EPJ Plus

Eur. Phys. J. Plus (2013) **128**: 2 DOI 10.1140/epjp/i2013-13002-8

Infinite face-centered-cubic network of identical resistors: Application to lattice Green's function

J.H. Asad, A.A. Diab, R.S. Hijjawi and J.M. Khalifeh

Società Italiana di Fisica

Springer

Eur. Phys. J. Plus (2013) **128**: 2
DOI 10.1140/epjp/i2013-13002-8

THE EUROPEAN
PHYSICAL JOURNAL PLUS

Regular Article

Infinite face-centered-cubic network of identical resistors: Application to lattice Green's function

J.H. Asad[1,a], A.A. Diab[2], R.S. Hijjawi[3], and J.M. Khalifeh[4]

[1] Department of Physics, Tabuk University - P. O Box 741- Tabuk 71491, Saudi Arabia
[2] General Studies Department, Yanbu Industrial College - P. O Box 30436, Yanbu Industrial City, KSA
[3] Department of Physics, Mutah University - Karak, Jordan
[4] Department of Physics, Jordan University - Amman 11942, Jordan

Received: 15 October 2012 / Revised: 19 November 2012
Published online: 9 January 2013 –

Abstract. The equivalent resistance between the origin and any other lattice site, in an infinite face-centered-cubic network consisting of identical resistors, has been expressed rationally in terms of the known value $f_o(3; 0, 0, 0)$ and π. The asymptotic behavior is investigated, and some calculated values for the equivalent resistance are presented.

1 Introduction

It is well known that the theory of Green's function plays an important role in physics. It has been used, *e.g.*, in statistical models of ferromagnetism such as Ising model [1], Heisenberg model [2], spherical model [3], random walks and percolation theories [4,5], diffusion [6], band structure [7], and other branches in physics. In condensed matter the Lattice Green's Function (LGF) is widely applied and considered to be a basic function [8]. It appears especially when impure solids are studied [9]. The LGF for several lattices has been widely studied. One can see from the literature that most of the studies on lattice Green's functions are based on an elliptic-type integral approach. In the literature, generally, the formulas for the lattice Green's functions have been given in terms of elliptic-type integrals and related functions.

The LGF for Face-Centered Cubic (FCC) lattice was studied well by Iwata [10]. He expressed $f_o(3; 0, 0, 0)$ (*i.e.*, the LGF at the origin of a FCC lattice) in a compact form as a product of complete elliptic integrals of the first kind. Much effort has been focused on the value of that function at the origin, although we need also the values at various lattice sites in many problems. Inoue [11] showed that $F(E; l, m, n)$ (*i.e.*, the LGF at an arbitrary lattice site (l, m, n) in FCC lattice) with nearest neighbor interactions can be expressed in terms of linear combinations of products of complete elliptic integrals of the first and second kinds. In his work, he introduced many recurrence formulae. Morita [12] showed that $F(E; l, m, n)$ can be calculated from the three known values at the lattice sites $(0, 0, 0)$, $(2, 0, 0)$, and $(2, 2, 0)$, with the aid of the recurrence formula presented by Inoue [11]. In a previous work, Hijjawi *et al.* [13] presented an expression for $F(E; l, m, n)$ in which it was evaluated analytically and numerically for a single impurity problem. The density of states, phase shift and scattering cross-section were expressed in terms of complete elliptic integrals of the first kind.

The calculation of the effective resistance in infinite networks of identical resistors is a classical problem in the circuit theory. It attracts the attention of many authors since Kirchhoff's [14] who formulated the study of electric networks more than 150 years ago. In addition, the electric circuit theory was discussed in details by Van der Pol and Bremmer [15], where they derived the resistance between nearby points on the square lattice. Many methods have been used to study different infinite networks consisting of identical resistors such as the random walk method [16,17], the current distribution method [18,19], and, recently, an important method based on the LGF has been introduced by Cserti [20,21]. Based on this method many studies have been carried out for many networks consisting of identical resistors [22–24]. More recently, one can see that this field is still of interest for many authors where many projects have been carried out by different authors [25–28].

[a] e-mail: jhasad1@yahoo.com

This paper is organized as follows: in sect. 2, a short revision is carried out on the origin of the problem (*i.e.*, an infinite d-dimensional hypercube network consisting of identical resistors). In sect. 3, we apply the formalism to the FCC network. In sect. 4, results and conclusions are presented.

2 Hypercubic lattices

Consider an infinite d-dimensional lattice consisting of identical resistors each of resistance R. We assume that all lattice points in the infinite lattice are specified by the position vector

$$\vec{r} = l_1\vec{a}_1 + l_2\vec{a}_2 + \ldots + l_d\vec{a}_d. \tag{1}$$

where $l_1, l_2, \ldots, l_d$ are integers (positive, negative or zero), and $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_d$ are independent primitive translation vectors.

When all $\vec{a}_i$'s have the same magnitude (*i.e.* $|\vec{a}_1| = |\vec{a}_2| = \ldots = |\vec{a}_d| = a$), then the d-dimensional lattice is called a hypercube.

Our aim is to find the equivalent resistance between the origin and any other lattice site $\vec{r}$. To do this, let the potential at the lattice site $\vec{r}\,'$ be defined as $V(\vec{r}\,')$, and assume the current in the network defined as

$$I(\vec{r}\,') = I\left[\delta_{\vec{r}\,',0} - \delta_{\vec{r}\,',\vec{r}}\right]. \tag{2}$$

According to Ohm's and Kirchhoff's laws we can write

$$I(\vec{r}\,')R_o(\vec{r}\,') = \sum_n \left[V(\vec{r}\,') - V(\vec{r}\,' + \vec{n}\,)\right], \tag{3}$$

where $\vec{n}$ are the vectors from the site $\vec{r}\,'$ to its nearest neighbors (*i.e.*, $\vec{n} = \pm\vec{a}_i, i = 1, 2, \ldots, d$).

Using the so-called lattice Laplacian defined on the hypercubic lattice [19] as

$$\Delta_{(\vec{r}\,')} f(\vec{r}\,) = \sum_n \left[f(\vec{r} + \vec{n}\,) - f(\vec{r}\,)\right], \tag{4}$$

one can rewrite eq. (3) as

$$\Delta_{(\vec{r}\,')} V(\vec{r}\,') = -I(\vec{r}\,')R_o(\vec{r}\,'). \tag{5}$$

From eqs. (2) and (5) we can write

$$R_o(\vec{r}\,) = \frac{V(0) - V(\vec{r}\,)}{I}. \tag{6}$$

To find the resistance we need to solve eq. (5), which is a Poisson-like equation and it may be solved using the LGF, so one may write (comparing with Poisson's equation)

$$V(\vec{r}\,) = R\sum_{\vec{r}\,'} G_o(\vec{r} - \vec{r}\,')I(\vec{r}\,'). \tag{7}$$

Using eqs. (2) and (7) into eq. (6) we get

$$R_o(\vec{r}\,) = 2R[G_o(0) - G_o(\vec{r}\,)], \tag{8}$$

where $G_o(\vec{r}\,)$ is the LGF of the d-dimensional hypercube at the lattice site $\vec{r}$, and $G_o(0)$ is the LGF of the d-dimensional hypercube at the origin. The last equation represents the basic result we search. Once the LGF values are known, one can easily obtain the required equivalent resistance.

Finally, the resistance in a d-dimensional hypercube can be written as [20]

$$R_o(l_1, l_2, \ldots, l_d) = R\int_{-\pi}^{\pi}\frac{\mathrm{d}x_1}{2\pi}\ldots\int_{-\pi}^{\pi}\frac{\mathrm{d}x_d}{2\pi}\frac{1 - \exp(il_1x_1 + il_2x_2 + \ldots + il_dx_d)}{\sum_{i=1}^{d}(1 - \cos x_i)}. \tag{9}$$

Also, the LGF for a d-dimensional hypercube can be written as [8]

$$G_o(l_1, l_2, \ldots, l_d) = \int_{-\pi}^{\pi}\frac{\mathrm{d}x_1}{2\pi}\ldots\int_{-\pi}^{\pi}\frac{\mathrm{d}x_d}{2\pi}\frac{\exp(il_1x_1 + il_2x_2 + \ldots + il_dx_d)}{2\sum_{i=1}^{d}(1 - \cos x_i)}. \tag{10}$$

3 Application: FCC network

Now we consider the infinite FCC network which consists of identical resistors each of resistance R. In this case the position vector $\vec{r}$ becomes

$$\vec{r} = l\vec{a}_1 + m\vec{a}_2 + n\vec{a}_3. \tag{11}$$

As a result, eq. (8) can be written as

$$R_o(3,l,m,n) = R[f_o(3,0,0,0) - F_o(3,l,m,n)], \tag{12}$$

where $f_0(3;0,0,0) = F(3;0,0,0)$ is the LGF of an infinite FCC lattice at the origin. The LGF for a FCC lattice is defined as

$$F(E;l,m,n) = \frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{\cos(lx)\cos(my)\cos(nz)}{E-\cos x\cos y-\cos y\cos z-\cos x\cos z}\mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{13}$$

for $E \geq 3$, and $l+m+n$ even ($F = 0$ for $l+m+n$ is odd) [29,30].

In a recent work [29], the value $F(3;l,m,n)$ has been expressed rationally in terms of the known value $f_o(3;0,0,0)$, and π as

$$F_o(3;l,m,n) = \rho_1 f_o(3,0,0,0) + \frac{\rho_o}{\pi^2 f_o(3,0,0,0)} + \rho_3. \tag{14}$$

Using eqs. (12) and (14) we can express the equivalent resistance as

$$\frac{R_o(3;l,m,n)}{R} = r_1 f_o(3,0,0,0) + \frac{r_2}{\pi^2 f_o(3,0,0,0)} + r_3, \tag{15}$$

Table 1. Values for r_1, r_2 and r_3.

The site lmn	r_1	r_2	r_3	$R(l,m,n)/R$
000	0	0	0	0
200	4/3	−1	0	0.371575
400	−16/9	16/3	0	0.408775
110	0	0	1/3	0.333333
310	8/3	−5	1/3	0.398327
510	−136/9	91/3	1/3	0.41714
211	4/3	2	−2/3	0.383065
411	64/9	−28/3	−2/3	0.410858
220	−8	−6	16/3	0.391257
420	−4/9	−275/3	64/3	0.412677
321	28/3	29	−31/3	0.405569
521	788/9	−95/3	−95/3	0.419201
222	4	−15	2	0.402099
422	−728/9	62/3	32	0.415695
330	−144	−186	107	0.410563
530	−688	−2316	2497/3	0.420924
431	2840/9	2101/3	−898/3	0.416958
332	−4	−120	88/3	0.414194
532	−8932/9	−1598/3	1697/3	0.422381
433	1712/9	502/3	−368/3	0.420872
440	−9056/3	−4960	7424/3	0.42004
541	82936/9	56735/3	−8405	0.423639
442	−8212/9	−9059/3	1092	0.42165
543	159056/45	12800/3	−7645/3	0.425684
444	−19024/15	464	464	0.425212
550	−218480/3	−133150	188225/3	0.425693
552	−1596212/45	−249428/3	34694	0.42654
554	−110992/9	−20534/3	21226/3	0.428607
600	1924/75	−49	0	0.421792
620	−56488/225	862/3	48	0.423101
640	−2155436/75	−67681	84544/3	0.426144
710	1425407/9600	−294	1/3	0.425719
730	−170742787/28800	−7201/3	9601/3	0.427291
800	−89212757/352800	73984/147	0	0.428583

where r_1, r_2 and r_3 are rational numbers related to ρ_1, ρ_2 and ρ_3 as

$$r_1 = 1 - \rho_1, r_2 = -\rho_2 \qquad \text{and} \qquad r_3 = \rho_3,$$

and ρ_1, ρ_2 and ρ_3 are rational numbers that can be obtained by means of Morita's [12] recurrence formulae.

The LGF at the origin for an infinite FCC lattice was first evaluated by Watson where he showed that [31]

$$f_0(3;0,0,0) = F(3;0,0,0) = \frac{\sqrt{3}}{2}[K(k)]^2 = \frac{3\Gamma^6(\frac{1}{3})}{2^{\frac{14}{3}}\pi^4} = 0.4482203944, \tag{16}$$

where $k = \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Various values for r_1, r_2 and r_3 can be obtained from table 2 of [29]. Below, we show some of these values in table 1.

The asymptotic case (*i.e.*, the separation between the origin and the site (l,m,n)) goes to a large value or infinity. In this case the resistance goes to a finite value. To explain this point, we note that from the theory of Fourier series (Riemann's Lemma) that $\lim_{n\to\infty}\int_a^b \Phi(x)\cos nx \mathrm{d}x \to 0$ for any integrable function $\Phi(x)$. Thus, $f_o(l,m,n) \to 0$ (corresponding to the boundary condition of Green's function at infinity), and, as a result, eq. (12) becomes

$$\frac{R_o(3,l,m,n)}{R} \to f_0(3,0,0,0), \tag{17}$$

when any of $l,m,n \to \infty$.

We have calculated additional values for r_1, r_2 and r_3 using the recurrence formulae for the LGF of the infinite FCC lattice [12] for the following lattice sites $(6,0,0)$, $(6,2,0)$, $(6,4,0)$, $(7,1,0)$, $(7,3,0)$ and $(8,0,0)$.

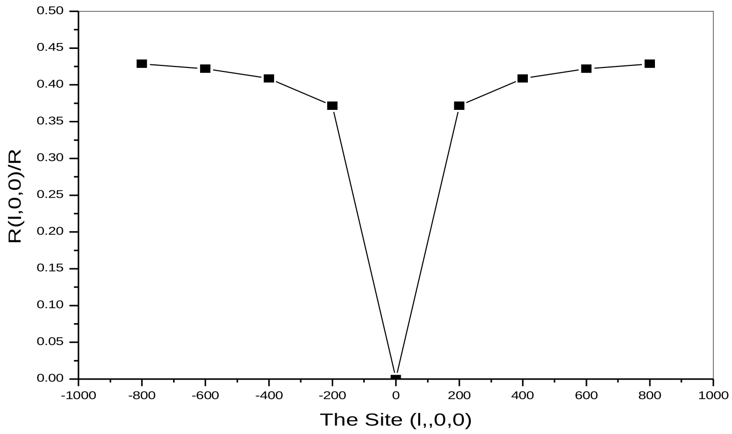

Fig. 1. The calculated resistance between the origin and the lattice site (l,m,n) along [100] direction of the infinite FCC lattice.

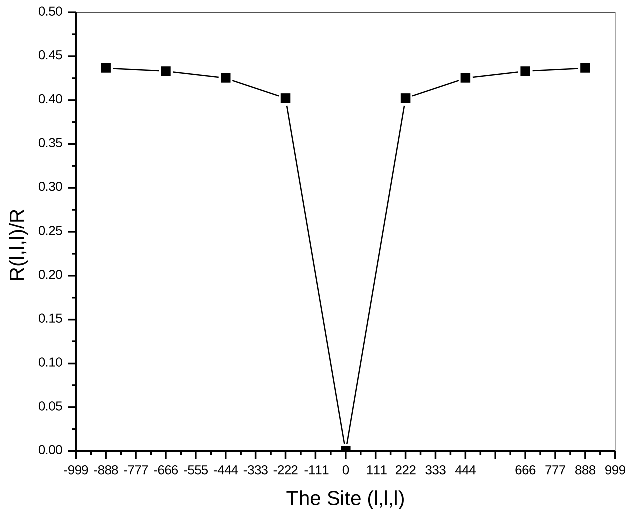

Fig. 2. The calculated resistance between the origin and the lattice site (l,m,n) along [111] direction of the infinite FCC lattice.

4 Results and discussion

We have expressed the equivalent resistance between the origin and the lattice site (l, m, n) in an infinite FCC network consisting of identical resistors each of resistance R. The equivalent resistance is plotted against the lattice site, and as shown in the figures the resistance in an infinite FCC lattice is symmetric under the transformation $(l, m, n) \rightarrow (-l, -m, -n)$ which is expected due to the inversion symmetry of the lattice.

Figure 1 shows the resistance in an infinite FCC lattice against the site (l, m, n) along the [100] direction. From this figure it is clear that the resistance is symmetric.

Figure 2 shows the resistance in an infinite FCC lattice against the site (l, m, n) along the [111] direction. From this figure it is clear that the resistance is symmetric.

The above figures indicate that as the separation between the origin and the lattice site (l, m, n) increases, the equivalent resistance approaches a finite value (*i.e.*, $f_o(3, 0, 0, 0) = 0.4482203944$) as explained above. A similar result was obtained for the resistance in an infinite SC network [23] where, as the separation between the origin and any other lattice site, the equivalent resistance approaches a finite value (*i.e.*, $g_o = 0.505462$) which is the LGF at the origin in an infinite SC lattice, while the resistance in an infinite square network diverges for large separation between the two sites [22].

References

1. B.M. McCoy, T.T. Wu, Phys. Rev. D **18**, 1253 (1978).
2. N.W. Dalton, D.W. Wood, Proc. Phys. Soc. (London) **90**, 4591 (1967).
3. M. Lax, Phys. Rev. **85**, 621 (1952).
4. E. Montroll, G. Weiss, J. Math. Phys. **4**, 241 (1965).
5. D. Hughes, *Random Walks and Random Environments*, Vol. **1** (Clarendon Press, Oxford, 1995).
6. G.L. Montet, Phys. Rev. B **7**, 650 (1975).
7. G.F. Koster, D.C. Slater, Phys. Rev. **96**, 1208 (1954).
8. E. Economou, *Green's Functions in Quantum Physics* (Springer, Berlin, 2006).
9. T. Morita, T. Horiguchi, J. Phys. A: Math. Gen. **5**, 67 (1972).
10. G. Iwata, Natur. Sci. Rep. Ochanomizu Univ. **20**, 13 (1969).
11. Inoue Michiko, J. Math. Phys. **15**, 704 (1974).
12. T. Morita, J. Phys. A: Math. Gen. **8**, 478 (1975).
13. R.S. Hijjawi, J.H. Asad, A.J. Sakaji, J.M. Khalifeh, Int. J. Theor. Phys. **43**, 11 (2004).
14. G. Kirchhoff, Ann. Phys. Chem. **72**, 497 (1847).
15. B. Van der Pol, H. Bremmer, *Operational Calculus Based on the Two-Sided Laplace Integral* (Cambridge University Press, England, 1955).
16. M. Jeng, Am. J. Phys. **68**, 37 (2000).
17. P. Doyle, L. Snell, *Random Walk and Electric Networks* (2006) `http://www.math.dartmouth.edu/~doyle/docs/walks/walks.pdf`.
18. R.E. Aitchison, Am. J. Phys. **32**, 566 (1964).
19. G. Venezian, Am. J. Phys. **62**, 1000 (1994).
20. J. Cserti, Am. J. Phys. **68**, 896 (2000).
21. J. Cserti, G. David, A. Piroth, Am. J. Phys. **70**, 153 (2002).
22. J.H. Asad, A.J. Sakaji, R.S. Hijjawi, J.M. Khalifeh, Eur. Phys. J. B **52**, 365 (2006).
23. J.H. Asad, R.S. Hijjawi, A. Sakaji, J.M. Khalifeh, Int. J. Theor. Phys. **43**, 2223 (2004).
24. R.S. Hijjawi, J.H. Asad, A.J. Sakaji, M. Al-Sabayleh, J.M. Khalifeh, Eur. Phys. J. Appl. Phys. **41**, 111 (2008).
25. J. Cserti, G. Szechenyi, G. David, J. Phys. A: Math. Gen. **44**, 215201 (2011).
26. J.H. Asad, R.S. Hijjawi, A. Sakaji, J.M. Khalifeh, Int. J. Theor. Phys. **44**, 471 (2004).
27. F.Y. Wu, J. Phys. A: Math. Gen. **37**, 6653 (2004).
28. N.Sh. Izmailian, Huang Ming-Chang, Phys. Rev. E **82**, 011125 (2010).
29. M.L. Glasser, J. Boersma, J. Phys. A: Math. Gen. **33**, 5017 (2000).
30. M. Inoue, J. Math. Phys. **15**, 704 (1974).
31. G.N. Watson, Q. J. Math. **10**, 266 (1939).