MOTION OF A SPHERICAL PARTICLE IN A ROTATING PARABOLA USING FRACTIONAL LAGRANGIAN

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In this work, the fractional Lagrangian of a particle moving in a rotating parabola is used to obtain the fractional Euler- Lagrange equations of motion where derivatives within it are given in Caputo fractional derivative. The obtained fractional Euler- Lagrange equations are solved numerically by applying the Bernstein operational matrices with Tau method. The results obtained are very good and when the order of derivative closes to 1, they are in good agreement with those obtained in Ref. [10] using Multi step- Differential Transformation Method (Ms-DTM).

Keywords: Caputo Fractional Derivatives, Riemann-Liouville fractional integral, Particle in a Rotating Parabola, Bernstein operational matrices.

1. Introduction

As it is well known from literature many techniques can be used to solve systems in classical mechanics. Two famous techniques are commonly used and give us similar results. The first technique (Newtonian mechanics) is based on the concept of force, while the second one (Lagrangian and Hamiltonian mechanics) is energy dependent technique [1, 2]. A wide range of physical systems have been modeled by these techniques especially the second technique, and as a result the equations of motion were derived. One of these interesting systems studied is the motion of a particle in a rotating parabola [2].

In classical mathematics, we deal with derivatives and integrals with integer order. The extension to any order is a branch of mathematics in which it is called Fractional calculus. Fractional calculus owes it origin to a question (nearly before 300 years) about the possibility of taking derivative to a non- integer order. For a long time it is considered as a pure branch of mathematics with no applications in

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real life, but it has been found that it is useful and powerful [3]. Nowadays fractional calculus plays an important role in many applications in sciences and engineering, and its application is spreading nearly in all branches of science and engineering especially where they appear in real systems [4]. In recent years, fractional calculus finds a wide range of applications in classical mechanics, where Lagrangian and Hamiltonian methods are used to study several systems. They were fractionalized and then the fractional Euler- Lagrange's equations or even the Fractional Hamilton's equations were obtained as seen in many works [5-7]. These equations contain right and left fractional derivatives. The analytical solution of the fractional equations obtained cannot be easily obtained, and as a result we seek numerical solution for them. In previous works we studied numerically these Fractional equations for many physical systems of certain interests. See for example [8, 9], and references within it. Now, we survey the Lagrangian of a spherical particle moving in a rotating parabola with Caputo fractional order.Various phenomena exist within the environment where a particle's motion can be observed on them, e.g. centrifugation, centrifugal filters, and industrial hopper. Motion surface possesses various shapes especially for rotating application as it can be circular, parabolic or conical, respectively. Analytical and numerical solutions can be of great use to analyze the motion of particles on these surfaces, as an example see [10] and references in it.
The structure of this work is as follows: In sec. 2, some definitions are listed briefly for definitions of fractional calculus. In sec. 3, the studied system is described classically and fractionally in details. In sec. 4, we present the approximate solutions based on the Bernstein polynomials, and finally we close the paper with conclusion in section 5.

2. Mathematical Tools

In this section, we only recall necessary definitions that are used in the next sections. We start from Riemann-Liouville integrals.

Definition 2.1. The left and right Riemann-Liouville fractional integrals are defined, respectively as follows [4, 11, 12]

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau , \tag{1}$$

$$ {}_tI_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) d\tau . \tag{2}$$

Next, we present definitions of fractional differential operators in Caputo sense by using the operators (1) and (2).

Definition 2.2. The left and right Caputo fractional derivatives are defined, respectively as follows [4, 11, 12]

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-\alpha-1} \frac{d^{n} f(\tau)}{d\tau^{n}} d\tau , \quad (3) $$

$$ {}_{t}^{C}D_{b}^{\alpha} f(t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{t}^{b} (\tau-t)^{n-\alpha-1} \frac{d^{n} f(\tau)}{d\tau^{n}} d\tau , \quad (4) $$

where $n-1<\alpha\leq n\,(n\in \mathrm{N})$. If $\alpha=n\,(n\in \mathrm{N})$ then ${}_{a}^{C}D_{t}^{n} \equiv \frac{d^{n}}{dt^{n}}$ and ${}_{t}^{C}D_{b}^{n} \equiv (-1)^{n} \frac{d^{n}}{dt^{n}}$.

3. Classical and Fractional Descriptions of the Physical Model

To give a classical description of the model, we consider a parabola with a shape defined by $z = cr^2$ [2, 10]. Let us assume that a spherical particle slides along its surface. The following assumptions are considered for the motion of the particle: the particle is in equilibrium, rotating in a circle of radius R, and finally, the surface is rotating about its vertical symmetry axis with angular velocity w, as indicated in Figure 1 below [2, 10].

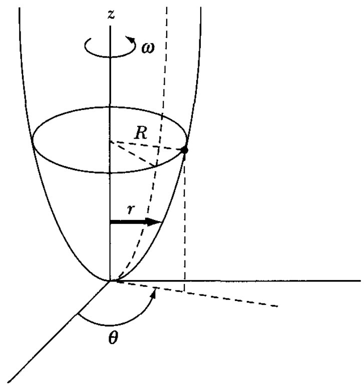

Fig. 1. Schematic view of a spherical particle on a rotating parabolic surface

Here we are going to apply the cylindrical coordinates system (r, θ, z) to describe the motion of the particle. Thus, the kinetic and potential energy of the particle respectively become:

$$ T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2), \quad (5) $$

$$ V = mgz . \quad (6) $$

As a result we have the following constraints: $z = cr^2$,

$$ \theta = wt \quad (7) $$

and the derivatives of the these constraints are: $\dot{z} = 2cr\,\dot{r}$,

$$ \dot{\theta} = w . \quad (8) $$

Making use of the above equations the classical Lagrangian can be written as:

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + 4c^2 r^2 \dot{r}^2 + r^2 w^2) - mgcr^2 . \tag{9}$$

Now, the fractional form of Eq. (9) in terms of Caputo fractional derivative is:

$$L^F = \frac{1}{2}m\left[({}_a^C D_t^\alpha r)^2 + 4c^2 r^2 ({}_a^C D_t^\alpha r)^2 + r^2 w^2\right] - mgcr^2 . \tag{10}$$

Thus, we get the fractional Euler Lagrange equation of motion as:

$$\frac{\partial L^F}{\partial q} + {}_t^C D_b^\alpha \frac{\partial L^F}{\partial {}_a^C D_t^\alpha q} + {}_a^C D_t^\beta \frac{\partial L^F}{\partial {}_t^C D_b^\beta q} = 0 . \tag{11}$$

It is important to indicate that Bourdin [13] derived some conditions that are sufficient to make one ensure that the fractional Euler- Lagrange equation admits a solution. Referring to Bourdin's paper [13] we make sure that in our case the fractional Euler- Lagrangian equation satisfies the conditions introduced in his paper (see for more details the Theorem 1 in 13,pp. 2-3).

Here we have one generalized coordinate $q = r$. Thus, we conclude:

$$4c^2 r({}_a^C D_t^\alpha r)^2 + r(w^2 - 2gc) + {}_t^C D_b^\alpha \, {}_a^C D_t^\alpha r + 4c^2 ({}_t^C D_b^\alpha r^2 \, {}_a^C D_t^\alpha r) = 0 . \tag{12}$$

In Ref. [14] it was considered that:

$$2gc - w^2 = \varepsilon^2 . \tag{13}$$

As a result the Eq. (12) reads:

$$4c^2 r({}_a^C D_t^\alpha r)^2 - r\varepsilon^2 + {}_t^C D_b^\alpha \, {}_a^C D_t^\alpha r + 4c^2 ({}_t^C D_b^\alpha r^2 \, {}_a^C D_t^\alpha r) = 0 . \tag{14}$$

As $\alpha \to 1$, then Eq. (14) reduces to the classical Euler- Lagrange equation.

In the next section, we are aiming to obtain the numerical solution for the Fractional Euler- Lagrange equation (14) for some initial conditions.

4. Numerical Solution Method and the Simulation Results

In this section, we propose the approximate analytical solution for the problem (14) with $0 < \alpha \leq 1$. We consider the initial conditions as follows

$$r(a) = r_0 \ , \tag{15}$$

$$\dot{r}(a) = r_1 . \tag{16}$$

Recently, we used the Bernstein operational matrices (BOM) of Caputo derivative, Riemann-Liouville fractional integral and product for solving some kinds of fractional differential equations and fractional optimal control problems [15-19]. Now, we apply the BOM method to solve the problem (14)-(16).

4.1. Primary concepts

Referring to Ref. [20], we can see details of the proposed concepts in here.

Let $\beta_{i,m}(t)=\binom{m}{i}\frac{(t-a)^i(b-t)^{m-i}}{(b-a)^m}$, $i=0,1,\cdots,m$ be the Bernstein polynomials of degree m on interval $[a,b]$ and $\Psi_m(t)=\left[\beta_{0,m}(t),\beta_{1,m}(t),\ldots,\beta_{m,m}(t)\right]^T$. Since the Bernstein polynomials form a basis on $[a,b]$, for any function $y\in C^{m+1}[a,b]$ we can approximate $y(t)$ as follows

$$y(t)\approx\sum_{i=0}^{m}c_i\,\beta_{i,m}(t)=c^T\Psi_m(t), \tag{17}$$

where $c^T=[c_0,c_2,\cdots,c_m]$ is obtained as

$c^T=\left(\int_a^b y(t)\,\Psi_m(t)^T\,dt\right)Q^{-1}, \quad Q=\left(Q_{i,j}\right)_{i,j=1}^{m+1}$ and

$$Q_{i+1,j+1}=\int_a^b\beta_{i,m}(t)\,\beta_{j,m}(t)\,dx=\frac{(b-a)\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}}, \quad i,j=0,1,\cdots m. \tag{18}$$

Also, we denote the operational matrix for the product of the vector c based on basis $\Psi_m(t)$ by $\hat{C}$ and define it as:

$$c^T\Psi_m(t)\,\Psi_m(t)^T\approx\Psi_m(t)^T\,\hat{C}. \tag{19}$$

On the other hand, the BOM of the standard derivative D, left Riemann-Liouville fractional integral F_{LRL}^{α} and right Caputo fractional derivative D_{RC}^{α} are introduced as follows

$$\frac{d}{dt}\Psi_m(t)\approx D\Psi_m(t), \tag{20}$$

$${}_aI_t^{\alpha}\Psi_m(t)\approx F_{LRL}^{\alpha}\Psi_m(t), \tag{21}$$

$${}_t^CD_b^{\alpha}\Psi_m(t)\approx D_{RC}^{\alpha}\Psi_m(t). \tag{22}$$

4.2. BOM for solving the proposed model

By using (17), we apply the following approximation:

$${}_a^CD_t^{\alpha}r(t)\approx\Upsilon^T\Psi_m(t), \tag{23}$$

where $\Upsilon=[\xi_0,\ldots,\xi_m]^T$ is an unknown vector. So, from Lemma 2.22 of [11, pp. 96]

$${}_aI_t^{\alpha}\,{}_a^CD_t^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(a)\frac{(t-a)^k}{k!}, \tag{24}$$

where $n-1<\alpha\le n\ (n\in \mathrm{N})$.
By (23), (15) , (21) and (24) we can get

$$r(t)\approx {}_aI_t^\alpha\left(\Upsilon^T\Psi_m(t)\right)+r(a)=\Upsilon^T F_{LRL}^\alpha\Psi_m(t)+r_0\mathbf{1}^T\Psi_m(t)=\Upsilon_\alpha^T\Psi_m(t), \qquad (25)$$

r_0

where $\mathbf{1}=\Big[\underbrace{1,1,\dots,1}_{m+1\ times}\Big]^T$, $\Upsilon_\alpha^T=\Upsilon^T F_{LRL}^\alpha+r_0\mathbf{1}^T$ (note $\sum_{i=0}^{m}\beta_{i,m}(t)=\mathbf{1}^T\Psi_m(t)=1$).

Now, we can approximate the problem (14) and (15) as follows

$$\begin{aligned}&4c^2\,\Upsilon_\alpha^T\Psi_m(t)\left(\underbrace{\Upsilon^T\Psi_m(t)\Psi_m(t)^T}_{\Psi_m(t)^T\,\hat{\Upsilon}}\Upsilon\right)-\varepsilon^2\,\Upsilon_\alpha^T\Psi_m(t)+\Upsilon^T D_{RC}^\alpha\Psi_m(t)\\&+4c^2\,{}_t^CD_b^\alpha\left(\Upsilon^T\Psi_m(t)\left(\underbrace{\Upsilon_\alpha^T\Psi_m(t)\Psi_m(t)^T}_{\Psi_m(t)^T\,\hat{\Upsilon}_\alpha}\Upsilon_\alpha\right)\right)=0\end{aligned} \qquad (26)$$

$$\begin{aligned}\Rightarrow\quad &4c^2\,\underbrace{\Upsilon_\alpha^T\Psi_m(t)\Psi_m(t)^T}_{\Psi_m(t)^T\,\hat{\Upsilon}_\alpha}\hat{\Upsilon}\Upsilon-\varepsilon^2\,\Upsilon_\alpha^T\Psi_m(t)+\Upsilon^T D_{RC}^\alpha\Psi_m(t)\\&+4c^2\,{}_t^CD_b^\alpha\left(\underbrace{\Upsilon^T\Psi_m(t)\Psi_m(t)^T}_{\Psi_m(t)^T\,\hat{\Upsilon}}\hat{\Upsilon}_\alpha\Upsilon_\alpha\right)=0\end{aligned} \qquad (27)$$

$$\begin{aligned}\Rightarrow\quad &4c^2\,\Psi_m(t)^T\,\hat{\Upsilon}_\alpha\,\hat{\Upsilon}\Upsilon-\varepsilon^2\,\Upsilon_\alpha^T\Psi_m(t)+\Upsilon^T D_{RC}^\alpha\Psi_m(t)\\&+4c^2\left(D_{RC}^\alpha\Psi_m(t)\right)^T\,\hat{\Upsilon}\,\hat{\Upsilon}_\alpha\Upsilon_\alpha=0\end{aligned}\ . \qquad (28)$$

We observe that from (26)-(28) the problem (14) and (15) is reduced to:

$$\Psi_m(t)^T\left(4c^2\ \hat{\Upsilon}_\alpha\,\hat{\Upsilon}\Upsilon-\varepsilon^2\,\Upsilon_\alpha+D_{RC}^{\alpha\ T}\Upsilon+4c^2\,D_{RC}^{\alpha\ T}\hat{\Upsilon}\,\hat{\Upsilon}_\alpha\Upsilon_\alpha\right)=0. \qquad (29)$$

Also, by (25) and (20) we can use the following approximation for the initial condition (16):

$$\Upsilon_\alpha^T\,D\,\Psi_m(a)=r_1. \qquad (30)$$

Now, by applying the Tau method (see for more details the subsection 6.4.4 of [21, pp. 367]) for (29), we obtain the following algebraic equations:

$$\int_a^b\left(4c^2\ \hat{\Upsilon}_\alpha\,\hat{\Upsilon}\Upsilon-\varepsilon^2\,\Upsilon_\alpha+D_{RRL}^{\alpha\ T}\Upsilon+4c^2\,D_{RC}^{\alpha\ T}\hat{\Upsilon}\,\hat{\Upsilon}_\alpha\Upsilon_\alpha\right)^T\Psi_m(t)\beta_{i,m}(t)\,dt=0.\quad i=0,\dots,m-1. \qquad (31)$$

Finally, the equations (31) with (30) produce a system of $m+1$ algebraic equations and $m+1$ variables $\xi_i, i=0,\dots,m$. For solving this system, we use the Newton's method [22-25]. Then, from (25) we obtain the approximation for $r(t)$.

Now, let $\varepsilon = 1$, $a = 0$, $b = 1$, $r_0 = 1$ and $r_1 = 0$. We can see the behaviors of $r(t)$ for $c = 1, 1.5, 2$ and $\alpha = 0.7, 0.8, 0.9, 1$ in figures 2-8.

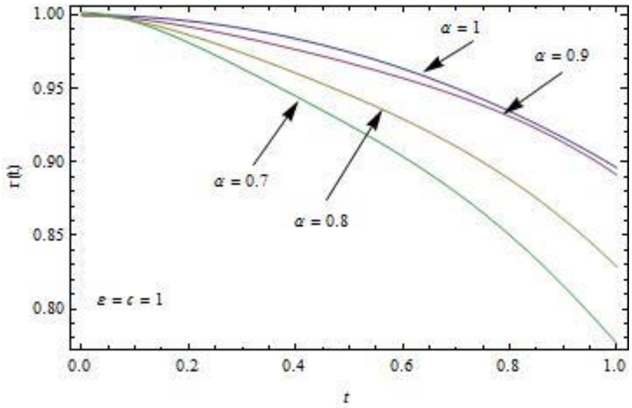

Fig. 2. A plot of $r(t)$ for $c = 1$ and $\alpha = 0.7, 0.8, 0.9, 1$.

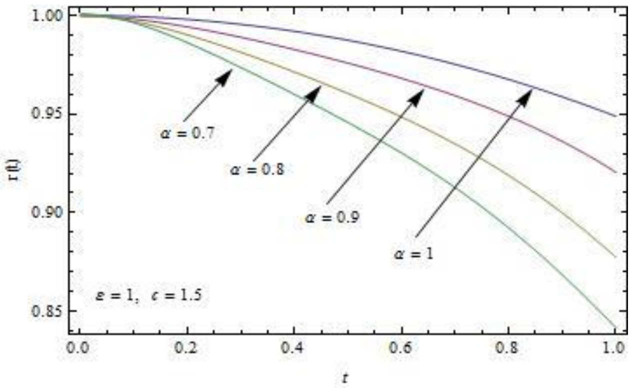

Fig. 3. A plot of $r(t)$ for $c = 1.5$ and $\alpha = 0.7, 0.8, 0.9, 1$.

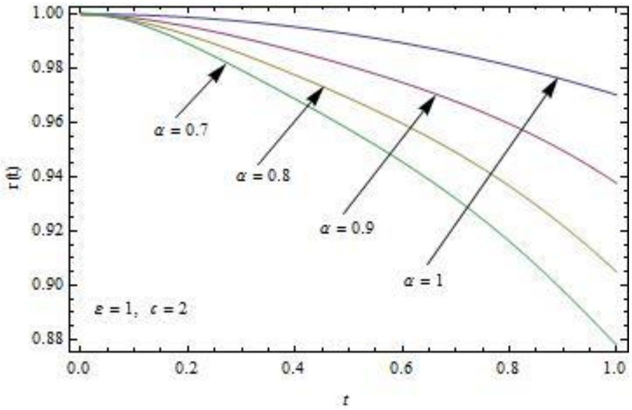

Fig. 4. A plot of $r(t)$ for $c = 2$ and $\alpha = 0.7, 0.8, 0.9, 1$.

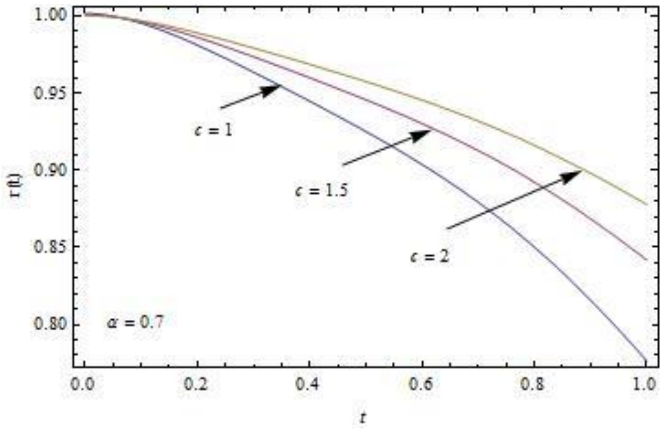

Fig. 5. A plot of $r(t)$ for $c = 1,1.5,2$ and $\alpha = 0.7$.

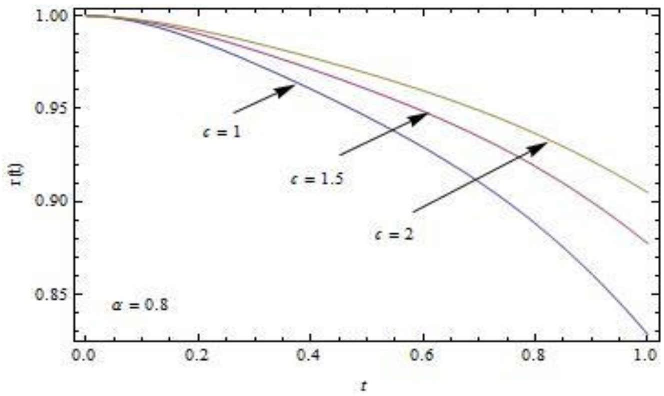

Fig. 6. A plot of $r(t)$ for $c = 1,1.5,2$ and $\alpha = 0.8$.

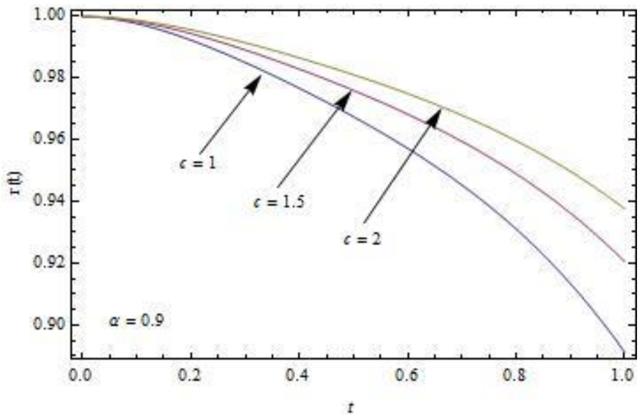

Fig. 7. A plot of $r(t)$ for $c = 1,1.5,2$ and $\alpha = 0.9$.

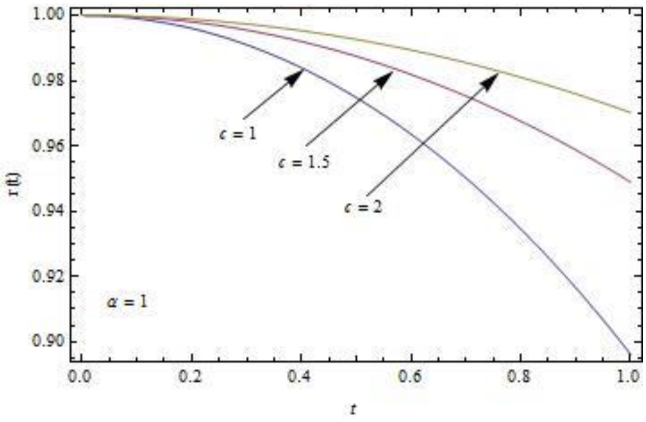

Fig. 8. A plot of $r(t)$ for $c = 1, 1.5, 2$ and $\alpha = 1$.

5. Conclusions

In this work, we have studied the motion of a spherical particle in a rotating parabola using fractional Lagrangian. For this aim, we used the derivatives in Caputo sense. We have applied the BOM for solving the proposed fractional model. The results show the approximate solutions close to the solutions for $\alpha = 1$ when α approaches to 1 and m be fixed.

As shown in the above figures we plot the position of the particle r(t) against t in the period [0, 1] for $\alpha = 0.7, 0.8, 0.9, 1$. It is clear from our figures that for $\alpha = 1$ (the classical case) the BOM is in a good agreement with results obtained by Ms- DTM used in Ref. [9] especially for the period of t from [0, 1].

Finally, it is of worth to mention that similar approaches have been used and applied in elasticity and thermoelasticity [25- 27].

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