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ELECTRICAL NETWORKS WITH INTERSTITIAL SINGLE CAPACITOR

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We investigate the equivalent capacitance between two arbitrary nodes in a perturbed network (i.e. an interstitial capacitor is introduced between two arbitrary points in the perfect lattice) based on the lattice Green's function approach. An explicit formula for the capacitance of the perturbed lattice is derived in terms of the capacitances of the perfect lattice by solving Dyson's equation exactly. Numerical results are presented for the infinite perturbed square network. Finally, the asymptotic behavior of the effective capacitance has been studied.

Keywords: Perturbed square lattice; lattice Green's function; interstitial capacitor.

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1. Introduction

The calculation of the resistance between two arbitrary lattice sites in infinite or finite networks is an old problem.[1–22] Besides being important in physics, it is also of interest in electrical engineering. Previous works include studying infinite perfect and perturbed networks. For the infinite square and simple cubic networks, experimental results are obtained for both perfect and perturbed cases. A comparison was carried out with the calculated values where excellent agreements have been found.[10–13]

As one can see from the literature, most of the previous attention has been paid to studying infinite networks of identical resistors. Studying infinite networks of identical capacitors is also of vital importance in electrical circuit theory. In recent works,[23–28] the infinite perfect and perturbed networks consisting of identical

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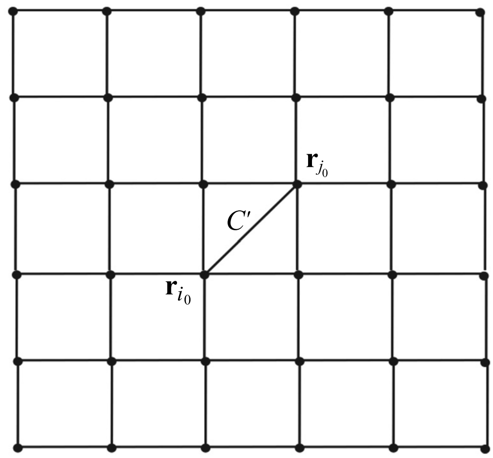

Fig. 1. A square lattice of identical capacitors with an interstitial capacitor C'.

capacitors are studied. Two methods have been employed in studying such networks: the lattice Green's function (LGF) and the charge distribution methods.

In this paper, the LGF method is applied to determine the effective capacitance between any two lattice sites in a perturbed lattice obtained by introducing an extra capacitor between two arbitrary points in a perfect network consisting of identical capacitors. This is illustrated in Fig. 1, an infinite square network of identical capacitors each of capacitances C. An extra capacitor (interstitial capacitor) C' is connected between the two sites $\mathbf{r}_0$ and $\mathbf{r}'_0$ in the infinite square perfect network which results in obtaining a perturbed network. The effective capacitance across this capacitor is equal to the parallel resultant of the capacitance between the sites $\mathbf{r}_0$ and $\mathbf{r}'_0$ in the perfect lattice and the interstitial capacitor.

The paper is organized as follows: In Sec. 2, some basic definitions for the effective capacitance in the infinite perfect network are introduced. In Sec. 3, the formalism of Green's function and the two-point capacitance in a perturbed network is given. Numerical results for the infinite perturbed square network are presented in Sec. 4. The paper is ended with a brief conclusion.

2. Basic Definitions and Preliminaries

First of all, consider an infinite d-dimensional network consisting of identical capacitors each of capacitance and assume that all lattice points are specified by the position vector

$$\mathbf{r}_\ell = \ell_1\mathbf{a}_1 + \ell_2\mathbf{a}_2 + \cdots + \ell_d\mathbf{a}_d\,, \tag{1}$$

where $\ell_1, \ell_2, \ldots, \ell_d$ are integers (positive, negative or zero) $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$ are independent primitive translation vectors, in the case of d-dimensional hypercube lattice $a_1 = a_2 = \cdots = a_d = a$ is the lattice constant.

For such an infinite network, the effective capacitance between sites $\mathbf{r}_i$ and $\mathbf{r}_j$ in the perfect lattice can be expressed in terms of the LGF as:[23]

$$C_o(\mathbf{r}_i, \mathbf{r}_j) = \frac{C}{2[G_o(\mathbf{r}_i, \mathbf{r}_i) - G_o(\mathbf{r}_i, \mathbf{r}_j)]} \,. \tag{2}$$

For the case of infinite perfect square network $\mathbf{r}_i = (i_x, i_y)$ and $\mathbf{r}_j = (j_x, j_y)$.

Also as the separation between the two sites $\mathbf{r}_i$ and $\mathbf{r}_j$ in the infinite square perfect network increases, the asymptotic behavior for the effective capacitance reads:[23]

$$C_o(\mathbf{r}_i, \mathbf{r}_j) \approx \frac{C}{\frac{1}{2\pi}[\ln((j_x - i_x)^2 + ((j_y - i_y)^2) + 2\gamma + \ln 8]} \,. \tag{3}$$

As the separation between the sites $\mathbf{r}_i$ and $\mathbf{r}_j$ goes to infinity then the above equation implies:

$$\frac{C_o(\mathbf{r}_i, \mathbf{r}_j)}{C} \to 0 \,. \tag{4}$$

3. Lattice Capacitance Function for Perturbed Lattice

In this section, we consider the problem of determining the effective capacitance for the perturbed network that is obtained by introducing an extra capacitor between any two nodes in the perfect lattice.

The electrical charge contribution due to bond $(\mathbf{r}_{i_0}, \mathbf{r}_{j_0})$ in the perfect lattice at site $\mathbf{r}_k$ is given by

$$\delta Q(\mathbf{r}_k) = C(\delta(\mathbf{r}_k, \mathbf{r}_{i_0}) - \delta(\mathbf{r}_k, \mathbf{r}_{j_0}))(V(\mathbf{r}_{i_0}) - V(\mathbf{r}_{j_0})) \,. \tag{5}$$

If an extra capacitor C' is placed between the nodes $\mathbf{r}_{i_0}$ and $\mathbf{r}_{j_0}$ in a perfect lattice, the effective capacitance across the interstitial capacitor will be equal to the parallel combination of the interstitial capacitor and the perfect capacitance $C_0(\mathbf{r}_{i_0}, \mathbf{r}_{j_0})$. Therefore, the charge contribution due to interstitial capacitor in the perturbed lattice at site $\mathbf{r}_k$ is

$$\frac{\delta Q'(\mathbf{r}_k)}{C} = \frac{C'}{C}\frac{\delta Q(\mathbf{r}_k)}{C} = \frac{C'}{C}(\delta(\mathbf{r}_k, \mathbf{r}_{i_0}) - \delta(\mathbf{r}_k, \mathbf{r}_{j_0}))(V(\mathbf{r}_{i_0}) - V(\mathbf{r}_{j_0})) \,. \tag{6}$$

In Dirac notation, the above equation can be written as

$$\frac{\delta Q'(\mathbf{r}_k)}{C} = \frac{C'}{C}\langle \mathbf{r}_k|(|\mathbf{r}_{i_0}\rangle - |\mathbf{r}_{j_0}\rangle)(\langle \mathbf{r}_{i_0}| - \langle \mathbf{r}_{j_0}|)|V\rangle = \langle \mathbf{r}_k|L'|V\rangle \,, \tag{7}$$

where the operator L' is the perturbation arising from the interstitial capacitor,

$$L' = \frac{C'}{C}|\alpha\rangle\langle\alpha| \tag{8}$$

where

$$|\alpha\rangle = |\mathbf{r}_{i_0}\rangle - |\mathbf{r}_{j_0}\rangle \,. \tag{9}$$

According to Kirchhoff's law, the charge at site $\mathbf{r}_k$ in the perturbed lattice is given by

$$(-L_0V)(\mathbf{r}_k) + \frac{1}{C}\delta Q'(\mathbf{r}_k) = \frac{1}{C}Q(\mathbf{r}_k)\,, \tag{10}$$

where L_o is the lattice Laplacian operator for perfect lattice. This equation can be written as

$$L\,|V\rangle = -\frac{1}{C}\,|Q\rangle\,, \tag{11}$$

where L is the lattice Laplacian operator for perturbed lattice:

$$L = L_0 - L'\,. \tag{12}$$

Similarly to the case of a perfect lattice, we would like to find the perturbed Green's function G corresponding to L defined by

$$G = -L^{-1}. \tag{13}$$

Combining Eqs. (8), (12) and (13) we obtain

$$G = (1 + G_0L')^{-1}G_0\,, \tag{14}$$

where $G_0 = -L_0^{-1}$ is the unperturbed Green's function (for the perfect lattice). The lattice Green's function G can be calculated by expanding $(1 + G_0L')^{-1}$ in Eq. (14) in a Dyson series[29]

$$G = G_0 - G_0L'G_0 + G_0L'G_0L'G_0 - G_0L'G_0L'G_0L'G_0 + \cdots\,. \tag{15}$$

The summation in the above equation can be performed exactly, because of the simple form of L', we obtain

$$G = G_0 - G_0|\alpha\rangle \frac{\frac{C'}{C}}{1 + \frac{C'}{C}\langle\alpha|G_0|\alpha\rangle}\langle\alpha|G_0\,. \tag{16}$$

Substituting Eq. (9) into (16), the Green's function $G(\mathbf{r}_i, \mathbf{r}_j)$ corresponding to L can be evaluated in terms of the matrix elements of G_0 and C', we have

$$G(\mathbf{r}_i,\mathbf{r}_j) = G_0(\mathbf{r}_i,\mathbf{r}_j) - \frac{C'(G_0(\mathbf{r}_i,\mathbf{r}_{i_0}) - G_0(\mathbf{r}_i,\mathbf{r}_{j_0}))(G_0(\mathbf{r}_j,\mathbf{r}_{i_0}) - G_0(\mathbf{r}_j,\mathbf{r}_{j_0}))}{C + 2C'(G_0(\mathbf{r}_{i_0},\mathbf{r}_{i_0}) - G_0(\mathbf{r}_{i_0},\mathbf{r}_{j_0}))}\,. \tag{17}$$

One can notice that the denominator in the above equation is never equal to zero and always positive. The capacitance for the perturbed lattice can be obtained in terms of the Green's function in the same way as for the perfect lattice. However, the capacitance between sites $\mathbf{r}_i$ and $\mathbf{r}_j$ in the perturbed network is given by

$$C(\mathbf{r}_i,\mathbf{r}_j) = \frac{Q}{V(\mathbf{r}_i) - V(\mathbf{r}_j)} = \frac{C}{G(\mathbf{r}_i,\mathbf{r}_i) + G(\mathbf{r}_j,\mathbf{r}_j) - G(\mathbf{r}_i,\mathbf{r}_j) - G(\mathbf{r}_j,\mathbf{r}_i)}\,. \tag{18}$$

From Eqs. (17) and (18) and using (2), we obtain the main result for the two-point capacitance in the perturbed lattice $C(\mathbf{r}_i, \mathbf{r}_j)$ in terms of the perfect one, C_0:

$$C(\mathbf{r}_i, \mathbf{r}_j) = \frac{1}{\dfrac{1}{C_0(\mathbf{r}_i, \mathbf{r}_j)} - \dfrac{\left(\dfrac{1}{C_0(\mathbf{r}_i, \mathbf{r}_{i_0})} - \dfrac{1}{C_0(\mathbf{r}_i, \mathbf{r}_{j_0})} - \dfrac{1}{C_0(\mathbf{r}_j, \mathbf{r}_{i_0})} + \dfrac{1}{C_0(\mathbf{r}_j, \mathbf{r}_{j_0})}\right)^2}{4\left(\dfrac{1}{C'} + \dfrac{1}{C_0(\mathbf{r}_{i_0}, \mathbf{r}_{j_0})}\right)}}\,. \tag{19}$$

It is simple to find the capacitance between the nodes $\mathbf{r}_{i_0}$ and $\mathbf{r}_{j_0}$. Using Eq. (19), we find that $C(\mathbf{r}_{i_0}, \mathbf{r}_{j_0}) = C' + C_0(\mathbf{r}_{i_0}, \mathbf{r}_{j_0})$, as mentioned previously. One can easily show from Eq. (19) that as C' goes to zero the problem reduces to the perfect lattice.

It is clear that the perfect lattice Laplacian L_0 was not used in the derivation of Eq. (19). Thus, our main result for the two-point capacitance in the perturbed lattice can be applied to any capacitor-lattice network of finite and infinite sizes in which each unit cell contains only one lattice point such as triangular lattices.

To study the asymptotic behavior of the capacitance in an infinite perturbed square lattice for large separation between the sites $\mathbf{r}_i$ and $\mathbf{r}_j$, one can easily show from Eq. (19) and from the fact that $C_o(\mathbf{r}_i, \mathbf{r}_j) \to 0$ also that $C(\mathbf{r}_i, \mathbf{r}_j) \to 0$.

4. Numerical Results

Below we shall present some numerical results for the infinite perturbed square lattice. Using the results for the perfect square lattice,[23] the capacitance between

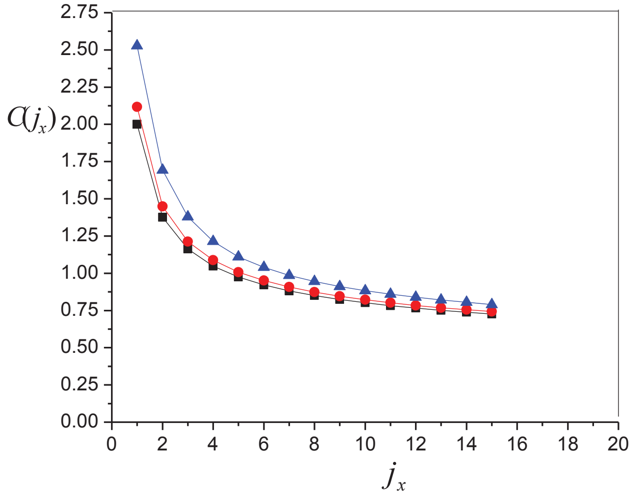

Fig. 2. The capacitance for an infinite square lattice, in units of C, on the perturbed lattices (*circles* for $C' = 3C$ and *triangles* for $C' = C/3$) and the perfect lattice (*squares*) are measured between the origin and $(j_x, 0)$.

Table 1. The values of $C(j_x, j_y)$ in units of C for a perfect and perturbed square lattices. Two cases are considered: the interstitial capacitor $C' = 3C$ is introduced between the sites $\mathbf{r}_{i_0} = (0, 0)$ and the $\mathbf{r}_{j_0} = (1, 1)$, and $C' = C/3$ is also introduced between the sites $\mathbf{r}_{i_0} = (0, 0)$ and $\mathbf{r}_{j_0} = (1, 1)$.

The site	The values of the capacitance $C(j_x, j_y)/C$		
(j_x, j_y)	Perfect lattice	Perturbed lattice $C' = 3C$	Perturbed lattice $C' = C/3$
0, 0	∞	∞	∞
1, 0	2	2.11802	2.52819
2, 0	1.37597	1.44833	1.69312
3, 0	1.16203	1.21281	1.3784
4, 0	1.04823	1.08773	1.21343
5, 0	0.974844	1.00771	1.11066
6, 0	0.922313	0.95088	1.03939
7, 0	0.882207	0.907765	0.986321
8, 0	0.850222	0.87355	0.944821
9, 0	0.823894	0.845499	0.911189
10, 0	0.801699	0.821926	0.88319
11, 0	0.782633	0.801732	0.859387
12, 0	0.766008	0.78416	0.83881
13, 0	0.751328	0.768675	0.820776
14, 0	0.738232	0.754884	0.804791
15, 0	0.726445	0.742488	0.790485
1, 1	1.5708	4.5708	1.90413
2, 2	1.1781	1.50802	1.2511
3, 3	1.02443	1.21105	1.06834
4, 4	0.937123	1.07451	0.970209
5, 5	0.878865	0.991209	0.906261
6, 6	0.836326	0.933367	0.860179
7, 7	0.803421	0.890053	0.824835
8, 8	0.776929	0.855969	0.796547∞
9, 9	0.754964	0.828188	0.773197
10, 10	0.736338	0.804941	0.753465
11, 11	0.72026	0.785088	0.73648
12, 12	0.706181	0.767857	0.721641
13, 13	0.693707	0.752702	0.708518
14, 14	0.682542	0.739224	0.696793
15, 15	0.672467	0.727128	0.686226

any two nodes for the perturbed square lattice can be computed using Eq. (19). Here we consider two cases. First, the interstitial capacitor $C' = 3C$ is introduced between the sites $\mathbf{r}_{i_0} = (0, 0)$ and $\mathbf{r}_{j_0} = (1, 1)$. Second, the interstitial capacitor $C' = C/3$ is also introduced between the sites $\mathbf{r}_{i_0} = (0, 0)$ and $\mathbf{r}_{j_0} = (1, 1)$. The results are listed in Table 1. Figure 2 shows the capacitances along j_x-axis for the perfect and the perturbed lattices. It can be seen from Fig. 2 that the equivalent capacitance between every pair of lattice points in the perturbed network is larger than that between them in the corresponding perfect network. This fact clearly follows from the negativity of the second term in the dominator of Eq. (19).

5. Conclusion

We have computed the equivalent capacitance between two arbitrary lattice sites in a perturbed network obtained by introducing an extra capacitor between two arbitrary nodes in the perfect lattice using Green's function method. The capacitance for the perturbed lattice is expressed in terms of that for the perfect one. Some numerical results for the capacitances are given for the perturbed square lattice along the x-axis. We found that the capacitance for the perturbed lattice is always larger than that for the perfect lattice, this is due to the negativity of the second term in the dominator of Eq. (19). A similar calculation can be performed for simple cubic and triangular lattices.

References

1. G. Kirchhoff, *Ann. Phys. Chem.* **72** (1847) 497–508.
2. B. Van der Pol and H. Bremmer, *Operational Calculus Based on the Two-Sided Laplace Integral* (Cambridge University Press, England, 1955).
3. M. Jeng, *Am. J. Phys.* **68**(1) (2000) 37–40.
4. P. Doyle and L. Snell, *Random Walk and Electric Networks* (2006), http://www.math.dartmouth.edu/~doyle/docs/walks/walks.pdf
5. R. E. Aitchison, *Am. J. Phys.* **32** (1964) 566.
6. G. Venezian, *Am. J. Phys.* **62** (1994) 1000–1004.
7. D. Atkinson and F. J. van Steenwijk, *Am. J. Phys.* **67** (1999) 486.
8. J. Cserti, *Am. J. Phys.* **68** (2000) 896–906.
9. J. Cserti, G. David and A. Piroth, *Am. J. Phys.* **70**(2) (2002) 153–159.
10. J. H. Asad, R. S. Hijjawi, A. Sakaji and J. M. Khaleh, *Int. J. Theor. Phys.* **43**(11) (2004) 2223.
11. J. H. Asad, R. S. Hijjawi, A. Sakaji and J. M. Khaleh, *Int. J. Theor. Phys.* **44**(4) (2004) 471.
12. J. H. Asad, A. J. Sakaji, R. S. Hijjawi and J. M. Khalifeh, *Eur. Phys. J. B* **52**(2) (2006) 365.
13. R. S. Hijjawi, J. H. Asad, A. J. Sakaji, M. Al-Sabayleh and J. M. Khalifeh, *EPJ Appl. Phys.* **41**(2) (2008) 111–114.
14. J. Cserti, G. Szechenyi and G. David, *J. Phys. A: Math. Gen.* **44**(21) (2011) 215201.
15. F. Y. Wu, *J. Phys. A: Math. Gen.* **37** (2004) 6653–6673.
16. J. W. Essam and F. Y. Wu, *J. Phys. A: Math. Theor.* **42** (2009) 025205.
17. N. Sh. Izmailian and M.-C. Huang, *Phys. Rev. E* **82** (2010) 011125.
18. J. H. Asad, A. A. Diab, R. S. Hijjawi and J. M. Khalifeh, *Eur. Phys. J. Plus* **128**(1) (2013) 1–5.
19. J. H. Asad, *J. Stat. Phys.* **150** (2013) 1177–1182.
20. M. Q. Owaidat, R. S. Hijjawi and J. M. Khalifeh, *Mod. Phys. Lett. B* **19** (2010) 2057–2068.
21. M. Q. Owaidat, R. S. Hijjawi and J. M. Khalifeh, *J. Phys. A: Math. Theor.* **43** (2010) 375204.
22. M. Q. Owaidat, R. S. Hijjawi and J. M. Khalifeh, *Int. J. Theor. Phys.* **51** (2012) 3152–3159.
23. J. H. Asad, R. S. Hijjawi, A. J. Sakaji and J. M. Khalifeh, *Int. J. Mod. Phys. B* **19**(24) (2005) 3713–3721.

24. R. S. Hijjawi, J. H. Asad, A. J. Sakaji and J. M. Khalifeh, *Int. J. Mod. Phys. B* **21**(2) (2007) 199–209.
25. J. H. Asad, A. J. Sakaji, R. S. Hijjawi and J. M. Khalifeh, *Eur. Phys. J. Appl. Phys.* **40**(3) (2007) 257–264.
26. J. H. Asad, R. S. Hijjawi, A. J. Sakaji and J. M. Khalifeh, *Eur. Phys. J. Appl. Phys.* **32**(3) (2005) 149–154.
27. J. H. Asad, A. J. Sakaji, R. Hijjawi and J. M. Khalifeh, *Mod. Phys. Lett. B* **24**(7) (2010) 695–705.
28. M. Q. Owaidat, *JJP* **5**(3) (2012) 113–118.
29. E. N. Economou, *Green's Functions in Quantum Physics*, 2nd edn. (Springer-Verlag, Berlin, Germany, 1983).