INTERNATIONAL JOURNAL OF CIRCUIT THEORY AND APPLICATIONS
Int. J. Circ. Theor. Appl. 2017; **45**:1942–1957
Published online 9 June 2017 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/cta.2366

Resistance formulae of a multipurpose *n*-step network and its application in *LC* network

Z.-Z. Tan[1,*,†], J.H. Asad[2,*,†] and M.Q. Owaidat[3]

[1]*Department of Physics, Nantong University, Nantong, China*
[2]*Department of Physics, Faculty of Arts and Sciences, Palestine Technical University, Tulkarm, Palestine*
[3]*Department of Physics, Al-Hussein Bin Talal University, Ma'an, Jordan*

SUMMARY

We consider a multipurpose *n*-step network with cross resistors that is a profound problem that has not been resolved before. This network contains a number of different types of resistor network model. This problem is resolved by three steps: First of all, we simplify a complex graphics into a simple equivalent model; next, we use Kirchhoff's laws to analyse the network and establish a nonlinear difference equation; and finally, we construct the method of equivalent transformation to obtain the general solution of the nonlinear difference equation. In this paper, we created a new concept of negative resistance for the needs of the equivalent conversion and obtain two general resistance formulae of a multipurpose ladder network of cross resistors. As applications, several interesting specific results are produced. In particular, an *n*-step impedance *LC* network is discussed. Our method and the results are suitable for the research of complex impedance network as well. Copyright © 2017 John Wiley & Sons, Ltd.

Received 9 August 2016; Revised 17 April 2017; Accepted 24 April 2017

KEY WORDS: complex network; equivalent transform; nonlinear difference equation; Kirchhoff's laws; negative resistance

1. INTRODUCTION

Resistor network is an important and useful model in applications of the modern natural science. The computation of the two-point resistance in networks is a classical problem in circuit theory and graph theory, which has been researched for more than 170 years [1–4]. The computation of resistances is relevant to a wide range of problems ranging from science and technology to engineering [5–38]. However, it is usually very difficult to obtain the exact resistance in complex resistor networks [7–29]; the construction of and research on the models of complex networks therefore make sense for theories and applications.

The main progress of research methods in resistor networks basically has several different approaches, such as Kirchhoff's laws [1], the lattice Green's function technique [7–12, 35], the Laplacian approach [13–18], the recursion-transform method [25–28], the equivalent transformation methods [4] and a method of nodal potentials [29]. Thus, many resistor networks have been resolved by these methods.

However, there are still some complex resistor networks unresolved because of the complexity of the structure conditions of the network in real life. Figure 1 is called a multipurpose *n*-step network of cross resistors, which has seven different resistors, and the right boundary is a single resistor R_0.

*Correspondence to: J. H. Asad, Department of Physics, Faculty of Arts and Sciences, PO Box 7, Palestine Technical University, Kadoorie, Tulkarm, Palestine; and Z.-Z. Tan, Department of Physics, Nantong University, Nantong, China.
†E-mail: drjasad@yahoo.com; tanzzh@163.com; tanz@ntu.edu.cn

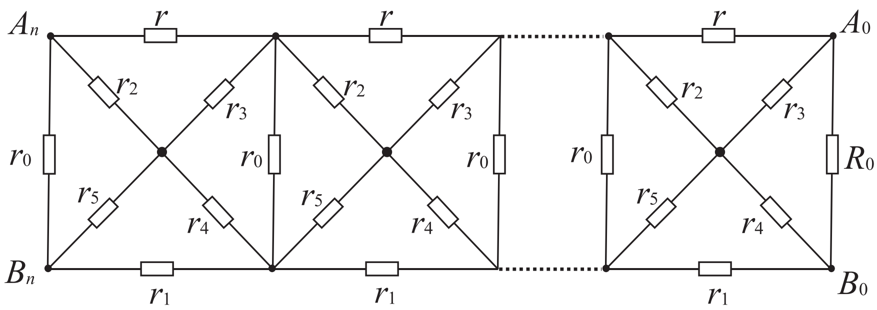

Figure 1. A multipurpose n-step network of cross resistors, which has seven different resistors, and the right boundary is a load resistor R_0.

This is a profound problem which has not been resolved before. Because the seven resistors are arbitrary, the network contains a number of different types of resistor network model, such as the regular ladder network [4] and the triangle ladder network. As is known to all, the two-port network is well studied and the transfer matrix approach is frequently applied for such problems [39, 40].

In this paper, we focus on the computation of resistance between two arbitrary nodes A_k and B_k by means of the equivalent transformation methods [4] as shown in Figure 1. This problem is resolved by three steps. The equivalent transformation methods split the derivation into three parts. The first part simplifies a complex graphics into a simple equivalent model. The second part creates a nonlinear difference equation model by using Kirchhoff's laws. The third part constructs the method of equivalent transformation to obtain the general solution of the nonlinear difference equation. In addition, we created a new concept of negative resistance for the needs of the equivalent conversion, and two general resistance formulae are derived. Because the multipurpose n-step network has seven arbitrary resistors, when we consider the particular values of the different resistors, several different types of resistor network model are produced automatically. Thus, we obtain several interesting new results by means of reducing the general formula. At the same time, the resistance results are applied to the complex impedance network. As is well known, the research of complex impedance network is more difficult than that of the resistor network because the equivalent impedance has many different properties from equivalent resistance. From previous studies of the equivalent impedance, we find that the results of the equivalent complex impedance are always curious and nonlinear [14,21,30,37]. Thus, complex impedance is an important problem worthy of study. In the lower part of the paper, we study the characteristics of the equivalent impedance of the LC network by the equivalent resistance formula.

The organization of this paper is as follows: In section 2, we present two exact formulae of resistance between two nodes in a multipurpose n-step network of cross resistors. In section 3, we set up an equivalent model by the Kirchhoff's laws. In section 4, we prove the resistance formulae by solving the nonlinear difference equation. In section 5, we deduce several specific results by the general formula. In section 6, we study the characteristics of the equivalent impedance of the LC network. Section 7 gives a summary and discussion of the method and results.

2. TWO GENERAL RESULTS

We call Figure 1 a general n-step network with cross resistors. This network has seven different resistors, and the right boundary is a load resistor R_0 which is different from other resistors (we simply call it as an arbitrary boundary because it is located at the right edge). Assuming that A_k is the kth node at the edge of A_0A_n and B_k is the kth node at the edge of B_0B_n, we have two main results as follows:

Case 1.
The equivalent resistance between two nodes A_n and B_n in the n-step network with cross resistors with anarbitrary right boundary can be written as

DOI: 10.1002/cta

$$R_{A_nB_n}(n) = \lambda - \alpha\beta\left(\frac{F_n + (R_0 - \lambda)F_{n-1}}{F_{n+1} + (R_0 - \lambda)F_n}\right). \tag{1}$$

where

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \lambda = \frac{b}{d}r_0 \tag{2}$$

and

$$\alpha\beta = \left(\frac{bc - ad}{d^2}\right)r_0 = \left(\frac{r_0\left(\frac{r_2r_3}{r+r_3+r_2} + \frac{r_4r_5}{r_4+r_1+r_5}\right)}{r_0 + \frac{r_2(r+r_3)}{r+r_3+r_2} + \frac{r_5(r_1+r_4)}{r_4+r_1+r_5}}\right)^2, \tag{3}$$

and

$$\alpha = \frac{1}{2d}\left(c + br_0 + \sqrt{(c - br_0)^2 + 4adr_0}\right), \beta = \frac{1}{2d}\left(c + br_0 - \sqrt{(c - br_0)^2 + 4adr_0}\right) \tag{4}$$

with

$$\begin{aligned}
a &= r_1r + \left(\frac{r_2r_5}{r_2 + r_5} + \frac{r_3r_4}{r_3 + r_4}\right)(r + r_1),\\
b &= \frac{r_2(r + r_3)(r_1 + r_4 + r_5) + r_5(r_4 + r_1)(r + r_3 + r_2)}{(r_2 + r_5)(r_3 + r_4)},\\
c &= a + r_0\frac{r_3(r_1 + r_5)(r + r_2 + r_4) + r_4(r + r_2)(r_1 + r_3 + r_5)}{(r_2 + r_5)(r_3 + r_4)},\\
d &= b + r_0\frac{(r_4 + r_1 + r_5)(r + r_3 + r_2)}{(r_2 + r_5)(r_3 + r_4)}.
\end{aligned} \tag{5}$$

Case 2.
The equivalent resistance between any two nodes A_k and B_k in the n-step network of cross resistors with an arbitrary right boundary can be written as

$$R_{A_kB_k} = \left(\frac{1}{R_{right}(k)} + \frac{1}{R_{left}(n-k)} - \frac{1}{r_0}\right)^{-1}, \tag{6}$$

or

$$R_{A_kB_k} = \frac{r_0R_r(k)R_l(n-k)}{r_0[R_r(k) + R_l(n-k)] - R_r(k)R_l(n-k)}. \tag{7}$$

where

$$R_{\text{left}}(n-k) = \lambda - \alpha\beta\left(\frac{F_{n-k} + (r_0 - \lambda)F_{n-k-1}}{F_{n-k+1} + (r_0 - \lambda)F_{n-k}}\right), \tag{8}$$

$$R_{\text{right}}(k) = \lambda - \alpha\beta\left(\frac{F_k + (R_0 - \lambda)F_{k-1}}{F_{k+1} + (R_0 - \lambda)F_k}\right). \tag{9}$$

3. EQUIVALENT MODEL AND RECURSIVE EQUATION

According to the structure characteristics of Figure 1, assuming that the equivalent resistance between two nodes A_n and B_n in the n-step network is R_n, the resistance in the $(n-1)$-step network corresponds

DOI: 10.1002/cta

to the resistance R_{n-1}. Thus, from Figure 1, we obtain an equivalent circuit model as shown in Figure 2.

Next, we use Kirchhoff's law to find the recursive relation between R_n and R_{n-1}. Assuming that the electric current I is constant and goes from the input node A_n to the output node B_n, we denote the branch currents in all segments of the network as shown in Figure 2. Using Kirchhoff's laws [1,4] to analyse the resistor network, the nodes' current equations and the meshes' voltage equations can be achieved from Figure 2.

In Figure 2, by Kirchhoff's law, the meshes' voltage equations can be written as

$$\begin{aligned} &I_5r_2 + I_7r_5 - I_3r_0 = 0, \\ &I_1r + I_6r_3 - I_5r_2 = 0, \\ &I_6r_3 + I_8r_4 - I_2R_{n-1} = 0, \\ &I_1r + I_2R_{n-1} + I_4r_1 - I_3r_0 = 0. \end{aligned} \tag{10}$$

By Kirchhoff's law, the nodes' current equations can be written as

$$\begin{aligned} &I_1 + I_3 + I_5 = I \text{ and } I_1 = I_2 + I_6, \\ &I_3 + I_4 + I_7 = I \text{ and } I_2 + I_8 = I_4. \end{aligned} \tag{11}$$

Solving the eight equations mentioned in the preceding texts, we finally obtain after some algebra and reduce the relation

$$\frac{I_3}{I} = \frac{a + bR_{n-1}}{c + dR_{n-1}}, \tag{12}$$

where a, b, c, d are given by (5). Because $R_n = U/I = I_3r_0/I$, substituting in (12) gives

$$R_n = \frac{a + bR_{n-1}}{c + dR_{n-1}} r_0. \tag{13}$$

Formula ((13)) is a key recurrence formula we are looking for, which belongs to the nonlinear difference equation. Next, we will study its general solution.

4. EQUIVALENT TRANSFORMATION AND SOLUTION

4.1. Derivation of result 1

We adopt the method of variable substitution to solve the nonlinear difference equation (13), assuming that there is sequence $\{x_n\}$ and it satisfies the following relation:

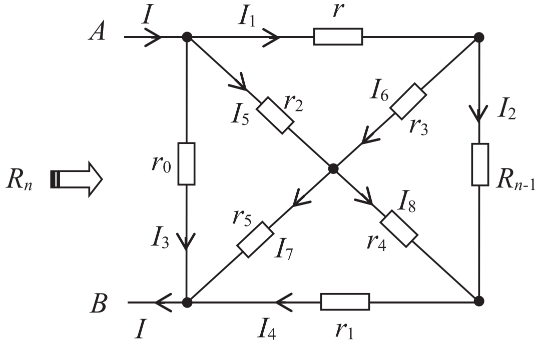

Figure 2. Equivalent circuit model of a two-terminal network with the current parameters.

DOI: 10.1002/cta

$$R_n = \frac{x_{n+1}}{x_n} - \frac{c}{d}. \tag{14}$$

We appoint the initial term $x_0 = 1$; from (14), we have

$$x_0 = 1, x_1 = R_0 + \frac{c}{d}. \tag{15}$$

Substituting (14) and its recursive formula R_{n-1} into (13), we obtain

$$x_{n+1} = \left(\frac{c + br_0}{d}\right)x_n - \frac{(bc - ad)r_0}{d^2}x_{n-1} \tag{16}$$

Supposing that α and β are the roots of the characteristic equation for x_k, solving Eqn (16), we obtain (4). Thus, from (16), we obtain

$$x_{n+1} = (\alpha + \beta)x_n - \alpha\beta x_{n-1}. \tag{17}$$

Solving Eqn (17), we obtain the general solution (one may refer to [4])

$$x_n = \frac{1}{\alpha - \beta}[(x_1 - \beta x_0)\alpha^n - (x_1 - \alpha x_0)\beta^n]. \tag{18}$$

Considering the initial term conditions in (15), we have

$$x_n = \frac{1}{\alpha - \beta}\left[\left(R_0 + \frac{c}{d} - \beta\right)\alpha^n - \left(R_0 + \frac{c}{d} - \alpha\right)\beta^n\right]. \tag{19}$$

From Eqns (16) and (17), we obtain

$$\alpha + \beta - \frac{c}{d} = \frac{c + br_0}{d} - \frac{c}{d} = \frac{b}{d}r_0 = \lambda. \tag{20}$$

Then, substituting Eqn (20) into Eqn (19) yields

$$x_n = \frac{1}{\alpha - \beta}[(R_0 + \alpha - \lambda)\alpha^n - (R_0 + \beta - \lambda)\beta^n]. \tag{21}$$

Substituting Eqn (21) and its recursive formula x_{n+1} into Eqn (14) gives

$$R_n = \frac{(R_0 + \alpha - \lambda)\alpha^{n+1} - (R_0 + \beta - \lambda)\beta^{n+1}}{(R_0 + \alpha - \lambda)\alpha^n - (R_0 + \beta - \lambda)\beta^n} - \frac{c}{d}. \tag{22}$$

From Eqn (20), we have $c/d = \alpha + \beta - \lambda$; substituting to Eqn (22) yields

$$R_n = \lambda - \alpha\beta\left(\frac{(R_0 + \alpha - \lambda)\alpha^{n-1} - (R_0 + \beta - \lambda)\beta^{n-1}}{(R_0 + \alpha - \lambda)\alpha^n - (R_0 + \beta - \lambda)\beta^n}\right), \tag{23}$$

where Eqn (4) is used. From Eqns (16) and (17), we have $\alpha\beta = r_0(bc - ad)/d^2$; thus, substituting Eqn (5) to $\alpha\beta$, Eqn (3) is derived. When we define $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, Eqn (23) reduces to Eqn (1). At this point, formula (1) is proved.

4.2. Derivation of result 2

4.2.1. Research ideas and methods. We created a new concept of negative resistance elements for the needs of the equivalent conversion, namely, we convert r_0 to be three resistors in parallel, which are

DOI: 10.1002/cta

two r_0 and an $-r_0$ as shown in Figure 3. According to the calculation method of parallel resistors, we have $\frac{1}{r_0}+\frac{1}{-r_0}+\frac{1}{r_0}=\frac{1}{r_0}$; obviously, the introduction of $-r_0$ is a very good method.

When we compute the resistance between two arbitrary nodes A_k and B_k, making use of Figure 3, we can convert Figure 1 into Figure 4.

According to the equivalent model of Figure 4, we obtain equivalent resistance between any two nodes A_k and B_k by means of the calculation of the parallel resistors,

$$\frac{1}{R_{A_kB_k}}=\frac{1}{R_{right}(k)}+\frac{1}{R_{left}(n-k)}-\frac{1}{r_0}, \tag{24}$$

where $R_{right}(k)$ and $R_{left}(n-k)$ are given by (8) and (9) in terms of formula (1). Thus, (6) is proved. Next, from (6), we immediately obtain (7) after the simple calculation.

Note that the idea of negative resistance elements is a great creation, which, resolving a profound resistance problem, is very interesting. In particular, when $k=n$, from (8), we have $R_{left}=r_0$; thus, from (6), we have $R_{A_nB_n}=R_0(n)$ which shows Eqn (1) included in Eqn (6).

5. SPECIAL CASES AND COMPARISONS

Case 1.
In Figure 1, when $r_3=r_4=r_5=r_2$, from Eqn (5), we have

$$\begin{aligned}a&=r_1r+r_2(r+r_1), b=r_2+\frac{3}{4}(r_1+r)+\frac{r_1r}{2r_2},\\ c&=a+r_0b, d=b+r_0\left(1+\frac{2r_2(r+r_1)+rr_1}{4r_2^2}\right).\end{aligned} \tag{25}$$

Then, from Eqn (4), we have

$$\alpha=\frac{1}{2d}\left(c+br_0+\sqrt{a^2+4adr_0}\right), \beta=\frac{1}{2d}\left(c+br_0-\sqrt{a^2+4adr_0}\right), \tag{26}$$

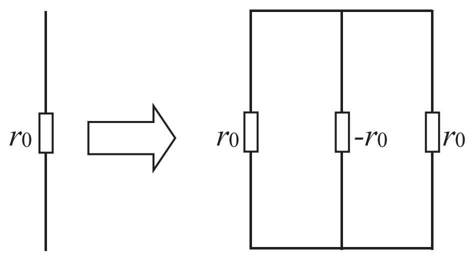

Figure 3. A resistor is equivalent to three resistors in parallel.

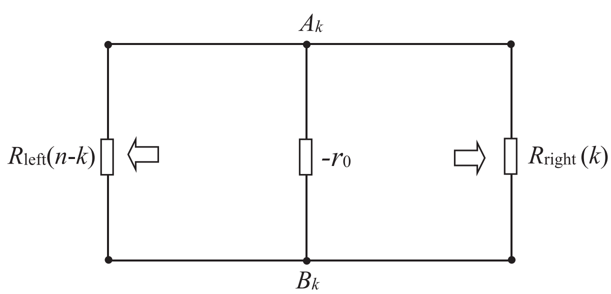

Figure 4. Equivalent conversion model of resistor network.

DOI: 10.1002/cta

and

$$\lambda = \frac{\left[4r_2^2 + 3(r_1 + r)r_2 + 2r_1 r\right] r_2 r_0}{r_2\left[4r_2^2 + 3(r_1 + r)r_2 + 2r_1 r\right] + r_0(r_1 + 2r_2)(r + 2r_2)}.$$

Thus, from Eqn (1), we have

$$R_{A_nB_n} = \lambda - \frac{r_0(bc - ad)}{d^2}\left(\frac{F_n + (R_0 - \lambda)F_{n-1}}{F_{n+1} + (R_0 - \lambda)F_n}\right). \tag{27}$$

where $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ defined in Eqn (2).

Case 2.
In Figure 1, when $r_1 = r_2 = r_3 = r_4 = r_5 = r$, from Eqn (5), we have

$$a = 3r^2, b = 3r, c = 3r(r + r_0), d = \frac{3}{4}(r + 3r_0), \tag{28}$$

and from Eqn (4), we have

$$\alpha = \frac{2r}{4r + 3r_0}\left(r + 2r_0 + \sqrt{r^2 + (4r + 3r_0)r_0}\right),$$
$$\beta = \frac{2r}{4r + 3r_0}\left(r + 2r_0 + \sqrt{r^2 + (4r + 3r_0)r_0}\right). \tag{29}$$

Thus, from Eqn (1), we have

$$R_{A_nB_n} = \lambda - \left(\frac{2rr_0}{4r + 3r_0}\right)^2\left(\frac{F_n + (R_0 - \lambda)F_{n-1}}{F_{n+1} + (R_0 - \lambda)F_n}\right). \tag{30}$$

where F_n is defined in Eqn (2) and $\lambda = 4r_0r/(4r + 3r_0)$ is deduced from Eqn (2).

Case 3.
When $r_3 \to \infty$, the cross network degrades into a three-triangular resistor network as shown in Figure 5.

When $r_3 \to \infty$, from Eqn (5), we have

$$a = r_1 r + \left(\frac{r_2 r_5}{r_2 + r_5} + r_4\right)(r + r_1), b = (r_4 + r_1) + \frac{r_2 r_5}{r_2 + r_5}$$
$$c = a + r_0\frac{(r_1 + r_5)(r + r_2 + r_4) + r_4(r + r_2)}{r_2 + r_5}, d = b + r_0\left(\frac{r_4 + r_1 + r_5}{r_2 + r_5}\right) \tag{31}$$

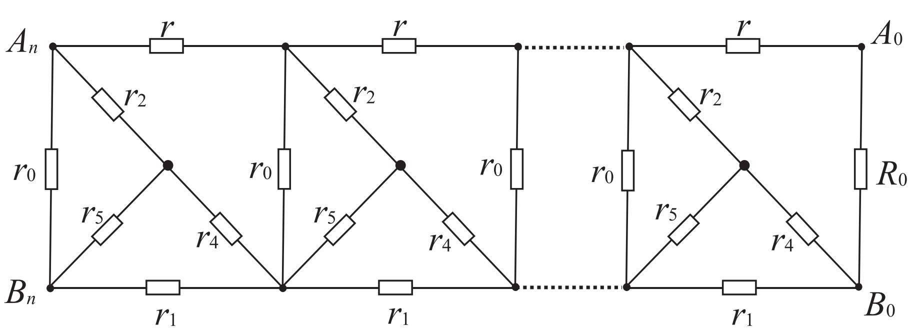

Figure 5. A multipurpose n-step network with three-triangle resistors, which has seven different resistors, and the right boundary is a load resistor R_0.

DOI: 10.1002/cta

Substituting Eqn (31) into Eqn (1), we obtain the resistance formula $R_{A_nB_n}$ (Eqn (1)).
In particular, when $r_3 \to \infty$, $r_1 = r_2 = r_4 = r_5 = r$, from Eqn (5), we have

$$a = 4r^2, b = \frac{5}{2}r, c = 4r(r + r_0), d = \frac{1}{2}(5r + 3r_0). \tag{32}$$

From Eqn (4), we have

$$\begin{aligned} \alpha &= \frac{r}{2(5r+3r_0)}\left(8r + 13r_0 + \sqrt{(8r+3r_0)^2 + 32(5r+3r_0)r_0}\right), \\ \beta &= \frac{r}{2(5r+3r_0)}\left(8r + 13r_0 - \sqrt{(8r+3r_0)^2 + 32(5r+3r_0)r_0}\right). \end{aligned} \tag{33}$$

Thus, from Eqn (1), we have

$$R_{A_nB_n} = \lambda - \left(\frac{4rr_0}{5r+3r_0}\right)^2 \left(\frac{F_n + (R_0 - \lambda)F_{n-1}}{F_{n+1} + (R_0 - \lambda)F_n}\right). \tag{34}$$

where $\lambda = 5r_0r/(5r + 3r_0)$ and F_n is defined in Eqn (2).

Case 4.
When $r_3, r_5 \to \infty$ and $r_4 = 0$, the cross network degrades into a two-triangular ladder resistor network as shown in Figure 6. In the following, we reduce (1) to be a simple result.

When $r_4 = 0$ and $r_3 = r_5 = \infty$, from Eqn (5), we obtain

$$a = r_1r + r_2(r + r_1), b = r_1 + r_2, c = (r_1 + r_0)(r + r_2) + rr_2, d = r_0 + r_1 + r_2. \tag{35}$$

Thus,

$$\lambda = \frac{b}{d}r_0 = \left(\frac{r_1 + r_2}{r_0 + r_1 + r_2}\right)r_0, \tag{36}$$

and

$$\begin{aligned} \alpha &= \frac{r_0(r_2 + r_1) + c + \sqrt{[r_0(r_2 + r_1) + c]^2 - 4r_0^2r_2^2}}{2(r_0 + r_2 + r_1)}, \\ \beta &= \frac{r_0(r_2 + r_1) + c - \sqrt{[r_0(r_2 + r_1) + c]^2 - 4r_0^2r_2^2}}{2(r_0 + r_2 + r_1)} \end{aligned} \tag{37}$$

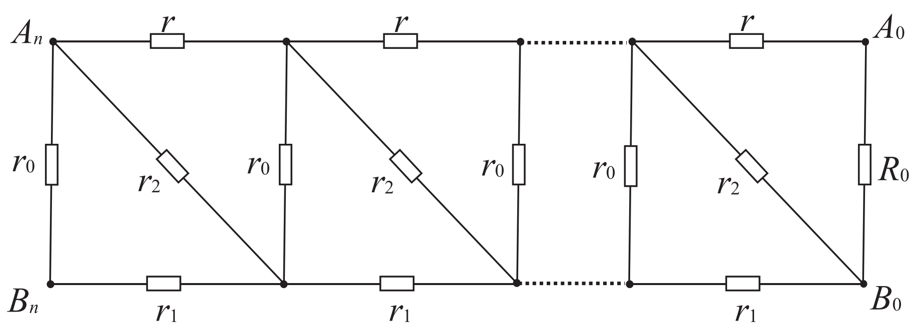

Figure 6. Ann-step triangular ladder network with arbitrary right boundary.

DOI: 10.1002/cta

where c is given by Eqn (35) and λ is given by Eqn (36). Thus, formula (1) reduces to

$$R_{A_nB_n} = \lambda - \left(\frac{r_0 r_2}{r_0 + r_1 + r_2}\right)^2 \left(\frac{F_n + (R_0 - \lambda)F_{n-1}}{F_{n+1} + (R_0 - \lambda)F_n}\right), \tag{38}$$

Case 5.
When $r_k \to \infty$ $(k=2,3,4,5)$, the n-step network of cross resistors degrades into a rectangular n-step ladder network as shown in Figure 7. Taking limits $r_2 \to \infty$ to Eqns (36)–(38), we have

$$\lambda = \lim_{r_2 \to \infty} \frac{r_1 + r_2}{r_0 + r_1 + r_2} r_0 = r_0 \tag{39}$$

and

$$\alpha = \frac{2r_0 + r + r_1 + \sqrt{(r + r_1)(4r_0 + r + r_1)}}{2}, \beta = \frac{2r_0 + r + r_1 - \sqrt{(r + r_1)(4r_0 + r + r_1)}}{2}. \tag{40}$$

Thus,

$$R_{\mathrm{n}} = r_0 - r_0^2 \left(\frac{F_n + (R_0 - r_0)F_{n-1}}{F_{n+1} + (R_0 - r_0)F_n}\right). \tag{41}$$

The network in Figure 7 has been resolved by [4]; comparing (41) with the result in [4], we find that the two results are completely the same, but their methods are different.

Case 6.
In Figure 1, when $R_0 = br_0/d$, from (1) and (2), we have

$$R_{\mathrm{A}_nB_n}(n) = \frac{b}{d} r_0 - \left(\frac{r_0\left(\frac{r_2 r_3}{r + r_3 + r_2} + \frac{r_4 r_5}{r_4 + r_1 + r_5}\right)}{r_0 + \frac{r_2(r + r_3)}{r + r_3 + r_2} + \frac{r_5(r_1 + r_4)}{r_4 + r_1 + r_5}}\right)^2 \frac{F_n}{F_{n+1}}. \tag{42}$$

where b and d are defined in Eqn (5).

Case 7.
When $n \to \infty$, from (4), we have $0 < \beta/\alpha < 1$; thus,

$$\lim_{n\to\infty} \left(\frac{\beta}{\alpha}\right)^n = \lim_{n\to\infty} \left(\frac{c + br_0 - \sqrt{(c - br_0)^2 + 4adr_0}}{c + br_0 + \sqrt{(c - br_0)^2 + 4adr_0}}\right)^n = 0, \tag{43}$$

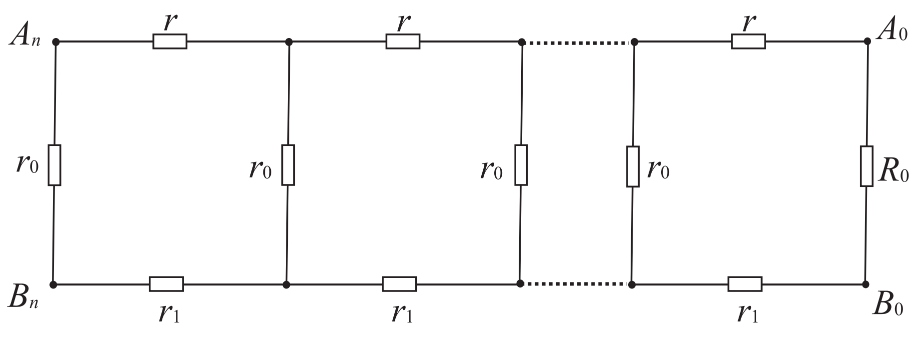

Figure 7. Ann-step ladder resistor network with arbitrary right boundary.

From Eqn (1), we have

$$R_{A_nB_n}(\infty) = \lambda - \beta. \tag{44}$$

Substituting Eqn (4) into (44) yields

$$R_{A_nB_n}(\infty) = \frac{1}{2d}\left(br_0 - c + \sqrt{(c - br_0)^2 + 4adr_0}\right). \tag{45}$$

Result (45) shows that formula (1) is bounded in the case of $n \to \infty$.

Case 8.
When $R_0 = \lambda - \beta$, from Eqn (22), we immediately obtain

$$R_{A_nB_n}(n) = \lambda - \beta. \tag{46}$$

Notice that we find that Eqn (46) is the same as Eqn (44), but their meanings are different from each other. Equation (46) is the characteristic resistance of the network, but Eqn (44) is the limiting resistance. When the load resistance satisfies the characteristic value, the equivalent resistance is a constant that equals limiting resistance.

6. EQUIVALENT IMPEDANCE OF THE N-STEP LC NETWORK

The research method and conclusion of this article are also applicable to the complex impedance network; we can obtain the equivalent impedance formula of the complex impedance network if we use the variable substitution method. We assume that the elements of $r_k (k = 0, 1, 2, 3, 4, 5)$ are composed of inductor, capacitor and resistor, then we can obtain the equivalent complex impedance by making use formula (1). In the following, we will study the LC network.

In Figure 1, consider an n-step impedance LC network; from formula (1), we can obtain the equivalent impedance of an n-step LC network. Assuming that the impedance elements in the network are $r_1 = r = i\omega L, r_3 = r_4 = r_5 = r_2 = \frac{1}{i\omega C}$ and $r_0 = R_0 = \frac{1}{i\omega C_0}$ as shown in Figure 8, where ω is the alternating current frequency, the equivalent complex impedance from (1) is given by

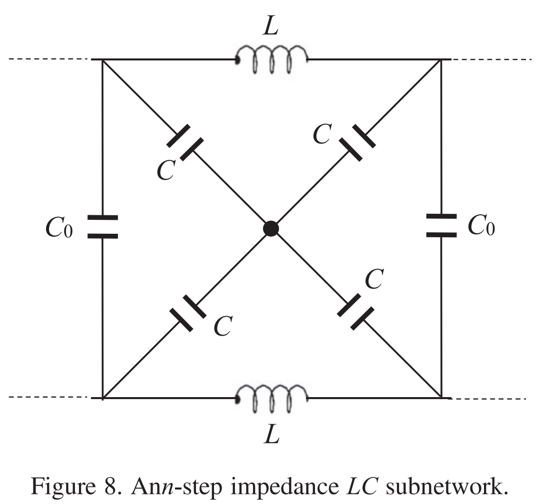

Figure 8. Ann-step impedance LC subnetwork.

DOI: 10.1002/cta

$$\frac{Z_{A_nB_n}}{r_0} = \frac{2h_1(h+1)}{(2h_1+1)h+2(h_1+1)} - \left(\frac{2h_1}{(2h_1+1)h+2(h_1+1)}\right)^2 \times \frac{[(2h_1+1)h+2(h_1+1)](\delta^n-\rho^n)+(h+2)\left(\delta^{n-1}-\rho^{n-1}\right)}{[(2h_1+1)h+2(h_1+1)]\left(\delta^{n+1}-\rho^{n+1}\right)+(h+2)(\delta^n-\rho^n)} \tag{47}$$

where

$$h = \frac{r}{r_2} = -\omega^2 LC, \qquad h_1 = \frac{r_2}{r_0} = \frac{r_2}{R_0} = \frac{C_0}{C}, \tag{48}$$

$$\delta = \frac{\alpha}{r_0} = \frac{2h_1\left(hh_1+h+1+\sqrt{h^2(h_1+1)^2+2h(h_1+1)}\right)}{2(1+h)h_1+(2+h)}, \quad \rho = \frac{\beta}{r_0} = \frac{2h_1\left(hh_1+h+1-\sqrt{h^2(h_1+1)^2+2h(h_1+1)}\right)}{2(1+h)h_1+(2+h)}. \tag{49}$$

Substituting Eqn (48) into Eqn (49), we obtain

$$\mathrm{i}Z_{A_nB_n} = \frac{2(1-\omega^2LC)}{\omega g} - \frac{4C_0}{\omega g^2} \times \frac{g(\delta^n-\rho^n)+C(2-\omega^2LC)\left(\delta^{n-1}-\rho^{n-1}\right)}{g\left(\delta^{n+1}-\rho^{n+1}\right)+C(2-\omega^2LC)(\delta^n-\rho^n)}, \tag{50}$$

where

$$g = C[(2h_1+1)h+2(h_1+1)] = 2(C_0+C)-(2C_0+C)\omega^2LC,$$
$$\text{and } \delta = \frac{2C_0\left(1-\omega^2L(C_0+C)+\sqrt{[1-\omega^2L(C_0+C)]^2-1}\right)}{2(C_0+C)-(2C_0+C)\omega^2LC}, \quad \rho = \frac{2C_0\left(1-\omega^2L(C_0+C)-\sqrt{[1-\omega^2L(C_0+C)]^2-1}\right)}{2(C_0+C)-(2C_0+C)\omega^2LC}. \tag{51}$$

It is clear that there are three parameters in the complex impedance; from Eqn (41), we should discuss three cases. By Eqn (48), we have

$$h^2(h_1+1)^2+2h(h_1+1) = \omega^2L(C_0+C)[\omega^2L(C_0+C)-2].$$

When $\omega^2 > 2/[L(C_0+C)]$, there are $\delta, \rho \in R$; the complex impedance is Eqn (50). There are five parameters (such as C_0, C, L, ω, n) in the complex impedance. In Eqn (50), the 3D graphic of the complex impedance changes with ω and n as shown in Figure 9. Figure 9 shows that the complex impedance of Eqn (50) is gradually decreasing with the increase of ω and n when $\omega^2 > 2/[L(C_0+C)]$. These situations do not appear oscillation phenomenon; they only gradually change.

When $\omega^2 = \frac{2}{L(C_0+C)}$, there are $hh_1+h+1 = -\omega^2L(C+C_0)+1 = -1$, then $\delta=\rho=-(C_0+C)/C_0$ and

$$\lim_{\rho\to\delta}\frac{\delta^n-\rho^n}{\delta-\rho} = n\delta^{n-1}. \tag{52}$$

Thus, from (50) and (52), we have

$$\mathrm{i}Z_{A_nB_n} = \frac{C_0+C}{\omega C_0^2}\left(1-\frac{2C}{C_0+C}+\frac{(C_0+C)n-C(n-1)}{(C_0+C)(n+1)-Cn}\right). \tag{53}$$

DOI: 10.1002/cta

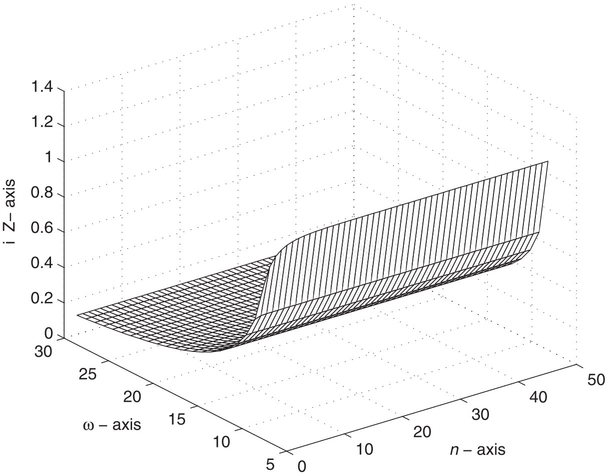

Figure 9. A 3D graph showing the impedance changes with ω ($\omega \in [8.2,30]$) and n in the case of $C_0 = 0.2$, $C = 0.1$ and $L = 0.1$.

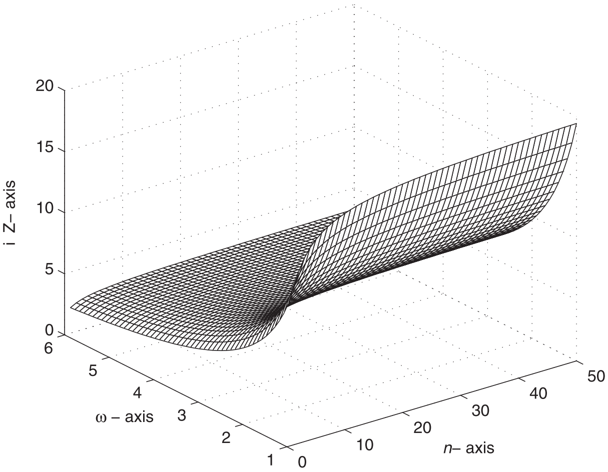

Figure 10. A 3D graph showing the impedance changes with ω and n in the case of $C_0 = 0.2$ and $C = 0.1$.

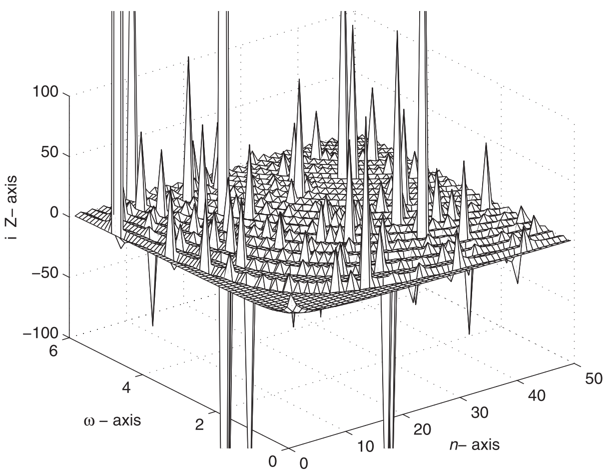

Figure 11. A 3D graph showing the impedance changes with ω and n in the case of $C_0 = 0.2$, $C = 0.1$ and $L = 0.1$.

DOI: 10.1002/cta

There are four parameters (such as C_0, C, ω, n) in the complex impedance; the 3D graphic of the complex impedance changes with ω and n as shown in Figure 10. Figure 10 shows that the complex impedance of Eqn (53) is gradually changing with the increase of ω when $\omega^2 = 2/[L(C_0 + C)]$. But the case is different from the cases of Figure 9.

When $\omega^2 < 2/[L(C_0 + C)]$, there are $\delta, \rho \in Z$; applying Eqn (48) to Eqn (49), we have

$$\delta = \frac{2h_1}{2(1+h)h_1 + (2+h)}(\cos\theta + \mathrm{i}\sin\theta), \rho = \frac{2h_1}{2(1+h)h_1 + (2+h)}(\cos\theta - \mathrm{i}\sin\theta) \tag{54}$$

where $\theta = \arccos(hh_1 + h + 1) = \arccos[1 - \omega^2 L(C_0 + C)]$. Applying Eqn (54) to Eqn (47), we have

$$Z_{A_nB_n} = \frac{2h_1 r_0}{(2h_1+1)h + 2(h_1+1)}\left(h + 1 - \frac{2h_1\sin(n\theta) + (h+2)\sin(n-1)\theta}{2h_1\sin(n+1)\theta + (h+2)\sin(n\theta)}\right). \tag{55}$$

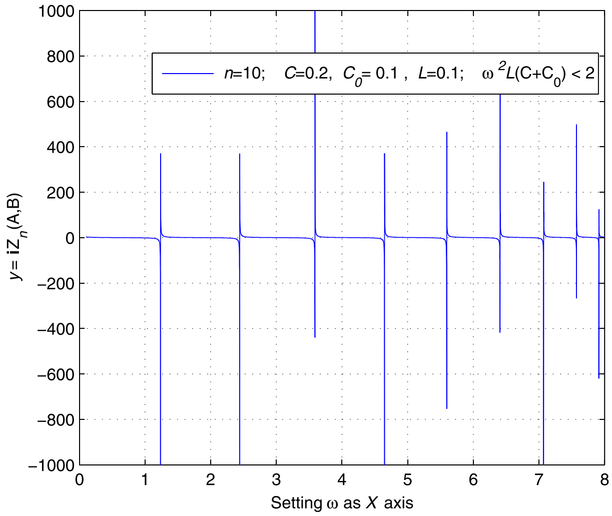

Figure 12. The impedance oscillatorily (with resonance) varying with ω, with setting $C_0 = 0.2$, $C = 0.1$, $L = 0.1$ and $n = 10$. [Colour figure can be viewed at wileyonlinelibrary.com]

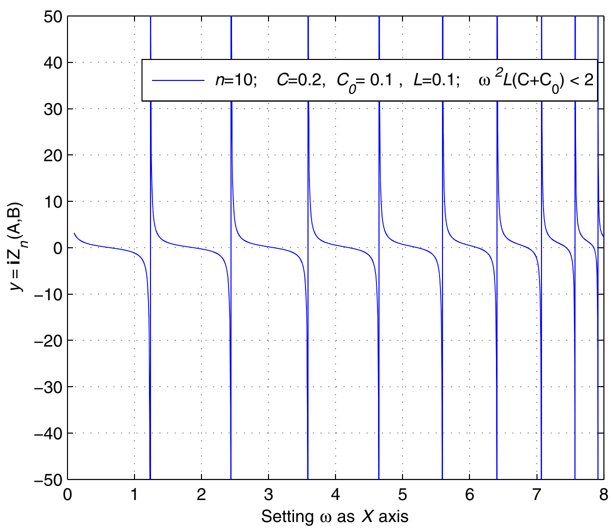

Figure 13. Magnified graph from Figure 13, showing the obvious oscillation features, with $C_0 = 0.2$, $C = 0.1$, $L = 0.1$ and $n = 10$. [Colour figure can be viewed at wileyonlinelibrary.com]

DOI: 10.1002/cta

Substituting Eqn (48) into Eqn (55), we obtain

$$iZ_{A_nB_n} = \frac{2}{g}\left(1 - \omega^2 LC - \frac{2C_0 \sin(n\theta) + C(2-\omega^2 LC)\sin(n-1)\theta}{2C_0 \sin(n+1)\theta + C(2-\omega^2 LC)\sin(n\theta)}\right), \tag{56}$$

where $g = \omega[2(C_0 + C) - \omega^2 LC(2C_0 + C)]$.

We set $C_0 = 0.2$, $C = 0.1$ and $L = 0.1$; the 3D graphic of the complex impedance changes with ω and n as shown in Figure 11. Figure 11 shows that impedance has resonance and oscillation. For clearly seeing the characteristics, we draw the 2D graphs. For example, setting $C_0 = 0.2$, $C = 0.1$, $L = 0.1$ and n = 10 and n = 20, the complex impedance changes with the alternating current frequency ω as shown in Figures 12–15.

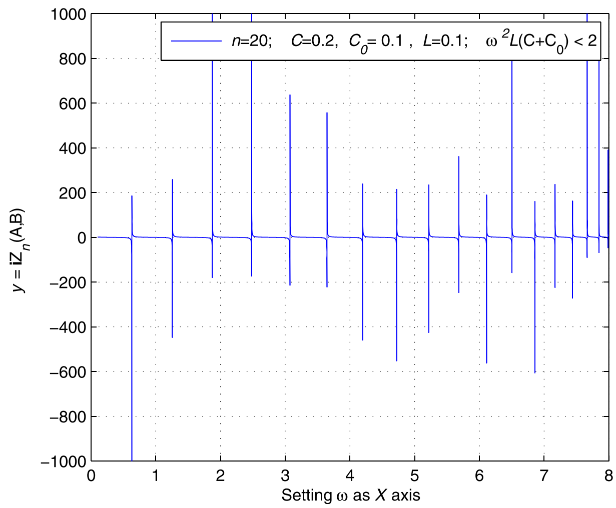

Figure 14. The impedance oscillatorily (with resonance) varying with ω, with setting C_0 = 0.2, C = 0.1, L = 0.1 and n = 20. [Colour figure can be viewed at wileyonlinelibrary.com]

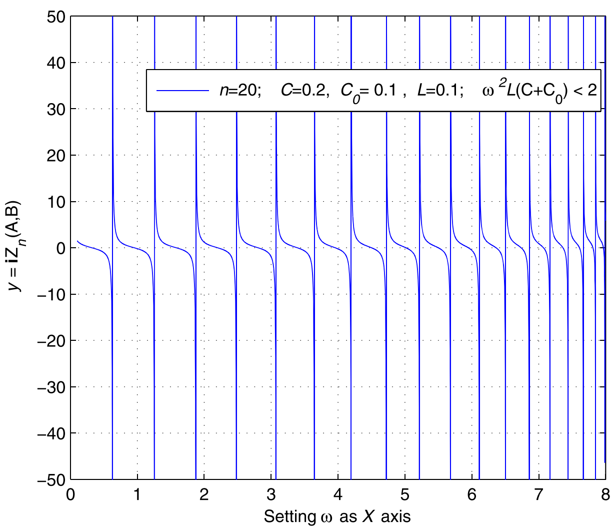

Figure 15. Magnified graph from Figure 15, showing the obvious oscillation features, with C_0 = 0.2, C = 0.1, L = 0.1 and n = 20. [Colour figure can be viewed at wileyonlinelibrary.com]

DOI: 10.1002/cta

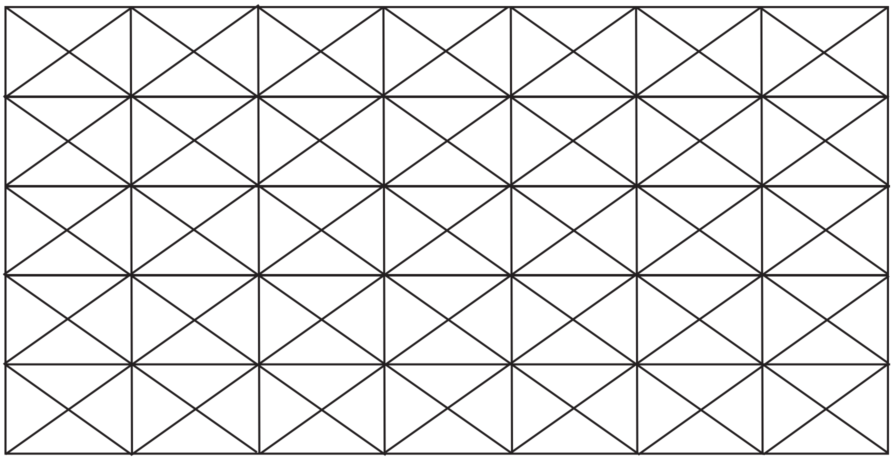

Figure 16. A 5 × 7 resistor network model in which a two-cross resistor is embedded in the rectangular network.

7. SUMMARY AND DISCUSSION

Complex resistor network with multiple parameters is a difficult problem, which is difficult to compute its resistance if there is no approach of innovation. Transform methods established in this paper solve an n-step complex network of cross resistors, which has never been solved before. The method of solving this problem mainly includes three steps. The first is to establish an equivalent network model; the second is the application of Kirchhoff's theorem to find the equivalent resistance of recursive relations; and finally, the third is adopting the transform method to give the general solution of differential equation. We obtain a concise conclusion of (1), although the network contains eight independent parameters. Further, we created a new concept of negative resistance elements for the needs of the equivalent conversion as shown in Figure 3. The negative resistance is a new concept which can be efficiently to solve some complex problems of resistor network. So, we obtained formula (6). As applications, a number of interesting results are deduced by the general formula when we take some parameters into special values. In particular, the research method and conclusion of this article are also applicable to the complex impedance network; we can obtain the equivalent impedance formula of the complex impedance network if we use variable substitution method. Meanwhile, the resonance behaviour is an interesting finding. For example, we consider an LC impedance network; the characteristics of resonance and oscillatory are discovered. Clearly, the complex impedance network is different from the resistor network. This somewhat curious result suggests the possibility of practical applications of our formulae to resonant circuits.

At the end of this paper, we propose a profound problem based on the previous research in the article [13–28, 37, 38]: how to derive the resistance of an $m \times n$ network of cross resistors as shown in Figure 16. We look forward to the solution of the problem.

ACKNOWLEDGEMENTS

This work was supported by funding from the Natural Science Foundation of Jiangsu Province (no. BK20161278).

REFERENCES

1. Kirchhoff G. Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. *Annals of Physical Chemistry* 1847; **72**:497–508.
2. Belevitch V. Classical network theory, Holden-Day, San Francisco, 1968.
3. Desoer Ch., Kuh E. Basic circuit theory, McGraw-Hill, 1969.
4. Tan Zhi-Zhong, Resistance network model. (Xidian Univ. Press, China, Xi'an). (2011).
5. Klein DJ, Randi M. Resistance distance. *Journal of Mathematical Chemistry* 1993; **12**:8195.
6. Xiao WJ, Gutman I. Resistance distance and Laplacian spectrum. *Theoretical Chemistry Accounts* 2003; **110**:284–289.

7. Cserti J. Application of the lattice Green's function for calculating the resistance of an infinite network of resistors. *American Journal of Physics* 2000; **68**:896–906.
8. Cserti J, David G, Piroth A. Perturbation of infinite networks of resistors. *American Journal of Physics* 2002; **70**:153–159.
9. Cserti J, Szechenyi G, David G. Uniform tiling with electrical resistors. *Journal of Physics A: Mathematical and Theoretical* 2011; **44**:215201.
10. Asad JH. Exact evaluation of the resistance in an infinite face-centered cubic network. *Journal of Statistical Physics* 2013; **150**:1177–1182.
11. Asad JH. Infinite simple 3D cubic network of identical capacitors. *Modern Physics Letters B* 2013; **27**:151350112.
12. Asad JH, Diab AA, Hijjawi RS, Khalifeh JM. Infinite face-centered-cubic network of identical resistors: application to lattice Green's function. *European Physical Journal Plus* 2013; **128**(1):1–5.
13. Wu FY. Theory of resistor networks: the two-point resistance. *Journal of Physics A: Mathematical and General* 2004; **37**:6653–6673.
14. Tzeng WJ, Wu FY. Theory of impedance networks: the two-point impedance and LC resonances. *Journal of Physics A: Mathematical and General* 2006; **39**:8579–8591.
15. Izmailian NS, Huang M-C. Asymptotic expansion for the resistance between two maximum separated nodes on a $M\times N$ resistor network. *Physical Review E* 2010; **82**:011125.
16. Izmailian NS, Kenna R, Wu FY. The two-point resistance of a resistor network: a new formulation and application to the cobweb network. *Journal of Physics A: Mathematical and Theoretical* 2014; **47**:035003.
17. Izmailian NS, Kenna R. A generalised formulation of the Laplacian approach to resistor networks. *Journal of Statistical Mechanics* 2014; **09**:1742–5468 P09016.
18. Essam J. W., Izmailyan N. Sh., Kenna Ralph, Tan Zhi-Zhong. *Royal Society open Science* 2015; **2**:140420. https://doi.org/10.1098/rsos.140420.
19. Tan Z-Z, Zhou L, Yang J-H. The equivalent resistance of a $3\times n$ cobweb network and its conjecture of an $m\times n$ cobweb network. *Journal of Physics A: Mathematical and Theoretical* 2013; **46**:195202.
20. Tan Z-Z, Zhou L, Luo D-F. Resistance and capacitance of $4\times n$ cobweb network and two conjectures. *International Journal of Circuit Theory and Applications* 2015; **43**:329–341.
21. Tan Z-Z. Theory on resistance of m×n cobweb network and its application. *International Journal of Circuit Theory and Applications* 2015; **34**:1687–1702.
22. Tan Z-Z, Essam JW, Wu FY. Two-point resistance of a resistor network embedded on a globe. *Physical Review E* 2014; **90**:012130.
23. Essam JW, Tan Z-Z, Wu FY. Resistance between two nodes in general position on an $m\times n$ fan network. *Physical Review E* 2014; **90**:032130.
24. Tan Z-Z, Fang J-H. Two-point resistance of a cobweb network with a $2r$ boundary. *Communications in Theoretical Physics* 2015; **63**:36–44.
25. Tan Z-Z. Recursion–transform approach to compute the resistance of a resistor network with an arbitrary boundary. *Chinese Physics B* 2015; **24**:020503.
26. Tan Z-Z. Recursion–transform method for computing resistance of the complex resistor network with three arbitrary boundaries. *Physical Review E* 2015; **91**:052122.
27. Tan Z-Z. Recursion–transform method to a non-regular m×n cobweb with an arbitrary longitude. *Scientific Reports* 2015; **5**:11 266. https://doi.org/10.1038/srep11266.
28. Tan Z-Z. Two-point resistance of a non-regular cylindrical network with a zero resistor axis and two arbitrary boundaries. *Communications in Theoretical Physics* 2017; **67**(3):280–288.
29. Kagan M. On equivalent resistance of electrical circuits. *American Journal of Physics* 2015; **83**:53–63.
30. Gabelli J, Fève G, Berroir J-M, Plaçais B, Cavanna A, Etienne B, Jin Y, Glattli DC. Violation of Kirchhoff's laws for a coherent RC circuit. *Science* 2006; **313**:499–502.
31. Tan Z-Z, Zhang Q-H. Formulae of resistance between two corner nodes on a common edge of the $m\times n$ rectangular network. *International Journal of Circuit Theory and Applications* 2015; **43**:944–958.
32. Whan CB, Lobb CJ. Complex dynamical behavior in *RCL* shunted Josephson tunnel junctions. *Physical Review E* 1996; **53**:405.
33. Zhuang J, Yu G-R, Nakayama K. A series *RCL* circuit theory for analyzing non-steady-state water uptake of maize plants. *Scientific Reports* 2015; **4**:6720. https://doi.org/10.1038/srep 06720.
34. Tan Z-Z. A universal formula of the n-th power of 2×2 matrix and its applications. *Journal of Nantong University* 2012; **11**:87–94.
35. Giordano S. Disordered lattice networks: general theory and simulations. *International Journal of Circuit Theory and Applications* 2005; **33**:519–540.
36. Bianco B, Giordano S. Electrical characterisation of linear and non-linear random networks and mixtures. *International Journal of Circuit Theory and Applications* 2003; **31**:199–218.
37. Tan Z-Z. Two-point resistance of an m×n resistor network with an arbitrary boundary and its application in RLC network. *Chinese Physics B* 2016; **25**:050504.
38. Tan Z-Z, Zhang Q-H. Calculation of the equivalent resistance and impedance of the cylindrical network based on RT method. *Acta Physica Sinica* 2017; **66**(7):070501.
39. Mlynek P, Misurec J, Koutny M, Silhavy P. Two-port network transfer function for power line topology modeling. *Radioengineering* 2012; **21**:356–363.
40. Nagashima HN, Onody RN, Faria RM. AC transport studies in polymers by resistor-network and transfer-matrix approaches: application to polyaniline. *Physical Review B* 1999; **59**:905.