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Journal of Statistical Physics
1

ISSN 0022-4715
Volume 150
Number 6

J Stat Phys (2013) 150:1177-1182
DOI 10.1007/s10955-013-0716-x

Springer

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Exact Evaluation of the Resistance in an Infinite Face-Centered Cubic Network

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Received: 21 November 2012 / Accepted: 12 February 2013 / Published online: 20 February 2013

Abstract The equivalent resistance between the origin and the lattice site $(2n, 0, 0)$, in an infinite Face Centered Cubic (FCC) network consisting from identical resistors each of resistance R, has been evaluated analytically and numerically. The asymptotic behavior of the equivalent resistance has been also investigated. Finally, some numerical values for the equivalent resistance are presented.

Keywords Lattice Green's function · Infinite FCC network · Resistors

1 Introduction

The calculation of the equivalent resistance in infinite networks of identical resistors is one of the classic and interesting problems in the electric circuit theory. Many approaches have been introduced to calculate the resistance in infinite networks, such as:

The superposition of current distribution has been used to calculate the effective resistance between adjacent sites on infinite networks [1–3].

A mapping between random walk problems and resistor networks problems have been used by Monwhea Jeng [4]. This method was used to calculate the effective resistance between any two sites in an infinite two-dimensional square lattice of unit resistors.

A third educational important method based on the Lattice Green's Function (LGF) of the lattices is used to calculate the equivalent resistance [5–11]. This method has been applied to both perfect and perturbed square, simple cubic (SC) networks.

The LGF plays a key role in the theory of solid state physics, and as seen from literature most studies on the lattice functions are based on elliptic integral and recurrence relation approaches [12–25]. The importance of the LGF comes from the fact that many quantities in solid state physics can be expressed in terms of it, for example, phase shift, density of states, scattering cross section and thermodynamic functions.

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The present work is oriented as follows: In Sect. 2, we briefly introduce the basic formulas of interest for the LGF of the FCC network. In Sect. 3, an application to the LGF of the FCC network is applied to calculate the equivalent resistance between the origin and the lattice site $(2n, 0, 0)$ in the infinite FCC network. Finally, we close the present work (i.e., Sect. 4) with a discussion to the results obtained in this study.

2 Preliminaries

The LGF of the FCC lattice appears in many statistical problems, and it is defined as [21]:

$$F(n,m,l;w) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\operatorname{Cos} n\theta_1 \operatorname{Cos} m\theta_2 \operatorname{Cos} l\theta_3}{w-(\operatorname{Cos}\theta_1\operatorname{Cos}\theta_2+\operatorname{Cos}\theta_2\operatorname{Cos}\theta_3+\operatorname{Cos}\theta_1\operatorname{Cos}\theta_3)}\,d\theta_1\,d\theta_2\,d\theta_3 \tag{1}$$

where $n+m+l=$ even integer, and $w=w_1+iw_2$ is a complex variable and (n,m,l) is any lattice site in the FCC lattice.

The LGF of the FCC at the site $(0,0,0)$ which represents the origin of the lattice for $w=3$ (i.e., $F(0,0,0;3)=f_o$), was evaluated by Watson [26], where he found:

$$F(0,0,0;3)=f_o=\frac{\sqrt{3}}{\pi^2}\big[K(k_3)\big]^2=0.4482203944 \tag{2}$$

where $K(k_3)$ is the complete elliptic integral of the first kind

$$k_3=\operatorname{Sin}\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

(i.e., the singular modulus of the elliptic integral).

Recently, Joyce and Delves [21] showed that at the site $(2n,0,0)$ the LGF of the FCC lattice can be written as:

$$F(2n,0,0;3)=(-1)^n\frac{\sqrt{3}}{3^n}\left\{\left[\frac{\breve{U}_n^{(1)}K_3}{\pi}\right]^2-\left[\frac{\breve{U}_n^{(2)}}{K_3}\right]^2\right\}. \tag{3}$$

Here $\{\breve{U}_n^{(j)}: j=1,2\}$ are rational numbers satisfying the following recurrence relation

$$(2n+1)\breve{U}_{n+1}^{(j)}-12n\breve{U}_n^{(j)}-3(2n-1)\breve{U}_{n-1}^{(j)}=0 \tag{4}$$

with $n=1,2,\ldots,$ and the following initial conditions

$$\breve{U}_0^{(1)}=1,\qquad \breve{U}_1^{(1)}=1,$$
$$\breve{U}_0^{(2)}\equiv 0,\qquad \breve{U}_1^{(2)}=1,$$
$$K_3=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k_3^2\right).$$

3 Application: Evaluation of the Resistance $R(2n, 0, 0; 3)$ in an Infinite FCC Network

The aim of this section is to express the equivalent resistance between the origin $(0, 0, 0)$ and the lattice site $(2n, 0, 0)$ in an infinite FCC network of identical resistors in terms of $F(2n, 0, 0; 3)$.

First of all, it has been showed that for a 3D infinite network consisting of identical resistors each of resistance R, the equivalent resistance between the origin and any other lattice site is [5]:

$$R(\vec{r}) = 2\big[G(\vec{0}) - G(\vec{r})\big]. \tag{5}$$

Here $\vec{r}$ is the position vector of the lattices point, and for a d-dimensional lattice it takes the form:

$$\vec{r} = l_1\vec{a}_1 + l_2\vec{a}_2 + \cdots + l_d\vec{a}_d \tag{6}$$

with $l_1, l_2, \dots, l_d$ are integers, and $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_d$ are independent primitive translation vectors.

Also, the equivalent resistance between the origin and any other lattice site is can be expressed in an integral form [5]:

$$R(l_1, l_2, \dots, l_d) = R\int_{-\pi}^{\pi}\frac{dx_1}{2\pi}\cdots\int_{-\pi}^{\pi}\frac{dx_d}{2\pi}\frac{1 - \exp(il_1x_1 + il_2x_2 + \cdots + il_dx_d)}{\sum_{i=1}^{d}(1 - \operatorname{Cos} x_i)}. \tag{7}$$

On the other hand, the LGF for a 3D hypercube read as [5]:

$$G(l_1, l_2, \dots, l_d) = \int_{-\pi}^{\pi}\frac{dx_1}{2\pi}\cdots\int_{-\pi}^{\pi}\frac{dx_d}{2\pi}\frac{\exp(il_1x_1 + il_2x_2 + \cdots + il_dx_d)}{2\sum_{i=1}^{d}(1 - \operatorname{Cos} x_i)}. \tag{8}$$

For cubic lattices $d = 3$. Then substituting $d = 3$ into Eqs. (7) and (8) and comparing them with Eq. (5) one get:

$$R(n, m, l) = R\big[f_o - F(n, m, l)\big]. \tag{9}$$

Now make use of Eq. (3) and Eq. (9) one yields

$$R(2n, 0, 0) = \left(\frac{\sqrt{3}}{\pi^2}\big[K(k_3)\big]^2 - (-1)^n\frac{\sqrt{3}}{3^n}\left\{\left[\frac{\breve{U}_n^{(1)}K_3}{\pi}\right]^2 - \left[\frac{\breve{U}_n^{(2)}}{K_3}\right]^2\right\}\right). \tag{10}$$

This is our basic relation. Now, using Eq. (4) and Eq. (10) with the initial conditions of $\{\breve{U}_n^{(j)}: j = 1, 2\}$, one can calculate the required equivalent resistance. In Table 1 we present some numerical calculated values for the resistance between the origin and the site $(2n, 0, 0)$. The results obtained in the present work are in exact agreement with those obtained recently [27] using the so-called the recurrence formulae for the LGF of the infinite FCC lattice presented by Morita [18].

Since the LGF is an even function (i.e., $F(2n, 0, 0; 3) = F(-2n, 0, 0; 3)$) then the resistance is also symmetric due to the fact that the FCC network is pure and symmetric, and also $R(2n, 0, 0) = R(-2n, 0, 0)$.

Finally, it is worth studying the asymptotic behavior of the resistance (i.e., as the separation between the origin $(0, 0, 0)$ and the site $(2n, 0, 0)$ goes to a large value or infinity). In this case the resistance goes to a finite value.

Table 1 Calculated values of $R(2n,0,0)$

The site $(2n,0,0)$	$\hat{U}_n^1$	$\hat{U}_n^2$	$R(2n,0,0)$
$(0,0,0)$	1	0	0
$(2,0,0)$	1	1	0.371575
$(4,0,0)$	5	4	0.408775
$(6,0,0)$	$\frac{129}{5}$	21	0.421792
$(8,0,0)$	$\frac{717}{5}$	$\frac{816}{7}$	0.428366
$(10,0,0)$	825	$\frac{4695}{7}$	0.432325
$(12,0,0)$	$\frac{266859}{55}$	$\frac{27612}{7}$	0.434969
$(14,0,0)$	$\frac{1593171}{55}$	$\frac{2142999}{91}$	0.43686
$(16,0,0)$	$\frac{9615591}{55}$	$\frac{12934080}{91}$	0.438278
$(18,0,0)$	$\frac{994789431}{935}$	$\frac{78712155}{91}$	0.439382
$(20,0,0)$	$\frac{1218673431}{187}$	$\frac{9160550820}{1729}$	0.440265
$(22,0,0)$	$\frac{3410853057}{85}$	$\frac{56405302965}{1729}$	0.440988
$(24,0,0)$	$\frac{5336440769529}{21505}$	$\frac{348809334480}{1729}$	0.441597
$(26,0,0)$	$\frac{1948213488537}{1265}$	$\frac{10824102013941}{8645}$	0.441895

This can be explained as follows: It is well known from the theory of Fourier series (Riemann's Lemma) that $\mathrm{Lim}_{n\to\infty}\int_a^b \Phi(x)\,\mathrm{Cos}\,nx\,dx \to 0$ for any integrable function $\Phi(x)$. Thus, $F(n,m,l)\to 0$ (corresponding to the boundary condition of Green's function at infinity), and as a result Eq. (9) becomes:

$$\frac{R(2n,0,0)}{R} \to f_o(3,0,0,0). \tag{11}$$

Or alternatively, it has been showed [25] that the asymptotic expansion of $F(2n,0,0;3)$ is:

$$F(2n,0,0;3) \approx \frac{1}{4\pi n}\hat{S}_o(n,3), \tag{12}$$

where

$$\hat{S}_o(n,3) = 1 - \frac{1}{32\pi^2} - \frac{37}{2048n^4} + \frac{1147}{65536n^6} + \frac{430163}{8388608n^8} - \frac{70774943}{268435456n^{10}} + \cdots. \tag{13}$$

This expansion approaches zero as $n\to\infty$.

As, a result Eq. (9) becomes:

$$R(2n,0,0) = f_o - \frac{1}{4\pi n}\hat{S}_o(n,3) \to f_o. \tag{14}$$

4 Results and Discussion

We have expressed the equivalent resistance between the origin $(0,0,0)$ and the lattice site $(2n,0,0)$ in an infinite FCC network consisting of identical resistors each of resistance R.

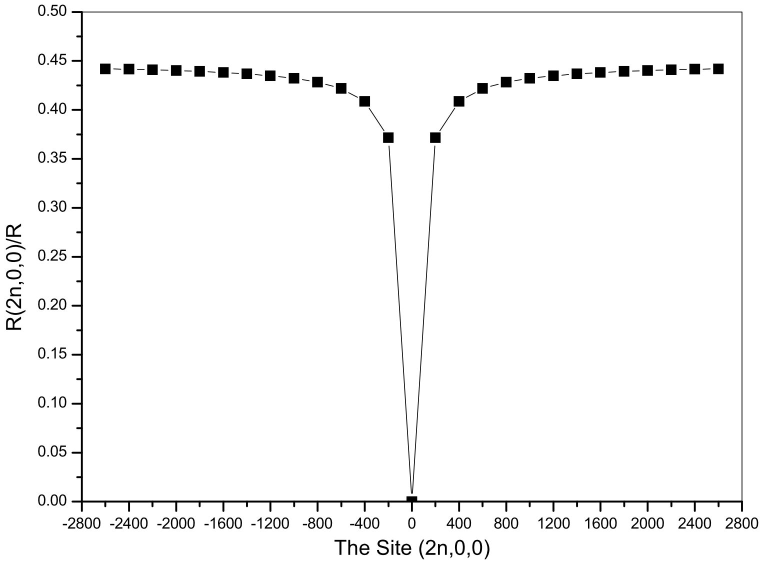

Fig. 1 Resistance $R(2n,0,0)$ in an infinite FCC against the site $(2n,0,0)$

The equivalent resistance is obtained in terms of rational numbers, π and the complete elliptic integral of the first kind. By means of Mathematica we obtained numerical values for these calculated resistance as presented in Fig. 1.

In Fig. 1 the calculated resistance in an infinite FCC lattice is plotted against the site $(2n,0,0)$ along the [100] direction. From this figure it is clear that the resistance is symmetric, and approaching a finite value (i.e., $f_o(3,0,0,0) = 0.4482203944$ as $n \to \infty$).

A similar result was obtained for the resistance in an infinite SC network [8] whereas the separation between the origin and any other lattice site the equivalent resistance approaches a finite value (i.e., $g_o = 0.505462$) which is the LGF at the origin in an infinite SC lattice, while the resistance in an infinite square network diverges for large separation between the two sites [9].

Acknowledgements The author would like to thanks the referees for their very useful comments and remarks.

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