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Resistance computation of generalized decorated square and simple cubic network lattices

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Keywords:	In the present work, the lattice Green's function technique has been used to investigate the equivalent two-site
Generalized decorated square lattice	resistance between arbitrary pairs of lattice sites in infinite, generalized decorated square and simple cubic
Two-site resistance	lattices with identical resistors. Some results for the resistance are presented. The results for the generalized
Green's function	decorated square lattice are numerically confirmed by commercial software (National Instruments software
	Multisim). The asymptotic values of the resistance for the generalized decorated simple cubic lattice are cal-
	culated numerically when the separation of the two lattice points goes to infinity.

Introduction

Infinite and finite resistive lattices have aroused the interest of numerous researchers in the past more than twenty years because they can be employed to model electrical systems as well as non-electrical systems although the analyzing and modeling of the electrical circuits were formulated by Kirchhoff more than 170 years ago [1]. As a real example, the resistor network of graphene has been investigated by Andre Geim and Konstantin Novoselov [2], which shows that the resistor network model is real and important in theoretical research and practical applications.

Various theoretical methods have been established in literature in calculating the resistance of infinite resistor lattice structures [3–6]. Venezian [4] studied the two-point resistance on an infinite square lattice based on the principle of the superposition of current distributions. By using complex Fourier transforms Atkinson and van Steenwijk [5] illuminated the method of Venezian and generalized it to infinite cubic and hypercubic lattices in three and more dimensions, as well as to infinite triangular and hexagonal lattices in two dimensions.

In [6], Cserti has used Green's function method to calculate the resistance for several infinite lattice structures of resistors. In [7], Cserti et al. have presented a general theory based on the Green's function for calculating two-point resistances in infinite *d*-dimensional uniform tilings with resistors. Based on this method considerable works has been performed to determine the resistance between two arbitrary lattice sites in infinite lattices of various topologies [8–14]. In [15] Kirkpatrick used the Green's function method to study the transport in inhomogeneous conductors and

the percolation in random lattice networks.

The Green function is an important mathematical tool in several areas of theoretical physics. It provides, for example, an efficient method for solving linear problems involving a differential equation. An excellent introduction to Green function and various applications can be found in Refs. [16–18].

Lattice Green functions appear in several problems in condensed matter physics, such as lattice vibration problems, luminescence, diffusion in solids and the dynamics of spin waves [19]. They are also used in statistical physics (theory of random walks) [20], theories of impurities in solids [21] and to the calculation of the capacitance of capacitor networks [22,23]. The Green's function method can be a very efficient way to study several types of defects, including a broken resistor (or capacitor), a replaced resistor (or capacitor) and an extra resistor (or capacitor) between two nonconnected lattice sites [24–32]. One of most popular models used to evaluate the lattice Green functions are the elliptic integrals or recurrence relations methods [33–42]. Such as application of Mahler measure theory to the face-centered cubic lattice Green function [33], and exact results for the diamond lattice Green function [34].

For finite resistor networks, different methods have been established, such as the Laplacian matrix approach [43–46], the recursion-transform method [47–53], and the equivalent transformation methods [54,55].

It is well-known that the connection between the electrical networks and random walks is the two-point resistance on an electrical network [3]. Many quantities of interests to know about random walks are first passage time and commute times can be calculated by the two-point resistance [56,57].

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In this paper, we follow the general Green's function theory presented in [7] to compute the effective resistance between any two of the lattice sites of infinite, generalized decorated square and simple cubic lattices of identical resistors, which have not been studied in the literature.

We believe that the Green's function method is a highly effective technique for the present problem, even in cases when other methods face extreme difficulties [3–5]. Further, this problem can be used in advanced mathematical methods course for undergraduate students in physics, and would provide an educational example for introducing the theory of Green's function, as well as other basic concepts (such as the unit cell, reciprocal lattice and the Brillouin zone) used in solid state physics. Another motivation is to computationally confirm the theoretical results. Such confirmation is applicable due to the difficulty of solving the problem by other methods [3–5].

The rest of the paper is organized as follows. In Section 2, the general theory of an infinite d- dimensional regular resistor network presented in Ref. [7] is briefly reviewed. The effective two-site resistance on infinite, generalized decorated square and simple cubic lattices with identical resistors are studied in Sections 3 and 4, respectively. A brief conclusion is given in Section 5.

A brief review of the general theory of an infinite periodic resistor lattice

In this section, we briefly recall the general theory given in Ref. [7] to compute the effective resistance between any two lattice sites in an infinite resistor lattice structure that is periodic tiling of *d*-dimensional space with resistors. Readers are referred to the Ref. [7] for detailed description of this method.

Consider an infinite lattice structure that is a uniform tiling of *d*dimensional space with identical resistors *R*. The lattice point can be represented by the vector $\mathbf{r} = \sum_{i=1}^{d} p_i \mathbf{a}_i$, where \mathbf{a}_i are the unit cell vectors in the *d*-dimensional space and p_i are arbitrary integers. If the unit cell contains *s* lattice points labeled $\alpha = 1$, 2, ...,*s*, then denote by $\{\mathbf{r}; \alpha\}$ any lattice point, where **r** and α specify the unit cell and the lattice point, and let $U_{\alpha}(\mathbf{r})$ and $I_{\alpha}(\mathbf{r})$ be the electric potential and current at lattice site $\{\mathbf{r}; \alpha\}$ respectively.

The electric potential and current at lattice site $\{r; \alpha\}$ in the *d*-dimensional real space can be represented by their Fourier transforms as the following.

$$U_{\alpha}(\mathbf{r}) = \frac{V_C}{(2\pi)^d} \int_{BZ} U_{\alpha}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d^d \mathbf{k},$$

$$I_{\alpha}(\mathbf{r}) = \frac{V_C}{(2\pi)^d} \int_{BZ} I_{\alpha}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d^d \mathbf{k}$$
(1)

where V_c is the volume of the unit cell and **k** is the wave vector in the *d*dimensional Fourier space (in the reciprocal lattice) and is confined to the first Brillouin zone (BZ) which is a *d*-dimensional hypercube with sides $k_i = 2\pi/a_i$.

According to Kirchhoff's and Ohm's laws, the currents entering the sites $\{r;1\}$, $\{r;2\}$, ..., $\{r;s\}$ in a one unit cell can be written formally in the matrix notation as.

$$LU = -I \tag{2}$$

where **L** is a *s* by *s* matrix usually called Laplacian matrix of the lattice, U(r) and I(r) are *s* by one column matrices:

$$\boldsymbol{U} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_s \end{pmatrix}, \qquad \boldsymbol{I} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_s \end{pmatrix}$$
(3)

In order to calculate the effective resistance $R_{\alpha\beta}(\mathbf{r})$ between the orgin $\{\mathbf{0}; \alpha\}$ and the site $\{\mathbf{r}; \beta\}$ we connect these sites to the two terminals of a battery and measure the current *I* going through the battery while no other sites are connected to batteries. Then, according Ohm's

law the desired resistance $R_{\alpha\beta}(\mathbf{r})$ is given by,

$$R_{\alpha\beta}(\mathbf{r}) = \frac{1}{I} [U_{\alpha}(\mathbf{0}) - U_{\beta}(\mathbf{r})]$$
(4)

The computation of the two-site resistance requires solving Eq. (4) for $U_{\alpha}(\mathbf{0})$ and $U_{\beta}(\mathbf{r})$ with the current distribution given by,

$$I_{\nu}(\mathbf{r}') = I(\delta_{\mathbf{r}',\mathbf{0}}\delta_{\alpha,\nu} - \delta_{\mathbf{r}',\mathbf{r}}\delta_{\beta,\nu})$$
(5)

The lattice Green's function, is also known resolvant matrix, is formally defined as,

$$G = -L^{-1} \tag{6}$$

Hence, Eq. (2) can be written as,

$$U_{\mu}(\boldsymbol{r}) = \sum_{\boldsymbol{r}',\nu} G_{\mu\nu}(\boldsymbol{r},\,\boldsymbol{r}')I_{\nu}(\boldsymbol{r}')$$
(7a)

Substituting Eq. (5) into Eq. (7a) gives,

$$U_{\mu}(\mathbf{r}') = I[G_{\mu\alpha}(\mathbf{r}', 0) - G_{\mu\beta}(\mathbf{r}', \mathbf{r})]$$
(7b)

Using Eq. (7b) in (4), the resistance in terms of lattice Green's functions can be obtained as,

$$R_{\alpha\beta}(\mathbf{r}) = G_{\alpha\alpha}(\mathbf{0}) + G_{\beta\beta}(\mathbf{0}) - G_{\alpha\beta}(\mathbf{r}) - G_{\beta\alpha}(\mathbf{r})$$
(8)

where the lattice Green's function $G_{\alpha\beta}(\mathbf{r})$ can be given by its Fourier transform $G_{\alpha\beta}(\mathbf{k})$ as

$$G_{\alpha}(\mathbf{r}) = \frac{V_C}{(2\pi)^d} \int_{BZ} G_{\alpha}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^d \mathbf{k},$$
(9a)

Or,

$$G_{\alpha\beta}(p_1, \dots, p_d) = \frac{a_1 \times \dots \times a_d}{(2\pi)^d} \int_{-\pi/a_1}^{\pi/a_1} dk_1 \cdot \\ \cdots \int_{-\pi/a_d}^{\pi/a_d} dk_d G_{\alpha\beta}(k_1, \dots, k_d) e^{-i(p_1 \mathbf{k} \cdot \mathbf{a}_1 + \dots + p_d \mathbf{k} \cdot \mathbf{a}_d)}$$
(9b)

By making the transformations $\mathbf{k} \cdot \mathbf{a}_i = \theta_i (i = 1, 2, ..., d)$ and substituting Eq. (9b) into (8), the general expression for the two-site resistance can be written as follows:

$$R_{\alpha\beta}(p_1, \dots, p_d) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \dots \int_{-\pi}^{\pi} \frac{d\theta_d}{2\pi} [G_{\alpha\alpha}(\theta_1, \dots, \theta_d) + G_{\beta\beta}(\theta_1, \dots, \theta_d) - G_{\alpha\beta}(\theta_1, \dots, \theta_d)e^{i(p_1\theta_1 + \dots + p_d\theta_d)} - G_{\beta\alpha}(\theta_1, \dots, \theta_d)e^{i(p_1\theta_1 + \dots + p_d\theta_d)}]$$

$$(10)$$

Generalized decorated square lattice

The well- studied decorated square lattice [58] is formed by introducing extra site in the middle of each side of a square lattice. Each line between a pair of sites represents a resistor. Here, we compute the two-site resistance on the *generalized* decorated square lattice obtained by introducing a resistor between the decorating sites (see Fig. 1). In Ref. [59], the antiferromagnetic Potts model has been studied on the generalized decorated square lattice. In each unit cell there are three lattice sites labeled by $\alpha = 1, 2, 3$ as shown in Fig. 1.

Equations of the unit cell current

Applying Kirchhoff 's current rule to the lattice sites $\{r; 1\}, \{r; 2\}, \{r; 3\}$ and using Ohm's law leads to the following system of current equations for the unit cell:

$$I_{1}(\mathbf{r}) = \frac{U_{1}(\mathbf{r}) - U_{2}(\mathbf{r})}{R} + \frac{U_{1}(\mathbf{r}) - U_{2}(\mathbf{r} - \mathbf{a}_{1})}{R} + \frac{U_{1}(\mathbf{r}) - U_{3}(\mathbf{r})}{R} + \frac{U_{1}(\mathbf{r}) - U_{3}(\mathbf{r} - \mathbf{a}_{2})}{R}$$
(11)

$$I_{2}(\mathbf{r}) = \frac{\frac{U_{2}(\mathbf{r}) - U_{1}(\mathbf{r})}{R} + \frac{U_{2}(\mathbf{r}) - U_{1}(\mathbf{r} + \mathbf{a}_{1})}{R} + \frac{U_{2}(\mathbf{r}) - U_{3}(\mathbf{r} + \mathbf{a}_{1} - \mathbf{a}_{2})}{R}}{+ \frac{U_{2}(\mathbf{r}) - U_{3}(\mathbf{r})}{R} + \frac{U_{2}(\mathbf{r}) - U_{3}(\mathbf{r} + \mathbf{a}_{1})}{R} + \frac{U_{2}(\mathbf{r}) - U_{3}(\mathbf{r} - \mathbf{a}_{2})}{R}}$$
(12)



Fig. 1. The generalized decorated square lattice of the resistor network.

$$I_{3}(\mathbf{r}) = \frac{\frac{U_{3}(\mathbf{r}) - U_{1}(\mathbf{r})}{R} + \frac{U_{3}(\mathbf{r}) - U_{1}(\mathbf{r} + \mathbf{a}_{2})}{R} + \frac{U_{3}(\mathbf{r}) - U_{2}(\mathbf{r} - \mathbf{a}_{1} + \mathbf{a}_{2})}{R} + \frac{U_{3}(\mathbf{r}) - U_{2}(\mathbf{r} - \mathbf{a}_{1})}{R} + \frac{U_{3}(\mathbf{r}) - U_{2}(\mathbf{r} + \mathbf{a}_{2})}{R}$$
(13)

Lattice Green's functions and resistances

Substituting Eq. (1) into (11)–(13) with d = 2, one can easily find out the Fourier transform of the Laplacian matrix of the generalized decorated square lattice:

$$L(\theta_1, \theta_2) = \frac{1}{R} \begin{pmatrix} -4 & 1 + e^{-i\theta_1} & 1 + e^{-i\theta_2} \\ 1 + e^{i\theta_1} & -6 & (1 + e^{i\theta_1})(1 + e^{-i\theta_2}) \\ 1 + e^{i\theta_2} & (1 + e^{-i\theta_1})(1 + e^{i\theta_2}) & -6 \end{pmatrix}$$
(14)

The Fourier transform of the Green's function $G(\theta_1, \theta_2)$ can be obtained from Eq. (6), we have,

$$G(\theta_{1}, \theta_{2}) = \frac{R}{3(8-3\cos\theta_{1}-3\cos\theta_{2}-2\cos\theta_{1}\cos\theta_{2})} \times \begin{pmatrix} 36-4(1 2(1+e^{-i\theta_{1}})(4 2(1+e^{-i\theta_{2}})(4) + \cos\theta_{1})(1 + \cos\theta_{2}) + \cos\theta_{1}) + \cos\theta_{2} \end{pmatrix} \times (15)$$

The equivalent resistance between the lattice sites $\{(0, 0); \alpha\}$ and $\{(p_1, p_2); \beta\}$ in the generalized decorated square lattice can be

determined from the general expression for the two-site resistance in Eq. (10) for the two-dimensional case (d = 2):

$$R_{\alpha\beta}(p_1, p_2) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \times [G_{\alpha\alpha}(\theta_1, \theta_2) + G_{\beta\beta}(\theta_1, \theta_2) - G_{\alpha\beta}(\theta_1, \theta_2) e^{-i(p_1\theta_1 + p_2\theta_2)} - G_{\beta\alpha}(\theta_1, \theta_2) e^{i(p_1\theta_1 + p_2\theta_2)}]$$
(16)

Since the lattice Green's function is 3 by 3 matrix we have nine types of resistance, $R_{\alpha\beta}(p_1, p_2)$. These types of resistance are given in Appendix A. It is interesting to note that from the symmetry $R_{33}(p_1, p_2) = R_{22}(p_2, p_1)$ and $R_{13}(p_1, p_2) = R_{12}(p_2, p_1)$.

For some small values of p_1 and p_2 , the resistance can be evaluated analytically as is shown below.

• The resistance between the lattice sites $\{(0, 0); \alpha = 2\}$ and $\{(0, 1); \beta = 2\}$ is given by.

$$R_{22}(0, 1) = \frac{R}{3} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \frac{(11 - \cos\theta_2)(1 - \cos\theta_2)}{8 - 3\cos\theta_1 - 3\cos\theta_2 - 2\cos\theta_1\cos\theta_2} = \frac{2R}{3\sqrt{5}\pi} [\sqrt{5} + 6\arcsin(1/\sqrt{6})]$$
(17)

• The resistance between the lattice sites $\{(0, 0); \alpha = 2\}$ and $\{(0, 2); \beta = 2\}$ is given by.

$$R_{22}(0, 2) = \frac{R}{3} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \frac{(11 - \cos\theta_2)(1 - \cos2\theta_2)}{8 - 3\cos\theta_1 - 3\cos\theta_2 - 2\cos\theta_1\cos\theta_2}$$
$$= \frac{8R}{3\sqrt{5}\pi} \left[-\sqrt{5} + 9\arcsin(1/\sqrt{6}) \right]$$
(18)

• The resistance between the lattice sites $\{(0, 0); \alpha = 2\}$ and $\{(0, 3); \beta = 2\}$ is given by.

$$R_{22}(0, 3) = \frac{R}{3} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \frac{(11 - \cos\theta_2)(1 - \cos3\theta_2)}{8 - 3\cos\theta_1 - 3\cos\theta_2 - 2\cos\theta_1\cos\theta_2}$$
$$= \frac{2R}{9\sqrt{5}\pi} [-197\sqrt{5} + 1098 \arcsin(1/\sqrt{6})]$$
(19)

where we have performed analytically the integration over θ_1 by the method of residues [60,61].

$$\int_{-\pi}^{\pi} \frac{d\theta}{a - b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$
(20)

and then over θ_2 by elementary ways.

In Table 1 we list the theoretical results for the resistance for small separations between lattice sites. In Figs. 2 and 3, the resistances (in units of R) are plotted as functions of p_1 .

We confirm these results by creating a virtual finite lattice of size 8 by 8 unit cells of identical resistors with National Instruments software Multisim. The numerical results are listed in Table 1. It can be seen in the table that the theoretical and numerical results of resistances are in adequate agreement near the origin of the finite lattice. It is well-known that the finiteness of the lattice causes the two-point resistances to be greater than the values for an infinite lattice, which is expected because the current has fewer paths.

Table 1

Theoretical and numerical values of two-site resistance $R_{\alpha\beta}(p_1, p_2)$ in units of *R*. The theoretical values are for an infinite lattice, and the numerical values are for finite lattice.

1

p_1, p_2	$R_{11}(p_1, p_2)$		$R_{22}(p_1, p_2)$		$R_{12}(p_1, p_2)$		$R_{23}(p_1, p_2)$	
	Infinite lattice	8×8 unit cells	Infinite lattice	8×8 unit cells	Infinite lattice	8×8 unit cells	Infinite lattice	8×8 unit cells
0, 0	0	0	0	0	0.393197	0.39451	0.356803	0.359447
1, 0	0.646978	0.652273	0.431975	0.437529	0.628088	0.640234	0.356803	0.359634
0, 1	0.646978	0.652273	0.451663	0.457156	0.569708	0.57649	0.535281	0.548882
1, 1	0.725732	0.737051	0.507877	0.520344	0.668151	0.68722	0.535281	0.550364
1, 2	0.821605	0.852105	0.609164	0.64334	0.738350	0.779288	0.636294	0.67905
2, 1	0.821605	0.852105	0.606593	0.641747	0.753742	0.800708	0.595643	0.627318
2, 0	0.797808	0.820128	0.582606	0.607982	0.738003	0.775282	0.535281	0.550308
0, 2	0.797808	0.820128	0.587913	0.610915	0.698965	0.723222	0.636294	0.675285



Fig. 2. Plot of the resistances $R_{\alpha\alpha}(p_1, 0)$, in units of *R*, between the sites $\{(0, 0); \alpha\}$ and $\{(p_1, 0); \alpha\}$ in a generalized decorated square lattice of resistors for $\alpha = 1, 2, 3$ and $0 \le p_1 \le 50$.

Generalized decorated simple cubic lattice

In this section, we calculate the two-point resistance on the generalized decorated simple cubic lattice shown in Fig. 4. The unit cell consists of four lattice points labeled by $\alpha = 1, 2, 3, 4$ with vectors $a_1 = a\hat{x}$, $a_2 = a\hat{y}$ and $a_3 = a\hat{z}$ along the edges of the cube (see Fig. 4).

Equations of the unit cell current

Using Kirchhoff's junction rule and Ohm's law, the currents at the lattice sites $\{r, 1\}, \{r, 2\}, \{r, 3\}$ and $\{r, 4\}$ in a one unit cell can be expressed as the following:

$$I_{1}(\mathbf{r}) = \frac{U_{1}(\mathbf{r}) - U_{2}(\mathbf{r})}{R} + \frac{U_{1}(\mathbf{r}) - U_{2}(\mathbf{r} - \mathbf{a}_{1})}{R} + \frac{U_{1}(\mathbf{r}) - U_{3}(\mathbf{r})}{R} + \frac{U_{1}(\mathbf{r}) - U_{3}(\mathbf{r} - \mathbf{a}_{3})}{R} + \frac{U_{1}(\mathbf{r}) - U_{4}(\mathbf{r})}{R} + \frac{U_{1}(\mathbf{r}) - U_{4}(\mathbf{r} - \mathbf{a}_{2})}{R}$$
(21)



Fig. 3. Plot of the resistances $R_{\alpha\beta}(p_1, 0)$, in units of *R*, between two sites $\{(0, 0); \alpha\}$ and $\{(p_1, 0); \beta\}$ in a generalized decorated square lattice of resistors for $\alpha \neq \beta$ and $0 \le p_1 \le 50$.



Fig. 4. The generalized simple cubic lattice of the resistor network.

$$I_{2}(\mathbf{r}) = \frac{U_{2}(\mathbf{r}) - U_{1}(\mathbf{r})}{R} + \frac{U_{2}(\mathbf{r}) - U_{1}(\mathbf{r} + \mathbf{a}_{1})}{R} + \frac{U_{2}(\mathbf{r}) - U_{3}(\mathbf{r})}{R} + \frac{U_{2}(\mathbf{r}) - U_{3}(\mathbf{r} + \mathbf{a}_{1})}{R} + \frac{U_{2}(\mathbf{r}) - U_{3}(\mathbf{r} - \mathbf{a}_{3})}{R} + \frac{U_{2}(\mathbf{r}) - U_{3}(\mathbf{r} + \mathbf{a}_{1} - \mathbf{a}_{3})}{R} + \frac{U_{2}(\mathbf{r}) - U_{4}(\mathbf{r})}{R} + \frac{U_{2}(\mathbf{r}) - U_{4}(\mathbf{r} + \mathbf{a}_{1})}{R} + \frac{U_{2}(\mathbf{r}) - U_{4}(\mathbf{r} - \mathbf{a}_{2})}{R} + \frac{U_{2}(\mathbf{r}) - U_{4}(\mathbf{r} + \mathbf{a}_{1} - \mathbf{a}_{2})}{R}$$
(22)

$$I_{3}(\mathbf{r}) = \frac{U_{3}(\mathbf{r}) - U_{1}(\mathbf{r})}{R} + \frac{U_{3}(\mathbf{r}) - U_{1}(\mathbf{r} + \mathbf{a}_{3})}{R} + \frac{U_{3}(\mathbf{r}) - U_{2}(\mathbf{r})}{R} + \frac{U_{3}(\mathbf{r}) - U_{2}(\mathbf{r} - \mathbf{a}_{1})}{R} + \frac{U_{3}(\mathbf{r}) - U_{2}(\mathbf{r} - \mathbf{a}_{1} + \mathbf{a}_{3})}{R} + \frac{U_{3}(\mathbf{r}) - U_{4}(\mathbf{r})}{R} + \frac{U_{3}(\mathbf{r}) - U_{4}(\mathbf{r} - \mathbf{a}_{2})}{R} + \frac{U_{3}(\mathbf{r}) - U_{4}(\mathbf{r} - \mathbf{a}_{2})}{R} + \frac{U_{3}(\mathbf{r}) - U_{4}(\mathbf{r} - \mathbf{a}_{2} + \mathbf{a}_{3})}{R}$$

$$(23)$$

$$I_{4}(\mathbf{r}) = \frac{U_{4}(\mathbf{r}) - U_{1}(\mathbf{r})}{R} + \frac{U_{4}(\mathbf{r}) - U_{1}(\mathbf{r} + a_{2})}{R} + \frac{U_{4}(\mathbf{r}) - U_{2}(\mathbf{r})}{R} + \frac{U_{4}(\mathbf{r}) - U_{2}(\mathbf{r} - a_{1})}{R} + \frac{U_{4}(\mathbf{r}) - U_{2}(\mathbf{r} - a_{1} + a_{2})}{R} + \frac{U_{4}(\mathbf{r}) - U_{2}(\mathbf{r} - a_{1} + a_{2})}{R} + \frac{U_{4}(\mathbf{r}) - U_{3}(\mathbf{r})}{R} + \frac{U_{4}(\mathbf{r}) - U_{3}(\mathbf{r} + a_{2})}{R} + \frac{U_{4}(\mathbf{r}) - U_{3}(\mathbf{r} - a_{3})}{R} + \frac{U_{4}(\mathbf{r}) - U_{3}(\mathbf{r} + a_{2} - a_{3})}{R}$$

$$(24)$$

Lattice Green's functions and resistances

In a similar way to the previous sub Section 3.2, it is easy to show that the Fourier transform of the Laplacian matrix $L(\theta_1, \theta_2, \theta_3)$ is given by

$$L(\theta_{1}, \theta_{2}, \theta_{3}) = \frac{1}{R} \begin{pmatrix} -6 & 1 + e^{-i\theta_{1}} & 1 + e^{-i\theta_{3}} & 1 + e^{-i\theta_{2}} \\ 1 + e^{i\theta_{1}} & -10 & (1 + e^{i\theta_{1}})(1 + e^{-i\theta_{3}}) & (1 + e^{i\theta_{1}})(1 + e^{-i\theta_{2}}) \\ 1 + e^{i\theta_{3}} & (1 + e^{-i\theta_{1}})(1 + e^{i\theta_{3}}) & -10 & (1 + e^{i\theta_{3}})(1 + e^{-i\theta_{2}}) \\ 1 + e^{i\theta_{2}} & (1 + e^{-i\theta_{1}})(1 + e^{i\theta_{2}}) & (1 + e^{-i\theta_{3}})(1 + e^{i\theta_{2}}) & -10 \end{pmatrix}$$

$$(29)$$

Using Eq. (6), the Green's function $G(\theta_1, \theta_2, \theta_3)$ can be calculated. The matrix elements of the Green's function for the generalized simple cubic lattice are listed in Appendix B.

The resistance $R_{\alpha\beta}(p_1, p_2, p_3)$ between the origin $\{0; \alpha\}$ and site $\{(p_1, p_2, p_3); \beta\}$ can be determined from Eq. (10) for the three – dimensional lattice (d = 3):

$$R_{\alpha\beta}(p_{1}, p_{2}, p_{3}) = \int_{-\pi}^{\pi} \frac{d\theta_{1}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_{2}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_{3}}{2\pi} \times [G_{\alpha\alpha}(\theta_{1}, \theta_{2}, \theta_{3}) + G_{\beta\beta}(\theta_{1}, \theta_{2}, \theta_{3}) - G_{\alpha\beta}(\theta_{1}, \theta_{2}, \theta_{3})e^{-i(p_{1}\theta_{1} + p_{2}\theta_{2} + p_{3}\theta_{3})} - G_{\beta\alpha}(\theta_{1}, \theta_{2}, \theta_{3})e^{i(p_{1}\theta_{1} + p_{2}\theta_{2} + p_{3}\theta_{3})}]$$
(30)

There are sixteen types of the resistance for the generalized simple

 $R_{\alpha\beta}(p_1,$

Table 2							
Numerical	values of	of two-site	resistance	in	units	of	R.

р	$R_{11}(p)$	$R_{22}(p)$	$R_{33}(p)$	$R_{44}(p)$	<i>R</i> ₁₂ (p)	<i>R</i> ₁₃ (p)	$R_{14}(p)$	<i>R</i> ₂₃ (p)	$R_{24}(p)$	$R_{34}(p)$
0, 0, 0	0	0	0	0	0.25487	0.25487	0.25487	0.20608	0.20608	0.20608
1, 0, 0	0.39248	0.22966	0.24252	0.24252	0.33547	0.32342	0.32342	0.20608	0.20608	0.25111
0, 1, 0	0.39248	0.24252	0.24252	0.22966	0.32342	0.32342	0.33547	0.25111	0.26007	0.26007
0, 0, 1	0.39248	0.24252	0.22966	0.24252	0.32342	0.33547	0.32342	0.26007	0.35179	0.20608
1, 1, 0	0.41434	0.25440	0.25954	0.25440	0.34462	0.33955	0.34462	0.25111	0.25111	0.26757
1, 0, 1	0.41434	0.25440	0.25440	0.25954	0.34462	0.34462	0.33955	0.26007	0.25111	0.25111
0, 1, 1	0.41434	0.25954	0.25440	0.25440	0.33955	0.34462	0.34462	0.26757	0.26757	0.26007
1, 1, 1	0.42285	0.26522	0.26522	0.26522	0.34989	0.34989	0.34989	0.26757	0.26757	0.26757
1, 2, 0	0.43071	0.27399	0.27364	0.27207	0.35509	0.35314	0.35667	0.27189	0.27616	0.27811
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~	0.45986	0.30145	0.30145	0.30145	0.38065	0.38065	0.38065	0.30145	0.30145	0.30145

cubic lattice and are listed in Appendix C. In Table 2, we report some numerical values of the resistance between nearby lattice sites.

It may be interesting to compute the resistance when the distance between the two lattice sites goes to infinity, i.e.,  $p_1, p_2, p_3 \rightarrow \infty$ . This resistance can be evaluated by using the well-known Riemann-Lebesgue *lemma*: If  $f(\theta)$  is an integrable function on  $[-\pi, \pi]$ , then,

$$\int_{-\pi}^{\pi} f(\theta) \cos p\theta \, d\theta = \int_{-\pi}^{\pi} f(\theta) \sin p\theta \, d\theta = 0 \quad \text{as } \mathbf{p} \to \infty$$
(31)

Therefore, the limit of the resistance  $R_{\alpha\beta}(p_1, p_2, p_3)$  in Eq. (10) as  $p_1, p_2, p_3 \rightarrow \infty$ , which is a finite value, can be written as,

$$p_{2}, p_{3}) \rightarrow G_{\alpha\alpha}(0, 0, 0) + G_{\beta\beta}(0, 0, 0)$$

$$= \int_{-\pi}^{\pi} \frac{d\theta_{1}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_{2}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_{3}}{2\pi} [G_{\alpha\alpha}(\theta_{1}, \theta_{2}, \theta_{3})$$

$$+ G_{\beta\beta}(\theta_{1}, \theta_{2}, \theta_{3})]$$
(32)

The numerical values for the limit of the resistance  $R_{\alpha\beta}(p_1, p_2, p_3)$  as

 $p_1, p_2, p_3 \rightarrow \infty$  are listed in Table 2 (last row).

#### Conclusion

In this work, we have used the general lattice Green's function technique developed in [7] for computing the effective resistance between two arbitrary lattice sites of infinite, generalized square and simple cubic lattices of identical electrical resistors. We have numerically evaluated the asymptotic values of the resistance for the generalized simple cubic lattice when the separation between the lattice sites tends infinity

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#### Appendix A. . The types of the resistance on the generalized decorated square lattice

In this appendix, we list the types of the resistance  $R_{\alpha\beta}(p_1, p_2)$  between the origin {0; $\alpha$ } and site { $(p_1, p_2)$ ;  $\beta$ } in a generalized decorated square lattice:

$$R_{11}(p_1, p_2) = \frac{2R}{3} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \frac{[9 - (1 + \cos\theta_1)(1 + \cos\theta_2)][1 - \cos p_1\theta_1 \cos p_2\theta_2]}{8 - 3\cos\theta_1 - 3\cos\theta_2 - 2\cos\theta_1 \cos\theta_2}$$
(A.1)

$$R_{22}(p_1, p_2) = \frac{R}{2} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \frac{[11 - \cos\theta_2][1 - \cos\theta_1\theta_1 \cos\theta_2\theta_2]}{2\cos\theta_1 - 2\cos\theta_1 - 2\cos\theta_1 \cos\theta_2}$$
(A.2)

$$R_{1}(r_{1},r_{2}) = \frac{R}{2\pi} \frac{c^{\pi}}{2\pi} \frac{d\theta_{1}}{d\theta_{2}} \frac{c^{\pi}}{[11 - \cos\theta_{1}][1 - \cos p_{1}\theta_{1}\cos p_{2}\theta_{2}]}{[11 - \cos\rho_{1}\theta_{1}\cos p_{2}\theta_{2}]}$$
(A.2)

$$R_{33}(p_1, p_2) = \frac{\pi}{3} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{1}{8} - 3\cos\theta_1 - 3\cos\theta_2 - 2\cos\theta_1\cos\theta_2$$
(A.3)

$$R_{12}(p_1, p_2) = \frac{R}{6} + \frac{R}{6} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \frac{19 + \cos\theta_1 - 2[4 + \cos\theta_2][\cos p_1\theta_1 + \cos(p_1 + 1)\theta_1]\cos p_2\theta_2}{8 - 3\cos\theta_1 - 3\cos\theta_2 - 2\cos\theta_1\cos\theta_2}$$
(A.4)

$$R_{13}(p_1, p_2) = \frac{R}{6} + \frac{R}{6} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \frac{19 + \cos\theta_2 - 2[4 + \cos\theta_1][\cos p_1\theta_1 + \cos(p_1 + 1)\theta_1]\cos p_2\theta_2}{8 - 3\cos\theta_1 - 3\cos\theta_2 - 2\cos\theta_1\cos\theta_2}$$
(A.5)

$$R_{23}(p_1, p_2) = \frac{R}{6} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \frac{22 - \cos\theta_1 - \cos\theta_2 - 5[\cos p_1\theta_1 + \cos(p_1 - 1)\theta_1][\cos p_2\theta_2 + \cos(p_2 + 1)\theta_2]}{8 - 3\cos\theta_1 - 3\cos\theta_2 - 2\cos\theta_1\cos\theta_2}$$
(A.6)

The remaining types of the resistance can be obtained from the lattice symmetry.

#### Appendix B. . The matrix elements of the Green's function for the generalized decorated simple cubic lattice

In this appendix, we list the matrix elements of the Fourier transform of the Green's function  $G(\theta_1, \theta_2, \theta_3)$  for a generalized decorated simple cubic lattice:

$$G_{11} = \frac{8}{\det L} [108 - 12\cos\theta_1 - 12\cos\theta_2 - 12\cos\theta_3 - 7\cos\theta_1\cos\theta_2 - 7\cos\theta_1\cos\theta_2 - 7\cos\theta_1\cos\theta_3 - 7\cos\theta_2\cos\theta_3 - 2\cos\theta_1\cos\theta_2\cos\theta_3]$$
(B.1)  

$$G_{12} = G_{21}^* = \frac{4}{\det L} [1 + e^{-i\theta_1}] [6 + \cos\theta_2] [6 + \cos\theta_3]$$
(B.2)

The authors wish to thank an anonymous reviewer for their helpful

$G_{13} = G_{31}^* = \frac{4}{\det L} [1 + e^{-i\theta_3}] [6 + \cos\theta_1] [6 + \cos\theta_2]$	(B.3)
$G_{14} = G_{41}^* = \frac{4}{\det L} [1 + e^{-i\theta_2}] [6 + \cos\theta_1] [6 + \cos\theta_3]$	(B.4)
$G_{22} = \frac{4}{\det L} [132 - 13\cos\theta_2 - 13\cos\theta_3 - 8\cos\theta_2\cos\theta_3]$	(B.5)
$G_{23} = G_{32}^* = \frac{14}{\det L} [1 + e^{i\theta_1}] [1 + e^{-i\theta_3}] [6 + \cos\theta_2]$	(B.6)
$G_{24} = G_{42}^* = \frac{14}{\det L} [1 + e^{i\theta_1}] [1 + e^{-i\theta_2}] [6 + \cos\theta_3]$	(B.7)

$$G_{33} = \frac{4}{\det L} [132 - 13\cos\theta_1 - 13\cos\theta_2 - 8\cos\theta_1\cos\theta_2]$$
(B.8)

$$G_{34} = G_{43}^* = \frac{14}{\det L} [1 + e^{i\theta_3}] [1 + e^{-i\theta_2}] [6 + \cos\theta_1]$$
(B.9)

$$G_{44} = \frac{4}{\det L} [132 - 13\cos\theta_1 - 13\cos\theta_3 - 8\cos\theta_1\cos\theta_3]$$
(B.10)

where

$$\det \mathbf{L} = \frac{40}{R} [108 - 24\cos\theta_1 - 24\cos\theta_2 - 24\cos\theta_3 - 11\cos\theta_1\cos\theta_2 - 11\cos\theta_1\cos\theta_3 - 11\cos\theta_2\cos\theta_3 - 3\cos\theta_1\cos\theta_2\cos\theta_3]$$
(B.11)

is the determinant of the Laplacian matrix  $L(\theta_1, \theta_2, \theta_3)$ .

#### Appendix C. . The types of the resistance on the generalized decorated simple cubic lattice

Here, we list the types of the resistance  $R_{\alpha\beta}(p_1, p_2, p_3)$  between the origin  $\{0; \alpha\}$  and site  $\{(p_1, p_2, p_3); \beta\}$  in a generalized decorated simple cubic lattice:

$$R_{\alpha\alpha}(p_1, p_2, p_3) = 2 \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} G_{\alpha\alpha}(\theta_1, \theta_2, \theta_3) [1 - \cos p_1 \theta_1 \cos p_2 \theta_2 \cos p_3 \theta_3]$$
(C.1)  
where  $\alpha = 1, 2, 3, 4.$ 

 $R_{12}(p_1, p_2, p_3) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \frac{4}{4\pi} \left[ 348 - 24\cos\theta_1 - 37\cos\theta_2 - 37\cos\theta_3 - 14\cos\theta_1\cos\theta_2 - 14\cos\theta_1\cos\theta_3 - 22\cos\theta_3 - 2\cos\theta_1\cos\theta_2\cos\theta_3 - 2\cos\theta_1\cos\theta_2\cos\theta_3 - 2\left[ 6 + \cos\theta_3 \right] \left[ \cos p_1\theta_1 + \cos(p_1 + 1)\theta_1 \right] \cos p_2\theta_2\cos p_3\theta_3 \right]$ (C.2)  $R_1(p_1, p_2, p_3) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \frac{4}{4\pi} \left[ 248 - 24\cos\theta_1 - 37\cos\theta_2 - 37\cos\theta_3 - 24\cos\theta_3 \right]$ (C.2)

$$\begin{aligned} \kappa_{13}(p_1, p_2, p_3) &= \int_{-\pi} \frac{1}{2\pi} \int$$

$$R_{14}(p_1, p_2, p_3) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \frac{4}{2\pi} \frac{d\theta_3}{\det L} [348 - 37\cos\theta_1 - 24\cos\theta_2 - 37\cos\theta_3 - 22\cos\theta_1\cos\theta_3 - 14\cos\theta_1\cos\theta_3 - 14\cos\theta_2\cos\theta_3 - 22\cos\theta_1\cos\theta_2\cos\theta_3 - 22\cos\theta_1\cos\theta_2\cos\theta_3 - 2[6 + \cos\theta_1][6 + \cos\theta_3]\cos p_1\theta_1[\cos p_2\theta_2 + \cos(p_2 + 1)\theta_2]\cos p_3\theta_3]$$
(C.4)

$$R_{23}(p_1, p_2, p_3) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \frac{4}{\det L} [264 - 13\cos\theta_1 - 26\cos\theta_2 - 13\cos\theta_3 - 8\cos\theta_1\cos\theta_2 - 8\cos\theta_2\cos\theta_3 - 7[6 + \cos\theta_2][\cos p_1\theta_1 + \cos(p_1 - 1)\theta_1]\cos p_2\theta_2[\cos p_3\theta_3 + \cos(p_3 + 1)\theta_3]]$$
(C.5)

$$R_{24}(p_1, p_2, p_3) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{4\pi} \left[ 264 - 13\cos\theta_1 - 13\cos\theta_2 - 26\cos\theta_3 - 8\cos\theta_1\cos\theta_3 - 8\cos\theta_1\cos\theta_3 - 8\cos\theta_1\cos\theta_3 - 8\cos\theta_2\cos\theta_3 - 7[6 + \cos\theta_3][\cos p_1\theta_1 + \cos(p_1 + 1)\theta_1][\cos p_2\theta_2 + \cos(p_2 - 1)\theta_2]\cos p_3\theta_3 \right]$$
(C.6)

$$R_{34}(p_1, p_2, p_3) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \frac{4}{detL} [264 - 26\cos\theta_1 - 13\cos\theta_2 - 13\cos\theta_3 - 8\cos\theta_1\cos\theta_2 - 8\cos\theta_1\cos\theta_3 - 7[6 + \cos\theta_1]\cos p_1\theta_1 [\cos p_2\theta_2 + \cos(p_2 + 1)\theta_2] [\cos p_3\theta_3 + \cos(p_3 - 1)\theta_3]]$$
(C.7)

The remaining types of the resistance can be obtained from the lattice symmetry.

#### Appendix D. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.rinp.2019.01.070.

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