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Two-Point Resistance on the Centered-Triangular Lattice

M. Q. Owaidat[1,2**], A. A. Al-Badawi[1], J. H. Asad[3], Mohammed Al-Twiessi[1]

[1]Department of Physics, Al-Hussein Bin Talal University, Ma'an 71111, Jordan

[2]Department of Physics, Faculty of Sciences, Taibah University, Yanbu, Kingdom of Saudi Arabia

[3]Department of Physics, Faculty of Arts and Sciences, Palestine Technical University, Kadoorie, Tulkarm, Palestine

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The resistance between any two lattice points in an infinite, centered-triangular lattice of equal resistors is determined using the lattice Green function method. It is shown that the two-point resistance on the centered-triangular lattice is expressed in terms of the resistance of a triangular lattice. Some exact values for the resistance near the origin of the lattice are presented. For large separation between lattice points the asymptotic forms of the resistance are calculated.

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A classical problem in electric circuit theory and graph theory is the calculation of the effective resistance between two arbitrary lattice sites in a resistor lattice network. This problem has received tremendous recent interest from numerous researchers in physics, mathematics and electrical engineering literature. Several theoretical advanced methods have been developed to investigate this problem.[1–8]

Recently, Cserti *et al.*[8] have generalized lattice Green function technique established in Ref. [7] to determine the two-point resistance on any infinite periodic lattice structure in d dimensions. They have obtained a general expression for the resistance between two arbitrary nodes and presented several examples of resistor networks in one, two and three dimensions. More recently, the lattice Green function approach has been applied to study a variety of resistance lattice structures.[9–14]

Also, the lattice Green function is used to study a capacitor lattice[15–17] and a resistor lattice with a missing bond[18] or with a substitutional resistor[19,20] or with an interstitial resistor.[21–23]

The problem of a nonlinear lattice is also interesting in the literature.[24–27] In Ref. [27] a bidirectional negative differential thermal resistance effect in the Frenkel–Kontorova nonlinear lattices has been investigated numerically. This phenomena is important in designing thermal devices via materials of nonlinear lattices, such as thermal transistors.[24]

In the present study, we apply the approach of Ref. [8] to the centered-triangular lattice, and compute the resistance between any two points in the lattice. The structure of centered-triangular lattice is more complicated than the triangular lattice, as there are three lattice points in each unit cell. We obtain resistance expressions for the centered-triangular lattice in terms of the resistance on the well-investigated triangular lattice. Other mappings between different resistor lattices can be found in Refs. [7,8].

The Green function is an ingenious technique to solve ordinary and partial differential equations under boundary conditions. Consider a linear differential operator L with the functions $f(\boldsymbol{r})$ and $\psi(\boldsymbol{r})$ such that

$$L\psi(\boldsymbol{r}) = f(\boldsymbol{r}). \tag{1}$$

The Green function of the operator L is defined as the function $G(\boldsymbol{r},\boldsymbol{r}')$ such that

$$LG(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{2}$$

In physics the Green function is often defined as

$$LG(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}'),$$

and at least formally $G = -L^{-1}$.

If L admits a complete, orthonormal set of eigenvectors, $\varphi_n(\boldsymbol{r})$ will be

$$L\varphi_n(\boldsymbol{r}) = \lambda_n\varphi_n(\boldsymbol{r}), \tag{3}$$

then $\psi(\boldsymbol{r})$ and $f(\boldsymbol{r})$ can be expanded onto this orthonormal set,

$$\psi(\boldsymbol{r}) = \sum_{n=0}^{\infty} a_n\varphi_n(\boldsymbol{r}), \tag{4}$$

$$f(\boldsymbol{r}) = \sum_{n=0}^{\infty} b_n\varphi_n(\boldsymbol{r}). \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (1), we have

$$a_n = \frac{b_n}{\lambda_n}. \tag{6}$$

The values of b are the projection onto the basis,

$$b_n = \int \varphi_n^*(\boldsymbol{r}')f'(\boldsymbol{r})d(\boldsymbol{r}'). \tag{7}$$

Therefore, one can write $\psi(\boldsymbol{r})$ as

$$\psi(\boldsymbol{r}) = \sum_{n=0}^{\infty} \frac{b_n}{\lambda_n}\varphi_n(\boldsymbol{r}) = \int \sum_{n=0}^{\infty} \frac{\varphi_n(\boldsymbol{r})\varphi_n^*(\boldsymbol{r}')}{\lambda_n} f'(\boldsymbol{r})d(\boldsymbol{r}'). \tag{8}$$

The Green function of the operator L is defined as

$$G(\boldsymbol{r},\boldsymbol{r}') = \sum_{n=0}^{\infty} \frac{\varphi_n(\boldsymbol{r})\varphi_n^*(\boldsymbol{r}')}{\lambda_n}. \tag{9}$$

For an infinite d-dimensional lattice under periodic boundary conditions, the functions are given in

**Corresponding author. Email: owaidat@ahu.edu.jo

the form of plane wave and the lattice Green function in the Fourier transform representation is given by

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{V_{\rm c}}{(2\pi)^d}\int_{\rm BZ} d^d\boldsymbol{k} G(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}, \tag{10}$$

where $G(\boldsymbol{k}) = \frac{1}{\lambda(\boldsymbol{k})}$ is the inverse Fourier transform of $G(\boldsymbol{r},\boldsymbol{r}')$, $V_{\rm c}$ is the volume of the unit cell, and $\boldsymbol{k}$ is the wave vector restricted to the first Brillouin zone.

The Green function is an important mathematical tool in several areas of theoretical physics. It provides, for example, an efficient method for solving linear problems involving a differential equation. An excellent introduction to Green function and different applications can be found in Refs. [28–30].

Lattice Green functions appear in various problems in condensed matter physics, such as lattice vibration problems, diffusion in solids, the dynamics of spin waves.[31] They are also used in statistical physics (theory of random walks)[32] and theories of impurities in solids.[33]

In the following we use the general Green function method[8] to calculate the effective resistance between two arbitrary lattice sites in an infinite centered-triangular lattice.

Consider the centered-triangular (CT) lattice made up of identical resistances R, as shown in Fig. 1. The CT lattice is a triangular lattice with a site inserted in each triangular face. This lattice has received considerable recent attention as instances of finite-temperature transitions in the antiferromagnetic Potts models.[34–37]

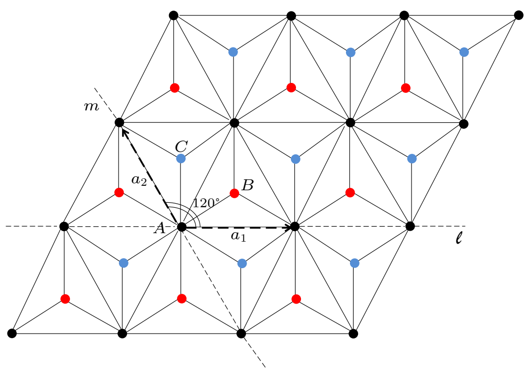

Fig. 1. The centered-triangular lattice of the resistor network.

The unit cell contains three sites of sublattices labeled by α=A, B and C. We choose the origin of the coordinate system at a site of type A. The position of a cell is specified by the vector $\boldsymbol{r} = \ell\boldsymbol{a}_1 + m\boldsymbol{a}_2 = (\ell, m)$, where $\boldsymbol{a}_i$ are the unit cell vectors forming an angle of 120° (see Fig. 1) and ℓ, m are any integers $(0, \pm1, \pm2, \ldots)$. Then, each site is characterized by the position of its cell, $\boldsymbol{r} = (\ell, m)$, and its position inside the cell, as α=A, B and C. Thus the complete coordinate of a lattice site is $(\ell, m; \alpha)$. The lattice constant is $a_1 = a_2 = a$. The distance between the origin (0, 0) and any site $\boldsymbol{r} = (\ell, m)$ is given by $r = \sqrt{\boldsymbol{r}\cdot\boldsymbol{r}} = \sqrt{\ell^2 + m^2 - \ell m}$.

Let $I_\alpha(\boldsymbol{r})$ and $V_\alpha(\boldsymbol{r})$ be the electric current and potential at lattice site $(\boldsymbol{r};\alpha)$, respectively. Using Kirchhoff's first rule and Ohm's law, the currents at the lattice sites $(\boldsymbol{r};{\rm A}), (\boldsymbol{r};{\rm B}), (\boldsymbol{r};{\rm C})$ can be written as

$$\begin{aligned} I_{\rm A}(\boldsymbol{r}) =& \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}+\boldsymbol{a}_1)}{R} + \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}-\boldsymbol{a}_1)}{R} \\ &+ \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}+\boldsymbol{a}_2)}{R} \\ &+ \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}-\boldsymbol{a}_2)}{R} \\ &+ \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}+\boldsymbol{a}_1+\boldsymbol{a}_2)}{R} \\ &+ \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}-\boldsymbol{a}_1-\boldsymbol{a}_2)}{R} \\ &+ \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm B}(\boldsymbol{r})}{R} + \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm B}(\boldsymbol{r}-\boldsymbol{a}_1)}{R} \\ &+ \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm B}(\boldsymbol{r}-\boldsymbol{a}_1-\boldsymbol{a}_2)}{R} \\ &+ \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm C}(\boldsymbol{r})}{R} + \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm C}(\boldsymbol{r}-\boldsymbol{a}_2)}{R} \\ &+ \frac{V_{\rm A}(\boldsymbol{r}) - V_{\rm C}(\boldsymbol{r}-\boldsymbol{a}_1-\boldsymbol{a}_2)}{R}, \end{aligned} \tag{11}$$

$$\begin{aligned} I_{\rm B}(\boldsymbol{r}) =& \frac{V_{\rm B}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r})}{R} + \frac{V_{\rm B}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}+\boldsymbol{a}_1)}{R} \\ &+ \frac{V_{\rm B}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}+\boldsymbol{a}_1+\boldsymbol{a}_2)}{R}, \end{aligned} \tag{12}$$

$$\begin{aligned} I_{\rm C}(\boldsymbol{r}) =& \frac{V_{\rm C}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r})}{R} + \frac{V_{\rm C}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}+\boldsymbol{a}_2)}{R} \\ &+ \frac{V_{\rm C}(\boldsymbol{r}) - V_{\rm A}(\boldsymbol{r}+\boldsymbol{a}_1+\boldsymbol{a}_2)}{R}. \end{aligned} \tag{13}$$

The electric potential and current at the site $(\boldsymbol{r};\alpha)$ are represented in their inverse Fourier transforms as

$$V_\alpha(\boldsymbol{r}) = \frac{A_0}{(2\pi)^2}\int_{\rm BZ} d^2\boldsymbol{k} V_\alpha(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{14}$$

$$I_\alpha(\boldsymbol{r}) = \frac{A_0}{(2\pi)^2}\int_{\rm BZ} d^2\boldsymbol{k}\ I_\alpha(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{15}$$

where $A_0 = |\boldsymbol{a}_1 \times \boldsymbol{a}_2| = \sqrt{3}a^2/2$ is the area of the unit cell, and the vector $\boldsymbol{k}$ is limited to the first Brillouin zone (BZ) of the lattice. Like the triangular network,[4,7] the CT lattice can be deformed to a square lattice without changing the resistance between two arbitrary lattices. Thus the Brillouin zone becomes a square with boundaries $-\pi/a \leqslant k_1, k_2 \leqslant \pi/a$, and the area of the unit cell is $A_0 = a^2$.

Substituting Eqs. (14) and (15) into (11), (12) and (13), we have

$$\boldsymbol{L}(\theta_1,\theta_2)\begin{pmatrix} V_{\rm A}(\theta_1,\theta_2) \\ V_{\rm B}(\theta_1,\theta_2) \\ V_{\rm C}(\theta_1,\theta_2)\end{pmatrix} = -R\begin{pmatrix} I_{\rm A}(\theta_1,\theta_2) \\ I_{\rm B}(\theta_1,\theta_2) \\ I_{\rm C}(\theta_1,\theta_2)\end{pmatrix}, \tag{16}$$

where we have made the transformations $\theta_i = \boldsymbol{k}\cdot\boldsymbol{a}_i (i = 1, 2)$, and $\boldsymbol{L}(\theta_1,\theta_2)$ is the Fourier transform of the Laplacian matrix, given by

$$\boldsymbol{L}(\theta_1,\theta_2) = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & -3 & 0 \\ L_{31} & 0 & -3\end{pmatrix}, \tag{17}$$

where $L_{11} = -12 + 2\cos\theta_1 + 2\cos\theta_2 + 2\cos(\theta_1+\theta_2)$, $L_{12} = 1 + e^{-i\theta_1} + e^{-i(\theta_1+\theta_2)}$, $L_{13} = 1 + e^{-i\theta_2} +$

$e^{-i(\theta_1+\theta_2)}$, $L_{21} = 1 + e^{i\theta_1} + e^{i(\theta_1+\theta_2)}$, and $L_{31} = 1 + e^{i\theta_2} + e^{i(\theta_1+\theta_2)}$.

The Fourier transform of the Green function can be determined from the definition $\boldsymbol{G} = -\boldsymbol{L}^{-1}$, we have

$$\boldsymbol{G}(\theta_1,\theta_2) = [90 - 30\cos\theta_1 - 30\cos\theta_2 - 30\cos(\theta_1+\theta_2)]^{-1} \times \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}, \tag{18}$$

where $G_{11} = 9$, $G_{12} = 3 + 3e^{-i\theta_1} + 3e^{-i(\theta_1+\theta_2)}$, $G_{13} = 3 + 3e^{-i\theta_2} + 3e^{-i(\theta_1+\theta_2)}$, $G_{21} = 3 + 3e^{i\theta_1} + 3e^{i(\theta_1+\theta_2)}$, $G_{22} = 33 - 8\cos\theta_1 - 8\cos\theta_2 - 8\cos(\theta_1+\theta_2)$, $G_{23} = 2 + 2e^{i\theta_1} + 2e^{-i\theta_2} + e^{i(\theta_1-\theta_2)} + 2\cos(\theta_1+\theta_2)$, $G_{31} = 3 + 3e^{i\theta_2} + 3e^{i(\theta_1+\theta_2)}$, $G_{32} = 2 + 2e^{-i\theta_1} + 2e^{i\theta_2} + e^{-i(\theta_1-\theta_2)} + 2\cos(\theta_1+\theta_2)$ and $G_{33} = 33 - 8\cos\theta_1 - 8\cos\theta_2 - 8\cos(\theta_1+\theta_2)$.

The lattice Green function $G_{\alpha\beta}(\ell,m)$ is given by its inverse Fourier transform

$$G_{\alpha\beta}(\ell,m) = \int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi} G_{\alpha\beta}(\theta_1,\theta_2)\, e^{i(\ell\theta_1+m\theta_2)}. \tag{19}$$

The resistance between the origin $(0,0;\alpha)$ and the lattice point $(\ell,m;\beta)$ in the CT lattice can be evaluated from the general resistance expression (2.13) in Ref. [8] with $d = 2$,

$$\frac{R_{\alpha\beta}(\ell,\ m)}{R} = \int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}[G_{\alpha\alpha}(\theta_1,\theta_2) + G_{\beta\beta}(\theta_1,\theta_2) - G_{\alpha\beta}(\theta_1,\theta_2)e^{-i(\ell\theta_1+m\theta_2)} - G_{\beta\alpha}(\theta_1,\theta_2)e^{i(\ell\theta_1+m\theta_2)}], \tag{20}$$

where $G_{\alpha\beta}(\theta_1,\theta_2)$ is given in Eq. (18).

Table 1. Values of the resistance (in units of R) of the CT lattice for small distances.

ℓ, m	$R_{\rm AA}(\ell,m)$	$R_{\rm BB}(\ell,m)$	$R_{\rm AB}(\ell,m)$	$R_{\rm AC}(\ell,m)$	$R_{\rm BC}(\ell,m)$
0, 0	0	0	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{31}{45}$
1, 0	$\frac{1}{5}$	$\frac{11}{15}$	$\frac{11}{15} - \frac{2\sqrt{3}}{5\pi}$	$\frac{4}{15} + \frac{2\sqrt{3}}{5\pi}$	$\frac{28}{45} + \frac{2\sqrt{3}}{15\pi}$
0, 1	$\frac{1}{5}$	$\frac{11}{15}$	$\frac{11}{15} - \frac{2\sqrt{3}}{5\pi}$	$\frac{4}{15} + \frac{2\sqrt{3}}{5\pi}$	$\frac{8}{15} + \frac{8\sqrt{3}}{15\pi}$
1, 1	$\frac{1}{5}$	$\frac{11}{15}$	$\frac{11}{15} - \frac{2\sqrt{3}}{5\pi}$	$\frac{4}{15} + \frac{2\sqrt{3}}{5\pi}$	$\frac{8}{9} - \frac{4\sqrt{3}}{15\pi}$
2, 0	$\frac{8}{5} - \frac{12\sqrt{3}}{5\pi}$	$\frac{32}{15} + \frac{12\sqrt{3}}{5\pi}$	$\frac{78}{15} - \frac{42\sqrt{3}}{5\pi}$	$-\frac{1}{3} + \frac{8\sqrt{3}}{5\pi}$	$\frac{22}{45} + \frac{4\sqrt{3}}{15\pi}$
0, 2	$\frac{8}{5} - \frac{12\sqrt{3}}{5\pi}$	$\frac{32}{15} + \frac{12\sqrt{3}}{5\pi}$	$\frac{78}{15} - \frac{42\sqrt{3}}{5\pi}$	$-\frac{1}{3} + \frac{8\sqrt{3}}{5\pi}$	$-\frac{11}{9} + \frac{56\sqrt{3}}{15\pi}$
2, 2	$\frac{8}{5} - \frac{12\sqrt{3}}{5\pi}$	$\frac{32}{15} + \frac{12\sqrt{3}}{5\pi}$	$\frac{78}{15} - \frac{42\sqrt{3}}{5\pi}$	$-\frac{1}{3} + \frac{8\sqrt{3}}{5\pi}$	$\frac{73}{45} - \frac{22\sqrt{3}}{15\pi}$
2, 1	$-\frac{2}{5} + \frac{10\sqrt{3}}{5\pi}$	$\frac{2}{15} + \frac{10\sqrt{3}}{5\pi}$	$\frac{-28}{15} + \frac{22\sqrt{3}}{5\pi}$	$-\frac{1}{3} + \frac{8\sqrt{3}}{5\pi}$	$\frac{22}{45} + \frac{8\sqrt{3}}{15\pi}$
1, 2	$-\frac{2}{5} + \frac{10\sqrt{3}}{5\pi}$	$\frac{2}{15} + \frac{10\sqrt{3}}{5\pi}$	$\frac{-28}{15} + \frac{22\sqrt{3}}{5\pi}$	$-\frac{28}{15} + \frac{22\sqrt{3}}{5\pi}$	$\frac{7}{3} - \frac{14\sqrt{3}}{5\pi}$
3, 0	$\frac{81}{5} - \frac{144\sqrt{3}}{5\pi}$	$\frac{251}{15} - \frac{144\sqrt{3}}{5\pi}$	$\frac{803}{15} - \frac{480\sqrt{3}}{5\pi}$	$-\frac{28}{3} + \frac{18\sqrt{3}}{\pi}$	$-\frac{89}{9} + \frac{292\sqrt{3}}{15\pi}$
0, 3	$\frac{81}{5} - \frac{144\sqrt{3}}{5\pi}$	$\frac{251}{15} - \frac{144\sqrt{3}}{5\pi}$	$\frac{803}{15} - \frac{480\sqrt{3}}{5\pi}$	$\frac{803}{15} - \frac{96\sqrt{3}}{\pi}$	$-\frac{1038}{15} + \frac{1906\sqrt{3}}{15\pi}$
3, 3	$\frac{81}{5} - \frac{144\sqrt{3}}{5\pi}$	$\frac{251}{15} - \frac{144\sqrt{3}}{5\pi}$	$\frac{803}{15} - \frac{480\sqrt{3}}{5\pi}$	$\frac{803}{15} - \frac{96\sqrt{3}}{\pi}$	$\frac{74}{3} - \frac{216\sqrt{3}}{5\pi}$

The resistance between any two lattice sites in the CT lattice can be calculated from Eq. (20). Moreover, the resistances $R_{\alpha\beta}(\ell,m)$ on the CT lattice can be expressed in terms of the well-known resistance $R^{\rm T}(\ell,m)$ on the triangular lattice.[4,7] We obtain the following explicit expressions

$$R_{\rm AA}(\ell,m) = \frac{3}{5}R^{\rm T}(\ell,m), \tag{21}$$

$$R_{\rm BB}(\ell,m) = R_{\rm CC}(\ell,m) = \frac{8R}{15}\Delta_{\ell,m} + \frac{3}{5}R^{\rm T}(\ell,m), \tag{22}$$

where

$$\Delta_{\ell m} = \begin{cases} 0, & \ell = m = 0, \\ 1, & \text{otherwise}, \end{cases} \tag{23}$$

$$R_{\rm AB}(\ell,m) = \frac{4}{15}R + \frac{1}{5}[R^{\rm T}(\ell,m) + R^{\rm T}(\ell+1,m) + R^{\rm T}(\ell+1,m+1)], \tag{24}$$

$$R_{\rm AC}(\ell,m) = \frac{4}{15}R + \frac{1}{5}[R^{\rm T}(\ell,m) + R^{\rm T}(\ell,m+1) + R^{\rm T}(\ell+1,m+1)] \tag{25}$$

$$R_{\rm BC}(\ell,m) = \frac{8}{15}R + \frac{2}{15}[R^{\rm T}(\ell,m) + R^{\rm T}(\ell,m+1) + R^{\rm T}(\ell-1,m)] + \frac{1}{15}[R^{\rm T}(\ell+1,m+1) + R^{\rm T}(\ell-1,m+1) + R^{\rm T}(\ell-1,m-1)]. \tag{26}$$

The resistance of the triangular lattice is given by[7]

$$R^{\rm T}(\ell,m) = R\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi} \cdot \frac{1-\cos(\ell\theta_1+m\theta_2)}{3-\cos\theta_1-\cos\theta_2-\cos(\theta_1+\theta_2)}. \tag{27}$$

The remaining resistances can be found from the lattice symmetry $R_{\beta\alpha}(\boldsymbol{r}) = R_{\alpha\beta}(-\boldsymbol{r})$.

The values of the resistance for small distances between lattice sites are listed in Table 1. In Fig. 2, the resistance $R_{\alpha\beta}(\ell,0)$ is plotted as a function of ℓ.

It is interesting to find the asymptotic forms of the resistance for large distance r between the lattice sites in the triangular and CT lattices. It was shown in Ref. [38] that the large-distance expansion of the Green function $G^{\rm T}(\ell,m)$ for the triangular lattice is given by

$$G^{\rm T}(\ell,m)=G^{\rm T}(0,0)-\frac{1}{2\sqrt{3}\pi}\Big[\ln r+\gamma+\frac{1}{2}\ln 12-\frac{1}{30}\frac{\cos 6\phi}{r^4}-\frac{5}{84}\frac{\cos 6\phi}{r^6}-\frac{7}{20}\frac{\cos 12\phi}{r^8}-\dots\Big],\tag{28}$$

where $r=\sqrt{\ell^2+m^2-\ell m}>1$, ϕ is the angle between the horizontal axis and r, and $\gamma\simeq 0.57722$ is the Euler–Mascheroni constant.

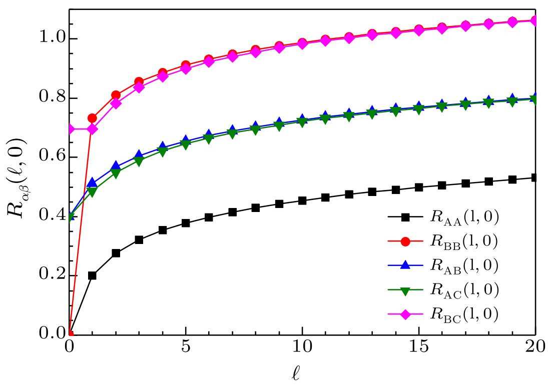

Fig. 2. The resistances $R_{\alpha\beta}(\ell,0)$ in units of R along the ℓ-axis for the CT lattice.

The resistance between the origin and site $\boldsymbol{r}=(\ell,m)$ in the triangular lattice is given by[7]

$$R^{\rm T}(\ell,m)=2R(G^{\rm T}(0,0)-G^{\rm T}(\ell,m)).\tag{29}$$

Inserting Eq. (28) into Eq. (29), the large-distance expansion of the resistance of the triangular lattice is

$$R^{\rm T}(\ell,m)=\frac{R}{\sqrt{3}\pi}\Big[\ln r+\gamma+\frac{1}{2}\ln 12-\frac{1}{30}\frac{\cos 6\phi}{r^4}-\frac{5}{84}\frac{\cos 6\phi}{r^6}-\frac{7}{20}\frac{\cos 12\phi}{r^8}-\dots\Big].\tag{30}$$

The resistance is logarithmically divergent for large values of ℓ and m. It is worth mentioning that Cserti[7] mentioned that the resistance of the triangular lattice goes to infinity as the separation between lattice sites tends to infinity without deriving the asymptotic form.

Now, using Eq. (30) into Eqs. (21)–(26), the asymptotic behaviors for the resistance on the CT lattice can be obtained. As a particular case, the asymptotic forms of the resistance on the line $\ell>1$, $m=0$ ($\phi=0$) are given by

$$R_{\rm AA}(\ell,0)=\frac{\sqrt{3}R}{5\pi}\Big(\ln|\ell|+\gamma+\frac{1}{2}\ln 12-\frac{1}{30\ell^4}-\frac{5}{84\ell^6}-\frac{7}{20\ell^8}-\dots\Big),\tag{31}$$

$$R_{\rm BB}(\ell,0)=R_{\rm CC}(\ell,0)=\frac{8R}{15}+\frac{\sqrt{3}R}{5\pi}\Big(\ln|\ell|+\frac{1}{2}\ln 12+\gamma-\frac{1}{30\ell^4}-\frac{5}{84\ell^6}-\frac{7}{20\ell^8}\dots\Big),\tag{32}$$

$$\begin{aligned}R_{\rm AB}(\ell,0)=\frac{4R}{15}+\frac{R}{5\sqrt{3}\pi}\Big[&\ln|\ell|+\ln|\ell+1|+\ln\sqrt{\ell^2+\ell+1}+3\gamma+\frac{3}{2}\ln 12\\&-\frac{1}{30\ell^4}-\frac{5}{84\ell^6}-\frac{7}{20\ell^8}\dots-\frac{1}{30(\ell+1)^4}-\frac{5}{84(\ell+1)^6}-\frac{7}{20(\ell+1)^8}\dots\\&-\frac{1}{30(\ell^2+\ell+1)^2}-\frac{5}{84(\ell^2+\ell+1)^3}-\frac{7}{20(\ell^2+\ell+1)^4}\dots\Big],\end{aligned}\tag{33}$$

$$\begin{aligned}R_{\rm AC}(\ell,0)=\frac{4R}{15}+\frac{R}{5\sqrt{3}\pi}\Big[&\ln|\ell|+\ln\sqrt{\ell^2-\ell+1}+\ln\sqrt{\ell^2+\ell+1}+3\gamma+\frac{3}{2}\ln 12-\frac{1}{30\ell^4}\\&-\frac{5}{84\ell^6}-\frac{7}{20\ell^8}\dots-\frac{1}{30(\ell^2-\ell+1)^2}-\frac{5}{84(\ell^2-\ell+1)^3}-\frac{7}{20(\ell^2-\ell+1)^4}\dots\\&-\frac{1}{30(\ell^2+\ell+1)^2}-\frac{5}{84(\ell^2+\ell+1)^3}-\frac{7}{20(\ell^2+\ell+1)^4}\dots\Big],\end{aligned}\tag{34}$$

$$\begin{aligned}&R_{\rm BC}(\ell,0)\\&=\frac{8R}{15}+\frac{2R}{15\sqrt{3}\pi}\Big[\ln|\ell|+\ln|\ell-1|+\ln\sqrt{\ell^2-\ell+1}+3\gamma+\frac{3}{2}\ln 12\\&\quad-\frac{1}{30\ell^4}-\frac{5}{84\ell^6}-\frac{7}{20\ell^8}\dots-\frac{1}{30(\ell-1)^4}-\frac{5}{84(\ell-1)^6}-\frac{7}{20(\ell-1)^8}\dots\\&\quad-\frac{1}{30(\ell^2-\ell+1)^2}-\frac{5}{84(\ell^2-\ell+1)^3}-\frac{7}{20(\ell^2-\ell+1)^4}\dots\Big]\\&\quad+\frac{R}{15\sqrt{3}\pi}\Big[\ln\sqrt{\ell^2-\ell+1}+\ln\sqrt{\ell^2-3\ell+3}+\ln\sqrt{\ell^2+\ell+1}\\&\quad+3\gamma+\frac{3}{2}\ln 12-\frac{1}{30(\ell^2-\ell+1)^2}-\frac{5}{84(\ell^2-\ell+1)^3}-\frac{7}{20(\ell^2-\ell+1)^4}\dots\\&\quad-\frac{1}{30(\ell^2-3\ell+3)^2}-\frac{5}{84(\ell^2-3\ell+3)^3}-\frac{7}{20(\ell^2-3\ell+3)^4}\dots-\frac{1}{30(\ell^2+\ell+1)^2}\\&\quad-\frac{5}{84(\ell^2+\ell+1)^3}-\frac{7}{20(\ell^2+\ell+1)^4}\dots\Big].\end{aligned}\tag{35}$$

In Table 2, the approximate values of the resistances $R_{\rm AA}(\ell,0)$ and $R_{\rm AB}(\ell,0)$, calculated using the asymptotic expansions (31) and (33), respectively, are compared with the exact values.

Table 2. Comparison between the exact and asymptotic results of the resistances $R_{\rm AA}(\ell,0)$ and $R_{\rm AB}(\ell,0)$ in units of R.

$\ell, 0$	Exact $R_{\rm AA}(\ell,0)$	Asymptotic $R_{\rm AA}(\ell,0)$	Exact $R_{\rm AB}(\ell,0)$	Asymptotic $R_{\rm AB}(\ell,0)$
2, 0	0.27681064569	0.276595082	0.5688372784569	0.5687143277
3, 0	0.32172777821	0.321726774	0.6057593728413	0.6057511430
4, 0	0.35349108478	0.353491975	0.6333649394453	0.6333636575
5, 0	0.37810667106	0.378107222	0.6554303758082	0.6554303068
6, 0	0.39821395661	0.398214450	0.6738128283294	0.6738130995
7, 0	0.41521291133	0.415213402	0.6895672915379	0.6895676804
8, 0	0.42993753635	0.429938013	0.7033516419120	0.7033520774
9, 0	0.44292531890	0.442925803	0.7156040197147	0.7156044750
10, 0	0.45454316214	0.454543661	0.7266309659817	0.7266314319

In summary, we have investigated the effective resistance between two arbitrary lattice sites in an infinite CT resistive lattice of identical resistors, using the lattice Green function method.[8] Explicit formulae are derived for the two-point resistance on the CT lattice in terms of the resistance on the triangular lattice. This problem could be of pedagogical interest for undergraduate physics students and would provide a good example for introducing the concept of Green function.

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