CLASSICAL AND FRACTIONAL ASPECTS OF TWO COUPLED PENDULUMS

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In this study, we consider two coupled pendulums (attached together with a spring) having the same length while the same masses are attached at their ends. After setting the system in motion we construct the classical Lagrangian, and as a result, we obtain the classical Euler-Lagrange equation. Then, we generalize the classical Lagrangian in order to derive the fractional Euler-Lagrange equation in the sense of two different fractional operators. Finally, we provide the numerical solution of the latter equation for some fractional orders and initial conditions. The method we used is based on the Euler method to discretize the convolution integral. Numerical simulations show that the proposed approach is efficient and demonstrate new aspects of the real-world phenomena.

Key words: Two coupled pendulums, Euler-Lagrange equation, fractional derivative, Euler method.

1. INTRODUCTION

Lagrangian Mechanics is a powerful method used in analyzing physical systems appeared especially in classical mechanics. This method is based on determining scalar quantities related to the system (*i.e.* kinetic and potential energies). In classical texts, one can find many exciting physical systems that have been solved *via* this method [1–3].

The main idea of the Lagrangian method is obtaining the so-called equations of motion by applying Euler-Lagrange equation to the Lagrangian of the system. In general, the obtained equations are of second-order and one has to solve them. In a few cases, the analytical solution is obtained while in many cases analytical solution is difficult to obtain. In these cases, we turn to use numerical techniques [4–7].

Fractional calculus has a long history and its origin goes back to about three hundred years. It was believed that this branch of mathematics has no applications. The last thirty years showed that in the real-world systems the fractional calculus can play an efficient role in analyzing these systems especially when numerical results are required [8–24].

Classical mechanics is a branch of physics where the fractional calculus has been widely applied. The first attempt to study systems within fractional Lagrangian and fractional Hamiltonian was carried out by Riewe [25, 26]. Later on, many researchers followed Riewe's work [27–29]. In these works, the researchers described the systems of interests by the fractional Lagrangian or the fractional Hamiltonian, and as a result, the fractional Euler-Lagrange equations (FELEs) or the fractional Hamilton equations are derived for the considered problems.

The obtained fractional equations cannot be solved analytically so easily in many cases; therefore, we seek for the numerical schemes used for solving fractional differential equations (FDEs). These methods include the Grünwald-Letnikov approximation [8], decomposition method [30–33], variational iteration method [34], Adams-Bashforth-Moulton technique [35], etc.

The rest of this work is organized as follows. In Sect. 2 some preliminaries concerning the fractional derivatives are presented. In Sect. 3, the classical and fractional studies have been carried out for the two coupled pendulum. Section 4 provides numerical solutions of the derived FELE for different values of fractional order and initial conditions. Finally, we close the paper by a conclusion in Sect. 5.

2. BASIC DEFINITIONS AND PRELIMINARIES

In this Section, we give in brief some preliminaries concerning the fractional derivatives. There are some definitions of the fractional derivatives including Riemann-Liouville, Weyl, Caputo, Marchaud, and Riesz [8]. Moreover, a new fractional derivative with Mittag-Leffler nonsingular kernel (ABC) was proposed recently and applied to some real-world models [36]. Below, we define the fractional derivatives in terms of classic Caputo and ABC. Starting with the classic Caputo, we present the following definitions.

Definition 2.1. *[8] Let $x : [a,b] \to \mathbb{R}$ be a time-dependent function. Then, the left and right Caputo fractional derivatives are defined as*

$$ {}_a^C D_t^\alpha x \triangleq \frac{1}{\Gamma(n-\alpha)} \int\limits_a^t \frac{x^{(n)}(\xi)}{(t-\xi)^{1+\alpha-n}} d\xi, \tag{1}$$

$$ {}_{t}^{C}D_{b}^{\alpha}x \triangleq \frac{1}{\Gamma(n-\alpha)}\int_{t}^{b}\frac{(-1)^{n}x^{(n)}(\xi)}{(t-\xi)^{1+\alpha-n}}d\xi, \tag{2} $$

respectively, where $\Gamma(\cdot)$ *denotes the Euler's Gamma function and* α *represents the fractional derivative order such that* $n-1<\alpha<n$.

Definition 2.2. *[36] For* $g \in H^{1}(a,b)$ *and* $0<\alpha<1$*, the left and right ABC fractional derivatives are defined as*

$$ {}_{a}^{ABC}D_{t}^{\alpha}g \triangleq \frac{B(\alpha)}{1-\alpha}\int_{a}^{t}E_{\alpha}\left(-\alpha\frac{(t-\xi)^{\alpha}}{1-\alpha}\right)\dot{g}(\xi)d\xi, \tag{3} $$

$$ {}_{t}^{ABC}D_{b}^{\alpha}g \triangleq -\frac{B(\alpha)}{1-\alpha}\int_{t}^{b}E_{\alpha}\left(-\alpha\frac{(\xi-t)^{\alpha}}{1-\alpha}\right)\dot{g}(\xi)d\xi, \tag{4} $$

respectively, where $B(\alpha)$ *is a normalization function obeying* $B(0)=B(1)=1$ *and the symbol* E_{α} *denotes the generalized Mittag-Leffler function*

$$ E_{\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\alpha k+1)}. \tag{5} $$

For more details on the new ABC fractional operator and its properties, the interested reader can refer to [36, 37].

3. DESCRIPTION OF THE TWO COUPLED PENDULUM

3.1. CLASSICAL DESCRIPTION

Consider two identical pendulums of length (l) and mass (m) coupled together with a spring stiffness (k) and hang as shown in Fig. 1 below [1]. In our system we consider the following two assumptions: firstly, the spring is connected halfway up the pendulums, and secondly, the spring is massless. Also, as it is clear from Fig. 1, $\theta_2>0$ while $\theta_1<0$. As a result, the kinetic energy (T) of the system reads

$$ T=\frac{ml^{2}}{2}\left(\dot{\theta}_{1}^{2}+\dot{\theta}_{2}^{2}\right), \tag{6} $$

while the potential energy takes the form

$$ V=V_{NC}+V_{C}, \tag{7} $$

where V_{NC} is the potential energy with no coupling spring between the pendulums, and it reads

$$ V_{NC}=\frac{mgl}{2}\left(\theta_{1}^{2}+\theta_{2}^{2}\right), \tag{8} $$

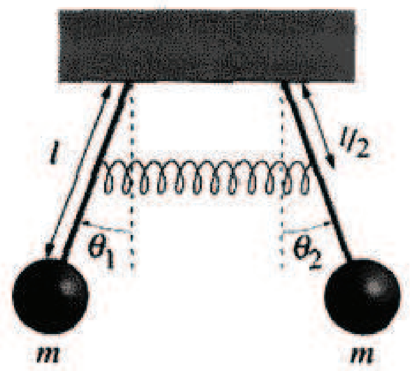

Fig. 1 – Two coupled pendulums.

and V_C is the potential energy due to the coupling spring, and this term takes the form

$$V_C = \frac{k}{2}\left(\frac{l}{2}\right)^2 (\theta_2 - \theta_1)^2. \tag{9}$$

Therefore, the classical Lagrangian is

$$L = T - V = \frac{ml^2}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{mgl}{2}\left(\theta_1^2 + \theta_2^2\right) - \frac{k}{2}\left(\frac{l}{2}\right)^2 (\theta_2 - \theta_1)^2. \tag{10}$$

To obtain the classical equations of motion of our system, we use Eq. (10) and the following equation

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{11}$$

Thus, for $q_1 = \theta_1$, we have

$$\ddot{\theta}_1 + \frac{g}{l}\theta_1 + \frac{k}{4m}(\theta_1 - \theta_2) = 0, \tag{12}$$

while for $q_2 = \theta_2$ we obtain

$$\ddot{\theta}_2 + \frac{g}{l}\theta_2 + \frac{k}{4m}(\theta_2 - \theta_1) = 0. \tag{13}$$

We can simplify the above two equations by introducing the dimensionless coupling parameter η, where $\eta = \frac{kl}{4mg}$, and let $w_0 = \sqrt{\frac{g}{l}}$. As a result, the above equations read

$$\ddot{\theta}_1 + w_0^2(1+\eta)\theta_1 - \eta w_0^2\theta_2 = 0, \tag{14}$$

$$\ddot{\theta}_2 + w_0^2(1+\eta)\theta_2 - \eta w_0^2\theta_1 = 0. \tag{15}$$

These two equations are a set of coupled second-order linear differential equations. If $\eta = 0$ (*i.e.* no coupling spring), we have two independent oscillating systems each of frequency $w_0 = \sqrt{\frac{g}{l}}$.

3.2. FRACTIONAL DESCRIPTION

Now, we will pay attention to the fractional case. The first step is fractionalizing Eq. (10). Thus, the fractional Lagrangian has the form

$$L^F = \frac{ml^2}{2}\left(\left({}_aD_t^\alpha\theta_1\right)^2+\left({}_aD_t^\alpha\theta_2\right)^2\right)-\frac{mgl}{2}\left(\theta_1^2+\theta_2^2\right)-\frac{k}{2}\left(\frac{l}{2}\right)^2(\theta_2-\theta_1)^2. \quad (16)$$

Using the following equation $\frac{\partial L^F}{\partial q_i}+{}_tD_b^\alpha\frac{\partial L^F}{{}_aD_t^\alpha q_i}+{}_aD_t^\alpha\frac{\partial L^F}{{}_tD_b^\alpha q_i}=0$ and Eq. (16), one can obtain the following two fractional equations of motion (for $q_1=\theta_1$ and $q_2=\theta_2$), respectively

$${}_tD_b^\alpha\left({}_aD_t^\alpha\theta_1\right)-w_0^2(1+\eta)\theta_1+\eta w_0^2\theta_2=0, \quad (17)$$

$${}_tD_b^\alpha\left({}_aD_t^\alpha\theta_2\right)+w_0^2(1+\eta)\theta_2-\eta w_0^2\theta_1=0. \quad (18)$$

Finally, as $\alpha\to 1$ the fractional equations of motion (*i.e.* Eqs. (17)-(18)) are reduced to the classical equations of motion defined in Eqs. (14)-(15). In the next Section, we are aiming to obtain the numerical solution for the fractional equations of motion for some initial conditions.

4. NUMERICAL SIMULATIONS

In this Section, an efficient numerical method is developed for solving the FELEs (17)-(18). For comparison purposes, the fractional operator in these equations is considered in the sense of Caputo or ABC. To provide the proposed scheme, we first reformulate Eqs. (17)-(18) in the way below. Suppose that new variables are defined as $\tilde{\theta}_1={}_aD_t^\alpha\theta_1$ and $\tilde{\theta}_2={}_aD_t^\alpha\theta_2$. Then, Eqs. (17)-(18) can be rewritten in the form of a system of fractional differential equations

$$\begin{cases} {}_aD_t^\alpha\theta_1=\tilde{\theta}_1, \\ {}_tD_b^\alpha\tilde{\theta}_1=w_0^2(l+\eta)\theta_1-\eta w_0^2\theta_2, \\ {}_aD_t^\alpha\theta_2=\tilde{\theta}_2, \\ {}_tD_b^\alpha\tilde{\theta}_1=-w_0^2(l+\eta)\theta_2+\eta w_0^2\theta_1. \end{cases} \quad (19)$$

Equation (19) is converted into the following fractional integral equations system by using the definition of fractional integral in the ABC sense [36] and assuming

$\tilde{\theta}_1(b) = \tilde{\theta}_1(b) = 0$

$$\begin{cases} \theta_1(t) = \theta_1(a) + \frac{1-\alpha}{B(\alpha)}\tilde{\theta}_1(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int\limits_a^t (t-\lambda)^{\alpha-1}\tilde{\theta}_1(\lambda)d\lambda, \\ \tilde{\theta}_1(t) = \frac{1-\alpha}{B(\alpha)}\left(w_0^2(l+\eta)\theta_1(t) - \eta w_0^2\theta_2(t)\right) \\ \qquad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int\limits_t^b (\lambda-t)^{\alpha-1}\left(w_0^2(l+\eta)\theta_1(\lambda) - \eta w_0^2\theta_2(\lambda)\right)d\lambda, \\ \theta_2(t) = \theta_2(a) + \frac{1-\alpha}{B(\alpha)}\tilde{\theta}_2(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int\limits_a^t (t-\lambda)^{\alpha-1}\tilde{\theta}_2(\lambda)d\lambda, \\ \tilde{\theta}_2(t) = \frac{1-\alpha}{B(\alpha)}\left(-w_0^2(l+\eta)\theta_2(t) + \eta w_0^2\theta_1(t)\right), \\ \qquad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int\limits_t^b (\lambda-t)^{\alpha-1}\left(-w_0^2(l+\eta)\theta_2(\lambda) + \eta w_0^2\theta_1(\lambda)\right)d\lambda. \end{cases} \tag{20}$$

Now, let us consider a uniform partition on $[a,b]$ with the time step length $h = \dfrac{b-a}{N}$, in which N is a positive integer. Suppose $t_k = a + kh$ is the time at node k for $0 \le k \le N$ and $\theta_{i,k}, \tilde{\theta}_{i,k}$ for $i = 1,2$ are the numerical approximations of $\theta_i(t_k), \tilde{\theta}_i(t_k)$, respectively. Then, by using the Euler method to discretize the convolution integrals in Eq. (20), a system of linear algebraic equations is obtained

$$\begin{cases} \Theta_1 - H_{N,\alpha}\tilde{\Theta}_1 = \Theta_{1,0}, \\ \tilde{\Theta}_1 - P_{N,\alpha}\left(w_0^2(l+\eta)\Theta_1 - \eta w_0^2\Theta_2\right) = 0, \\ \Theta_2 - H_{N,\alpha}\tilde{\Theta}_2 = \Theta_{2,0}, \\ \tilde{\Theta}_2 - P_{N,\alpha}\left(-w_0^2(l+\eta)\Theta_2 + \eta w_0^2\Theta_1\right) = 0, \end{cases} \tag{21}$$

where

$$\Theta_i = \begin{bmatrix} \theta_{i,0} \\ \vdots \\ \theta_{i,N} \end{bmatrix}, \ \tilde{\Theta}_i = \begin{bmatrix} \tilde{\theta}_{i,0} \\ \vdots \\ \tilde{\theta}_{i,N} \end{bmatrix}, \ \Theta_{i,0} = \begin{bmatrix} \theta_{i,0} \\ \vdots \\ \theta_{i,0} \end{bmatrix}, \quad i = 1,2, \tag{22}$$

$$H_{N,\alpha} = \frac{1-\alpha}{B(\alpha)}I_{N+1} + \frac{\alpha}{B(\alpha)}B_{N,\alpha}, \tag{23}$$

$$P_{N,\alpha} = \frac{1-\alpha}{B(\alpha)}I_{N+1} + \frac{\alpha}{B(\alpha)}B_{N,\alpha}^T, \tag{24}$$

$$B_{N,\alpha} = h^\alpha \begin{bmatrix} \omega_{0,\alpha} & 0 & \dots & 0 \\ \omega_{1,\alpha} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \omega_{N,\alpha} & \dots & \omega_{1,\alpha} & \omega_{0,\alpha} \end{bmatrix}, \tag{25}$$

$$\omega_{0,\alpha} = 1, \ \ \omega_{k,\alpha} = \left(\frac{\alpha+k-1}{k}\right)\omega_{k-1,\alpha}, \ \ k = 1,2,\dots. \tag{26}$$

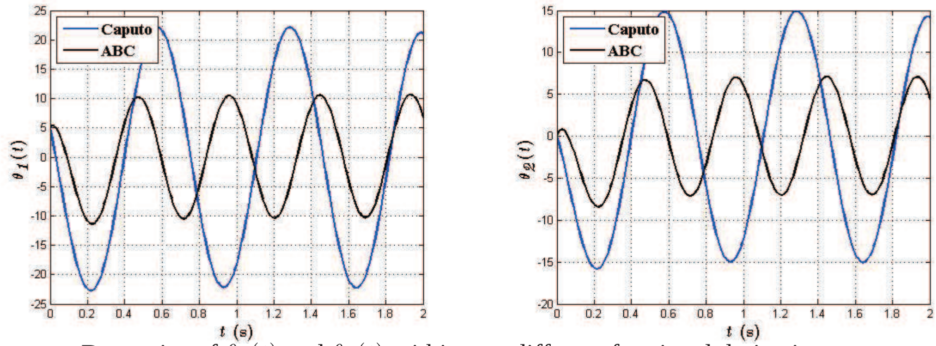

Fig. 2 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 0.9$, $m = 0.2$, $l = 1$, $k = 100$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = 0$.

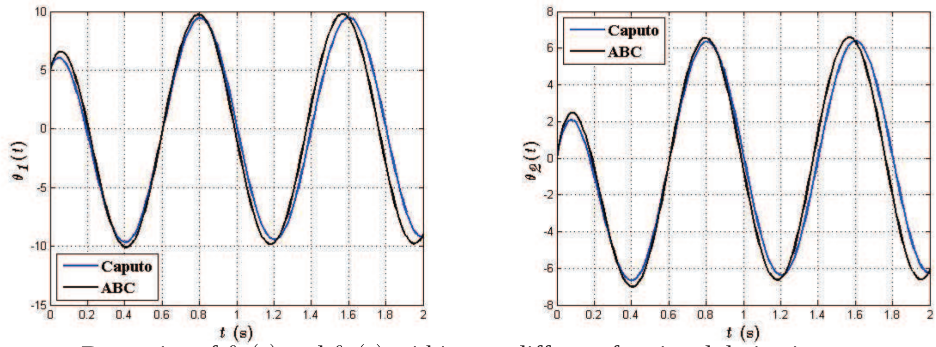

Fig. 3 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 0.95$, $m = 0.2$, $l = 1$, $k = 100$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = 0$.

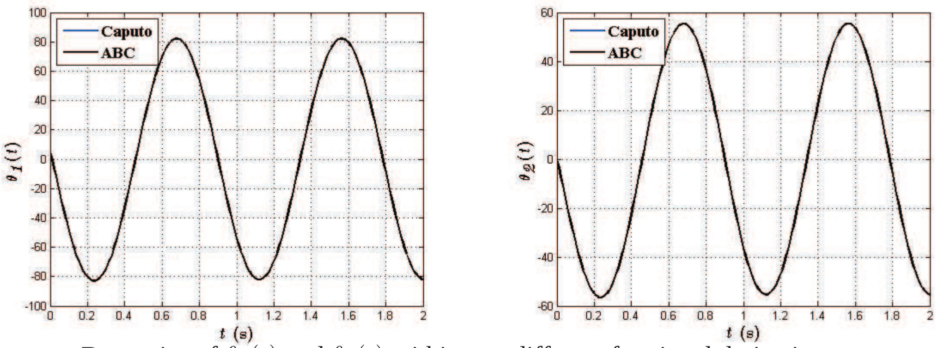

Fig. 4 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 1$, $m = 0.2$, $l = 1$, $k = 100$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = 0$.

Note that, using the Caputo fractional integral [8] instead of the ABC in Eq. (20) and the repetition of the discretization process above, the results can be generalized

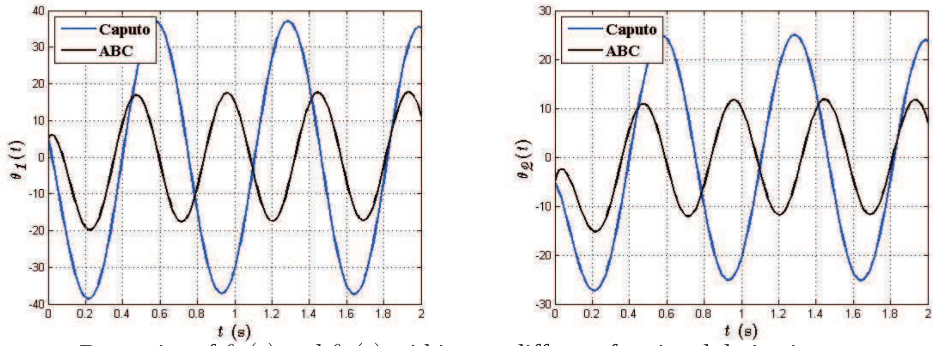

Fig. 5 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 0.9$, $m = 0.2$, $l = 1$, $k = 100$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = -5$.

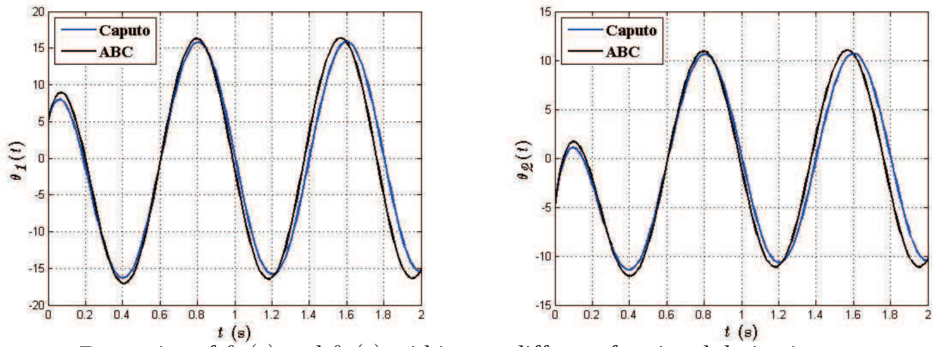

Fig. 6 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 0.95$, $m = 0.2$, $l = 1$, $k = 100$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = -5$.

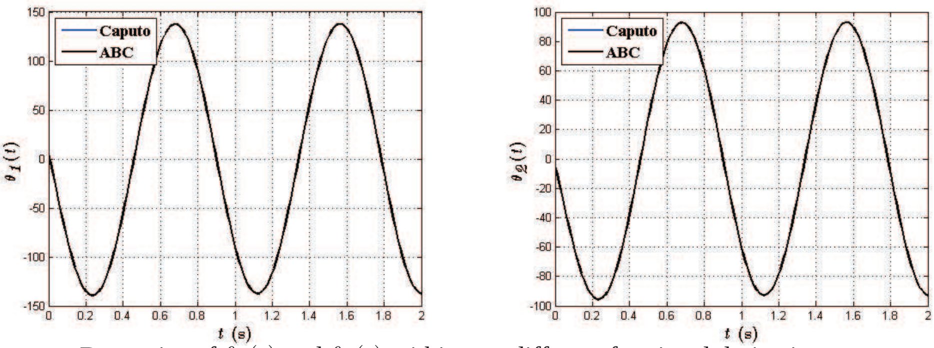

Fig. 7 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 1$, $m = 0.2$, $l = 1$, $k = 100$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = -5$.

to the Caputo derivative case.

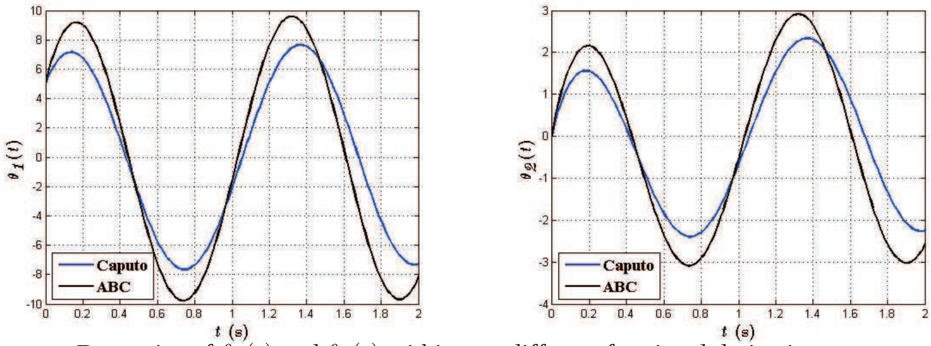

Fig. 8 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 0.9$, $m = 0.2$, $l = 1$, $k = 10$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = 0$.

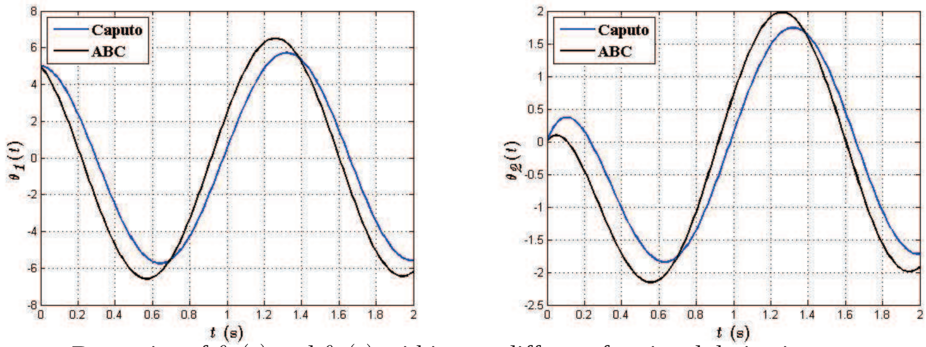

Fig. 9 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 0.95$, $m = 0.2$, $l = 1$, $k = 10$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = 0$.

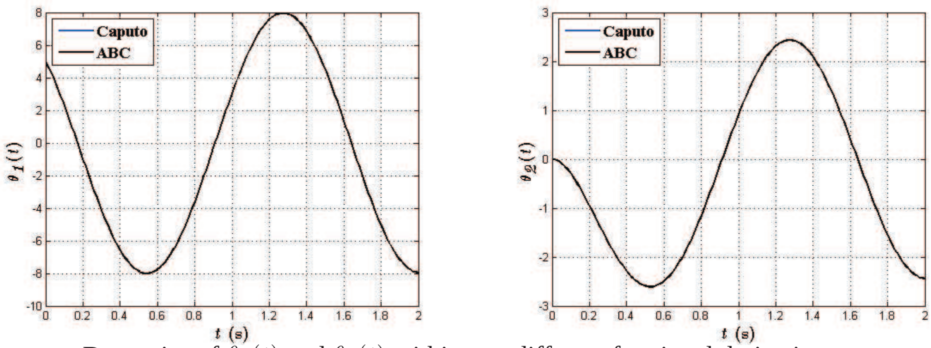

Fig. 10 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 1$, $m = 0.2$, $l = 1$, $k = 10$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = 0$.

4.1. NUMERICAL SIMULATIONS RESULTS

In this Section, we examine the dynamic behaviors of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional operators as well as different values of fractional derivative

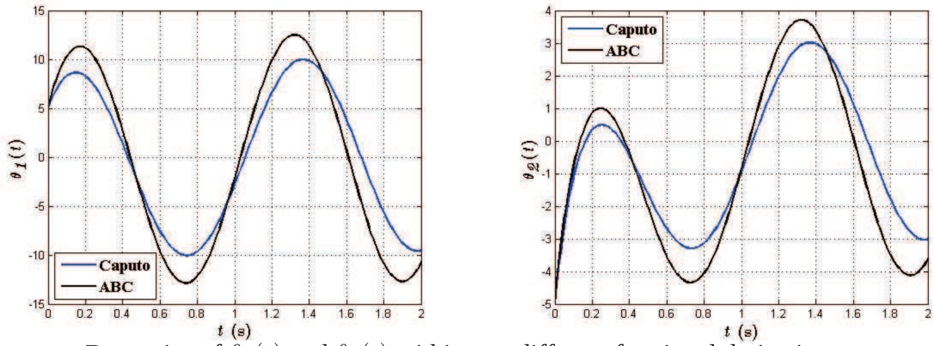

Fig. 11 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 0.9$, $m = 0.2$, $l = 1$, $k = 10$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = -5$.

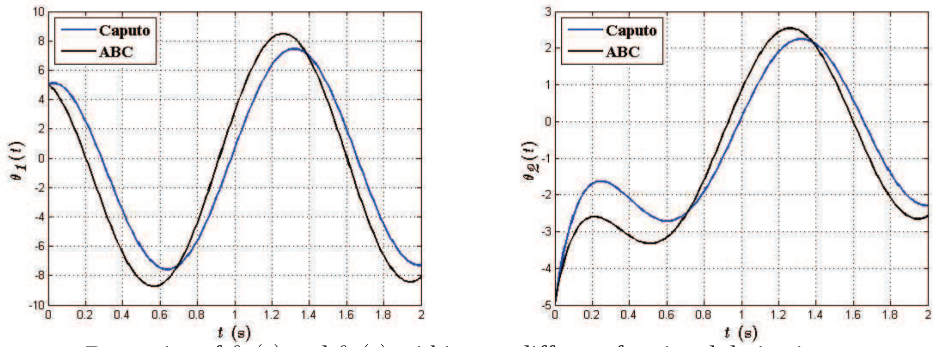

Fig. 12 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 0.95$, $m = 0.2$, $l = 1$, $k = 10$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = -5$.

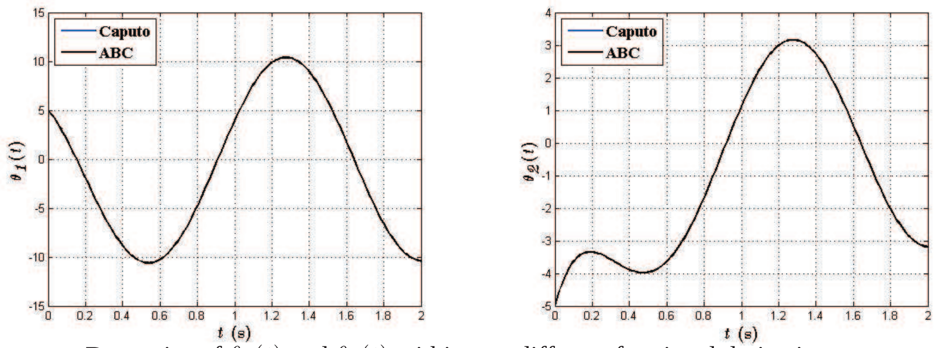

Fig. 13 – Dynamics of $\theta_1(t)$ and $\theta_2(t)$ within two different fractional derivative operators when $\alpha = 1$, $m = 0.2$, $l = 1$, $k = 10$, $g = 9.81$, $\theta_1(0) = 5$ and $\theta_2(0) = -5$.

order and system parameters. The plots are depicted in Figs. 2-13. As it is shown in these figures, the numerical response of the Euler-Lagrange equations depends on the values of derivative order α. Also, these equations have different dynamic behaviors for different fractional derivative operators. Therefore, taking into account the new definitions of the fractional derivatives leads to finding more flexible models that help us to better understand the complex behaviors of the real-world systems.

5. CONCLUSIONS

In this work, we investigated the model of two coupled pendulums by using the fractional Lagrangian. For this aim, we generalized the classical Lagrangian to the fractional case and derived the FELEs in the Caputo and ABC sense. Then, we solved the proposed models within these two fractional operators by using a numerical method based on the discretization of convolution integral by the Euler convolution quadrature rule. The results reported in Figs. 2-13 indicated that the behaviors of the FELEs depend on the fractional operators. Thus, the recently investigated features of the fractional calculus provide more realistic models that help us to adjust better the dynamical behaviours of the real-world phenomena.

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