a_1(\alpha)$ or $a_3(\alpha)$ are not linear functions in α.

Example 1.5.6 ([18] page 146): If $\bar{A} = (a_1, a_2, a_3, a_4)$ and $\bar{B} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers and the α −cuts of $\bar{A}$ and $\bar{B}$ are:

$$\bar{A}[\alpha] = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4],$$

$$\bar{B}[\alpha] = [(b_2 - b_1)\alpha + b_1, -(b_4 - b_3)\alpha + b_4].$$

Then,

1. $\bar{A}[\alpha] + \bar{B}[\alpha] = [(a_2 - a_1 + b_2 - b_1)\alpha + a_1 + b_1, -(a_4 - a_3 + b_4 - b_3)\alpha + a_4 + b4$

2. $\bar{A}[\alpha] - \bar{B}[\alpha] = [(a_2 - a_1 + b_4 - b_3)\alpha + a_1 - b_4, -(a_4 - a_3 + b_2 - b_1)\alpha + a_4 - b1$

3. $\bar{A}[\alpha] \cdot \bar{B}[\alpha] = [m(\alpha), M(\alpha)]$ where

$$m(\alpha) = min\{a_1(\alpha)b_1(\alpha),\ a_1(\alpha)b_4(\alpha), a_4(\alpha)b_1(\alpha), a_4(\alpha)b_4(\alpha)\}$$

$$M(\alpha) = max\{a_1(\alpha)b_1(\alpha),\ a_1(\alpha)b_4(\alpha), a_4(\alpha)b_1(\alpha), a_4(\alpha)b_4(\alpha)\}$$

and,

$$a_1(\alpha) = (a_2 - a_1)\alpha + a_1, a_4(\alpha) = -(a_4 - a_3)\alpha + a_4,$$

$$b_1(\alpha) = (b_2 - b_1)\alpha + b_1, b_4(\alpha) = -(b_4 - b_3)\alpha + b_4.$$

It is clear from Example 1.5.6 that addition and subtraction of two trapezoidal fuzzy numbers is also a trapezoidal fuzzy number. That is,

$$\bar{A} + \bar{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$$

and

$$\bar{A} - \bar{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1).$$

While $m(\alpha)$ and $M(\alpha)$ as functions of α are not linear so, multiplication of two trapezoidal fuzzy numbers is a trapezoidal shaped fuzzy number. That is,

$\bar{A} \cdot \bar{B} \approx (m(0), m(1), M(0), M(0))$ where

$m(0) = min\{a_1b_1,\ a_1b_4, a_4b_1, a_4b_4\}$, $M(0) = max\{a_1b_1,\ a_1b_4, a_4b_1, a_4b_4\}$

$m(1) = min\{a_2b_2,\ a_2b_3, a_3b_2, a_3b_3\}$, $M(1) = max\{a_2b_2,\ a_2b_3, a_3b_2, a_3b_3\}$.

1.6 Special Fuzzy Numbers

In this section we define a new fuzzy number and introduce its properties, then we give illustrations of other fuzzy numbers that can be easily defined.

Definition 1.6.1: Consider a fuzzy number that is determined by five real numbers a_1, a_2, a_3, a_4 and c such that $a_1 < a_2 < a_3 < a_4$ and $0 < c < 1$, denoted by $\overline{N}_c = (a_1/a_2/a_3/a_4)_c$ or $(a_1, a_2, a_3, a_4; c)$ whose membership function is given by

$$\overline{N}_c(x) = \begin{cases} 0 & x \leq a_1 \\ \dfrac{c}{a_2 - a_1}(x - a_1) & a_1 < x < a_2 \\ 1 - \dfrac{1 - c}{(a_2 - a_3)^2}(2x - a_2 - a_3)^2 & a_2 \leq x \leq a_3 \\ \dfrac{-c}{a_4 - a_3}(x - a_4) & a_3 < x < a_4 \\ 0 & x \geq a_4 \end{cases}$$

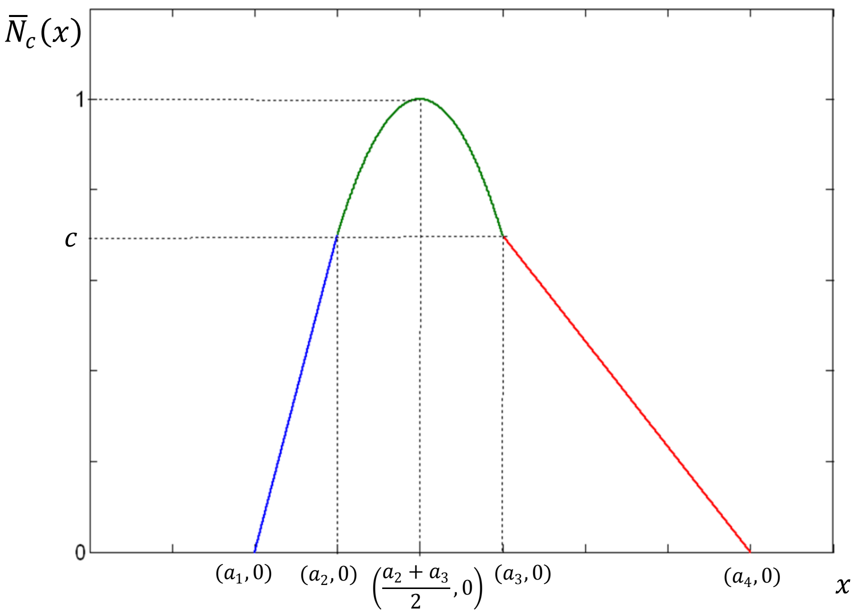

Figure 1.6.1

Remarks 1.6.2:

1. $\overline{N}_c(x)$ is linear on the intervals $[a_1, a_2]$ and $[a_3, a_4]$.
2. $\overline{N}_c(x)$ is a parabola on the interval $[a_2, a_3]$ whose vertex is $\left(\frac{a_2+a_3}{2}, 1\right)$ and focus is $\left(\frac{a_2+a_3}{2}, 1 - \frac{(a_3-a_2)^2}{16(1-c)}\right)$.
3. If the graph of $\overline{N}_c(x)$ in any of the intervals $[a_1, a_2], [a_2, a_3]$, or $[a_3, a_4]$ is not as prescribed above then $\overline{N}_c$ is denoted by $\overline{N}_c \approx (a_1/a_2/a_3/a_4)_c$ or $\overline{N}_c \approx (a_1, a_2, a_3, a_4; c)$.

Remark 1.6.3: Let $\overline{N}_c = (a_1/a_2/a_3/a_4)_c$. If $0 \le \alpha \le c$, then $\alpha = \frac{c}{a_2-a_1}(x - a_1)$, and $\alpha = \frac{-c}{a_4-a_3}(x - a_4)$ and write x in terms of α to get

$$\overline{N}_c[\alpha] = \left[\frac{(a_2-a_1)}{c}\alpha + a_1,\ \frac{-(a_4-a_3)}{c}\alpha + a_4\right] \text{ for } 0 \le \alpha \le c$$

and if $c \le \alpha \le 1$, then $\alpha = 1 - \frac{1-c}{(a_2-a_3)^2}(2x - a_2 - a_3)^2$ and write x in terms of α to get

$$\overline{N}_c[\alpha] = \left[\frac{-(a_3-a_2)}{2}\sqrt{\frac{1-\alpha}{1-c}} + \frac{a_2+a_3}{2},\ \frac{(a_3-a_2)}{2}\sqrt{\frac{1-\alpha}{1-c}} + \frac{a_2+a_3}{2}\right] \text{ for } c \le \alpha \le 1.$$

Remark 1.6.4: If $\overline{N}_c = (a_1/a_2/a_3/a_4)_c$, and $\overline{M}_c = (b_1/b_2/b_3/b_4)_c$ then

1. $\overline{N}_c + \overline{M}_c = (a_1 + b_1/a_2 + b_2/a_3 + b_3/a_4 + b_4)_c$.

2. $\overline{N}_c - \overline{M}_c = (a_1 - b_4/a_2 - b_3/a_3 - b_2/a_4 - b_1)_c$.

Here we give an example to illustrate different operations on the fuzzy number that are given in Definition 1.1.6.

Example 1.6.5: Let $\bar{A}_{0.2} = (1/2/3/4)_{0.2}$ find $\bar{B} = \bar{A}_{0.2} \, . \bar{A}_{0.2}$ and $\bar{C} = \bar{A}_{0.2}/\bar{A}_{0.2}$.

$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$ and $c = 0.2$. So the α −cuts of $\bar{A}_{0.2}$ will be

$$\bar{A}_{0.2}[\alpha] = [5\alpha + 1, -5\alpha + 4] \text{ for } 0 \leq \alpha \leq 0.2$$

and,

$$\bar{A}_{0.2}[\alpha] = \left[\frac{-1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2},\ \frac{1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}\right] \text{ for } 0.2 \leq \alpha \leq 1 .$$

Therefore,

$$\bar{B}[\alpha] = [(5\alpha + 1)^2, (-5\alpha + 4)^2] \text{ for } 0 \leq \alpha \leq 0.2,$$

and,

$\bar{B}[\alpha] = \left[\left(\frac{-1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}\right)^2, \left(\frac{1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}\right)^2\right]$ for $0.2 \leq \alpha \leq 1$. Hence, $\bar{B}[0] = [1, 16], \bar{B}[0.2] = [4,9]$ and $\bar{B}[1] = 6.25$ and so $\bar{B} \approx (1/4/9/16)_{0.2}$.

Also we have,

$$\bar{C}[\alpha] = \left[\frac{5\alpha+1}{-5\alpha+4}, \frac{-5\alpha+4}{5\alpha+1}\right] \text{ for } 0 \leq \alpha \leq 0.2$$

and,

$$\bar{C}[\alpha] = \left[\frac{\frac{-1}{2}\sqrt{1.25(1-\alpha)}+\frac{5}{2}}{\frac{1}{2}\sqrt{1.25(1-\alpha)}+\frac{5}{2}}, \frac{\frac{1}{2}\sqrt{1.25(1-\alpha)}+\frac{5}{2}}{\frac{-1}{2}\sqrt{1.25(1-\alpha)}+\frac{5}{2}}\right] \text{ for } 0.2 \leq \alpha \leq 1 .$$

Hence, $\bar{C}[0] = \left[\frac{1}{4}, 4\right], \bar{C}[0.2] = \left[\frac{2}{3}, \frac{3}{2}\right]$ and $\bar{C}[1] = 1$and so $\bar{C} \approx \left(\frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4\right)_{0.2}$.

Remark 1.6.6: Let $\overline{N}_{0.2} = (1/2/3/4)_{0.2}$ and $\overline{N}_{0.4} = (0/1/2/3)_{0.4}$, then $\overline{N}_{0.2} + \overline{N}_{0.4}$ and $\overline{N}_{0.2} - \overline{N}_{0.4}$ can be computed as follows:

For $0 \leq \alpha \leq 0.2$, $\overline{N}_{0.2}[\alpha] = [5\alpha + 1, -5\alpha + 4]$.

For $0.2 \leq \alpha \leq 1$, $\overline{N}_{0.2}[\alpha] = \left[\frac{-1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2},\ \frac{1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}\right]$.

For $0 \leq \alpha \leq 0.4$, $\overline{N}_{0.4}[\alpha] = \left[\frac{5}{2}\alpha, -\frac{5}{2}\alpha + 3\right]$.

For $0.4 \leq \alpha \leq 1$, $\overline{N}_{0.4}[\alpha] = \left[\frac{-1}{2}\sqrt{\frac{5}{3}(1-\alpha)} + \frac{3}{2},\ \frac{1}{2}\sqrt{\frac{5}{3}(1-\alpha)} + \frac{3}{2}\right]$.

$\overline{N}_{0.2}[\alpha] + \overline{N}_{0.4}[\alpha] = [7.5\alpha + 1, -7.5\alpha + 7]$, for $0 \leq \alpha \leq 0.2$.

$\overline{N}_{0.2}[\alpha] + \overline{N}_{0.4}[\alpha] = \left[\frac{-1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}\alpha + \frac{5}{2},\ \frac{1}{2}\sqrt{1.25(1-\alpha)} - \frac{5}{2}\alpha + \frac{11}{2}\right]$,

for $0.2 \leq \alpha \leq 0.4$.

$\overline{N}_{0.2}[\alpha] + \overline{N}_{0.4}[\alpha] =$

$$= \left[\frac{-1}{2}\left(\sqrt{1.25(1-\alpha)} + \sqrt{\frac{5}{3}(1-\alpha)}\right) + 4,\ \frac{1}{2}\left(\sqrt{1.25(1-\alpha)} + \sqrt{\frac{5}{3}(1-\alpha)}\right) + 4\right],$$

for $0.4 \leq \alpha \leq 1$.

Therefore, $\overline{N}_{0.2} + \overline{N}_{0.4}$ is not of the same type of the fuzzy number that is given in Definition 1.1.6.

On the other hand,

$\overline{N}_{0.2}[\alpha] - \overline{N}_{0.4}[\alpha] = [7.5\alpha - 2, -7.5\alpha + 4]$, for $0 \leq \alpha \leq 0.2$.

$\overline{N}_{0.2}[\alpha] - \overline{N}_{0.4}[\alpha] = \left[\frac{-1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}\alpha - \frac{1}{2},\ \frac{1}{2}\sqrt{1.25(1-\alpha)} - \frac{5}{2}\alpha + \frac{5}{2}\right]$,

for $0.2 \leq \alpha \leq 0.4$.

$\overline{N}_{0.2}[\alpha] - \overline{N}_{0.4}[\alpha] =$

$$= \left[\frac{-1}{2}\left(\sqrt{1.25(1-\alpha)} + \sqrt{\frac{5}{3}(1-\alpha)}\right) + 1,\ \frac{1}{2}\left(\sqrt{1.25(1-\alpha)} + \sqrt{\frac{5}{3}(1-\alpha)}\right) + 1\right],$$

for $0.4 \leq \alpha \leq 1$.

Therefore, $\overline{N}_{0.2} - \overline{N}_{0.4}$ is not of the same type of the fuzzy number that given in Definition 1.1.6.

The fuzzy number defined in this section gives a conception of other types of fuzzy numbers that could be defined in a similar way. Below we give some of them.

$$\bar{N}_1(x) = \begin{cases} 0, & x \le a_1 \\ \dfrac{c}{a_2 - a_1}(x - a_1), & a_1 \le x \le a_2 \\ \dfrac{1-c}{a_3 - a_2}(x - a_2) + c, & a_2 \le x \le a_3 \\ 1, & a_3 \le x \le a_4 \\ \dfrac{-(1-c)}{a_5 - a_4}(x - a_5) + c, & a_4 \le x \le a_5 \\ \dfrac{-c}{a_6 - a_5}(x - a_6), & a_5 \le x \le a_6 \\ 0, & x \ge a_6 \end{cases}$$

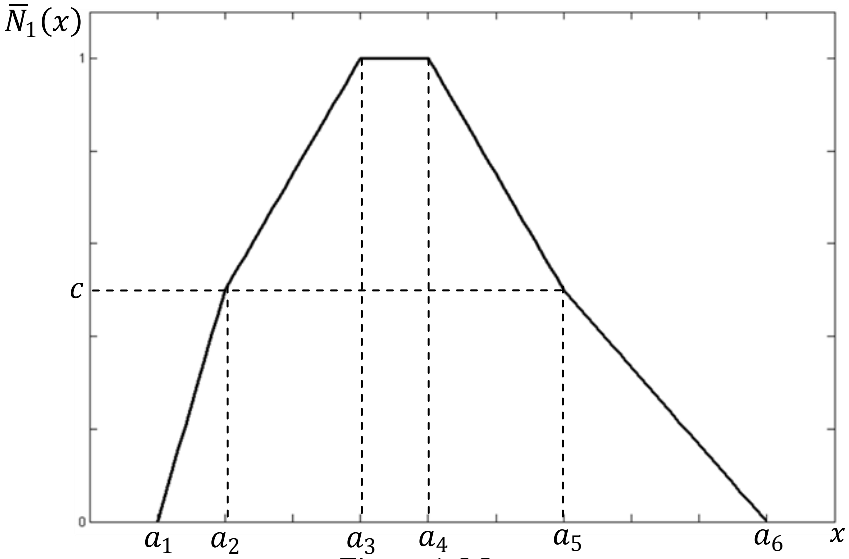

Figure 1.6.2

$$\bar{N}_2(x) = \begin{cases} 0, & x \le a_1 \\ \frac{c_1}{a_2 - a_1}(x - a_1), & a_1 \le x \le a_2 \\ \frac{c_2 - c_1}{a_3 - a_2}(x - a_2) + c_1, & a_2 \le x \le a_3 \\ \frac{1 - c_2}{a_4 - a_3}(x - a_3) + c_2, & a_3 \le x \le a_4 \\ \frac{-(1-c_2)}{a_5 - a_4}(x - a_5) + c_2, & a_4 \le x \le a_5 \\ \frac{-(c_2 - c_1)}{a_6 - a_5}(x - a_6) + c_1, & a_5 \le x \le a_6 \\ \frac{-c_1}{a_7 - a_6}(x - a_7), & a_6 \le x \le a_7 \\ 0, & x \ge a_7 \end{cases}$$

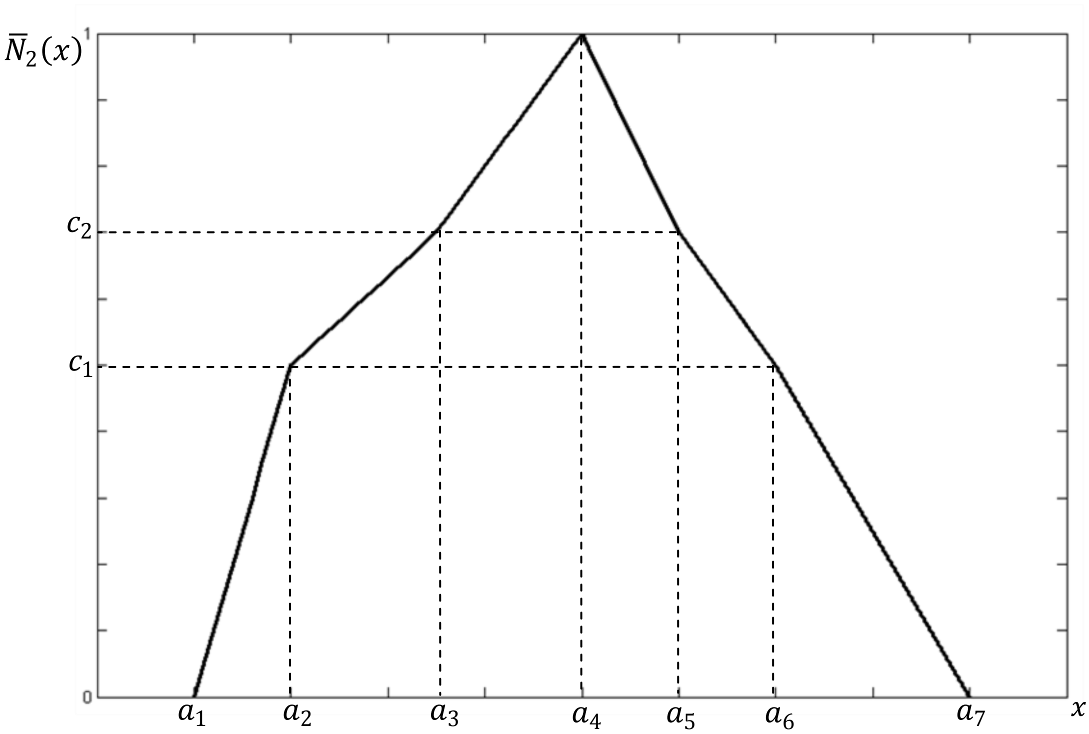

Figure 1.6.3

$$\overline{N}_3(x) = \begin{cases} 0, & x \le a_1 \\ \dfrac{c_1}{a_2 - a_1}(x - a_1), & a_1 \le x \le a_2 \\ \dfrac{c_2 - c_1}{a_3 - a_2}(x - a_2) + c_1, & a_2 \le x \le a_3 \\ \dfrac{c_3 - c_2}{a_4 - a_3}(x - a_3) + c_2, & a_3 \le x \le a_4 \\ \dfrac{1 - c_3}{a_5 - a_4}(x - a_4) + c_3, & a_4 \le x \le a_5 \\ \dfrac{-(1 - c_3)}{a_6 - a_5}(x - a_6) + c_3, & a_5 \le x \le a_6 \\ \dfrac{-(c_3 - c_2)}{a_7 - a_6}(x - a_7) + c_2, & a_6 \le x \le a_7 \\ \dfrac{-(c_2 - c_1)}{a_8 - a_7}(x - a_8) + c_1, & a_7 \le x \le a_8 \\ \dfrac{-c_1}{a_9 - a_8}(x - a_9), & a_8 \le x \le a_9 \\ 0, & x \ge a_9 \end{cases}$$

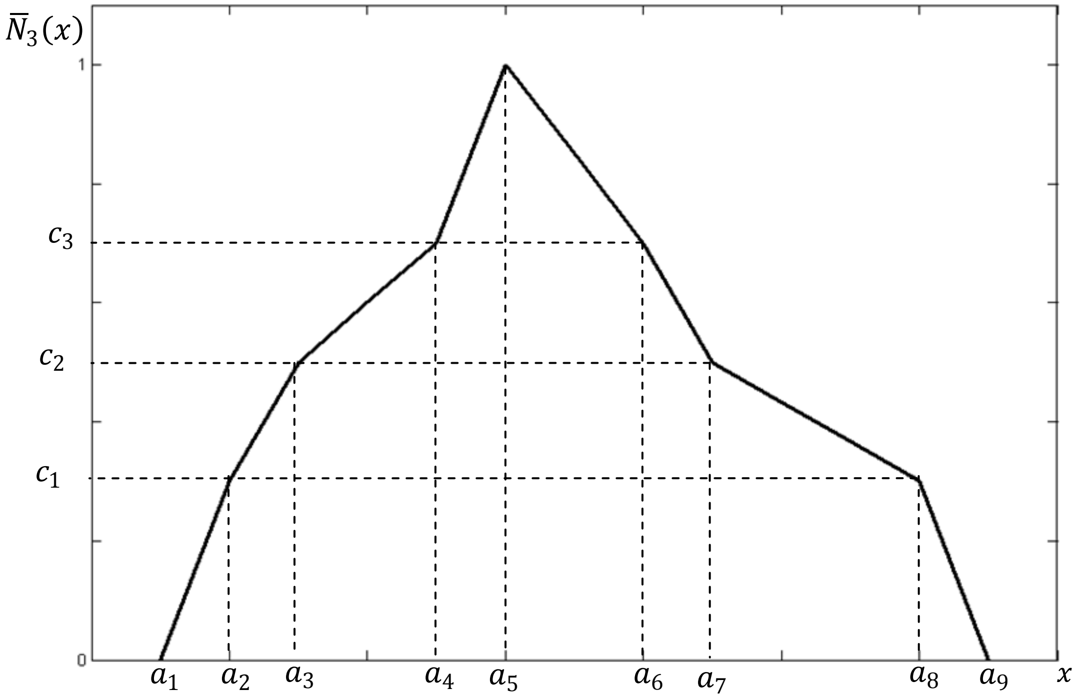

Figure 1.6.4

$$\overline{N}_4(x) = \begin{cases} 0, & x \le a_1 \\ \dfrac{c_1}{a_2 - a_1}(x - a_1), & a_1 \le x \le a_2 \\ \dfrac{c_2 - c_1}{a_3 - a_2}(x - a_2) + c_1, & a_2 \le x \le a_3 \\ 1 - 4\dfrac{(1 - c_2)}{(a_4 - a_3)^2}\left(x - \frac{a_3 + a_4}{2}\right)^2, & a_3 \le x \le a_4 \\ \dfrac{-(c_2 - c_1)}{a_5 - a_4}(x - a_5) + c_1, & a_4 \le x \le a_5 \\ \dfrac{-c_1}{a_6 - a_5}(x - a_6), & a_5 \le x \le a_6 \\ 0, & x \ge a_6 \end{cases}$$

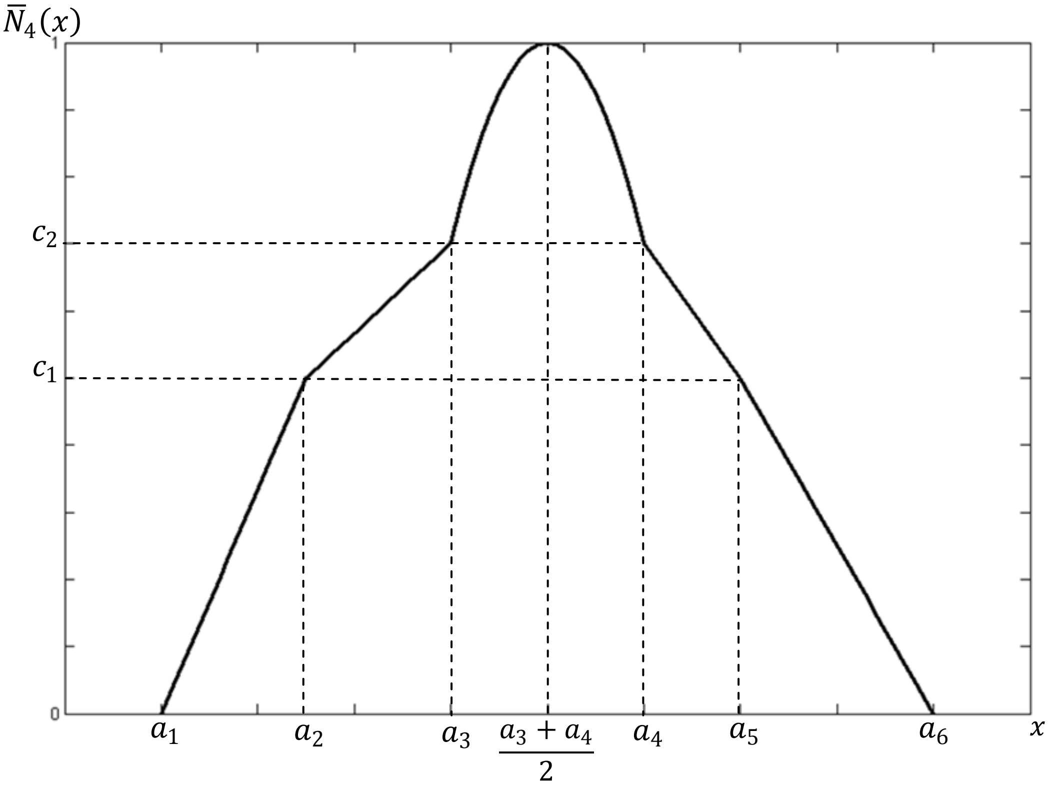

Figure 1.6.5

$$\overline{N}_5(x)=\begin{cases} 0, & x\le a_1\\ \frac{c_1}{a_2-a_1}(x-a_1), & a_1\le x\le a_2\\ \frac{c_2-c_1}{a_3-a_2}(x-a_2)+c_1, & a_2\le x\le a_3\\ \frac{c_3-c_2}{a_4-a_3}(x-a_3)+c_2, & a_3\le x\le a_4\\ \frac{c_4-c_3}{a_5-a_4}(x-a_4)+c_3, & a_4\le x\le a_5\\ 1-4\frac{(1-c_4)}{(a_6-a_5)^2}\left(x-\frac{a_5+a_6}{2}\right)^2, & a_5\le x\le a_6\\ \frac{-(c_4-c_3)}{a_7-a_6}(x-a_7)+c_3, & a_6\le x\le a_7\\ \frac{-(c_3-c_2)}{a_8-a_7}(x-a_8)+c_2, & a_7\le x\le a_8\\ \frac{-(c_2-c_1)}{a_9-a_8}(x-a_9)+c_1, & a_8\le x\le a_9\\ \frac{-c_1}{a_{10}-a_9}(x-a_{10}), & a_9\le x\le a_{10}\\ 0, & x\ge a_{10} \end{cases}$$

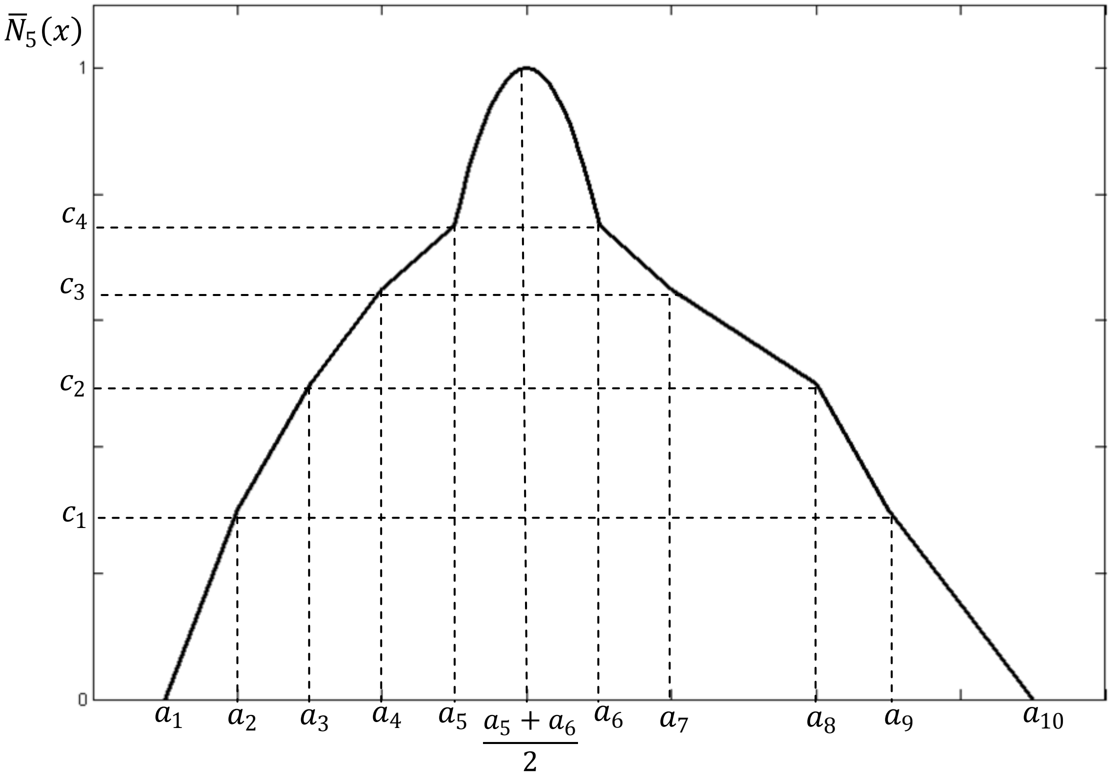
$\overline{N}_5(x)$
1
c_4
c_3
c_2
c_1
0
a_1
a_2
a_3
a_4
a_5
$\frac{a_5+a_6}{2}$
a_6
a_7
a_8
a_9
a_{10}

Figure 1.6.6

Chapter 2

Finite Markov Chains

This chapter consists of three sections. In section one; we present basic definitions of Markov chains. Classifications of states and Markov chains are presented in section two. We end up with some examples illustrating the several types of Markov chains in section three. [5], [6], [11], [14], [23], [28].

2.1 Markov Chains

Definition 2.1.1 ([5] page 111): Let S be a countable set. Suppose that to each i and j in S there is assigned a nonnegative number p_{ij} and that these numbers satisfy the constraint

$$\sum_{j \in S} p_{ij} = 1, \ \ \forall i, j \in S.$$

Let $X_0, X_1, X_2, \dots$ be a sequence of random variables whose ranges contained in S. This sequence is a *Markov chain* if

$$P[X_{n+1} = j | X_0 = i_0, \dots, X_n = i_n] = P[X_{n+1} = j | X_n = i_n] = p_{i_n j}$$

for every n and every sequence $i_0, \dots, i_n$ in S for which $P[X_0 = i_0, \dots, X_n = i_n] > 0$.

The set S is called the *state space* or *phase space* of the Markov process, and its elements are the *states* of the process. The probabilities $p_{ij} = P[X_{n+1} = j|X_n = i]$ are called the *transition probabilities*. The elements p_{ij} form the *matrix of transition probabilities* or the *transition matrix* $P = [p_{ij}]$, so if S is a finite state space with cardinality $m > 1$, then the transition matrix P is an $m \times m$ matrix. Here the transition probabilities $p_{ij} = P[X_{n+1} = j|X_n = i]$ are assumed to be independent of n, in this case the chain is said to have *stationary transition probabilities* ([5] pages 111and112, [11] page 374).

Definition 2.1.2 ([5] page 111): The *initial distribution of the chain* $a_i^{(0)} = P[X_0 = i]$, where $a_i^{(0)} \geq 0$ and $\sum_i a_i^{(0)} = 1$.

Definition 2.1.3 ([5] page 111): A square matrix *P* with nonnegative elements and unit row sums is called a *regular stochastic matrix*.

Definition 2.1.4 ([5] page 115): Let $P = [p_{ij}]$ be the transition matrix of a Markov chain $\{X_n, n \geq 0\}$, the n^{th} power of P, is $P^n = \left[p_{ij}^{(n)}\right]$ where $p_{ij}^{(n)}$represents the probability of a transition from state i to state j in n steps, $p_{ij}^{(n)}$is called the n *−step transition probability* for the Markov chain.

Since $P = [p_{ij}]$ is the transition matrix of a Markov chain then by Definition 2.1.1 we have $\sum_{j \in S} p_{ij} = 1, i \in S$ and this implies that $\sum_{j \in S} p_{ij}^{(n)} = 1, i \in S$.

Proposition 2.1.5 ([23] page 73): For all $n > 1$, and i, j in the state space S, $p_{ij}^{(n)} = P[X_n = j|X_0 = i]$.

Proposition 2.1.6 ([23] page 75): The *unconditional probabilities* $P[X_n = i]$ are computed from $a_j^{(n)} = P[X_n = j] = \sum_i a_j^{(0)} p_{ij}^{(n)}$. In the matrix form, $a^n = a^0 P^n$.

2.2 Classifications of the States

If S is a finite state space then:

1. ([28] page 646): A state $j \in S$ is *transient* if it can reach another state but cannot itself be reached back from another state. Mathematically, this happens if $\lim_{n\to\infty} p_{ij}^{(n)} = 0$, for all i.

2. ([14] page 811, [11] page 389): A state $j \in S$ is *persistent (or recurrent)* if, upon entering this state, the process definitely will return to this state again. This can happen if, and only if the state is not transient.

3. ([5] page 125): A state $j \in S$ is *periodic* with period t if $p_{jj}^{(n)} > 0$ implies that t divides n and t the largest integer with this property. In other words, the period of j is the greatest common divisor of the set of integers $\left\{n: n \geq 1, p_{jj}^{(n)} > 0\right\}$. If $t = 1$, then the state is *aperiodic (or nonperiodic).*

4. ([5] page 119, [6]): A Markov chain is called *irreducible (or regular)* if $\exists n \in \mathbb{N}$ such that $p_{ij}^{(n)} > 0, \forall i, j \in S$, otherwise it is called *reducible (or irregular)*. That is, a Markov chain is irreducible if and only if every state can be reached from every other state in a finite number of steps.

5. ([14] page 812): An aperiodic persistent state $j \in S$, is called *ergodic*. Therefore, a Markov chain is ergodic if all its states are ergodic.

6. ([6]): A state $j \in S$ is called *absorbing* if $p_{jj} = 1$. The Markov chain is called an *absorbing Markov chain* if it has at least one absorbing state and from every non-absorbing state it is possible to reach some absorbing state in a finite number of steps.

Definition 2.2.1 ([5] pages 124and125): A set of probabilities $(\pi_j), j \in S$ satisfying $\sum_{i \in S} \pi_i p_{ij} = \pi_j$, is called a *stationary distribution*.

Remark 2.2.2 ([5] page 125): If $(\pi_j), j \in S$ is a stationary distribution then $\sum_{i \in S} \pi_i p_{ij}^{(n)} = \pi_j, j \in S, = 0,1,2, \ldots$.

Theorem 2.2.3 ([5] page 125): Suppose of an irreducible aperiodic chain that there exists a stationary distribution, that is a solution of $\sum_{i \in S} \pi_i p_{ij} = \pi_j$, $j \in S$ satisfying

$\pi_i \geq 0$ and $\sum_i \pi_i = 1$. Then the chain is persistent, $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j \;\; \forall i,j \in S, \pi_j > 0$, and the stationary distribution is unique.

If S is finite then Theorem 2.2.3 implies that in order to find the limit of P^n we first find the unique left eigenvector π of P corresponding to eigenvalue 1 (i.e. solving the system $\pi P = \pi$) where π is a row vector whose components are π_j with $\pi_j > 0$, and $\sum_j \pi_j = 1$. Then, P^n converges to the matrix Π whose rows are identical and each of which is π [6].

Theorem 2.2.4 ([5] page 131): If the state space S is finite and the chain is irreducible and aperiodic, then there is a stationary distribution (π_i), and $\left|p_{ij}^{(n)} - \pi_j\right| \leq A\rho^n$ where $A \geq 0,\; 0 \leq \rho < 1$.

2.3 Examples of Finite Markov Chains

Example 2.3.1: Consider a Markov chain whose transition matrix is $= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $p_{ij} > 0, \forall i,j = 1,2$. Such a chain is ergodic, to find the unique stationary distribution we let $\pi = [\pi_1 \; \pi_2]$ with $\pi_1, \pi_2 > 0$ and $\pi_1 + \pi_2 = 1$, then we solve $\pi P = \pi$ from which we have $\pi_1 = \frac{p_{21}}{p_{21}+p_{12}}, \pi_2 = \frac{p_{12}}{p_{21}+p_{12}}$.

Example 2.3.2: Let A and B be two transition matrices of two Markov chains, where:

$$A = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

Then $A^2 = A, B^2 = B$, and in general $A^n = A, B^n = B$, for any n. So, both A and B are transition matrices of non-ergodic chains ([23] page 81).

Example 2.3.3: Let $A = \begin{bmatrix} 0 & 1 \\ p & q \end{bmatrix}$ be a transition matrix of a Markov chain, then $A^2 = \begin{bmatrix} p & q \\ pq & p + q^2 \end{bmatrix}$, so A is irreducible (regular) and aperiodic. Hence, the Markov chain is ergodic.

Example 2.3.4: Let $P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ be a transition matrix of a Markov chain, then in

P^2 we have $p_{22}^{(2)} = p_{33}^{(2)} = 1$ and in general $p_{22}^{(2n)} = p_{33}^{(2n)} = 1$ for $n = 1,2,3, \dots$. So, the second and third states are periodic with period 2. Therefore, the Markov chain is not ergodic.

Example 2.3.5 ([28] page 347): Let $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}$ be a transition matrix of a Markov chain then states 1 and 2 are transient because they can not be reentered once the system is trapped in states 3 and 4. States 3 and 4 are persistent states since if the system starts in either of theses states, and moves from one of these states to the other one, it always will return to the original state eventually.

Example 2.3.6 ([28] page 647): Let $P = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$ be a transition matrix of a Markov chain. Then, states 1 and 2 are transient because they reach state 3 but can never be reached back. State 3 is absorbing since $p_{33} = 1$.

Example 2.3.7: If $P = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0 & 1 & 0 \\ 0.6 & 0.4 & 0 \end{bmatrix}$ is a transition matrix of a Markov chain then, states 1 and 3 are periodic with each of period 2.

Solution.

We need to show that $p_{11}^{(n)} = p_{33}^{(n)} = 0$ for odd values of n. That is, $p_{11}^{(2k-1)} = p_{33}^{(2k-1)} = 0$ for $k \in \mathbb{N}$. We prove this by induction.

For $k = 1$: $p_{11}^{(1)} = p_{11} = 0$ and $p_{33}^{(1)} = p_{33} = 0$, so it is true for $k = 1$.

Suppose that, $p_{11}^{(2k-1)} = p_{33}^{(2k-1)} = 0$ for some $k \in \mathbb{N}$, then we show it is true for $k+1$.

That is we want to prove that $p_{11}^{(2k+1)} = p_{33}^{(2k+1)} = 0$.

$$p_{11}^{(2k+1)} = \sum_{i=1}^{3} p_{1i}^{(2k-1)} p_{i1}^{(2)} = p_{11}^{(2k-1)} p_{11}^{(2)} + p_{12}^{(2k-1)} p_{21}^{(2)} + p_{13}^{(2k-1)} p_{31}^{(2)},$$

$$p_{33}^{(2k+1)} = \sum_{i=1}^{3} p_{3i}^{(2k-1)} p_{i3}^{(2)} = p_{31}^{(2k-1)} p_{13}^{(2)} + p_{32}^{(2k-1)} p_{23}^{(2)} + p_{33}^{(2k-1)} p_{33}^{(2)}.$$

But $P^2 = \begin{bmatrix} 0.24 & 0.76 & 0 \\ 0 & 1 & 0 \\ 0 & 0.76 & 0.24 \end{bmatrix}$, so $p_{21}^{(2)} = p_{31}^{(2)} = 0$ and $p_{13}^{(2)} = p_{23}^{(2)} = 0$, together with the induction hypothesis we have $p_{11}^{(2k+1)} = p_{33}^{(2k+1)} = 0$. Therefore, $p_{11}^{(n)} = p_{33}^{(n)} = 0$ for odd values of n. Hence, states 1 and 3 are periodic with each of period 2.

Example 2.3.8: Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ be a transition matrix of a Markov chain. Then, the Markov chain corresponding to P fails to be absorbing because even though state 1 is an absorbing state, it is not possible to reach it from the nonabsorbing states 2 and 3.

Chapter 3
Finite Regular Fuzzy Markov Chains:
Uncertain Probabilities

This chapter consists of four sections. In Section 3.1 we introduce the restricted fuzzy matrix multiplication which will be used intensively throughout this chapter to study fuzzy Markov chains [6]. In Section 3.2 we present the Karush-Kuhn-Tucker (KKT) Method in optimization problems with inequality constraints [28]. In Section 3.3 we study explicitly examples on finite regular fuzzy Markov chains [6]. In Section 3.4 we study deeply the limit of powers of 2×2 regular fuzzy transition matrices and we give three propositions concerning the uniqueness of this limit.

3.1 Restricted Fuzzy Matrix Multiplication

We consider finite Markov chains where there are uncertainties in some/all of the transition probabilities. These uncertainties are modeled by fuzzy numbers. Using a restricted fuzzy matrix multiplication we investigate the properties of regular fuzzy Markov chains and show that the basic properties of regular classical Markov chains generalize to them.

Let $Q = [q_{ij}]$ be a $r \times r$ transition matrix of a Markov chain. If a $q_{ij} = 0$ or $q_{ij} = 1$ then we assume that there is no uncertainty in this value, otherwise we assume there is uncertainty in the transition probability q_{ij} i.e. when $0 < q_{ij} < 1$. In the last case we replace each of q_{ij} by a fuzzy number $\bar{p}_{ij}$ where $0 < \bar{p}_{ij} < 1$ also, with the restriction that there are $p_{ij} \in \bar{p}_{ij}[1]$ such that $P = [p_{ij}]$ is a transition matrix, and we define the fuzzy transition matrix $\bar{P} = [\bar{p}_{ij}]$, with the understanding that $\bar{p}_{ij} = 0$ when $q_{ij} = 0$ and $\bar{p}_{ij} = 1$ when $q_{ij} = 1$. The restriction that there are $p_{ij} \in \bar{p}_{ij}[1]$ such that $P = [p_{ij}]$ is a transition matrix, guarantees that $p_{ij} \in \bar{p}_{ij}[\alpha]$ for all $0 \leq \alpha \leq 1$. Since $\bar{p}_{ij}$ is a fuzzy number, then $\bar{p}_{ij}[\alpha]$ is a closed and bounded interval for all $0 \leq \alpha \leq 1$, so we let $\bar{p}_{ij}[\alpha] = \left[p_{ij1}(\alpha), p_{ij2}(\alpha)\right]$.

In order to compute $\overline{P}^n$ for $n = 2,3, \ldots$, we need the definition of *the restricted fuzzy matrix multiplication*.

Let $S = \{x = (x_1, \ldots, x_r) | x_i \geq 0, \sum_{i=1}^{r} x_i = 1\}$,

The i^{th} domain of α, denoted by $Dom_i[\alpha]$ is

$$Dom_i[\alpha] = \left(\prod_{j=1}^{r} \bar{p}_{ij}[\alpha]\right) \cap S = \left\{(p_{i1}, \ldots, p_{ir}) \middle| p_{i1}, \ldots, p_{ir} \geq 0, \sum_{j=1}^{r} p_{ij} = 1\right\},$$

for $0 \leq \alpha \leq 1$ and $i = 1, \ldots, r$. Then

$$Dom[\alpha] = \prod_{i=1}^{r} Dom_i[\alpha] = \left\{(p_{11}, \ldots, p_{rr}) \middle| p_{ij} \geq 0, \forall i,j \ and \ \sum_{j=1}^{r} p_{ij} = 1, i = 1, \ldots, r\right..$$

If $M = \left\{P = [p_{ij}] \middle| (p_{11}, \ldots, p_{rr}) \in Dom[\alpha]\right\}$, then, $Q \in M$.

Next, set $\overline{P}^n = \left[\bar{p}_{ij}^{(n)}\right]$ where we will define $\bar{p}_{ij}^{(n)}$ and show that they are fuzzy numbers. Let $P \in M$, and consider $P^n = \left[p_{ij}^{(n)}\right]$. We know that $p_{ij}^{(n)} = f_{ij}^{(n)}(p_{11}, \ldots, p_{rr})$, for some function $f_{ij}^{(n)}$. That is the elements of P^n are just some function of the elements of P. Now consider $f_{ij}^{(n)}$ a function of $p = (p_{11}, \ldots, p_{rr}) \in Dom[\alpha]$. Let $\Gamma_{ij}^{(n)}[\alpha] =$

$f_{ij}^{(n)}(Dom[\alpha])$ be the range of $f_{ij}^{(n)}$. Since $f_{ij}^{(n)}$ is continuous and $Dom[\alpha]$ is connected, closed and bounded (compact), which implies that $\Gamma_{ij}^{(n)}[\alpha]$ is a closed and bounded interval for all α, i, j and n. We set $\bar{p}_{ij}^{(n)}[\alpha] = \Gamma_{ij}^{(n)}[\alpha]$, giving the $\alpha -$cuts of the $\bar{p}_{ij}^{(n)}$ in $\bar{P}^n$. Now we show that the resulting $\bar{p}_{ij}^{(n)}$ is a fuzzy number. First, $\Gamma_{ij}^{(n)}[\alpha]$ is closed, bounded, interval. Second, $Dom_i[1] = \left(\prod_{j=1}^{r} \bar{p}_{ij}[1]\right) \cap S \neq \emptyset$ as $\bar{p}_{ij}[1] \neq \emptyset$ (this is guaranteed by the restriction on $\bar{p}_{ij}$) so $Dom[1] = \prod_{i=1}^{r} Dom_i[1] \neq \emptyset$, and surely $\Gamma_{ij}^{(n)}[1] = f_{ij}^{(n)}(Dom[1]) \neq \emptyset$, this implies that $\bar{p}_{ij}^{(n)}$ is normalized. Therefore, $\bar{p}_{ij}^{(n)}$ is a fuzzy number whose $\alpha -$cuts are:

$$\bar{p}_{ij}^{(n)}[\alpha] = \left[p_{ij1}^{(n)}(\alpha),\ p_{ij2}^{(n)}(\alpha)\right] \text{ for all } 0 \leq \alpha \leq 1,$$

where

$$p_{ij1}^{(n)}(\alpha) = min\left\{f_{ij}^{(n)}(p) | p \in Dom[\alpha]\right\},$$

$$p_{ij2}^{(n)}(\alpha) = max\left\{f_{ij}^{(n)}(p) | p \in Dom[\alpha]\right\}.$$

3.2 Optimization Problems with Inequality Constrains

According to the restricted fuzzy matrix multiplication, we need to maximize and minimize $f_{ij}^{(n)}$ on $Dom[\alpha]$ to find the endpoints of the α −cuts of $\bar{p}_{ij}^{(n)}$. So we specify this section to introduce the Karush-Kuhn-Tucker (KKT) Method in optimization.

Definition 3.2.1: Let $f(\boldsymbol{X})$ be a function where $\boldsymbol{X} = (x_1, x_2, \dots, x_n)$, then a point $\boldsymbol{X}_0 = (x_1^0, x_2^0, \dots, x_n^0)$ is a *maximum* if $f(\boldsymbol{X}_0 + \boldsymbol{h}) \leq f(\boldsymbol{X}_0)$ for all $\boldsymbol{h} = (h_1, h_2, \dots, h_n)$ where $|h_j|$ is sufficiently small for all j. In a similar manner $\boldsymbol{X}_0$ is a *minimum* if $f(\boldsymbol{X}_0 + \boldsymbol{h}) \geq f(\boldsymbol{X}_0)$. An *extreme* point of a function $f(\boldsymbol{X})$ defines either a maximum or a minimum of the function.

Consider the problem

$$\text{Maximize } z = f(\boldsymbol{X})$$

Subject to

$$g(\boldsymbol{X}) \leq \boldsymbol{0}$$

The inequality constraints may be converted into equations by using nonnegative slack variables. Let $S_i^2 (\geq 0)$ be the slack quantity added to the i^{th} constraint $g_i(\boldsymbol{X}) \leq 0$ and define

$$\boldsymbol{S} = (S_1, S_2, \dots, S_m)^T, \boldsymbol{S}^2 = (S_1^2, S_2^2, \dots, S_m^2)^T$$

where m is the total number of inequality constraints. The Lagrangian function is thus given by

$$L(\boldsymbol{X}, \boldsymbol{S}, \boldsymbol{\lambda}) = f(\boldsymbol{X}) - \boldsymbol{\lambda}[g(\boldsymbol{X}) + \boldsymbol{S}^2]$$

given the constraints

$$g(\boldsymbol{X}) \leq \boldsymbol{0}$$

A necessary condition for optimality is that $\boldsymbol{\lambda}$ be nonnegative (nonpositive) for maximization (minimization) problems. This result is justified by noting that the vector $\boldsymbol{\lambda}$ measures the rate of variation of f with respect to g- that is,

$$\boldsymbol{\lambda} = \frac{\partial f}{\partial g}$$

In the maximization case, as the right-hand side of the constraint $g(\boldsymbol{X}) \leq 0$ increases from 0 to the vector ∂g, the solution space becomes less constrained and hence f cannot decrease, meaning that $\boldsymbol{\lambda} \geq \boldsymbol{0}$. Similarly for minimization, as the right-hand side of the constraints increases, f cannot increase, which implies that $\boldsymbol{\lambda} \leq \boldsymbol{0}$. If the constraints are equalities, that is, $g(\boldsymbol{X}) = 0$, then $\boldsymbol{\lambda}$ becomes unrestricted in sign.

The restrictions on $\boldsymbol{\lambda}$ hold as part of the KKT necessary conditions. The remaining conditions will now be developed.

Taking the partial derivatives of L with respect to $\boldsymbol{X}, \boldsymbol{S}$, and $\boldsymbol{\lambda}$, we obtain

$$\frac{\partial L}{\partial \boldsymbol{X}} = \nabla f(\boldsymbol{X}) - \boldsymbol{\lambda} \nabla g(\boldsymbol{X}) = \boldsymbol{0}$$

$$\frac{\partial L}{\partial S_i} = -2\lambda_i S_i = 0, i = 1,2, \ldots, m$$

$$\frac{\partial L}{\partial \boldsymbol{\lambda}} = -(g(\boldsymbol{X}) + \boldsymbol{S}^2) = \boldsymbol{0}$$

The second set of equations reveals the following results:

1. If $\lambda_i \neq 0$, then ${S_i}^2 = 0$, which means that the corresponding resource is abundant, and, hence, it is consumed completely (equality constraint).

2. If ${S_i}^2 > 0$, then $\lambda_i = 0$. This means resource i is not scarce and, consequently, it has no effect on the value of f (i.e., $\lambda_i = \frac{\partial f_i}{\partial g_i}$).

From the second and third sets of equations, we obtain

$$\lambda_i g_i(\boldsymbol{X}) = 0, i = 1,2, \ldots, m$$

This new condition essentially repeats the foregoing argument, because if $\lambda_i > 0$, $g_i(\boldsymbol{X}) = 0$ or ${S_i}^2 = 0$; and if $g_i(\boldsymbol{X}) < 0, {S_i}^2 > 0$, and $\lambda_i = 0$.

The KKT necessary conditions for maximization problem are summarized as:

$$\boldsymbol{\lambda} \geq \boldsymbol{0}$$

$$\nabla f(\boldsymbol{X}) - \boldsymbol{\lambda} \nabla g(\boldsymbol{X}) = \boldsymbol{0}$$

$$\lambda_i g_i(\boldsymbol{X}) = 0, i = 1,2, \ldots, m$$

$$g(\boldsymbol{X}) \leq \boldsymbol{0}$$

These conditions apply to the minimization case as well, except that $\boldsymbol{\lambda}$ must be nonpositive. In both maximization and minimization, the Lagrange multipliers corresponding to equality constraints are unrestricted in sign.

3.3 Finite Regular Fuzzy Markov Chains

As presented in Chapter 2, if P is a $r \times r$ crisp transition matrix for a regular Markov chain then $\lim_{n\to\infty} P^n = \Pi$ where each row in Π is $w = (w_1, \dots, w_r)$, $w_i > 0$ and $\sum_{i=1}^{r} w_i = 1$. Here w is the solution of $wP = w$ satisfying $w_i > 0$ and $\sum_{i=1}^{r} w_i = 1$.

If $Q = [q_{ij}]$ is a $r \times r$ crisp transition matrix for a regular Markov chain, then consider $\bar{P} = [\bar{p}_{ij}]$ where $\bar{p}_{ij}$ gives the uncertainty (if any) in q_{ij}.

If $(p_{11}, \dots, p_{rr}) \in Dom[\alpha]$, then $P = [p_{ij}]$ is also transition matrix for a regular Markov chain. Let $\overline{P}^n \to \overline{\Pi}$ where each row in $\overline{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1, \dots, \bar{\pi}_r)$. Also let

$$\bar{\pi}_j[\alpha] = [\pi_{j1}(\alpha), \pi_{j2}(\alpha)], j = 1, \dots, r.$$

We show how to compute the α $-$cuts of $\bar{\pi}_j$.

For each $(p_{11}, \dots, p_{rr}) \in Dom[\alpha]$, set $P = [p_{ij}]$ and we get $P^n \to \Pi$. Let

$\Gamma(\alpha) = \{w | w \ a \ row \ in \ \Pi, (p_{11}, \dots, p_{rr}) \in Dom[\alpha]\}$. Then

$$\pi_{j1}(\alpha) = min\{w_j | w \in \Gamma(\alpha)\}$$

$$\pi_{j2}(\alpha) = max\{w_j \,|\, w \in \Gamma(\alpha)\}$$

where w_j is the j^{th} component in the vector w [6].

The following example shows how we find the limit of 2×2 regular fuzzy Markov chains

using the restricted fuzzy matrix multiplication.

Example 3.3.1([6]): Let $Q = [q_{ij}]$ be a 2×2 transition matrix of a regular Markov chain, then consider $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix}$ where $\bar{p}_{ij}$ gives the uncertainty (if any) in q_{ij} for $i, j = 1,2$. If $(p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]$, then $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ is a regular transition matrix and so P^n is convergent. We solve $[w_1\ w_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [w_1\ w_2]$ where $w_1, w_2 > 0$ and $w_1 + w_2 = 1$. It follows from Example 2.3.1 that $w_1 = \frac{p_{21}}{p_{21}+p_{12}}$ and $w_2 = \frac{p_{12}}{p_{21}+p_{12}}$. Now,

$\frac{\partial w_1}{\partial p_{21}} = \frac{p_{12}}{(p_{21}+p_{12})^2} > 0,$ $\frac{\partial w_1}{\partial p_{12}} = \frac{-p_{21}}{(p_{21}+p_{12})^2} < 0,$ $\frac{\partial w_2}{\partial p_{21}} = \frac{-p_{12}}{(p_{21}+p_{12})^2} < 0,$ and $\frac{\partial w_2}{\partial p_{12}} = \frac{p_{21}}{(p_{21}+p_{12})^2} > 0.$

If $\bar{p}_{21}[\alpha] = [p_{211}(\alpha), p_{212}(\alpha)]$ and $\bar{p}_{12}[\alpha] = [p_{121}(\alpha), p_{122}(\alpha)]$, then

by restricted matrix multiplication $\overline{P}^n \to \overline{\Pi}$ where each row in $\overline{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2)$, where

$\bar{\pi}_1[\alpha] = [\pi_{11}(\alpha), \pi_{12}(\alpha)]$ and $\bar{\pi}_2[\alpha] = [\pi_{21}(\alpha), \pi_{22}(\alpha)]$.

$$\pi_{11}(\alpha) = min\left\{\frac{p_{21}}{p_{21}+p_{12}} \middle| (p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]\right\} = \frac{p_{211}(\alpha)}{p_{211}(\alpha)+p_{122}(\alpha)}$$

$$\pi_{12}(\alpha) = max\left\{\frac{p_{21}}{p_{21}+p_{12}} \middle| (p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]\right\} = \frac{p_{212}(\alpha)}{p_{212}(\alpha)+p_{121}(\alpha)}$$

$$\pi_{21}(\alpha) = min\left\{\frac{p_{12}}{p_{21}+p_{12}} \middle| (p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]\right\} = \frac{p_{121}(\alpha)}{p_{121}(\alpha)+p_{212}(\alpha)}$$

$$\pi_{22}(\alpha) = max\left\{\frac{p_{12}}{p_{21}+p_{12}} \middle| (p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]\right\} = \frac{p_{122}(\alpha)}{p_{122}(\alpha)+p_{211}(\alpha)}$$

Therefore,

$$\bar{\pi}_1[\alpha] = \left[\frac{p_{211}(\alpha)}{p_{211}(\alpha) + p_{122}(\alpha)}, \frac{p_{212}(\alpha)}{p_{212}(\alpha) + p_{121}(\alpha)}\right] \qquad 3.3.1$$

$$\bar{\pi}_2[\alpha] = \left[\frac{p_{121}(\alpha)}{p_{121}(\alpha) + p_{212}(\alpha)}, \frac{p_{122}(\alpha)}{p_{122}(\alpha) + p_{211}(\alpha)}\right] \qquad 3.3.2$$

for all $0 \le \alpha \le 1$.

Now, we show how $\pi_{11}(\alpha)$ can be derived using KKT conditions:

We want to minimize $f(p_{21}, p_{12}) = \frac{p_{21}}{p_{21}+p_{12}}$ subject to

$$g_1(p_{21}, p_{12}) = p_{21} - p_{212}(\alpha) \le 0$$

$$g_2(p_{21}, p_{12}) = p_{211}(\alpha) - p_{21} \le 0$$

$$g_3(p_{21}, p_{12}) = p_{12} - p_{122}(\alpha) \leq 0$$

$$g_4(p_{21}, p_{12}) = p_{121}(\alpha) - p_{12} \leq 0$$

The KKT conditions will be

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \leq \mathbf{0}$$

$$\nabla f(p_{21}, p_{12}) - \boldsymbol{\lambda} \nabla g(p_{21}, p_{12}) = \mathbf{0}$$

$$\lambda_i g_i(p_{21}, p_{12}) = 0, i = 1,2,3,4$$

$$g(p_{21}, p_{12}) = \begin{pmatrix} g_1(p_{21}, p_{12}) \\ g_2(p_{21}, p_{12}) \\ g_3(p_{21}, p_{12}) \\ g_3(p_{21}, p_{12}) \end{pmatrix} \leq \mathbf{0}$$

Or,

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \leq \mathbf{0},$$

$$\left(\frac{\partial f}{\partial p_{21}}, \frac{\partial f}{\partial p_{12}}\right) - (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \begin{pmatrix} \frac{\partial g_1}{p_{21}} & \frac{\partial g_1}{p_{12}} \\ \frac{\partial g_2}{p_{21}} & \frac{\partial g_2}{p_{12}} \\ \frac{\partial g_3}{p_{21}} & \frac{\partial g_3}{p_{12}} \\ \frac{\partial g_4}{p_{21}} & \frac{\partial g_4}{p_{12}} \end{pmatrix} = \mathbf{0},$$

$$\Rightarrow \frac{\partial f}{\partial p_{21}} - \lambda_1 + \lambda_2 = 0 \; and \; \frac{\partial f}{\partial p_{12}} - \lambda_3 + \lambda_4 = 0.$$

$$\lambda_1\big(p_{21} - p_{212}(\alpha)\big) = 0,$$

$$\lambda_2(p_{211}(\alpha) - p_{21}) = 0,$$

$$\lambda_3\big(p_{12} - p_{122}(\alpha)\big) = 0,$$

$$\lambda_4(p_{121}(\alpha) - p_{12}) = 0.$$

If $\lambda_2 = 0$ then $\frac{\partial f}{\partial p_{21}} = \lambda_1 \leq 0$ which is not possible since $\frac{\partial f}{\partial p_{21}} = \frac{p_{12}}{(p_{21}+p_{12})^2} > 0$, so $p_{21} = p_{211}(\alpha), \lambda_1 = 0$, and $\lambda_2 = -\frac{\partial f}{\partial p_{21}}$.

If $\lambda_3 = 0$ then $\frac{\partial f}{\partial p_{12}} = -\lambda_4 \geq 0$ which is not possible since $\frac{\partial f}{\partial p_{12}} = \frac{-p_{21}}{(p_{21}+p_{12})^2} < 0$, so $p_{12} = p_{122}(\alpha)$, $\lambda_4 = 0$, and $\lambda_3 = \frac{\partial f}{\partial p_{12}}$.

Since all constraints are satisfied we have $min\, \frac{p_{21}}{p_{21}+p_{12}} = \frac{p_{211}(\alpha)}{p_{211}(\alpha)+p_{122}(\alpha)}$ where $p_{21} \in [p_{211}(\alpha), p_{212}(\alpha)]$ and $p_{12} \in [p_{121}(\alpha), p_{122}(\alpha)]$.

Similarly, $\pi_{12}(\alpha), \pi_{21}(\alpha)$, and $\pi_{22}(\alpha)$ are derived using KKT conditions.

In the next example, we apply relations 3.3.1 and 3.3.2 on triangular fuzzy numbers.

Example 3.3.2 ([6]): Let $Q = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$ be a crisp transition matrix. As we mentioned above we have uncertainties in all the entries, so we model these uncertainties by fuzzy numbers between 0 and 1. So, we may take $\bar{p}_{11} = (0.6/0.7/0.8)$, $\bar{p}_{12} = (0.2/0.3/0.4)$, $\bar{p}_{21} = (0.3/0.4/0.5)$ and $\bar{p}_{22} = (0.5/0.6/0.7)$. Hence,

$$\bar{p}_{11}[\alpha] = [0.6 + 0.1\alpha, 0.8 - 0.1\alpha]$$

$$\bar{p}_{12}[\alpha] = [0.2 + 0.1\alpha, 0.4 - 0.1\alpha]$$

$$\bar{p}_{21}[\alpha] = [0.3 + 0.1\alpha, 0.5 - 0.1\alpha]$$

$$\bar{p}_{22}[\alpha] = [0.5 + 0.1\alpha, 0.7 - 0.1\alpha]$$

Then $\overline{P}^n \rightarrow \overline{\Pi}$ where each row in $\overline{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2)$ and it follows from relations 3.3.1 and 3.3.2 that

$$\bar{\pi}_1[\alpha] = \left[\frac{3}{7} + \frac{1}{7}\alpha, \frac{5}{7} - \frac{1}{7}\alpha\right] \text{ and } \bar{\pi}_2[\alpha] = \left[\frac{2}{7} + \frac{1}{7}\alpha, \frac{4}{7} - \frac{1}{7}\alpha\right], \text{ for all } 0 \leq \alpha \leq 1.$$

Note that the endpoints of the α −cuts are linear functions of α, and hence we have

$\bar{\pi}_1[0] = \left[\frac{3}{7}, \frac{5}{7}\right], \bar{\pi}_1[1] = \frac{4}{7}$ so $\bar{\pi}_1 = (\frac{3}{7}/\frac{4}{7}/\frac{5}{7})$ is a triangular fuzzy number.

$\bar{\pi}_2[0] = \left[\frac{2}{7}, \frac{4}{7}\right], \bar{\pi}_2[1] = \frac{3}{7}$ so $\bar{\pi}_2 = (\frac{2}{7}/\frac{3}{7}/\frac{4}{7})$ is a triangular fuzzy number.

3.4 A Deeper Look on the Limit of Powers of 2×2 Regular Fuzzy Transition Matrices

For a crisp transition matrix $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ with $0 < q_{ij} < 1$, let $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix}$ be a fuzzy transition matrix where $0 < \bar{p}_{ij} < 1$ representing the uncertainty in the q_{ij} . Let $\bar{p}_{ij}[\alpha] = [p_{ij1}(\alpha), p_{ij2}(\alpha)]$ for all $0 \leq \alpha \leq 1$. If $(p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]$, then $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ is a regular transition matrix. Let $w = (w_1, w_2)$ be the steady state vector of P, then we have the following four cases:

1. If $w_1 = \frac{p_{21}}{p_{21}+p_{12}}$ and $w_2 = \frac{p_{12}}{p_{21}+p_{12}}$, then $\frac{\partial w_1}{\partial p_{21}} = \frac{p_{12}}{(p_{21}+p_{12})^2} > 0$, $\frac{\partial w_1}{\partial p_{12}} = \frac{-p_{21}}{(p_{21}+p_{12})^2} < 0$, $\frac{\partial w_2}{\partial p_{21}} = \frac{-p_{12}}{(p_{21}+p_{12})^2} < 0$, $\frac{\partial w_2}{\partial p_{12}} = \frac{p_{21}}{(p_{21}+p_{12})^2} > 0$.

   It follows by restricted fuzzy matrix multiplication that $\overline{P}^n \to \overline{\Pi}_1$, where each row in $\overline{\Pi}_1$ is $\bar{\pi}_1 = (\bar{\pi}_{11}, \bar{\pi}_{12})$ where

$$\bar{\pi}_{11}[\alpha] = \left[\frac{p_{211}(\alpha)}{p_{211}(\alpha)+p_{122}(\alpha)}, \frac{p_{212}(\alpha)}{p_{212}(\alpha)+p_{121}(\alpha)}\right] \qquad 3.4.1$$

$$\bar{\pi}_{12}[\alpha] = \left[\frac{p_{121}(\alpha)}{p_{121}(\alpha)+p_{212}(\alpha)}, \frac{p_{122}(\alpha)}{p_{122}(\alpha)+p_{211}(\alpha)}\right] \qquad 3.4.2$$

   for all $0 \leq \alpha \leq 1$.

2. If $w_1 = \frac{1-p_{22}}{1-p_{22}+p_{12}}$ and $w_2 = \frac{p_{12}}{p_{21}+1-p_{22}}$, then $\frac{\partial w_1}{\partial p_{22}} = \frac{-p_{12}}{(1-p_{22}+p_{12})^2} < 0$, $\frac{\partial w_1}{\partial p_{12}} = \frac{-(1-p_{22})}{(1-p_{22}+p_{12})^2} < 0$, $\frac{\partial w_2}{\partial p_{22}} = \frac{p_{12}}{(1-p_{22}+p_{12})^2} > 0$, and $\frac{\partial w_2}{\partial p_{12}} = \frac{1-p_{22}}{(1-p_{22}+p_{12})^2} > 0$.

   It follows by restricted fuzzy matrix multiplication that $\overline{P}^n \to \overline{\Pi}_2$, where each row in $\overline{\Pi}_2$ is $\bar{\pi}_2 = (\bar{\pi}_{21}, \bar{\pi}_{22})$ where

$$\bar{\pi}_{21}[\alpha] = \left[\frac{1-p_{222}(\alpha)}{1-p_{222}(\alpha)+p_{122}(\alpha)}, \frac{1-p_{221}(\alpha)}{1-p_{221}(\alpha)+p_{121}(\alpha)}\right] \qquad 3.4.3$$

$$\bar{\pi}_{22}[\alpha] = \left[\frac{p_{121}(\alpha)}{p_{121}(\alpha)+1-p_{221}(\alpha)}, \frac{p_{122}(\alpha)}{p_{122}(\alpha)+1-p_{222}(\alpha)}\right] \qquad 3.4.4$$

   for all $0 \leq \alpha \leq 1$.

3. If $w_1 = \frac{p_{21}}{1-p_{11}+p_{21}}$ and $w_2 = \frac{1-p_{11}}{1-p_{11}+p_{21}}$, then $\frac{\partial w_1}{\partial p_{21}} = \frac{1-p_{11}}{(1-p_{11}+p_{21})^2} > 0$, $\frac{\partial w_1}{\partial p_{11}} = \frac{p_{21}}{(1-p_{11}+p_{21})^2} > 0$, $\frac{\partial w_2}{\partial p_{21}} = \frac{-(1-p_{11})}{(1-p_{11}+p_{21})^2} < 0$, and $\frac{\partial w_2}{\partial p_{11}} = \frac{-p_{21}}{(1-p_{11}+p_{21})^2} < 0$.

   It follows by restricted fuzzy matrix multiplication that $\bar{P}^n \to \overline{\Pi}_3$, where each row in $\overline{\Pi}_3$ is $\bar{\pi}_3 = (\bar{\pi}_{31}, \bar{\pi}_{32})$ where

$$\bar{\pi}_{31}[\alpha] = \left[\frac{p_{211}(\alpha)}{p_{211}(\alpha)+1-p_{111}(\alpha)}, \frac{p_{212}(\alpha)}{p_{212}(\alpha)+1-p_{112}(\alpha)}\right] \qquad 3.4.5$$

$$\bar{\pi}_{32}[\alpha] = \left[\frac{1-p_{112}(\alpha)}{1-p_{112}(\alpha)+p_{212}(\alpha)}, \frac{1-p_{111}(\alpha)}{1-p_{111}(\alpha)+p_{211}(\alpha)}\right] \qquad 3.4.6$$

   for all $0 \le \alpha \le 1$.

4. If $w_1 = \frac{1-p_{22}}{2-p_{11}-p_{22}}$ and $w_2 = \frac{1-p_{11}}{2-p_{11}-p_{22}}$, then $\frac{\partial w_1}{\partial p_{11}} = \frac{1-p_{22}}{(2-p_{11}-p_{22})^2} > 0$, $\frac{\partial w_1}{\partial p_{22}} = \frac{-(1-p_{11})}{(2-p_{11}-p_{22})^2} < 0$, $\frac{\partial w_2}{\partial p_{11}} = \frac{-(1-p_{22})}{(2-p_{11}-p_{22})^2} < 0$, and $\frac{\partial w_2}{\partial p_{22}} = \frac{1-p_{11}}{(2-p_{11}-p_{22})^2} > 0$.

   It follows by restricted fuzzy multiplication that $\overline{P}^n \to \overline{\Pi}_4$, where each row in $\overline{\Pi}_4$ is $\bar{\pi}_4 = (\bar{\pi}_{41}, \bar{\pi}_{42})$ where

$$\bar{\pi}_{41}[\alpha] = \left[\frac{1-p_{222}(\alpha)}{2-p_{111}(\alpha)-p_{222}(\alpha)}, \frac{1-p_{221}(\alpha)}{2-p_{112}(\alpha)-p_{221}(\alpha)}\right] \qquad 3.4.7$$

$$\bar{\pi}_{42}[\alpha] = \left[\frac{1-p_{112}(\alpha)}{2-p_{112}(\alpha)-p_{221}(\alpha)}, \frac{1-p_{111}(\alpha)}{2-p_{111}(\alpha)-p_{222}(\alpha)}\right] \qquad 3.4.8$$

   for all $0 \le \alpha \le 1$.

Below we study $\overline{\Pi}_1, \overline{\Pi}_2, \overline{\Pi}_3$, and $\overline{\Pi}_4$ when the entries are triangular fuzzy numbers.

Theorem 3.4.1: If $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ is a crisp transition matrix with $0 < q_{ij} < 1$, and $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix}$ is a fuzzy transition matrix, where $\bar{p}_{ij} = \left(q_{ij} - \delta_{ij} / q_{ij} / q_{ij} + \acute{\delta_{ij}}\right)$ such that $\delta_{ij}, \acute{\delta_{ij}} > 0$ and $0 < \bar{p}_{ij} < 1$, $i, j = 1,2$. Then for distinct values of δ_{ij} and $\acute{\delta_{ij}}$, we have $\overline{\Pi}_i \neq \overline{\Pi}_j$ $i \neq j, i, j = 1,2,3,4$.

Proof. The fuzzy probability $\bar{p}_{ij}$ represents the uncertainty in q_{ij}. Now $\bar{p}_{ij}[\alpha] = \left[p_{ij\,1}(\alpha), p_{ij\,2}(\alpha)\right] = \left[q_{ij} - \delta_{ij} + \delta_{ij}\alpha, q_{ij} + \acute{\delta_{ij}} - \acute{\delta_{ij}}\alpha\right]$. Then according to relations $3.4.1 - 3.4.8$ we have:

$$\bar{\pi}_{11}[\alpha] = \left[\frac{q_{21} - \delta_{21} + \delta_{21}\alpha}{q_{21} + q_{12} + \acute{\delta_{12}} - \delta_{21} + \left(\delta_{21} - \acute{\delta_{12}}\right)\alpha}, \frac{q_{21} + \acute{\delta_{21}} - \acute{\delta_{21}}\alpha}{q_{21} + q_{12} + \acute{\delta_{21}} - \delta_{12} + \left(\delta_{12} - \acute{\delta_{21}}\right)\alpha}\right]$$

$$\bar{\pi}_{11}[0] = \left[\frac{q_{21} - \delta_{21}}{q_{21} + q_{12} + \acute{\delta_{12}} - \delta_{21}}, \frac{q_{21} + \acute{\delta_{21}}}{q_{21} + q_{12} + \acute{\delta_{21}} - \delta_{12}}\right], \bar{\pi}_{11}[1] = \frac{q_{21}}{q_{21} + q_{12}}$$

$$\bar{\pi}_{12}[\alpha] = \left[\frac{q_{12} - \delta_{12} + \delta_{12}\alpha}{q_{21} + q_{12} + \acute{\delta_{21}} - \delta_{12} + \left(\delta_{12} - \acute{\delta_{21}}\right)\alpha}, \frac{q_{12} + \acute{\delta_{12}} - \acute{\delta_{12}}\alpha}{q_{21} + q_{12} + \acute{\delta_{12}} - \delta_{21} + \left(\delta_{21} - \acute{\delta_{12}}\right)\alpha}\right]$$

$$\bar{\pi}_{12}[0] = \left[\frac{q_{12} - \delta_{12}}{q_{21} + q_{12} + \acute{\delta_{21}} - \delta_{12}}, \frac{q_{12} + \acute{\delta_{12}}}{q_{21} + q_{12} + \acute{\delta_{12}} - \delta_{21}}\right], \bar{\pi}_{12}[1] = \frac{q_{12}}{q_{21} + q_{12}}$$

$$\bar{\pi}_{21}[\alpha] = \left[\frac{q_{21} - \acute{\delta_{22}} + \acute{\delta_{22}}\alpha}{q_{21} + q_{12} - \acute{\delta_{22}} + \acute{\delta_{12}} + \left(\acute{\delta_{22}} - \acute{\delta_{12}}\right)\alpha}, \frac{q_{21} + \delta_{22} - \delta_{22}\alpha}{q_{21} + q_{12} + \delta_{22} - \delta_{12} + (\delta_{12} - \delta_{22})\alpha}\right]$$

$$\bar{\pi}_{21}[0]=\left[\frac{q_{21}-\acute{\delta}_{22}}{q_{21}+q_{12}-\acute{\delta}_{22}+\acute{\delta}_{12}},\frac{q_{21}+\delta_{22}}{q_{21}+q_{12}+\delta_{22}-\delta_{12}}\right],\bar{\pi}_{21}[1]=\frac{q_{21}}{q_{21}+q_{12}}$$

$$\bar{\pi}_{22}[\alpha]=\left[\frac{q_{12}-\delta_{12}+\delta_{12}\alpha}{q_{21}+q_{12}+\delta_{22}-\delta_{12}\ +(\delta_{12}-\delta_{22})\alpha},\frac{q_{12}+\acute{\delta}_{12}-\acute{\delta}_{12}\alpha}{q_{21}+q_{12}+\acute{\delta}_{12}-\acute{\delta}_{22}\ +\left(\acute{\delta}_{22}-\acute{\delta}_{12}\right)\alpha}\right]$$

$$\bar{\pi}_{22}[0]=\left[\frac{q_{12}-\delta_{12}}{q_{21}+q_{12}+\delta_{22}-\delta_{12}},\frac{q_{12}+\acute{\delta}_{12}}{q_{21}+q_{12}+\acute{\delta}_{12}-\acute{\delta}_{22}}\right],\bar{\pi}_{22}[1]=\frac{q_{12}}{q_{21}+q_{12}}$$

$$\bar{\pi}_{31}[\alpha]=\left[\frac{q_{21}-\delta_{21}+\delta_{21}\alpha}{q_{21}+q_{12}+\delta_{11}-\delta_{21}+(\delta_{21}-\delta_{11})\alpha},\frac{q_{21}+\acute{\delta}_{21}-\acute{\delta}_{21}\alpha}{q_{21}+q_{12}+\acute{\delta}_{21}-\acute{\delta}_{11}\ +\left(\acute{\delta}_{11}-\acute{\delta}_{21}\right)\alpha}\right]$$

$$\bar{\pi}_{31}[0]=\left[\frac{q_{21}-\delta_{21}}{q_{21}+q_{12}+\delta_{11}-\delta_{21}},\frac{q_{21}+\acute{\delta}_{21}}{q_{21}+q_{12}+\acute{\delta}_{21}-\acute{\delta}_{11}}\right],\bar{\pi}_{31}[1]=\frac{q_{21}}{q_{21}+q_{12}}$$

$$\bar{\pi}_{32}[\alpha]=\left[\frac{q_{12}-\acute{\delta}_{11}+\acute{\delta}_{11}\alpha}{q_{21}+q_{12}+\acute{\delta}_{21}-\acute{\delta}_{11}\ +\left(\acute{\delta}_{11}-\acute{\delta}_{21}\right)\alpha},\frac{q_{12}+\delta_{11}-\delta_{11}\alpha}{q_{21}+q_{12}+\delta_{11}-\delta_{21}+(\delta_{21}-\delta_{11})\alpha}\right]$$

$$\bar{\pi}_{32}[0]=\left[\frac{q_{12}-\acute{\delta}_{11}}{q_{21}+q_{12}+\acute{\delta}_{21}-\acute{\delta}_{11}},\frac{q_{12}+\delta_{11}}{q_{21}+q_{12}+\delta_{11}-\delta_{21}}\right],\bar{\pi}_{32}[1]=\frac{q_{12}}{q_{21}+q_{12}}$$

$$\bar{\pi}_{41}[\alpha]=\left[\frac{q_{21}-\acute{\delta}_{22}+\acute{\delta}_{22}\alpha}{q_{21}+q_{12}+\delta_{11}-\acute{\delta}_{22}+\left(\acute{\delta}_{22}-\delta_{11}\right)\alpha},\frac{q_{21}+\delta_{22}-\delta_{22}\alpha}{q_{21}+q_{12}+\delta_{22}-\acute{\delta}_{11}\ +\left(\acute{\delta}_{11}-\delta_{22}\right)\alpha}\right]$$

$$\bar{\pi}_{41}[0]=\left[\frac{q_{21}-\acute{\delta}_{22}}{q_{21}+q_{12}+\delta_{11}-\acute{\delta}_{22}},\frac{q_{21}+\delta_{22}}{q_{21}+q_{12}+\delta_{22}-\acute{\delta}_{11}}\right],\bar{\pi}_{41}[1]=\frac{q_{21}}{q_{21}+q_{12}}$$

$$\bar{\pi}_{42}[\alpha]=\left[\frac{q_{12}-\acute{\delta}_{11}+\acute{\delta}_{11}\alpha}{q_{21}+q_{12}-\acute{\delta}_{11}+\delta_{22}\ +\left(\acute{\delta}_{11}-\delta_{22}\right)\alpha},\frac{q_{12}+\delta_{11}-\delta_{11}\alpha}{q_{21}+q_{12}+\delta_{11}-\acute{\delta}_{22}+\left(\acute{\delta}_{22}-\delta_{11}\right)\alpha}\right]$$

$$\bar{\pi}_{42}[0] = \left[\frac{q_{12} - \acute{\delta}_{11}}{q_{21} + q_{12} - \acute{\delta}_{11} + \delta_{22}}, \frac{q_{12} + \delta_{11}}{q_{21} + q_{12} + \delta_{11} - \acute{\delta}_{22}}\right], \bar{\pi}_{42}[1] = \frac{q_{12}}{q_{21} + q_{12}}$$

Therefore, for distinct values of δ_{ij} and $\acute{\delta}_{ij}$, we get $\bar{\pi}_{i1}[\alpha]$ are distinct and similarly

for $\bar{\pi}_{i2}[\alpha]$ where $0 \leq \alpha < 1$, $i, j = 1,2,3,4,$ and this implies that $\overline{\Pi}_i \neq \overline{\Pi}_j$ for $i \neq j, i, j = 1,2,3,4$, it is clear that $\bar{\pi}_{ij}$'s are triangular shaped fuzzy numbers, and this completes the proof.

Similarly, if triangular fuzzy numbers are replaced by trapezoidal fuzzy numbers or any fuzzy number, we get the same conclusion. Moreover, these conclusions can be generalized for any $n \times n$ fuzzy transition matrix since for each component w_j of the steady state vector w, we may replace every p_{ik} in w_j by $1 - \sum_{\substack{l=1 \\ l \neq k}}^{n} p_{il}$, and hence we get distinct limits.

Our main purpose is to find minimal conditions that are needed to guarantee the uniqueness of the limit of the 2×2 regular fuzzy transition matrices. Therefore, we have the following three propositions:

Proposition 3.4.2: Let $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ be a regular crisp transition matrix with $0 < q_{ij} < 1$ for $i, j = 1,2$. Let $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix}$ where $\bar{p}_{ij} = (q_{ij} - \delta_i / q_{ij} / q_{ij} + \delta_i)$ - triangular fuzzy numbers- for $i, j = 1,2,$ and $\delta_1, \delta_2 > 0$ such that $0 < \bar{p}_{ij} < 1$ $i, j = 1,2$

. Then by restricted fuzzy matrix multiplication $\overline{P}^n$ converges to the unique limit $\overline{\Pi}$, where each row in $\overline{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2)$, with

$$\bar{\pi}_1 \approx \left(\frac{q_{21} - \delta_2}{q_{21} + q_{12} + \delta_1 - \delta_2} / \frac{q_{21}}{q_{21} + q_{12}} / \frac{q_{21} + \delta_2}{q_{21} + q_{12} + \delta_2 - \delta_1}\right)$$

$$\bar{\pi}_2 \approx \left(\frac{q_{12} - \delta_1}{q_{21} + q_{12} + \delta_2 - \delta_1} / \frac{q_{12}}{q_{21} + q_{12}} / \frac{q_{12} + \delta_1}{q_{21} + q_{12} + + \delta_1 - \delta_2}\right)$$

for $\delta_1 \neq \delta_2$ (both are triangular shaped fuzzy numbers),

and

$$\bar{\pi}_1 = \left(\frac{q_{21} - \delta}{q_{21} + q_{12}} / \frac{q_{21}}{q_{21} + q_{12}} / \frac{q_{21} + \delta}{q_{21} + q_{12}}\right)$$

$$\bar{\pi}_2 = \left(\frac{q_{12} - \delta}{q_{21} + q_{12}} / \frac{q_{12}}{q_{21} + q_{12}} / \frac{q_{12} + \delta}{q_{21} + q_{12}}\right)$$

for $\delta_1 = \delta_2 = \delta$ (both are triangular fuzzy numbers).

Proof. The alpha cuts of $\bar{p}_{ij}$ are $\bar{p}_{ij}[\alpha] = [q_{ij} - \delta_i + \delta_i\alpha, q_{ij} + \delta_i - \delta_i\alpha]$, $i, j = 1,2$ for all $0 \leq \alpha \leq 1$. In this case, $\delta_{1j} = \acute{\delta}_{1j} = \delta_1, \delta_{2j} = \acute{\delta}_{2j} = \delta_2$, for $j = 1,2$, so from the previous discussion $\bar{P}^n$ converges to the unique limit $\overline{\Pi}$, where each row in $\overline{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2)$, with

$$\bar{\pi}_1[\alpha] = \left[\frac{q_{21} - \delta_2 + \delta_2\alpha}{q_{21} + q_{12} + \delta_1 - \delta_2 + (\delta_2 - \delta_1)\alpha}, \frac{q_{21} + \delta_2 - \delta_2\alpha}{q_{21} + q_{12} + \delta_2 - \delta_1 + (\delta_1 - \delta_2)\alpha}\right], \text{ for } \delta_1 \neq \delta_2$$

So, $\bar{\pi}_1[0] = \left[\frac{q_{21} - \delta_2}{q_{21} + q_{12} + \delta_1 - \delta_2}, \frac{p_{21} + \delta_2}{q_{21} + q_{12} + \delta_2 - \delta_1}\right]$ and $\bar{\pi}_1[1] = \frac{q_{21}}{q_{21} + q_{12}}$.

$$\bar{\pi}_2[\alpha] = \left[\frac{q_{12} - \delta_1 + \delta_1\alpha}{q_{21} + q_{12} + \delta_2 - \delta_1 + (\delta_1 - \delta_2)\alpha}, \frac{q_{12} + \delta_1 - \delta_1\alpha}{q_{21} + q_{12} + \delta_1 - \delta_2 + (\delta_2 - \delta_1)\alpha}\right], \text{ for } \delta_1 \neq \delta_2$$

So, $\bar{\pi}_2[0] = \left[\frac{q_{12}-\delta_1}{q_{21}+q_{12}+\delta_2-\delta_1}, \frac{q_{12}+\delta_1}{q_{21}+q_{12}+\delta_1-\delta_2}\right]$ and $\bar{\pi}_2[1] = \frac{q_{12}}{q_{21}+q_{12}}$.

In this case, $\bar{\pi}_1[\alpha]$ and $\bar{\pi}_2[\alpha]$ are not linear functions of α, so both $\bar{\pi}_1$ and $\bar{\pi}_2$ are triangular shaped fuzzy numbers with the form given in the proposition. If $\delta_1 = \delta_2 = \delta$, then it is clear that $\bar{\pi}_1[\alpha]$ and $\bar{\pi}_2[\alpha]$ are linear functions of α, so, both $\bar{\pi}_1$ and $\bar{\pi}_2$ are triangular fuzzy numbers with the form given in the present proposition, and this completes the proof.

Proposition 3.4.3: Let $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ be a regular crisp transition matrix with $0 < q_{ij} < 1$ for $i,j = 1,2$. Let $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix}$, where $\bar{p}_{ij} = (q_{ij} - \delta_i, q_{ij} - \acute{\delta}_i, q_{ij} + \acute{\delta}_i, q_{ij} + \delta_i)$, -trapezoidal fuzzy numbers- for $i,j = 1,2$ and $0 < \acute{\delta}_1 < \delta_1$, $0 < \acute{\delta}_2 < \delta_2$ such that $0 < \bar{p}_{ij} < 1$ $i,j = 1,2$. Then, by restricted fuzzy matrix multiplication $\overline{P}^n$ converges to the unique limit $\overline{\Pi}$, where each row in $\overline{\Pi}$ is $\pi = (\bar{\pi}_1, \bar{\pi}_2)$, with

$$\bar{\pi}_1 \approx \left(\frac{q_{21}-\delta_2}{q_{21}+q_{12}-\delta_2+\delta_1}, \frac{q_{21}-\acute{\delta}_2}{q_{21}+q_{12}+\acute{\delta}_1-\acute{\delta}_2}, \frac{q_{21}+\acute{\delta}_2}{q_{21}+q_{12}+\acute{\delta}_2-\acute{\delta}_1}, \frac{q_{21}+\delta_2}{q_{21}+q_{12}+\delta_2-\delta_1}\right)$$

$$\bar{\pi}_2 \approx \left(\frac{q_{12}-\delta_1}{q_{21}+q_{12}+\delta_2-\delta_1}, \frac{q_{12}-\acute{\delta}_1}{q_{21}+q_{12}-\acute{\delta}_1+\acute{\delta}_2}, \frac{q_{12}+\acute{\delta}_1}{q_{21}+q_{12}-\acute{\delta}_2+\acute{\delta}_1}, \frac{q_{12}+\delta_1}{q_{21}+q_{12}-\delta_2+\delta_1}\right)$$

for $\delta_2 - \acute{\delta}_2 \neq \delta_1 - \acute{\delta}_1$(both are trapezoidal shaped fuzzy numbers),

and

$$\bar{\pi}_1 = \left(\frac{q_{21}-\delta_2}{q_{21}+q_{12}-\delta_2+\delta_1}, \frac{q_{21}-\acute{\delta}_2}{q_{21}+q_{12}-\delta_2+\delta_1}, \frac{q_{21}+\acute{\delta}_2}{q_{21}+q_{12}+\delta_2-\delta_1}, \frac{q_{21}+\delta_2}{q_{21}+q_{12}+\delta_2-\delta_1}\right)$$

$$\bar{\pi}_2 = \left(\frac{q_{12}-\delta_1}{q_{21}+q_{12}+\delta_2-\delta_1}, \frac{q_{12}-\acute{\delta}_1}{q_{21}+q_{12}+\delta_2-\delta_1}, \frac{q_{12}+\acute{\delta}_1}{q_{21}+q_{12}-\delta_2+\delta_1}, \frac{q_{12}+\delta_1}{q_{21}+q_{12}-\delta_2+\delta_1}\right)$$

for $\delta_2 - \acute{\delta}_2 = \delta_1 - \acute{\delta}_1$(both are trapezoidal fuzzy numbers).

Proof. The alpha cuts of $\bar{p}_{ij}$ are $\bar{p}_{ij}[\alpha] = \left[(\delta_i - \acute{\delta}_i)\alpha + q_{ij} - \delta_i, -(\delta_i - \acute{\delta}_i)\alpha + q_{ij} + \delta_i\right]$, $i, j = 1,2$, for all $0 \le \alpha \le 1$. According to relations $3.4.1 - 3.4.8$ we have $\bar{P}^n$ converges to the unique limit $\bar{\Pi}$, where each row in $\bar{\Pi}$ is $\pi = (\bar{\pi}_1, \bar{\pi}_2)$, with

$$\bar{\pi}_1[\alpha] = \left[\frac{(\delta_2-\acute{\delta}_2)\alpha+q_{21}-\delta_2}{q_{21}+q_{12}+(\delta_2-\acute{\delta}_2)\alpha-(\delta_1-\acute{\delta}_1)\alpha-\delta_2+\delta_1}, \frac{-(\delta_2-\acute{\delta}_2)\alpha+q_{21}+\delta_2}{q_{21}+q_{12}-(\delta_2-\acute{\delta}_2)\alpha+(\delta_1-\acute{\delta}_1)\alpha+\delta_2-\delta_1}\right].$$

So, for $\delta_2 - \acute{\delta}_2 \neq \delta_1 - \acute{\delta}_1$, $\bar{\pi}_1[0] = \left[\frac{q_{21}-\delta_2}{q_{21}+q_{12}-\delta_2+\delta_1}, \frac{q_{21}+\delta_2}{q_{21}+q_{12}+\delta_2-\delta_1}\right]$,

and

$$\bar{\pi}_1[1] = \left[\frac{q_{21}-\acute{\delta}_2}{q_{21}+q_{12}+\acute{\delta}_1-\acute{\delta}_2}, \frac{q_{21}+\acute{\delta}_2}{q_{21}+q_{12}+\acute{\delta}_2-\acute{\delta}_1}\right].$$

Also,

$$\bar{\pi}_2[\alpha] = \left[\frac{(\delta_1-\acute{\delta}_1)\alpha+q_{12}-\delta_1}{q_{21}+q_{12}-(\delta_2-\acute{\delta}_2)\alpha+(\delta_1-\acute{\delta}_1)\alpha+\delta_2-\delta_1}, \frac{-(\delta_1-\acute{\delta}_1)\alpha+q_{12}+\delta_1}{q_{21}+q_{12}+(\delta_2-\acute{\delta}_2)\alpha-(\delta_1-\acute{\delta}_1)\alpha-\delta_2+\delta_1}\right].$$

So, for $\delta_2 - \acute{\delta}_2 \neq \delta_1 - \acute{\delta}_1$, $\bar{\pi}_2[0] = \left[\frac{q_{12}-\delta_1}{q_{21}+q_{12}+\delta_2-\delta_1}, \frac{q_{12}+\delta_1}{q_{21}+q_{12}-\delta_2+\delta_1}\right]$,

and

$$\bar{\pi}_2[1] = \left[\frac{q_{21}-\acute{\delta}_1}{q_{21}+q_{12}-\acute{\delta}_1+\acute{\delta}_2}, \frac{q_{21}+\acute{\delta}_1}{q_{21}+q_{12}-\acute{\delta}_2+\acute{\delta}_1}\right].$$

In this case, $\bar{\pi}_1[\alpha]$ and $\bar{\pi}_2[\alpha]$ are not linear functions of α, so, both $\bar{\pi}_1$ and $\bar{\pi}_2$ are trapezoidal shaped fuzzy numbers with the form given in the proposition. If $\delta_2 - \acute{\delta}_2 = \delta_1 - \acute{\delta}_1$, then it is clear that $\bar{\pi}_1[\alpha]$ and $\bar{\pi}_2[\alpha]$ are linear functions of α, so, both $\bar{\pi}_1$ and $\bar{\pi}_2$ are trapezoidal fuzzy numbers with the form given in the present proposition, and this completes the proof.

The following proposition deals with the fuzzy number that given in Definition 1.1.6.

Proposition 3.4.4: Let $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ be a regular crisp transition matrix with $0 < q_{ij} < 1$ for $i, j = 1,2$. Let $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix}$ where $\bar{p}_{ij} = (q_{ij} - \delta_i, q_{ij} - \acute{\delta}_i, q_{ij} + \acute{\delta}_i, q_{ij} + \delta i)c$, $0<c<1$ and $0<\delta 1<\delta 1$, $0<\delta 2<\delta 2$ such that $0<pij<1$ for $i,j=1,2$. Then, by restricted fuzzy matrix multiplication $\overline{P}^n$ converges to the unique limit $\overline{\Pi}$, where each row in $\overline{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2)$, with

$$\bar{\pi}_1 \approx \left(\frac{q_{21}-\delta_2}{q_{21}+q_{12}-\delta_2+\delta_1}, \frac{q_{21}-\acute{\delta}_2}{q_{21}+q_{12}+\acute{\delta}_1-\acute{\delta}_2}, \frac{q_{21}+\acute{\delta}_2}{q_{21}+q_{12}+\acute{\delta}_2-\acute{\delta}_1}, \frac{q_{21}+\delta_2}{q_{21}+q_{12}+\delta_2-\delta_1}\right)_c$$

$$\bar{\pi}_2 \approx \left(\frac{q_{12}-\delta_1}{q_{21}+q_{12}+\delta_2-\delta_1}, \frac{q_{12}-\acute{\delta}_1}{q_{21}+q_{12}-\acute{\delta}_1+\acute{\delta}_2}, \frac{q_{12}+\acute{\delta}_1}{q_{21}+q_{12}-\acute{\delta}_2+\acute{\delta}_1}, \frac{q_{12}+\delta_1}{q_{21}+q_{12}-\delta_2+\delta_1}\right)_c.$$

for $\delta_2 - \acute{\delta}_2 \neq \delta_1 - \acute{\delta}_1$ or $\acute{\delta}_2 \neq \acute{\delta}_1$,

and

$$\bar{\pi}_1 = \left(\frac{q_{21}-\delta_2}{q_{21}+q_{12}}, \frac{q_{21}-\acute{\delta}_2}{q_{21}+q_{12}}, \frac{q_{21}+\acute{\delta}_2}{q_{21}+q_{12}}, \frac{q_{21}+\delta_2}{q_{21}+q_{12}}\right)_c$$

$$\bar{\pi}_2 = \left(\frac{q_{12}-\delta_1}{q_{21}+q_{12}}, \frac{q_{12}-\acute{\delta}_1}{q_{21}+q_{12}}, \frac{q_{12}+\acute{\delta}_1}{q_{21}+q_{12}}, \frac{q_{12}+\delta_1}{q_{21}+q_{12}}\right)_c.$$

for $\delta_2 = \delta_1$ and $\acute{\delta}_2 = \acute{\delta}_1$.

Proof. $\bar{p}_{ij}[\alpha] = \left[\frac{1}{c}\left(\delta_i - \acute{\delta}_i\right)\alpha + q_{ij} - \delta_i, -\frac{1}{c}\left(\delta_i - \acute{\delta}_i\right)\alpha + q_{ij} + \delta_i\right]$ for all $0 \leq \alpha \leq c$,

and $\bar{p}_{ij}[\alpha] = \left[-\acute{\delta}_i \sqrt{\frac{1-\alpha}{1-c}} + q_{ij}, \acute{\delta}_i \sqrt{\frac{1-\alpha}{1-c}} + q_{ij}\right]$ for all $c \leq \alpha \leq 1$, $i, j = 1,2$. According to relations $3.4.1 - 3.4.8$ we have $\bar{P}^n$ converges to the unique limit $\bar{\Pi}$, where each row in $\bar{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2)$, with

$$\bar{\pi}_1[\alpha] = \left[\frac{\frac{1}{c}(\delta_2-\acute{\delta}_2)\alpha+q_{21}-\delta_2}{q_{21}+q_{12}+\frac{1}{c}(\delta_2-\acute{\delta}_2)\alpha-\frac{1}{c}(\delta_1-\acute{\delta}_1)\alpha-\delta_2+\delta_1}, \frac{-\frac{1}{c}(\delta_2-\acute{\delta}_2)\alpha+q_{21}+\delta_2}{q_{21}+q_{12}-\frac{1}{c}(\delta_2-\acute{\delta}_2)\alpha+\frac{1}{c}(\delta_1-\acute{\delta}_1)\alpha+\delta_2-\delta_1}\right],$$

$0 \leq \alpha \leq c$

$$\bar{\pi}_1[\alpha] = \left[\frac{-\acute{\delta}_2\sqrt{\frac{1-\alpha}{1-c}}+q_{21}}{q_{21}+q_{12}-\acute{\delta}_2\sqrt{\frac{1-\alpha}{1-c}}+\acute{\delta}_1\sqrt{\frac{1-\alpha}{1-c}}}, \frac{\acute{\delta}_2\sqrt{\frac{1-\alpha}{1-c}}+q_{21}}{q_{21}+q_{12}+\acute{\delta}_2\sqrt{\frac{1-\alpha}{1-c}}-\acute{\delta}_1\sqrt{\frac{1-\alpha}{1-c}}}\right], \text{ for } c \leq \alpha \leq 1.$$

So, for $\delta_2 - \acute{\delta}_2 \neq \delta_1 - \acute{\delta}_1$ or $\acute{\delta}_2 \neq \acute{\delta}_1$, $\bar{\pi}_1[0] = \left[\frac{q_{21}-\delta_2}{q_{21}+q_{12}-\delta_2+\delta_1}, \frac{q_{21}+\delta_2}{q_{21}+q_{12}+\delta_2-\delta_1}\right]$, and

$$\bar{\pi}_1[c] = \left[\frac{q_{21}-\acute{\delta}_2}{q_{21}+q_{12}+\acute{\delta}_1-\acute{\delta}_2}, \frac{q_{21}+\acute{\delta}_2}{q_{21}+q_{12}+\acute{\delta}_2-\acute{\delta}_1}\right].$$

Also,

$$\bar{\pi}_2[\alpha] = \left[\frac{\frac{1}{c}(\delta_1-\acute{\delta}_1)\alpha+q_{12}-\delta_1}{q_{21}+q_{12}-\frac{1}{c}(\delta_2-\acute{\delta}_2)\alpha+\frac{1}{c}(\delta_1-\acute{\delta}_1)\alpha+\delta_2-\delta_1}, \frac{-\frac{1}{c}(\delta_1-\acute{\delta}_1)\alpha+q_{12}+\delta_1}{q_{21}+q_{12}+\frac{1}{c}(\delta_2-\acute{\delta}_2)\alpha-\frac{1}{c}(\delta_1-\acute{\delta}_1)\alpha+\delta_2-\delta_1}\right], 0 \leq$$

$\alpha \leq c$

$$\bar{\pi}_2[\alpha] = \left[\frac{-\acute{\delta}_1\sqrt{\frac{1-\alpha}{1-c}}+q_{12}}{q_{21}+q_{12}+\acute{\delta}_2\sqrt{\frac{1-\alpha}{1-c}}-\acute{\delta}_1\sqrt{\frac{1-\alpha}{1-c}}}, \frac{\acute{\delta}_1\sqrt{\frac{1-\alpha}{1-c}}+q_{12}}{q_{21}+q_{12}-\acute{\delta}_2\sqrt{\frac{1-\alpha}{1-c}}+\acute{\delta}_1\sqrt{\frac{1-\alpha}{1-c}}}\right], \text{ for } c \leq \alpha \leq 1.$$

So, for $\delta_2 - \acute{\delta}_2 \neq \delta_1 - \acute{\delta}_1$ or $\acute{\delta}_2 \neq \acute{\delta}_1$, $\bar{\pi}_2[0] = \left[\frac{q_{12}-\delta_1}{q_{21}+q_{12}+\delta_2-\delta_1}, \frac{q_{12}+\delta_1}{q_{21}+q_{12}-\delta_2+\delta_1}\right]$, and

$$\bar{\pi}_2[c] = \left[\frac{q_{12}-\acute{\delta}_1}{q_{21}+q_{12}-\acute{\delta}_1+\acute{\delta}_2}, \frac{q_{12}+\acute{\delta}_1}{q_{21}+q_{12}-\acute{\delta}_2+\acute{\delta}_1}\right].$$

Therefore, $\bar{\pi}_1$ and $\bar{\pi}_2$ are the fuzzy numbers with the form given in the present proposition. Similarly, if $\delta_2 = \delta_1$ and $\acute{\delta}_2 = \acute{\delta}_1$ we have

$\bar{\pi}_1[0] = \left[\frac{q_{21}-\delta_2}{q_{21}+q_{12}}, \frac{q_{21}+\delta_2}{q_{21}+q_{12}}\right]$ and $\bar{\pi}_1[c] = \left[\frac{q_{21}-\acute{\delta}_2}{q_{21}+q_{12}}, \frac{q_{21}+\acute{\delta}_2}{q_{21}+q_{12}}\right]$.

$\bar{\pi}_2[0] = \left[\frac{q_{12}-\delta_1}{q_{21}+q_{12}}, \frac{q_{12}+\delta_1}{q_{21}+q_{12}}\right]$ and $\bar{\pi}_2[c] = \left[\frac{q_{12}-\acute{\delta}_1}{q_{21}+p_{12}}, \frac{q_{12}+\acute{\delta}_1}{q_{21}+p_{12}}\right]$.

Therefore, $\bar{\pi}_1$ and $\bar{\pi}_2$ are the fuzzy numbers with the form given in the present proposition and this completes the proof.

Chapter 4
Finite Fuzzy Markov Chains

In Chapter 3, we start with a crisp Markov chain, then the uncertainties in its transition matrix are replaced by fuzzy numbers and the powers of the resultant fuzzy transition matrix are computed by the restricted fuzzy matrix multiplication.

Throughout this chapter we study the finite fuzzy Markov chains in a completely different way than that presented in Chapter 3. Here the states will be fuzzy sets and the fuzzy transition matrix is a fuzzy relation on a finite state space, so the entries are numbers between *zero* and *one*, and the row sum need not be *one*. Max-min composition is used to find the powers of the fuzzy transition matrices. This chapter consists of four sections. In Section 4.1 we give basic definitions concerning finite fuzzy Markov chains [2] and [3]. In Section 4.2 we make a comparison between crisp and fuzzy Markov chains. In Sections 4.3 and 4.4 we study the ergodicity of a particular class of finite fuzzy Markov chains.

Throughout this chapter we denote the finite state space $\{1, \ldots, n\}$ by S.

4.1 Basic Definitions

Definition 4.1.1 ([2] and [3]): A *(finite) fuzzy set* or a *fuzzy distribution*, on S, is defined by the membership function x from S into $[0,1]$, represented by a vector $x = (x_1, \ldots, x_n)$, with

x_i denoting the image of i under x, i.e. $x(i)$, $0 \leq x_i \leq 1, i \in S$. The set of all fuzzy sets on S is denoted by $\mathcal{F}(S)$.

Definition 4.1.2 ([2] and [3]): A *fuzzy relation* $\bar{P}$ is defined as a fuzzy set on the Cartesian product $S \times S$. $\bar{P}$ is represented by a matrix $[\bar{p}_{ij}]$, with $\bar{p}_{ij}$ denoting $\bar{P}(i,j)$, $0 \leq \bar{p}_{ij} \leq 1,\ i,j \in S$.

Definition 4.1.3 ([2] and [3]): At each time instant $t,\ t = 0,1,\ldots$, the *state* of the system is described by the fuzzy set (or distribution) $\bar{x}^{(t)} \in \mathcal{F}(S)$. The *transition law of the fuzzy Markov chain* given by the fuzzy relation P as follows, at time instant $t,\ t = 1,2,\ldots$

$$\bar{x}_j^{(t+1)} = max\left\{min\left\{\bar{x}_1^{(t)}, \bar{p}_{1j}\right\}, min\left\{\bar{x}_2^{(t)}, \bar{p}_{2j}\right\}, \ldots, min\left\{\bar{x}_n^{(t)}, \bar{p}_{nj}\right\}\right\}, \qquad j \in S.$$

We refer to $\bar{x}^{(0)}$ as the initial fuzzy set (or the initial distribution).

It is natural to define the powers of the fuzzy transition matrix. Namely,

$$\bar{p}_{ij}^{(t)} = max\left\{min\left\{\bar{p}_{i1}, \bar{p}_{1j}^{(t-1)}\right\}, min\left\{\bar{p}_{i2}, \bar{p}_{2j}^{(t-1)}\right\}, \ldots, min\left\{\bar{p}_{in}, \bar{p}_{nj}^{(t-1)}\right\}\right\}, \qquad \bar{p}_{ij}^{(1)} = \bar{p}_{ij},$$

$$\bar{p}_{ij}^{(0)} = \delta_{ij}\ .$$

where δ_{ij} is a Kronecker delta.

Note that the fuzzy state $\bar{x}_k^{(t)}, k = 1, \dots, n$ at time instant $t,\ t = 1,2, \dots$ can be calculated by the formula

$$\bar{x}_k^{(t)} = max\left\{min\left\{\bar{x}_1^{(0)}, \bar{p}_{1k}^{(t)}\right\}, min\left\{\bar{x}_2^{(0)}, \bar{p}_{2k}^{(t)}\right\}, \dots, min\left\{\bar{x}_n^{(0)}, \bar{p}_{nk}^{(t)}\right\}\right\} \qquad 4.1.1$$

Formula 4.1.1 has a similar structure to that given in Proposition 2.1.6, the only difference between them is in the employed operations and, of course, the meaning of the terms as fuzzy grades, instead of probabilities. Equation 4.1.1 is obtained from that given in Proposition 2.1.6, by changing the algebraic summation to the max-operation and the algebraic multiplication to the min-operation ([2] and [3]).

Theorem 4.1.4 ([13] and [29]): The powers of the fuzzy transition matrix $\bar{P} = \left[\bar{p}_{ij}\right]$ either converge to idempotent $\overline{P}^{\tau} = \left[\bar{p}_{ij}^{(\tau)}\right]$, where τ is a finite number, or oscillate with a finite period υ starting from some finite power.

Definition 4.1.5 ([2] and [3]): Let the powers of fuzzy transition matrix converge in τ steps to a non periodic solution, then the associated fuzzy Markov chain is called *nonperiodic (or aperiodic)* and $\bar{P}^* = \bar{P}^{\tau}$ is called a *limiting fuzzy transition matrix.*

Definition 4.1.6 ([2] and [3]): The fuzzy Markov chain is called *ergodic* if it is aperiodic and the limiting transition matrix has identical rows.

In this chapter we call this matrix: ergodic fuzzy transition matrix.

4.2 Comparison between Crisp and Fuzzy Markov Chains

Definition 4.2.1: A fuzzy state $j \in S$ is *transient* if it can reach another state but cannot itself be reached back from another state. Mathematically, this happens if $\lim_{n\to\infty} \bar{p}_{ij}^{(n)} = 0$, for all i.

Definition 4.2.2: A fuzzy state $j \in S$ is *persistent (or recurrent)* if, upon entering this state, the process definitely will return to this state again. This can happen if, and only if the state is not transient.

Definition 4.2.3: A fuzzy Markov chain is called *irreducible (or regular)* if $\exists m \in \mathbb{N}$ such that $\bar{p}_{ij}^{(m)} > 0$, $\forall i, j \in S$, otherwise it is called *reducible (or irregular)*.

Example 4.2.4: Let $\bar{P} = \begin{bmatrix} 0.5 & 0.7 \\ 1 & 0 \end{bmatrix}$ be a fuzzy transition matrix of a fuzzy Markov chain.Then,

$$\overline{P}^2 = \bar{P} \circ \bar{P} = \begin{bmatrix} max\{min\{0.5,0.5\}, min\{0.7,1\}\} & max\{min\{0.5,0.7\}, min\{0.7,0\}\} \\ max\{min\{1,0.5\}, min\{0,1\}\} & max\{min\{1,0.7\}, min\{0,0\}\} \end{bmatrix}$$

$= \begin{bmatrix} 0.7 & 0.5 \\ 0.5 & 0.7 \end{bmatrix}$, $\overline{P}^3 = \begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 0.5 \end{bmatrix}$, $\overline{P}^4 = \begin{bmatrix} 0.7 & 0.5 \\ 0.5 & 0.7 \end{bmatrix}$. Note that $\bar{p}_{11}^{(n)} = \bar{p}_{22}^{(n)} = 0.7$ for n even and $\bar{p}_{11}^{(n)} = \bar{p}_{22}^{(n)} = 0.5$ for n odd. This example shows the definition for a periodic state in the classical sense is not applicable in the fuzzy sense; therefore, the periodicity is related to the matrix not to the states. Hence, this fuzzy transition matrix corresponds to a periodic fuzzy Markov chain with period 2.

The above example also clarifies why ergodicity was defined for fuzzy transition matrices not for fuzzy states.

Example 4.2.5 ([2] and [3]): Let $\bar{P} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$ be a fuzzy transition matrix of a fuzzy Markov chain, then P is a transition matrix of a crisp Markov chain since the row sums is 1. It is clear that this transition matrix in the classical sense corresponds to an irreducible, aperiodic Markov chain which is certainly ergodic. Moreover, $\lim_{n\to\infty} \overline{P}^n = \begin{bmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{bmatrix}$ and this limiting matrix is independent of $\bar{P}$.

In the fuzzy sense we have $\overline{P}^2 = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$ and so $\overline{P}^n = \bar{P}$ for $= 1,2,3,\ldots$. This result shows that in general, a fuzzy Markov chain which is irreducible and aperiodic need not be ergodic. In fact, this a crucial difference between fuzzy and crisp Markov chains. Moreover, the limiting fuzzy transition matrix is definitely depends on $\bar{P}$.

4.3 A Particular Ergodic 2×2 and 3×3 Fuzzy Transition Matrices

In this section we consider particular 2×2 and 3×3 fuzzy transition matrices, and we determine conditions that guarantee achievement of ergodicity. But before that we give the following example to show that a fuzzy transition matrix whose rows are identical is ergodic, which agrees with our intuition.

Example 4.3.1: Let $\bar{P} = \begin{bmatrix} \bar{\pi}_1 & \bar{\pi}_2 & \cdots & \bar{\pi}_n \\ \bar{\pi}_1 & \bar{\pi}_2 & \cdots & \bar{\pi}_n \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\pi}_1 & \bar{\pi}_2 & \cdots & \bar{\pi}_n \end{bmatrix}$ be a fuzzy transition matrix of a fuzzy Markov chain. Let $\bar{\pi}^* = max\{\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_n\}$ and $\bar{\pi}_* = min\{\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_n\}$.Then, in $\overline{P}^2 = \left[\bar{p}_{ij}^{(2)}\right]$, we have, $\bar{p}_{ij}^{(2)} = [\bar{\pi}_1 \quad \bar{\pi}_2 \quad \cdots \quad \bar{\pi}_n] \begin{bmatrix} \bar{\pi}_j \\ \bar{\pi}_j \\ \vdots \\ \bar{\pi}_j \end{bmatrix}$, where $i, j = 1,2, \dots, n$,

$$= max\left\{min\{\bar{\pi}_1, \bar{\pi}_j\}, min\{\bar{\pi}_2, \bar{\pi}_j\}, \dots, min\{\bar{\pi}_j, \bar{\pi}_j\}, \dots, min\{\bar{\pi}_n, \bar{\pi}_j\}\right\}.$$

If $\bar{\pi}_j = \bar{\pi}^*$ or $\bar{\pi}_j = \bar{\pi}_*$ then it is clear that $\bar{p}_{ij}^{(2)} = \bar{\pi}_j$. For otherwise, we have $\bar{\pi}_j \geq \bar{\pi}_{j_k}$ for $k = 1,2, \dots, l$ and $\bar{\pi}_j \leq \bar{\pi}_{j_k}$ for $k = l+1, \dots, n$ where $1 \leq l < n$, also $\left\{\bar{\pi}_{j_k} \middle| k = 1, \dots, n\right\} = \{\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_n\}$. Now, $min\{\bar{\pi}_{j_k}, \bar{\pi}_j\} = \bar{\pi}_{j_k}$ for $k = 1,2, \dots, l$, and $min\{\bar{\pi}_{j_k}, \bar{\pi}_j\} = \bar{\pi}_j$ for $k = l+1, \dots, n$. Since, $\bar{\pi}_j \geq \bar{\pi}_{j_k}$ for $k = 1,2, \dots, l$ we have $\bar{p}_{ij}^{(2)} = \bar{\pi}_j$. Hence, in all cases $\overline{P}^2 = \bar{P}$, and so $\overline{P}^m = \bar{P}$ for $m = 1,2,3, \dots$. Therefore, $\bar{P}$ is ergodic.

Theorem 4.1.4 does not give us information about fuzzy Markov chains having the ergodic behavior. Also, J. C. F. Garcia et al. [12] have done a simulation study on fuzzy Markov chains from which they have shown that most of fuzzy Markov chains are not ergodic. Besides, in [2] and [3] Avrachenkov and Sanchez introduced an open problem about the general conditions that guarantee the ergodicity of fuzzy Markov chains. These results together with Example 4.2.5 motivate us to search much deeper in the structure of fuzzy Markov chains which are ergodic. For this purpose we put the following assumption on the fuzzy transition matrices to be considered throughout this section and the subsequent section.

Assumption 4.3.2: For $1 \leq k \leq n$, let $\bar{P} = [\bar{p}_{ij}]$ be an $n \times n$ fuzzy transition matrix. Suppose that $\bar{p}_{ij} = 0$ or 1 for $i \in \{1, \dots, n\} - \{k\}$, $j = 1, \dots, n$ and in each of these rows –all rows except possibly the k^{th} one– exactly one entry is 1.

First: We consider 2×2 and 3×3 fuzzy transition matrices satisfying Assumption 4.3.2 for $k = 1$.

For 2×2 fuzzy transition matrices we have the following cases with a prescribe condition for each case:

1. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{11} \leq \bar{p}_{12} \leq 1$.

   $\overline{P}^2 = \begin{bmatrix} \bar{p}_{12} & \bar{p}_{11} \\ \bar{p}_{11} & \bar{p}_{12} \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{12} & \bar{p}_{11} \end{bmatrix}, \overline{P}^4 = \begin{bmatrix} \bar{p}_{12} & \bar{p}_{11} \\ \bar{p}_{11} & \bar{p}_{12} \end{bmatrix}$. Therefore,

   $\overline{P}^n = \begin{cases} \overline{P}^2, for\ n\ even \\ \overline{P}^3, for\ n\ odd \end{cases}$.

2. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1$.

$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} \end{bmatrix}$. So $\overline{P}^n = \overline{P}^2$ for $= 2,3,4, \ldots$. According to Theorem 4.1.4, $\tau = 2$.

3. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 0 & 1 \end{bmatrix}$ with $0 \leq \bar{p}_{11} \leq \bar{p}_{12} \leq 1$.

   $\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 0 & 1 \end{bmatrix} = \bar{P}$, so $\overline{P}^n = \bar{P}$ for $= 1,2,3, \ldots$. According to Theorem 4.1.4, $\tau = 1$.

4. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 0 & 1 \end{bmatrix}$ with $0 \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1$.

   $\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ 0 & 1 \end{bmatrix} = \bar{P}$, so $\overline{P}^n = \bar{P}$ for $= 1,2,3, \ldots$. According to Theorem 4.1.4, $\tau = 1$.

We conclude that case 2 above is the only ergodic one, from which we have $\bar{p}_{11} \geq \bar{p}_{12}$ and $\bar{p}_{22} \neq 1$.

Next we consider the 3×3 fuzzy transition matrices but applying the following two assumptions:

1. $\bar{p}_{11}$ is the maximum entry in the first row.
2. $\bar{p}_{22} \neq 1$ and $\bar{p}_{33} \neq 1$.

So we have the following cases with a prescribe condition for each case:

1. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{13} \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1$.

$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}$. So $\overline{P}^n = \overline{P}^2$ for $= 2,3,4, \dots$.

According to Theorem 4.1.4, $\tau = 2$.

2. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \le \bar{p}_{12} \le \bar{p}_{13} \le \bar{p}_{11} \le 1$.

$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}$. So $\overline{P}^n = \overline{P}^2$ for $= 2,3,4, \dots$.

According to Theorem 4.1.4, $\tau = 2$.

3. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \le \bar{p}_{13} \le \bar{p}_{12} \le \bar{p}_{11} \le 1$.

$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \overline{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}$.

So $\overline{P}^n = \overline{P}^3$ for $= 3,4,5, \dots$. According to Theorem 4.1.4, $\tau = 3$.

4. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \le \bar{p}_{12} \le \bar{p}_{13} \le \bar{p}_{11} \le 1$.

$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \overline{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \end{bmatrix}$,

$\overline{P}^5 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \end{bmatrix}$. So $\overline{P}^n = \overline{P}^4$ for $= 4,5,6, \dots$. According to Theorem 4.1.4, $\tau = 4$.

5. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \le \bar{p}_{13} \le \bar{p}_{12} \le \bar{p}_{11} \le 1$.

$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \overline{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \end{bmatrix},$

$\overline{P}^5 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \end{bmatrix}$. So $\overline{P}^n = \overline{P}^4$ for $= 4,5,6, \ldots$. According to Theorem 4.1.4,

$\tau = 4$.

6. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{12} \leq \bar{p}_{13} \leq \bar{p}_{11} \leq 1$.

$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}, \overline{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \end{bmatrix}.$

So $\overline{P}^n = \overline{P}^3$ for $= 3,4,5, \ldots$. According to Theorem 4.1.4, $\tau = 3$.

7. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{13} \leq \bar{p}_{12} \leq \bar{p}_{11} \leq 1$.

$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \overline{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Therefore, $\overline{P}^n = \begin{cases} \overline{P}^2, for\ n\ even \\ \overline{P}^3, for\ n\ odd \end{cases}.$

8. $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{12} \leq \bar{p}_{13} \leq \bar{p}_{11} \leq 1$.

$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \overline{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore,

$\overline{P}^n = \begin{cases} \overline{P}^2, for\ n\ even \\ \overline{P}^3, for\ n\ odd \end{cases}.$

It is clear that matrices in cases 1-6 are ergodic, while in cases 7 and 8 they are not.

Second: If we consider 2×2 and 3×3 fuzzy transition matrices satisfying Assumption 4.3.2 for $k = 2, k = 3$ respectively, then we have the following ergodic cases with a prescribe condition for each case:

1. $\bar{P} = \begin{bmatrix} 0 & 1 \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix}$ with $0 \leq \bar{p}_{21} \leq \bar{p}_{22} \leq 1$.

2. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \bar{p}_{31} & \bar{p}_{32} & \bar{p}_{33} \end{bmatrix}$ with $0 \leq \bar{p}_{31} \leq \bar{p}_{32} \leq \bar{p}_{33} \leq 1$.

3. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \bar{p}_{31} & \bar{p}_{32} & \bar{p}_{33} \end{bmatrix}$ with $0 \leq \bar{p}_{32} \leq \bar{p}_{31} \leq \bar{p}_{33} \leq 1$.

4. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \bar{p}_{31} & \bar{p}_{32} & \bar{p}_{33} \end{bmatrix}$ with $0 \leq \bar{p}_{31} \leq \bar{p}_{32} \leq \bar{p}_{33} \leq 1$.

5. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \bar{p}_{31} & \bar{p}_{32} & \bar{p}_{33} \end{bmatrix}$ with $0 \leq \bar{p}_{32} \leq \bar{p}_{31} \leq \bar{p}_{33} \leq 1$.

6. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \bar{p}_{31} & \bar{p}_{32} & \bar{p}_{33} \end{bmatrix}$ with $0 \leq \bar{p}_{31} \leq \bar{p}_{32} \leq \bar{p}_{33} \leq 1$.

7. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \bar{p}_{31} & \bar{p}_{32} & \bar{p}_{33} \end{bmatrix}$ with $0 \leq \bar{p}_{32} \leq \bar{p}_{31} \leq \bar{p}_{33} \leq 1$.

We conclude from cases 2-7 above that in addition to Assumption 4.3.2 the following conditions are satisfied:

1. $\bar{p}_{33}$ is the maximum among the entries in the last row.
2. $\bar{p}_{ii} \neq 1$ for $i = 1,2$.
3. If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \neq j, i, j = 1,2$.

Third: If we consider 3×3 fuzzy transition matrices satisfying Assumption 4.3.2 for $k = 2$, then we have the following ergodic cases with a prescribe condition for each case:

1. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{21} \leq \bar{p}_{23} \leq \bar{p}_{22} \leq 1$.
2. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{23} \leq \bar{p}_{21} \leq \bar{p}_{22} \leq 1$.
3. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{21} \leq \bar{p}_{23} \leq \bar{p}_{22} \leq 1$.
4. $\bar{P} = \begin{bmatrix} 0 & 0 & 1 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 0 & 1 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{23} \leq \bar{p}_{21} \leq \bar{p}_{22} \leq 1$.
5. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{21} \leq \bar{p}_{23} \leq \bar{p}_{22} \leq 1$.
6. $\bar{P} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ 1 & 0 & 0 \end{bmatrix}$ with $0 \leq \bar{p}_{23} \leq \bar{p}_{21} \leq \bar{p}_{22} \leq 1$.

We conclude from the above cases that in addition to Assumption 4.3.2 the following conditions are satisfied:

1. $\bar{p}_{22}$ is the maximum among the entries in the second row.
2. $\bar{p}_{ii} \neq 1$ for $i = 1,3$.
3. If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \neq j, i, j = 1,3$.

4.4 A Particular Class of Ergodic Finite Fuzzy Markov Chains

In this section we consider an $n \times n$ fuzzy transition matrix $\bar{P} = [\bar{p}_{ij}]$, $n \geq 4$ satisfying Assumption 4.3.2, and determine what conditions needed to guarantee the ergodic behavior. But first we need the following lemma which follows directly from the definition of the max-min composition of fuzzy matrices.

Lemma 4.4.1: Let $\bar{P} = [\bar{p}_{ij}]$ be an $n \times n$ fuzzy transition matrix. Then, by the max-min

composition $e_{nk}\bar{P}$ is the k^{th} row of $\bar{P}$, where $e_{nk} = [\delta_{jk}]$ is a $1 \times n$ matrix, and δ_{jk} is a Kronecker delta, and $j = 1, \dots n$, $k \in \{1, \dots, n\}$.

Now, we present the main theorem of this chapter. We consider an $n \times n$ fuzzy transition matrix $\bar{P}$ and under certain conditions we prove by the max-min composition that $\bar{P}$ is ergodic.

Theorem 4.4.2: For $n \geq 4$ let $\bar{P} = [\bar{p}_{ij}]$ be an $n \times n$ fuzzy transition matrix, such that $\bar{p}_{ij} = 0$ or 1 for $i = 2, \dots n$, $j = 1, \dots, n$ and in each row except possibly the first one, exactly one entry is 1. If the following conditions hold:

1. $\bar{p}_{11}$ is the maximum among the entries in the first row.

2. $\bar{p}_{ii} \neq 1$ for $i = 2, \dots, n$.

3. If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \neq j$, $i, j = 2, \dots, n$.

4. $\bar{p}_{i_1 1} = \bar{p}_{i_2 1} = \cdots = \bar{p}_{i_k 1} = 1$ with $k \in \{n-3, n-2, n-1\}$ and $i_1, i_2, \dots, i_k \in \{2,3, \dots, n\}$.

Then, by max-min composition, $\bar{P}$ is ergodic.

Proof. See the Appendix.

In the following, we show by examples that all conditions of Theorem 4.4.2 are sufficient.

Examples and Comments 4.4.3: We discuss the conditions of Theorem 4.4.2

1. If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ with $\bar{p}_{12} \geq \bar{p}_{11} \geq \bar{p}_{13} \geq \bar{p}_{14}$.

   Then by max-min composition we have

$$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix},$$

$$\overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}, \overline{P}^4 = \begin{bmatrix} \bar{p}_{12} & \bar{p}_{11} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}.$$

Therefore, $\overline{P}^{2n} = \overline{P}^2$ for $n = 1,2,3,\ldots$ and $\overline{P}^{2n+1} = \overline{P}^3$ for $n = 1,2,3,\ldots$. Hence, $\bar{P}$ is not ergodic. Here conditions 2, 3 and 4 are satisfied but condition 1 is not satisfied.

2. If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ with $\bar{p}_{11} \geq \bar{p}_{12} \geq \bar{p}_{13} \geq \bar{p}_{14}$, (note $\bar{p}_{22} = 1$).

   Then by max-min composition we have

   $$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}, \text{ so}$$

   $\overline{P}^n = \overline{P}^2$ for $n = 2,3,4,\ldots$. Therefore, $\bar{P}$ is not ergodic. Here conditions 1,3 and 4 are satisfied but condition 2 is not satisfied.

3. If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ with $\bar{p}_{11} \geq \bar{p}_{12} \geq \bar{p}_{13} \geq \bar{p}_{14}$,(note that $\bar{p}_{23} =$ 1and $\bar{p}_{32} = 1$). Then by max-min composition we have

   $$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \end{bmatrix},$$

$$\overline{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \end{bmatrix}, \overline{P}^5 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{12} & \bar{p}_{14} \end{bmatrix}.$$

So, $\overline{P}^{2n} = \overline{P}^4$ for $n = 2,3,4, \ldots$ and $\overline{P}^{2n+1} = \overline{P}^3$ for $= 1,2,3, \ldots$. Therefore, $\bar{P}$ is not ergodic. Here conditions 1,2 and 4 are satisfied but condition 3 is not satisfied.

4. If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ with $\bar{p}_{11} \geq \bar{p}_{12} \geq \bar{p}_{13} \geq \bar{p}_{14} \geq \bar{p}_{15}$,(note $\bar{p}_{21} = 1$ only). Then by max-min composition we have

$$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{14} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{14} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\overline{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \overline{P}^5 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\overline{P}^6 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \overline{P}^7 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\overline{P}^8 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \overline{P}^9 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{13} & \bar{p}_{13} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, $\overline{P}^{3n+1} = \overline{P}^4$, $\overline{P}^{3n+2} = \overline{P}^5$, and $\overline{P}^{3n+3} = \overline{P}^6$ for $n = 1,2,3, \dots$. Therefore, $\bar{P}$ is not ergodic. Here conditions 1,2 and 3 are satisfied but condition 4 is not satisfied.

5. If $\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ with $\bar{p}_{11} \geq \bar{p}_{12} \geq \bar{p}_{13} \geq \bar{p}_{14} \geq \bar{p}_{15}$,(note $\bar{p}_{21} = 1$ only). Then by max-min composition we have

$$\overline{P}^2 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \overline{P}^3 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{P}^4 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \overline{P}^5 = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \\ \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} & \bar{p}_{14} & \bar{p}_{15} \end{bmatrix},$$

$\overline{P}^5 = \overline{P}^6$. Hence, $\overline{P}^m = \overline{P}^5$ for $m = 5,6,7, \dots$. Therefore, $\bar{P}$ is ergodic.

We can notice from 5 above that even though condition 4 of Theorem 4.4.2 does not hold the result is satisfied.

We can notice from the above examples that conditions 1,2, and 3 of Theorem 4.4.2 can not be reduced, and condition 4 can be modified in a way that guarantees the result of Theorem 4.4.2.

Corollary 4.4.4: For $n \geq 4$ let $\bar{P} = [\bar{p}_{ij}]$ be an $n \times n$ fuzzy transition matrix, such that $\bar{p}_{ij} = 0$ or 1 for $i = 1, \dots n-1$, $j = 1, \dots, n$ and in each row except possibly the last one, exactly one entry is 1. If the following conditions hold:

1. $\bar{p}_{nn}$ is the maximum among the entries in the last row.

2. $\bar{p}_{ii} \neq 1$ for $i = 1, \dots, n-1$.

3. If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \neq j$, $i, j = 1, \dots, n-1$.

4. $\bar{p}_{i_1 n} = \bar{p}_{i_2 n} = \cdots = \bar{p}_{i_k n} = 1$ with $k \in \{n-3, n-2, n-1\}, i_1, i_2, \dots, i_k \in \{1,2, \dots, n-1\}$ Then, by max-min composition $\bar{P}$ is ergodic.

Proof. If $k = n-3$ then $\bar{p}_{i_1 n} = \bar{p}_{i_2 n} = \cdots = \bar{p}_{i_{n-3} n} = 1$, $i_1, i_2, \dots, i_{n-3} \in \{1,2, \dots, n-$ *1,* and *pin−1j1=pin−2j2=1* for *in−1,in−2∈1,2,…,n−1−i1,i2,…,in−3* for $j_1, j_2 \in \{1,2, \dots, n-1\}$. Either $i_{n-1} < i_{n-2}$ or $i_{n-1} > i_{n-2}$ we may assume that $i_{n-1} < i_{n-2}$.

Case 1: If $j_1 = j_2$ then

$$\bar{P} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ \bar{p}_{n1} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{n(n-1)} & \bar{p}_{nn} \end{bmatrix} \begin{array}{l} \\ \\ \\ \longleftarrow row\ i_{n-1} \\ \\ \\ \\ \longleftarrow row\ i_{n-2} \\ \\ \\ \\ \\ \end{array}$$

$$\text{Let } E_n = \begin{array}{r} \\ \\ \\ \\ row\ n-k+1 \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} e_{nn} \\ e_{n(n-1)} \\ \vdots \\ \vdots \\ e_{nk} \\ \vdots \\ e_{n1} \end{bmatrix} = \begin{array}{c} \begin{array}{ccccccc} & & & column\ n-k+1 & & & \\ & & & \downarrow & & & \end{array} \\ \begin{bmatrix} e_{nn}^T & e_{n(n-1)}^T & \cdots & e_{nk}^T & \cdots & e_{n2}^T & e_{n1}^T \end{bmatrix} \end{array}. \text{ Then,}$$

E_n is an $n \times n$ permutation matrix and $E_n E_n = I_n$. Consider $\bar{T} = E_n \bar{P} E_n$.

$$E_n\bar{P} = \begin{bmatrix} \bar{p}_{n1} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{n(n-1)} & \bar{p}_{nn} \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ \ n-i_{n-2}+1 \\ \\ \\ \\ \leftarrow row\ \ n-i_{n-1}+1 \\ \\ \\ \\ \end{array}$$

$$\bar{T} = E_n\bar{P}E_n = \begin{array}{cl} \begin{array}{cccccc} & & & \overset{column\, n-j_1+1}{\downarrow} & & \end{array} & \\ \left[\begin{array}{cccccc} \bar{p}_{nn} & \bar{p}_{n(n-1)} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{n_1} \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ \ n-i_{n-2}+1 \\ \\ \\ \\ \leftarrow row\ \ n-i_{n-1}+1 \\ \\ \\ \\ \end{array} \end{array}$$

Therefore, $\bar{T}$ satisfies the conditions of Theorem 4.4.2 and there is $k \in \mathbb{N}$ such that $\bar{T}^m = \bar{T}^K$, for $= k, k+1, k+2, \dots$, where the rows are identical in $\bar{T}^K$. Therefore, $\bar{T}^m = \bar{T}^K \Longrightarrow (E_n\bar{P}E_n)^m = \bar{T}^K$

$$\underbrace{(E_n\bar{P}E_n)(E_n\bar{P}E_n)\cdots\cdots(E_n\bar{P}E_n) = \bar{T}^K}_{m-times}$$

$$E_n\bar{P}I_n\bar{P}\cdots\cdots I_n\bar{P}E_n = \bar{T}^K \Longrightarrow E_n\bar{P}^m E_n = \bar{T}^K \Longrightarrow \bar{P}^m = E_n\bar{T}^K E_n\ ,\ \ m = k, k+1, \cdots \quad .$$

Since E_n is a permutation matrix we conclude that the rows are identical in $E_n\bar{T}^K E_n$. Hence, $\bar{P}$ is ergodic.

Case 2 : $j_1 \neq j_2$then either $j_1 < j_2$ or $j_1 > j_2$ we may assume that $j_1 < j_2$.

$$\bar{P} = \left[\begin{array}{cccccccc} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cdots & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \bar{p}_{n1} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{nj_2} & \cdots & \bar{p}_{n(n-1)} & \bar{p}_{nn} \end{array}\right] \begin{array}{l} \\ \\ \\ \leftarrow row\ \ i_{n-1} \\ \\ \\ \\ \leftarrow row\ \ i_{n-2} \\ \\ \\ \\ \\ \end{array}$$

We use the same E_n as in Case 1and consider the composition $\bar{T} = E_n\bar{P}E_n$

$$E_n\bar{P} = \left[\begin{array}{cccccccc} \bar{p}_{n1} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{nj_2} & \cdots & \bar{p}_{n(n-1)} & \bar{p}_{nn} \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ \ n - i_{n-2} + 1 \\ \\ \\ \\ \leftarrow row\ \ n - i_{n-1} + 1 \\ \\ \\ \\ \end{array}$$

$$\bar{T} = E_n\bar{P}E_n = \begin{array}{cc} \begin{array}{cccccccc} & & & \mathit{column}\ n-j_2+1 & & \mathit{column}\ n-j_1+1 & & \\ & & & \downarrow & & \downarrow & & \end{array} & \\ \begin{bmatrix} \bar{p}_{nn} & \bar{p}_{n(n-1)} & \cdots & \bar{p}_{nj_2} & \cdots & \bar{p}_{nj_1} & \cdots & \bar{p}_{n1} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow \mathit{row}\ \ n-i_{n-2}+1 \\ \\ \\ \\ \leftarrow \mathit{row}\ \ n-i_{n-1}+1 \\ \\ \\ \\ \end{array} \end{array}$$

Therefore $\bar{T}$ satisfies the conditions of Theorem 4.4.2 and as in Case 1 before we conclude that $\bar{P}$ is ergodic.

Similar argument applies for $k = n-2$ and $k = n-1$.

Corollary 4.4.5: For $n \geq 4$ let $\bar{P} = [\bar{p}_{ij}]$ be an $n \times n$ fuzzy transition matrix such that, $\bar{p}_{ij} = 0$ or 1 for $i \in \{1, \dots, n\} - \{k\}, j = 1, \dots, n,$ where $1 < k < n,$ and in each row except possibly the k^{th} one, exactly one entry is 1. If the following conditions hold:

1. $\bar{p}_{kk}$ is the maximum among the entries in the k^{th} row.

2. $\bar{p}_{ii} \neq 1$ for, $i \in \{1, \dots, n\} - \{k\}$.

3. If $\bar{p}_{ij} = 1$then $\bar{p}_{ji} = 0$ for $i \neq j, i, j \in \{1, \dots, n\} - \{k\}$.

4.$\bar{p}_{i_1k} = \bar{p}_{i_2k} = \cdots = \bar{p}_{i_lk} = 1$ where $l \in \{n-3, n-2, n-1\}$ and $i_1, i_2, \dots, i_l \in \{1, \dots, n\} - \{k\}$.

Then, by max-min composition $\bar{P}$ is ergodic.

Proof. If $l = n - 3$ then $\bar{p}_{i_1 k} = \bar{p}_{i_2 k} = \cdots = \bar{p}_{i_{n-3} k} = 1$, $i_1, i_2, \dots, i_{n-3} \in \{1, \dots, n\} - \{k\}$, and $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{1, \dots, n\} - \{i_1, i_2, \dots, i_{n-3}, k\}$ for $j_1, j_2 \in \{1, \dots, n\} - \{k\}$. Either $i_{n-1} < i_{n-2}$ or $i_{n-1} > i_{n-2}$ we may assume that $i_{n-1} < i_{n-2}$.

Case 1: If $j_1 = j_2$ then

$$\bar{P} = \left[\begin{array}{ccccccccc}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\bar{p}_{k1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\
\end{array}$$

Let $E_n = \left[\begin{array}{c}
e_{nk} \\ e_{n2} \\ e_{n3} \\ \vdots \\ \vdots \\ e_{n(k-1)} \\ e_{n1} \\ e_{n(k+1)} \\ e_{n(k+2)} \\ \vdots \\ e_{nn}
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ \\ \\ \leftarrow k^{th} row \\ \\ \\ \\ \\
\end{array}$

$$
\begin{array}{cccccccccc}
 & & & & & k^{th}column & & & & \\
 & & & & & \downarrow & & & & \\
=\Big[e_{nk}^T & e_{n2}^T & e_{n3}^T & \cdots & e_{n(k-1)}^T & e_{n1}^T & e_{n(k+1)}^T & e_{n(k+2)}^T & \cdots & e_{nn}^T\Big],
\end{array}
$$

then E_n is an $n \times n$ permutation matrix and $E_n E_n = I_n$. Consider the composition $\bar{T} = E_n \bar{P} E_n$.

$$
E_n\bar{P} = \left[\begin{array}{ccccccccc}
\bar{p}_{k1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\
\leftarrow row\ \ i_{n-1} \\
\\ \\
\leftarrow k^{th}row \\
\\ \\
\leftarrow row\ \ i_{n-2} \\
\\ \\ \\
\end{array}
$$

Hence, $\bar{T} = E_n \bar{P} E_n$

$$
= \left[\begin{array}{cccccccccc}
\bar{p}_{kk} & \bar{p}_{k2} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{k1} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\
\leftarrow row\ \ i_{n-1} \\
\\ \\
\leftarrow k^{th}\ row \\
\\ \\
\leftarrow row\ \ i_{n-2} \\
\\ \\ \\
\end{array}
$$

Therefore $\bar{T}$ satisfies the conditions of Theorem 4.4.2 and as in Case 1 of Corollary 4.4.4 before we conclude that $\bar{P}$ is ergodic.

Case 2: $j_1 \neq j_2$then either $j_1 < j_2$ or $j_1 > j_2$ we may assume that $j_1 < j_2$.

We use the same E_n as in Case 1 and consider the composition $\bar{T} = E_n \bar{P} E_n$

$$\bar{P} = \begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\bar{p}_{k1} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\begin{matrix}
\\ \\ \\
\leftarrow row\ \ i_{n-1} \\
\\ \\ \\ \\ \\ \\ \\
\leftarrow row\ \ i_{n-2} \\
\\ \\ \\
\end{matrix}$$

$$E_n\bar{P} = \begin{bmatrix}
\bar{p}_{k1} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\begin{matrix}
\\ \\ \\ \\
\leftarrow row\ \ i_{n-1} \\
\\ \\
\leftarrow k^{th}\ row \\
\\ \\
\leftarrow row\ \ i_{n-2} \\
\\ \\ \\
\end{matrix}$$

Hence, $\bar{T} = E_n \bar{P} E_n$

$$
= \begin{array}{c}
\begin{array}{ccccccccccc} & & & & & k^{th} column & & & & & \\ & & & & & \downarrow & & & & & \end{array} \\
\left[\begin{array}{ccccccccccc}
\bar{p}_{kk} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{k1} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\
1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\end{array}
\begin{array}{l}
\\ \\ \\ \\ \\ \\ \leftarrow row\ \ i_{n-1} \\ \\ \\ \leftarrow k^{th}\ row \\ \\ \\ \leftarrow row\ \ i_{n-2} \\ \\ \\ \\
\end{array}
$$

Therefore $\bar{T}$ satisfies the conditions of Theorem 4.4.2 and as in Case 1 of Corollary 4.4.4 before we conclude that $\bar{P}$ is ergodic.

Similar argument applies for $l = n - 2$ and $l = n - 1$.

Remark 4.4.6: We prove Corollary 4.4.5 using another permutation matrix E_n as follows:

If $l = n - 3$ then $\bar{p}_{i_1 k} = \bar{p}_{i_2 k} = \cdots = \bar{p}_{i_{n-3} k} = 1$, $i_1, i_2, \dots, i_{n-3} \in \{1, \dots, n\} - \{k\}$, and $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{1, \dots, n\} - \{i_1, i_2, \dots, i_{n-3}, k\}$ for $j_1, j_2 \in \{1, \dots, n\} - \{k\}$. Either $i_{n-1} < i_{n-2}$ or $i_{n-1} > i_{n-2}$ we may assume that $i_{n-1} < i_{n-2}$.

Case 1: If $j_1 = j_2$ then

$$\bar{P} = \left[\begin{array}{ccccccccc}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\bar{p}_{k1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \leftarrow row \;\; i_{n-1} \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow row \;\; i_{n-2} \\ \\ \\ \\
\end{array}$$

$$\text{Let } E_n = \left[\begin{array}{c}
e_{nk} \\ e_{n(n-1)} \\ e_{n(n-2)} \\ \vdots \\ \vdots \\ e_{n(n-k+2)} \\ e_{n1} \\ e_{n(n-k)} \\ \vdots \\ e_{n2} \\ e_{nn}
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ \\ \\ \leftarrow k^{th}\, row \\ \\ \\ \\ \\
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & & & & k^{th} column & & & & & \\
 & & & & & \downarrow & & & & & \\
= \big[e_{nk}^T & e_{n(n-1)}^T & e_{n(n-2)}^T & \cdots & e_{n(n-k+2)}^T & e_{n1}^T & e_{n(n-k)}^T & \cdots & e_{n2}^T & e_{nn}^T\big].
\end{array}$$

Then, E_n is an $n \times n$ permutation matrix and $E_n E_n = I_n$.

Consider the composition $\bar{T} = E_n \bar{P} E_n$.

$$
E_n\bar{P} = \begin{array}{cl}
\begin{bmatrix}
\bar{p}_{k1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
&
\begin{array}{l}
\\ \\ \\ \\
\leftarrow row\ \ n - i_{n-2} + 1 \\
\\ \\
\leftarrow k^{th}\ row \\
\\ \\
\leftarrow row\ \ n - i_{n-1} + 1 \\
\\ \\ \\
\end{array}
\end{array}
$$

$$
\bar{T} = \begin{array}{cl}
\begin{array}{ccc}
column\, n - j_1 + 1 & & k^{th}\, column \\
\downarrow & & \downarrow
\end{array} & \\
\begin{bmatrix}
\bar{p}_{kk} & \bar{p}_{k(n-1)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(n-k+2)} & \bar{p}_{k1} & \bar{p}_{k(n-k)} & \cdots & \bar{p}_{k2} & \bar{p}_{kn} \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
&
\begin{array}{l}
\\ \\ \\ \\
\leftarrow^{1} \\
\\ \\
\leftarrow^{2} \\
\\ \\
\leftarrow^{3} \\
\\ \\ \\
\end{array}
\end{array}
$$

1: $row\ \ n - i_{n-2} + 1$,

2: $k^{th}\ row$,

3: $row\ \ n - i_{n-1} + 1$.

Therefore, $\bar{T}$ satisfies the conditions of Theorem 4.4.2 and as in Case 1 of Corollary 4.4.4 before we conclude that $\bar{P}$ is ergodic.

Case 2: $j_1 \neq j_2$then either $j_1 < j_2$ or $j_1 > j_2$ we may assume that $j_1 < j_2$.

We use the same E_n as in Case 1 and consider the composition $\bar{T} = E_n \bar{P} E_n$.

$$\bar{P} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{k1} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{matrix}$$

$$E_n\bar{P} = \left[\begin{array}{ccccccccccc}
\bar{p}_{k1} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k(k-1)} & \bar{p}_{kk} & \bar{p}_{k(k+1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{kn} \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ \leftarrow^{1} \\ \\ \\ \leftarrow^{2} \\ \\ \\ \leftarrow^{3} \\ \\ \\ \\
\end{array}$$

1: $row\ \ n - i_{n-2} + 1$,

2: k^{th} row,

3: $row\ \ n - i_{n-1} + 1$.

$$
\bar{T} = \begin{array}{c}
\begin{matrix} & & & \text{column } n-j_2+1 & & & k^{th}\text{column} & & & \text{column } n-j_1+1 & & & \\ & & & \downarrow & & & \downarrow & & & \downarrow & & & \end{matrix} \\
\begin{bmatrix}
\bar{p}_{kk} & \bar{p}_{k(n-1)} & \cdots & \bar{p}_{kj_2} & \cdots & \bar{p}_{k(n-k+2)} & \bar{p}_{k1} & \bar{p}_{k(n-k)} & \cdots & \bar{p}_{kj_1} & \cdots & \bar{p}_{k2} & \bar{p}_{kn} \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{bmatrix}
\end{array}
\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \leftarrow^{1} \\ \\ \\ \leftarrow^{2} \\ \\ \\ \leftarrow^{3} \\ \\ \\ \\ \end{matrix}
$$

Therefore $\bar{T}$ satisfies the conditions of Theorem 4.4.2 and as in Case 1 of Corollary 4.4.4 before we conclude that $\bar{P}$ is ergodic.

Similar argument applies for $l = n-2$ and $l = n-1$.

1: $row\ n - i_{n-2} + 1$,

2: $k^{th}\ row$,

3: $row\ n - i_{n-1} + 1$.

Remark 4.4.7: We use Corollary 4.4.4 to prove Corollary 4.4.5 using either the permutation matrix E_n or P_n, where,

E_n given by:

$$\begin{array}{cc} E_n = \begin{bmatrix} e_{n1} \\ e_{n2} \\ \vdots \\ \vdots \\ e_{n(k-1)} \\ e_{nn} \\ e_{n(k+1)} \\ \vdots \\ \vdots \\ e_{n(n-1)} \\ e_{nk} \end{bmatrix} & \begin{array}{c} \\ \\ \\ \\ \\ \leftarrow k^{th}\ row \\ \\ \\ \\ \\ \\ \end{array} \end{array}$$

$$\begin{array}{cccccccccccc} & & & & & k^{th} column & & & & & \\ & & & & & \downarrow & & & & & \\ = \big[e_{n1}^T & e_{n2}^T & \cdots & \cdots & e_{n(k-1)}^T & e_{nn}^T & e_{n(k+1)}^T & \cdots & \cdots & e_{n(n-1)}^T & e_{nk}^T\big], \end{array}$$ and $E_n E_n = I_n$.

P_n given by:

$$\begin{array}{cc} P_n = \begin{bmatrix} e_{n1} \\ e_{n(n-1)} \\ e_{n(n-2)} \\ \vdots \\ \vdots \\ e_{n(n-k+2)} \\ e_{nn} \\ e_{n(n-k)} \\ \vdots \\ e_{n2} \\ e_{nk} \end{bmatrix} & \begin{array}{c} \\ \\ \\ \\ \\ \\ \leftarrow k^{th}\ row \\ \\ \\ \\ \\ \end{array} \end{array}$$

$$\begin{array}{cccccccccc} & & & & & k^{th} column & & & & \\ & & & & & \downarrow & & & & \\ = \big[e_{n1}^T & e_{n(n-1)}^T & e_{n(n-2)}^T & \cdots & e_{n(n-k+2)}^T & e_{nn}^T & e_{n(n-k)}^T & \cdots & e_{n2}^T & e_{nk}^T\big] \end{array}$$ and

$P_n P_n = I_n$.

Similar to the above arguments in corollaries 4.4.4, 4.4.5, and 4.4.6, we consider the max-min composition $E_n \bar{P} E_n$ or $P_n \bar{P} P_n$ in either case the resulting matrix satisfies the conditions of Corollary 4.4.4, and as in Case 1 of Corollary 4.4.4 before we conclude that $\bar{P}$ is ergodic.

Conclusions:

In this work, we studied finite fuzzy Markov chains concentrating on their ergodic behavior. The limit of powers of 2×2 regular fuzzy transition matrices was studied. The uniqueness of that limit under certain conditions was proved for a special fuzzy number in addition to the triangular and trapezoidal cases. On the other hand, we classified the fuzzy states similar to the crisp states and we presented the similarities and the differences between them. Then, we studied the ergodicity of a particular class of finite fuzzy Markov chains where exactly one row of the transition matrices consists of arbitrary values (between zero and one) while the other rows' entries are one in one place and zero elsewhere.

In this work, we studied fuzzy Markov chains in two ways, one by considering the classical (crisp) Markov chains, and replacing the uncertainties in the transition matrix by fuzzy numbers, then using the restricted fuzzy matrix multiplication to find the powers of the resulted matrix. Another way is by considering the fuzzy transition matrix of a fuzzy Markov chain as a fuzzy relation on a finite state space; in this case the states are fuzzy sets. In fact, there is a third way to study the fuzzy Markov chains using the concept of *Possibility Measure [32]*, from which the *transition fuzzy possibility* and the *transition fuzzy possibility matrix* were defined [28].

Finally, in this work we studied the *finite* and *stationary (homogeneous)* fuzzy Markov chains. As a future work, we recommend studying the possibility of generalizing the basic properties of classical Markov chains to the *infinite* fuzzy Markov chains and to the *non-stationary (non-homogeneous)* fuzzy Markov chains.

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Appendix:

In this appendix we give the proof of Theorem 4.4.2.

Proof. If $k = n-3$ then $\bar{p}_{i_1 1} = \bar{p}_{i_2 1} = \cdots = \bar{p}_{i_{n-3} 1} = 1$, $i_1, i_2, \dots, i_{n-3} \in \{2,3,\dots,n\}$, and $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{2,3,\dots,n\} - \{i_1, i_2, \dots, i_{n-3}\}$ for $j_1, j_2 \in \{2,3,\dots,n\}$. Either $i_{n-1} < i_{n-2}$ or $i_{n-1} > i_{n-2}$ we may assume that $i_{n-1} < i_{n-2}$.

Let $R_i^{(m)}$ denote the i^{th} row in $\overline{P}^m$ (the m^{th} power of $\bar{P}$), then $R_i^{(m+1)} = R_i^{(1)} \overline{P}^m$. During the proof $R_1^{(m+1)}$ will be computed by $R_1^{(m+1)} = R_1^{(m)} \bar{P}$ and $R_i^{(m+1)} = R_i^{(1)} \overline{P}^m$ for $= 2, \dots, n$.

Now we consider two cases:

Case 1: $j_1 = j_2$ then $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_1} = 1$ for $i_{n-1}, i_{n-2} \in \{2,3,\dots,n\} - \{i_1, i_2, \dots, i_{n-3}\}$, and $j_1 = i_k$ for some $k \in \{1,2,\dots,n-3\}$ otherwise (i.e. $j_1 = i_{n-1}$ or $j_1 = i_{n-2}$) we have $\bar{p}_{j_1 j_1} = 1$ which contradicts condition 2.

$$\bar{P} = \begin{array}{c} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array}$$

Consider, $\overline{P}^2$ then $R_{i_1}^{(2)} = R_{i_2}^{(2)} = \cdots = R_{i_{n-3}}^{(2)} = R_1^{(1)}$, $R_{i_{n-1}}^{(2)} = R_{i_{n-2}}^{(2)} = R_{j_1}^{(1)} = R_{i_k}^{(1)}$ by Lemma 4.4.1. $R_1^{(2)} = \left[\bar{p}_{1j}^{(2)}\right]$, for $j \neq j_1$ $\bar{p}_{1j}^{(2)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(2)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}, \bar{p}_{1i_{n-2}}\}$ by condition 1. Therefore,

$$\overline{P}^2 = \begin{array}{cl} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Consider, $\overline{P}^3$ then $R_{i_1}^{(3)} = R_{i_2}^{(3)} = \cdots = R_{i_{n-3}}^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(3)} = R_{i_{n-2}}^{(3)} = R_{j_1}^{(2)} = R_{i_k}^{(2)} = R_1^{(1)}$ by Lemma 4.4.1. $R_1^{(3)} = \left[\bar{p}_{1j}^{(3)}\right]$, $\forall j \neq j_1$ $\bar{p}_{1j}^{(3)} = \bar{p}_{1j}$, and $\bar{p}_{1j_1}^{(3)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_1}^{(2)}$ by condition 1. So, $R_1^{(3)} = R_1^{(2)}$. If $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ then $R_1^{(2)} = R_1^{(1)}$, so in $\overline{P}^3$ we have $R_1^{(3)} = R_2^{(3)} = \cdots = R_n^{(3)} = R_1^{(1)}$. It is obvious that $\overline{P}^4 = \overline{P}^3$. Hence, $\overline{P}^m = \overline{P}^3$ for $m = 3,4,5, \ldots$. Therefore, $\bar{P}$ is ergodic. If $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$ then

$$\overline{P}^3 = \begin{array}{c} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1n} \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array}$$

Consider, $\overline{P}^4$ then

$R_{i_1}^{(4)} = R_{i_2}^{(4)} = \cdots = R_{i_{n-3}}^{(4)} = R_1^{(3)} = R_1^{(2)}, R_{i_{n-1}}^{(4)} = R_{i_{n-2}}^{(4)} = R_{j_1}^{(3)} = R_{i_k}^{(3)} = R_1^{(2)}$ by Lemma 4.4.1. $R_1^{(4)} = \left[\bar{p}_{1j}^{(4)}\right]$, for $j \neq j_1$, $\bar{p}_{1j}^{(4)} = \bar{p}_{1j}$, and $\bar{p}_{1j_1}^{(4)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$, by condition1. So $R_1^{(4)} = R_1^{(2)}$, and so in $\overline{P}^4$ we have, $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(2)}$. It is obvious that $\overline{P}^5 = \overline{P}^4$. Hence, $\overline{P}^m = \overline{P}^4$ for $= 4,5,6, \ldots$. Therefore, $\bar{P}$ is ergodic. Similarly if $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-2}}$ then $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(2)}$. Hence, $\overline{P}^m = \overline{P}^4$ for $= 4,5,6, \ldots$. Therefore, $\bar{P}$ is ergodic.

Case 2: $j_1 \neq j_2$ then either $j_1 < j_2$ or $j_1 > j_2$ we may assume that $j_1 < j_2$.

So $\bar{p}_{i_{n-1}j_1} = \bar{p}_{i_{n-2}j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{2,3, \ldots, n\} - \{i_1, i_2, \ldots, i_{n-3}\}$, and $j_1, j_2 = \{2,3, \ldots, n\}$.

Now we have the following subcases:

(1) $j_1 = i_k, j_2 = i_m$, where $k, m \in \{1,2,\dots,n-3\}$.

(2) $j_1 = i_k, j_2 = i_{n-1}$, where $k \in \{1,2,\dots,n-3\}$.

(3) $j_1 = i_{n-2}, j_2 = i_k$, where $k \in \{1,2,\dots,n-3\}$.

Note that the cases $j_1 = i_{n-1}, j_2 = i_{n-2}$ and $j_1 = i_{n-2}, j_2 = i_{n-1}$ are not taken into account since they contradict conditions 2 and 3 respectively. Again we keep in mind that $i_{n-1} < i_{n-2}$, and continue with this assumption throughout the proof.

We first deal with the subcase (1):

$$\bar{P} = \begin{array}{cl} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Consider, $\overline{P}^2$ then $R_{i_1}^{(2)} = R_{i_2}^{(2)} = \cdots = R_{i_{n-3}}^{(2)} = R_1^{(1)}$, $R_{i_{n-1}}^{(2)} = R_{j_1}^{(1)} = R_{i_k}^{(1)}$, and $R_{i_{n-2}}^{(2)} = R_{j_2}^{(1)} = R_{i_m}^{(1)}$ by Lemma 4.4.1 . $R_1^{(2)} = \left[\bar{p}_{1j}^{(2)}\right]$, for $j \neq j_1, j_2$, $\bar{p}_{1j}^{(2)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(2)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\}, \bar{p}_{1j_2}^{(2)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\}$, by condition 1.

$$\overline{P}^2 = \begin{array}{c@{}l} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Consider, $\overline{P}^3$ then

$R_{i_1}^{(3)} = R_{i_2}^{(3)} = \cdots = R_{i_{n-3}}^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(3)} = R_{j_1}^{(2)} = R_{i_k}^{(2)} = R_1^{(1)}$, $R_{i_{n-2}}^{(3)} = R_{j_2}^{(2)} = R_{i_m}^{(2)} = R_1^{(1)}$ by Lemma 4.4.1 . $R_1^{(3)} = \left[\bar{p}_{1j}^{(3)}\right]$, for $j \neq j_1, j_2$, $\bar{p}_{1j}^{(3)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(3)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = \bar{p}_{1j_1}^{(2)}$, $\bar{p}_{1j_2}^{(3)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_2}^{(2)}$, by condition 1. So $R_1^{(3)} = R_1^{(2)}$. We have the following subcases:

i. $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1j_2}$.

ii. $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

iii. $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1j_2}$.

iv. $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

For the subcase i, $R_1^{(3)} = R_2^{(3)} = \cdots = R_n^{(3)} = R_1^{(1)}$ and it is obvious that $\overline{P}^4 = \overline{P}^3$. Hence, $\overline{P}^m = \overline{P}^3$ for $= 3,4,5, \ldots$. Therefore, $\bar{P}$ is ergodic.

For the subcases ii and iii we need to find $\overline{P}^4$ from which we have –as previously shown in Case 1- $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(2)}$ and it is obvious that $\overline{P}^5 = \overline{P}^4$. Hence, $\overline{P}^m = \overline{P}^4$ for $= 4,5,6,\ldots$. Therefore, $\bar{P}$ is ergodic.

For the subcase iv:

$$\overline{P}^3 = \begin{array}{cl} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Consider $\overline{P}^4$, then $R_{i_1}^{(4)} = R_{i_2}^{(4)} = \cdots = R_{i_{n-3}}^{(4)} = R_1^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(4)} = R_{j_1}^{(3)} = R_{i_k}^{(3)} = R_1^{(2)}$, $R_{i_{n-2}}^{(4)} = R_{j_2}^{(3)} = R_{i_m}^{(3)} = R_1^{(2)}$ by Lemma 4.4.1 . $R_1^{(4)} = \left[\bar{p}_{1j}^{(4)}\right]$, for $j \neq j_1, j_2$, $\bar{p}_{1j}^{(4)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(4)} = max\{\bar{p}_{1i_{n-1}}, \bar{p}_{1j_1}\} = \bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$, $\bar{p}_{1j_2}^{(4)} = max\{\bar{p}_{1i_{n-2}}, \bar{p}_{1j_2}\} = \bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-2}}$. In $\overline{P}^4$ we have $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(2)}$ and it is obvious that $\overline{P}^5 = \overline{P}^4$. Hence, $\overline{P}^m = \overline{P}^4$ for $= 4,5,6,\ldots$. Therefore, $\bar{P}$ is ergodic.

Next, we deal with the subcase (2) in which $j_1 = i_k, j_2 = i_{n-1}$, where $k \in \{1,2,\ldots,n-3\}$.

$$\bar{P} = \left[\begin{array}{cccccccc}
\bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\
1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\
\end{array}$$

Consider, $\overline{P}^2$ then,

$R_{i_1}^{(2)} = R_{i_2}^{(2)} = \cdots = R_{i_{n-3}}^{(2)} = R_1^{(1)}$, $R_{i_{n-1}}^{(2)} = R_{j_1}^{(1)} = R_{i_k}^{(1)}$, $R_{i_{n-2}}^{(2)} = R_{j_2}^{(1)} = R_{i_{n-1}}^{(1)}$ by Lemma 4.4.1 . $R_1^{(2)} = \left[\bar{p}_{1j}^{(2)}\right]$, for $\neq j_1, j_2$ $\bar{p}_{1j}^{(2)} = \bar{p}_{1j}$, but condition 1 implies that

$$\bar{p}_{1j_1}^{(2)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = max\{\bar{p}_{1j_1}, \bar{p}_{1j_2}\},$$

$$\bar{p}_{1j_2}^{(2)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\} = max\{\bar{p}_{1i_{n-1}}, \bar{p}_{1i_{n-2}}\}.$$

$$\overline{P}^2 = \left[\begin{array}{cccccccc}
\bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\
\bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\
1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\
0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\
\bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n}
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\
\end{array}$$

Consider, $\overline{P}^3$ then $R_{i_1}^{(3)} = R_{i_2}^{(3)} = \cdots = R_{i_{n-3}}^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(3)} = R_{j_1}^{(2)} = R_{i_k}^{(2)} = R_1^{(1)}$, $R_{i_{n-2}}^{(3)} = R_{j_2}^{(2)} = R_{i_{n-1}}^{(2)} = R_{j_1}^{(1)} = R_{i_k}^{(1)}$by Lemma 4.4.1 . $R_1^{(3)} = \left[\bar{p}_{1j}^{(3)}\right]$, for $j \neq j_1, j_2$ $\bar{p}_{1j}^{(3)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(3)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = \bar{p}_{1j_1}^{(2)}$, $\bar{p}_{1j_2}^{(3)} = max\{\bar{p}_{1i_{n-2}}, \bar{p}_{1j_2}\} = \bar{p}_{1j_2}^{(2)}$. So, $R_1^{(3)} =$ *R12*.

$$\overline{P}^3 = \begin{array}{cl} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1}^{(2)} & \cdots & \bar{p}_{1j_2}^{(2)} & \cdots & \bar{p}_{1n} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array} \end{array}$$

Consider, $\overline{P}^4$ then $R_{i_1}^{(4)} = R_{i_2}^{(4)} = \cdots = R_{i_{n-3}}^{(4)} = R_1^{(3)} = R_1^{(2)}$, $R_{i_{n-1}}^{(4)} = R_{j_1}^{(3)} = R_{i_k}^{(3)} = R_1^{(2)}$, $R_{i_{n-2}}^{(4)} = R_{j_2}^{(3)} = R_{i_{n-1}}^{(3)} = R_{j_1}^{(2)} = R_{i_k}^{(2)} = R_1^{(1)}$ by Lemma 4.4.1 . $R_1^{(4)} = \left[\bar{p}_{1j}^{(4)}\right]$, for $j \neq j_1, j_2$ $\bar{p}_{1j}^{(4)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(4)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = \bar{p}_{1j_1}^{(2)}$, $\bar{p}_{1j_2}^{(4)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_2}^{(2)}$, by condition 1.

So, $R_1^{(4)} = R_1^{(2)}$. As before, we have the following subcases:

i. $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1j_2}$.

ii. $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1j_1}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

iii. $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1j_2}$.

iv. $\bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$ and $\bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

For the subcase i, $R_1^{(4)} = R_2^{(4)} = \cdots = R_n^{(4)} = R_1^{(1)}$ and it is obvious that $\overline{P}^5 = \overline{P}^4$. Hence, $\overline{P}^m = \overline{P}^4$ for $m = 4,5,6,\dots$. Therefore, $\bar{P}$ is ergodic.

For the subcases ii and iii we need to find $\bar{P}^5$ from which we have $R_1^{(5)} = R_2^{(5)} = \cdots = R_n^{(5)} = R_1^{(4)} = R_1^{(2)}$ and it is obvious that $\overline{P}^6 = \overline{P}^5$. Hence, $\overline{P}^m = \overline{P}^5$ for $= 5,6,7,\dots$. Therefore, $\bar{P}$ is ergodic.

For the subcase iv:

$$\overline{P}^4 = \begin{array}{c} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1j_1} & \cdots & \bar{p}_{1j_2} & \cdots & \bar{p}_{1n} \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1i_{n-1}} & \cdots & \bar{p}_{1i_{n-2}} & \cdots & \bar{p}_{1n} \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \leftarrow row\ i_{n-1} \\ \\ \\ \\ \leftarrow row\ i_{n-2} \\ \\ \\ \\ \end{array}$$

Consider, $\overline{P}^5$ then $R_{i_1}^{(5)} = R_{i_2}^{(5)} = \cdots = R_{i_{n-3}}^{(5)} = R_1^{(4)} = R_1^{(2)}$, $R_{i_{n-1}}^{(5)} = R_{j_1}^{(4)} = R_{i_k}^{(4)} = R_1^{(2)}$, $R_{i_{n-2}}^{(5)} = R_{j_2}^{(4)} = R_{i_{n-1}}^{(4)} = R_{j_1}^{(3)} = R_{i_k}^{(3)} = R_1^{(2)}$ by Lemma 4.4.1 . $R_1^{(5)} = \left[\bar{p}_{1j}^{(5)}\right]$, for $j \neq j_1, j_2$ $\bar{p}_{1j}^{(5)} = \bar{p}_{1j}$ and $\bar{p}_{1j_1}^{(5)} = max\{\bar{p}_{1j_1}, \bar{p}_{1i_{n-1}}\} = \bar{p}_{1j_1}^{(2)} = \bar{p}_{1i_{n-1}}$, $\bar{p}_{1j_2}^{(5)} = max\{\bar{p}_{1j_2}, \bar{p}_{1i_{n-2}}\} = \bar{p}_{1j_2}^{(2)} = \bar{p}_{1i_{n-2}}$.

So, $R_1^{(5)} = R_1^{(2)}$. In $\overline{P}^5$, $R_1^{(5)} = R_2^{(5)} = \cdots = R_n^{(5)} = R_1^{(3)} = R_1^{(2)}$ and it is obvious that $\overline{P}^6 = \overline{P}^5$.

Hence, $\overline{P}^m = \overline{P}^5$ for $m = 5,6,7,\dots$. Therefore, $\bar{P}$ is ergodic.

For the subcase (3) in which we have $j_1 = i_{n-2}, j_2 = i_k$, where $k \in \{1,2,\dots,n-3\}$, we deal with it similar to the subcase (2) before.

We have proved the result when $k = n-3$ so $\bar{p}_{i_1 1} = \bar{p}_{i_2 1} = \cdots = \bar{p}_{i_{n-3} 1} = 1$, $i_1, i_2, \dots, i_{n-3} \in \{2,3,\dots,n\}$, and $\bar{p}_{i_{n-1} j_1} = \bar{p}_{i_{n-2} j_2} = 1$ for $i_{n-1}, i_{n-2} \in \{2,3,\dots,n\} - \{i_1, i_2, \dots, i_{n-3}\}$, $j_1, j_2 \in \{2,3,\dots,n\}$.

Similarly we can prove the theorem when $k = n-2$ so $\bar{p}_{i_1 1} = \bar{p}_{i_2 1} = \cdots = \bar{p}_{i_{n-2} 1} = 1$, $i_1, i_2, \dots, i_{n-2} \in \{2,3,\dots,n\}$, and $\bar{p}_{i_{n-1} j_1} = 1$ for $i_{n-1} \in \{2,3,\dots,n\} - \{i_1, i_2, \dots, i_{n-2}\}$, $j_1 \in \{2,3,\dots,n\}$.

Finally, for the case $k = n-1$, we have $\bar{p}_{21} = \bar{p}_{31} = \cdots = \bar{p}_{n1} = 1$ and by considering $\overline{P}^2$ we get $R_1^{(2)} = R_2^{(2)} = \cdots = R_n^{(2)} = R_1^{(1)}$. It is obvious that $\overline{P}^m = \overline{P}^2$ for $m = 2,3,4,\dots$. Therefore, $\bar{P}$ is ergodic, and this completes the proof.