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INFINITE SIMPLE 3D CUBIC NETWORK OF IDENTICAL CAPACITORS

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In this paper, the effective capacitance between the origin $(0, 0, 0)$ and any other lattice site (l_1, l_2, l_3), in an infinite simple cubic (SC) network consisting of identical capacitors each of capacitance C, has been expressed rationally in terms of the known value g_o and π. The asymptotic behavior is also investigated, and some numerical values for the effective capacitance are presented.

Keywords: Infinite networks; simple cubic; lattice Green's function; capacitors.

1. Introduction

Analysis of electric networks that are composed of identical resistors attracts the attention of both physicists and electrical engineers.[1–17] The main problem is to find the equivalent resistance between two different sites. One can see from previous studies that perfect and perturbed networks have been studied well for different lattices.

On the other hand, the calculation of the effective capacitance in infinite networks consisting of identical capacitors is of the same interests, as calculation of equivalent resistance. Little attention has been paid on this problem. In a previous work, we used the lattice Green's function (LGF) method to study the effective capacitance in both the perfect and perturbed infinite square network consisting of identical capacitors.[18–20] Later on, we used the superposition method which is based on charge distribution to calculate the effective capacitance in both an infinite perfect square and simple cubic (SC) network consisting of identical capacitors.[21,22] The results obtained using these two methods where in excellent agreement.

The LGF plays an important role in the theory of condensed matter physics. It was a field of interests for many years.[23–48] The LGF for cubic lattices has been investigated by many authors,[41–48] and the so-called recurrence formulae which are often used to calculate the LGF of the SC at different sites are presented.[44,45] The values of the LGF for the SC lattice have been recently exactly evaluated,[38] where these values are expressed in terms of the known value of the LGF at the origin. The LGF defined in our work is related to the Green's function (GF) of the tight-binding Hamiltonian (TBH).[30]

The outline of this work is oriented as follows. In Sec. 2, some basic definitions are introduced. In Sec. 3, an application to the LGF of the SC network has been applied to calculate the effective capacitance between the origin and any lattice site (l_1, l_2, l_3) in the infinite SC network. We close this work (i.e. Sec. 4) with a discussion to the results obtained in this study.

2. Basic Definitions and Preliminaries

First of all, consider an infinite d-dimensional network consisting of identical capacitors each of capacitance C and assuming that all lattice points are specified by the position vector

$$\mathbf{r} = l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + \cdots + l_d\mathbf{a}_d\,, \tag{1}$$

where $l_1, l_2, \ldots, l_d$ are integers (positive, negative or zero) $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$ are independent primitive translation vectors, in the case that all these independent primitive translation vectors have the same magnitude (i.e. $|\mathbf{a}_1| = |\mathbf{a}_2| = \cdots = |\mathbf{a}_d| = a$), then the d-dimensional lattice is called a hypercube.

For such an infinite network the effective capacitance between the origin and any other lattice site $(l_1, l_2, \ldots, l_d)$ reads:[18]

$$C(l_1, l_2, \ldots, l_d) = \frac{C}{\displaystyle\int_{-\pi}^{\pi}\frac{dx_1}{2\pi}\cdots\int_{-\pi}^{\pi}\frac{dx_d}{2\pi}\frac{1-\exp(il_1x_1+\cdots+il_dx_d)}{\sum_{i=1}^{d}(1-\cos x_i)}}\,. \tag{2}$$

On the other hand, the LGF for a d-dimensional hypercube has the following form:[30]

$$G(l_1, l_2, \ldots, l_d) = \int_{-\pi}^{\pi}\frac{dx_1}{2\pi}\cdots\int_{-\pi}^{\pi}\frac{dx_d}{2\pi}\frac{\exp(il_1x_1+il_2x_2+\cdots+il_dx_d)}{2\sum_{i=1}^{d}(1-\cos x_i)}\,. \tag{3}$$

Now, the energy dependent LGF of the tight-binding Hamiltonian defined for a SC lattice is defined as:[30]

$$G(E; l_1, l_2, l_3) = \int_{-\pi}^{\pi}\frac{dx_1}{2\pi}\int_{-\pi}^{\pi}\frac{dx_2}{2\pi}\int_{-\pi}^{\pi}\frac{dx_3}{2\pi}\frac{\cos(l_1x_1+l_2x_2+l_3x_3)}{E-\cos x_1-\cos x_2-\cos x_3}\,. \tag{4}$$

This is a generalization of the LGF defined above where we introduced a new variable E instead of the value 3 in the denominator in Eq. (3) for $d = 3$.

The LGF for the infinite SC at the origin (i.e. $G(3;0,0,0) = g_o$) was first evaluated by Watson[40] where he expressed it in a closed form in terms of elliptic integrals as:

$$G(3;0,0,0) = g_o = \left(\frac{2}{\pi}\right)^2 (18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})[K(k_o)]^2 = 0.5054620197\,, \quad (5)$$

where $K(k_o)$ is the complete elliptic integral of the first kind and k_o is its modulus (i.e. $k_o = (2-\sqrt{3})(\sqrt{3}-\sqrt{2})$).

Finally, the LGF for the infinite SC network has been expressed rationally in terms of the known value g_o and π as:[38]

$$G(3;l_1,l_2,l_3) = r_1 g_o(3,0,0,0) + \frac{r_2}{\pi^2 g_o(3,0,0,0)} + r_3\,. \quad (6)$$

3. Application: Infinite SC Network of Identical Capacitors

In this section, we will apply the main results mentioned in Sec. 2 to the infinite SC network which consists of identical capacitors each of capacitance C. In this case $d = 3$, so the position vector becomes:

$$\mathbf{r} = l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + l_3\mathbf{a}_3\,. \quad (7)$$

Using Eq. (2) above, the effective capacitance, in the infinite SC network, between the origin $(0,0,0)$ and any other lattice site (l_1,l_2,l_3) reads:

$$C(l_1,l_2,l_3) = \frac{C}{\displaystyle\int_{-\pi}^{\pi}\frac{dx_1}{2\pi}\int_{-\pi}^{\pi}\frac{dx_2}{2\pi}\int_{-\pi}^{\pi}\frac{dx_3}{2\pi}\,\frac{1-\cos(l_1x_1+l_2x_2+l_3x_3)}{3-\cos x_1-\cos x_2-\cos x_3}}\,. \quad (8)$$

We can easily calculate the exact value of the effective capacitance between two adjacent lattice sites, in the infinite SC network, from the above equation (due to symmetry reason) as:

$$\frac{C}{C(1,0,0)} + \frac{C}{C(0,1,0)} + \frac{C}{C(0,0,1)} = \int_{-\pi}^{\pi}\frac{dx_1}{2\pi}\int_{-\pi}^{\pi}\frac{dx_2}{2\pi}\int_{-\pi}^{\pi}\frac{dx_3}{2\pi} = 1\,. \quad (9)$$

Therefore, the effective capacitance between adjacent sites is $3C$. A similar result was obtained using charge distribution method.[22]

Now, comparing Eq. (8) with Eq. (4) for $E = 3$, the effective capacitance in the infinite SC network, between the origin $(0,0,0)$ and any other lattice site (l_1,l_2,l_3), can be written as:

$$C(l_1,l_2,l_3) = \frac{C}{[G(3;0,0,0) - G(3;l_1,l_2,l_3)]}\,. \quad (10)$$

Finally, making use of Eqs. (6) and (10), one gets:

$$C(l_1,l_2,l_3) = \frac{C}{\sigma_1 g_o + \dfrac{\sigma_2}{\pi^2 g_o} + \sigma_3}\,, \quad (11)$$

Table 1. Selected values for rational numbers σ_1, σ_2, σ_3 and effective capacitance.

The site $(\sigma_1, \sigma_2, \sigma_3)$	σ_1	σ_2	σ_3	$C_{l_1,l_2,l_3} = \frac{C}{\sigma_1 g_0 \frac{\sigma_2}{\pi^2 g_0} + \pi_3}$
(6, 0, 0)	−34937/6	−313079/25	5454	2.08885
(6, 1, 0)	71939/24	160009/20	−9355/3	2.0871
(6, 1, 1)	−10958/9	−532244/105	1632	2.08545
(6, 2, 0)	−48043/36	−6635509/7350	856	2.08242
(6, 2, 1)	1205/2	11979/35	−1118/3	2.08099
(6, 2, 2)	−13799/36	140621/210	60	2.07709
(6, 3, 0)	−22857/32	47767227/19600	−127	2.07594
(6, 3, 1)	6164/9	−7071296/3675	40	2.07477
(6, 3, 2)	−3773/12	56831/70	−10/3	2.07153
(6, 3, 3)	773/3	−249218/385	0	2.06684
(6, 4, 0)	−949423/432	96920323/17640	10	2.06869
(6, 4, 1)	57667/32	−88855401/19600	−5/3	2.06777
(6, 4, 2)	−36869/36	208988183/80850	0	2.06518
(6, 4, 3)	34823/96	−28097771/30800	0	2.06135
(6, 4, 4)	−383051/1872	23950043/46200	0	2.05677
(6, 5, 0)	−1842661/288	948892421/58800	−1/3	2.06143
(6, 5, 1)	580963/108	−657893021/48510	0	2.06072
(6, 5, 2)	−25545/	434122377/53900	0	2.05868
(6, 5, 3)	631967/468	−25009387/7350	0	2.05562
(6, 5, 4)	−159139/416	41699619/43120	0	2.05187
(6, 5, 5)	13157/78	−5698667/13475	0	2.04779

where σ_1, σ_2 and σ_3 are rational numbers related to Duffin and Shelly's[38] parameters λ_1, λ_2 and λ_3 as:

$$\sigma_1 = 1 - r_1 = 1 - \left(\lambda_1 + \frac{15}{12}\lambda_2\right); \quad \sigma_2 = -r_2 = \frac{1}{2}\lambda_2; \quad \sigma_3 = -r_3 = \frac{1}{3}\lambda_3 . \quad (12)$$

Various values for σ_1, σ_2 and σ_3 can be obtained from Table 1[38] for (l_1, l_2, l_3) ranging from $(0, 0, 0)$–$(5, 5, 5)$.

We calculated different values of σ_1, σ_2 and σ_3 for the sites from $(6, 0, 0)$–$(6, 5, 5)$ using the following recurrence relation:[44]

$$G(l_1 + 1, l_2, l_3) + G(l_1 - 1, l_2, l_3) + G(l_1, l_2 + 1, l_3) + G(l_1, l_2 - 1, l_3) + G(l_1, l_2, l_3 + 1) + G(l_1, l_2, l_3 - 1) = -2\delta_{l_1 0}\delta_{l_2 0}\delta_{l_3 0} + 2EG(l_1, l_2, l_3) . \quad (13)$$

Here E represents the energy.

To study the asymptotic behavior of the effective capacitance — as the separation between the origin and the site (l_1, l_2, l_3) goes to a large value or infinity — in this case [from Eq. (10)], the effective capacitance goes to a finite value. To explain this point, we note that from the theory of Fourier series (Riemann's Lemma) that $\lim_{n\to\infty} \int_a^b \Phi(x) \cos nx dx \to 0$ for any integrable function $\Phi(x)$. Thus, $G(l_1, l_2, l_3) \to 0$ (corresponding to the boundary condition of Green's function at infinity), and as a result Eq. (10) becomes:

$$C(l_1, l_2, l_3) = \frac{C}{g_o} = \frac{C}{0.5054620197} = 1.9783880C\,. \tag{14}$$

When any of $l_1, l_2, l_3 \to \infty$.

4. Results and Discussion

We have expressed the effective capacitance between the origin $(0, 0, 0)$ and any other lattice site (l_1, l_2, l_3), in an infinite SC network consisting of identical capacitors each of capacitance C, rationally in terms of the known value g_o and π. The effective capacitance is plotted against the lattice site, and from the figures shown the effective capacitance in the infinite SC network is symmetric under the transformation $(l_1, l_2, l_3) \to (-l_1, -l_2, -l_3)$ which is expected due to the inversion symmetry of the perfect infinite network.

The calculated effective capacitance in an infinite SC network has been plotted in Fig. 1 against the site $(l, 0, 0)$. While in Fig. 2, the calculated effective capacitance has been plotted against the site (l, l, l). From these figures, it is clear that the calculated effective capacitance is symmetric. This is due to the fact that the infinite SC network itself is symmetric and in addition the fact that the LGF is an even function.

It also clear from the two figures that as the separation between the two sites increases then the calculated effective capacitance approaches a finite value as shown in Eq. (14) above.

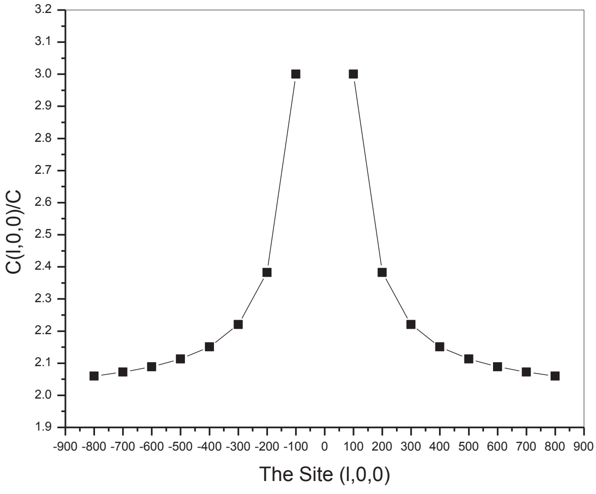

Fig. 1. Capacitance between the origin $(0, 0, 0)$ and the site $(l, 0, 0)$ along [100] direction for an infinite SC network.

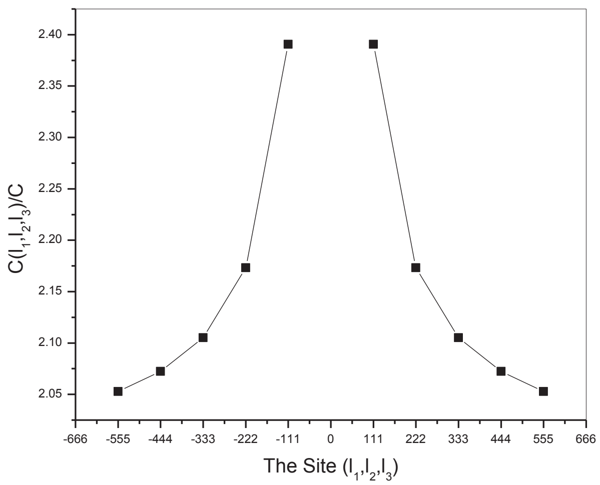

Fig. 2. Capacitance between the origin $(0, 0, 0)$ and the site (l, l, l) along [111] direction for an infinite SC network.

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