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New features of the fractional Euler-Lagrange equations for a physical system within non-singular derivative operator

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Abstract. Free motion of a fractional capacitor microphone is investigated in this paper. First, a capacitor microphone is introduced and the Euler-Lagrange equations are established. Due to the fractional derivative's the history independence, the fractional order displacement and electrical charge are used in the equations. Fractional differential equations involve in the right- and left-hand–sided derivatives which is reduced to a boundary value problem. Finally, numerical simulations are obtained and dynamical behaviors are numerically discussed.

1 Introduction

In the classical mechanics, we have mainly two approaches to study the dynamical systems and get their equation of motion: the Newtonian approach, which is a force-based one, and the energy approach, which has been invented by the French mathematician Joseph Louis Lagrange. In many cases, we face with some difficulties in applying the force-based approach, since we have to set up all forces acting the system, while sometimes they are not clear. The second approach (*i.e.*, Lagrangian) is a very elegant and useful method for finding the equation of motion for all dynamical systems [1]. In the literature, one can find many interesting systems that can be solved using the Lagrangian method; for instance, the spring pendulum, the coupled pendulum, Atwood's machine, and many others. In recent years, many efforts have been put extending the classical mathematics by using the concept of the fractional calculus. As is known, the fractional calculus goes back to a very long time, say about 300 years ago. It found many applications in all branches of science and engineering; see for example [2–12] and references therein. Riewe was the first physicist who studied the non-conservative Lagrangian systems within the framework of the fractional calculus [13,14]. He defined conjugate momenta and derived the Hamilton's equations via the fractional and higher-order derivatives. A fractional path integral technique on the path of the Lévy flights was introduced by Laskin [15,16]. He established the statistical mechanics, fractional quantum, and a fractional generalization of the Schrödinger equation via the proposed fractional path integral scheme. Using the Fox's H function, he also derived a free particle quantum-mechanical kernel, a fractional motion equation for the relevant density matrix, and a fractional generalization for the related Heisenberg uncertainty. As a consequence, Laskin in [17] demonstrated the Hermiticity of the fractional Hamilton operator, investigated the fractional quantum mechanics, and also instituted the parity conservation for it. Later on, many scientists followed the works by Riewe and Laskin; hence, one can find in the literature many published materials on this topic such as [18–21]. As can be seen from these works, the fractional Lagrangian (or the fractional Hamiltonian) is used to investigate the physical systems, and this results the fractional Euler-Lagrange equations (FELEs). The next step is to seek the numerical schemes for solving these equations effectively. These methods include the Grünwald-Letnikov approximation [22], decomposition method [23,24], variational iteration method [25], and Adams-Bashforth-Moulton technique [26].

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Recent investigations have shown that the representation of many physical systems via the fractional calculus can provide new features of their complex dynamics with memory effects; however, due to the singularity of the classical fractional derivatives, the nonlocal dynamics of the real-world phenomena may not be illustrated accurately. Therefore, modelling and analyzing the nonlocal processes is a significant issue needs to be investigated. To overcome this drawback, several types of the fractional operators with nonsingular kernel have been presented, which characterize the nonlocal dynamics appropriately. One of the most important candidate among the existing ones is a new differential operator with Mittag-Leffler (ML) kernel (ABC) [27]. The ML function is the queen of the fractional calculus, as was noted by many researchers. We can point out here the studies done by Srivastava, Tomowski, Hilfer, Gorenflo, Mainardi, and many other researchers [28–30]. The importance of the ML function in the integral transforms has also been mentioned by Kilbas, Samko, and many others [31,32]. Despite of many high level researches about the integral transforms on the basis of the ML function, just the above-mentioned new ABC was founded for the fractional derivatives and applied to some practical cases [27]. Many new works has been reported verifying that this new kind of calculus comes up with the satisfactory results when we encounter with the realistic systems [33–35]. Indeed, the nonsingular kernel used in the ABC derivative can capture the nonlocality of the complex phenomena more accurately than the classical fractional derivatives. Hence, compared to the standard fractional calculus, this new operator has quite different properties, for example in the transient state, and can represent different features of the real-world dynamics more precisely. Another important issue here is to design a suitable approximation scheme solving the FELEs in the ABC sense. Note that, due to memory effects, the numerical schemes in fractional sense are not a direct generalization of their classical counterparts. Hence, a substantial numerical analysis should be developed to solve the new fractional models related to the real-world dynamics. Motivated by the above discussion, the main contribution of this research is to investigate the free motion of a capacitor microphone by using its new formulation in fractional sense. The main features of the new thinks provided in this manuscript are summarized as follows:

- Due to the history independence of the fractional derivative, the fractional-order displacement and electrical charge are used in the equations.
- To characterize the nonlocal dynamics, the ABC fractional derivative is employed possessing the ML nonsingular kernel. To the best of our knowledge, this is the first time to employ a nonlocal and nonsingular derivative operator for the free motion of a capacitor microphone.
- A theoretical analysis is given to derive the related FELEs in the sense of ABC for the capacitor microphone.
- The new equations involve the right- and left-hand–sided derivatives, which makes them more complicated to solve in practice. Hence, a new and powerful numerical technique is suggested to solve these equations effectively.
- According to the obtained results in this paper, the fractional calculus provides more flexible models than the standard classical calculus, based on the fractional derivative order and the fractional operator itself. This feature plays a remarkable role to extract new hidden aspects of physical system under consideration.

Consequently, the FELEs in the ABC sense and their solution method presented in this paper for the capacitor microphone are new and comprise quite different information than their corresponding standard fractional equations. Due to these features, we believe that the obtained results in this paper are valuable from both mathematical and physical points of view.

The rest of this paper is organized as follows. In sect. 2, some preliminaries concerning the fractional derivatives are presented. In sect. 3, the classical and fractional descriptions of the capacitor microphone are carried out. In sect. 4, a numerical approach is established for solving the derived FELEs. In sect. 5, simulation results and their discussions are given. Finally, we conclude the paper in the last section.

2 Preliminaries

In this section, some preliminary definitions are presented for the fractional operators. In the following, we define the fractional derivatives in terms of the Riemann-Liouville [22], classic Caputo [22], and the ABC with ML kernel [27].

For a time-dependent function z: $[a,b] \to \mathbb{R}^n$ and $0 < \alpha < 1$, the left and right α-th–order Riemann-Liouville fractional derivatives are, respectively, described by [22]

$$ {}^{RL}_{a}D^{\alpha}_{t} z(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\nu)^{-\alpha} z(\nu) \mathrm{d}\nu, \tag{1} $$

$$ {}^{RL}_{t}D^{\alpha}_{b} z(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b (\nu-t)^{-\alpha} z(\nu) \mathrm{d}\nu, \tag{2} $$

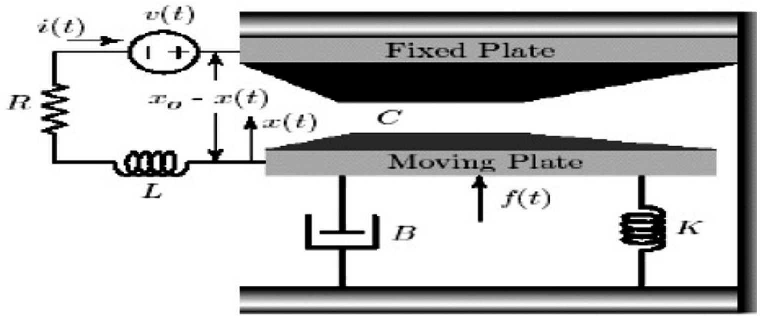

Fig. 1. The capacitor microphone.

where $\Gamma(\cdot)$ is the Euler's Gamma function. Also, the left and right α-th–order Caputo fractional derivatives are, respectively, determined as [22]

$$ {}_a^C D_t^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\nu)^{-\alpha} \dot{z}(\nu) \mathrm{d}\nu, \tag{3} $$

$$ {}_t^C D_b^\alpha z(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\nu-t)^{-\alpha} \dot{z}(\nu) \mathrm{d}\nu. \tag{4} $$

For $z \in \mathbb{H}^1(a,b)$ and $0 < \alpha < 1$, the left and right α-th–order ABC fractional derivatives are, respectively, defined by [27]

$$ {}_a^{ABC} D_t^\alpha z(t) = \frac{N(\alpha)}{1-\alpha} \int_a^t E_\alpha[-\beta(t-\nu)^\alpha] \dot{z}(\nu) \mathrm{d}\nu, \tag{5} $$

$$ {}_t^{ABC} D_b^\alpha z(t) = -\frac{N(\alpha)}{1-\alpha} \int_t^b E_\alpha[-\beta(\nu-t)^\alpha] \dot{z}(\nu) \mathrm{d}\nu, \tag{6} $$

where $\beta = \frac{\alpha}{1-\alpha}$, $N(\alpha)$ is a normalization function with $N(0) = N(1) = 1$ and $E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}$ is the ML function.

Note that, here the normalization function in eqs. (5), (6) is considered as $N(\alpha) = 1$ for the simplicity in the simulations. For more details on the above-mentioned fractional operators, the readers are referred to [22, 27].

3 Classical and fractional descriptions of the capacitor microphone

In this section, we investigate a capacitor microphone and present a fully description of its dynamical equation of motion. As is shown in fig. 1, the capacitance C is changed according to the displacement $x(t)$ of the bottom plate, which is attached to a damper with constant $B > 0$ and a spring with constant $K > 0$ [36]. The air pressure caused by the sound is modeled by a mechanical force $f(t)$, which is applied to the bottom plate. Moreover, $v(t)$ is an external voltage source. The kinetic and potential energies of this system are, respectively, computed by

$$ V = \frac{1}{2} K x^2 + \frac{1}{2\varepsilon A}(x_0 - x) q^2, \tag{7} $$

$$ K_e = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} L \dot{q}^2, \tag{8} $$

where q is the charge of capacitor, $x_0 - x$ is the distance between two plates, ε is the air dielectric constant, A is the area of each plate, m is the mass of bottom plate, and L is the inductance. Furthermore, the power function P, which is half of the amount of energy that is dissipated, is as follows:

$$ P = \frac{1}{2} B \dot{x}^2 + \frac{1}{2} R \dot{q}^2, \tag{9} $$

where R is the resistance. For the physical system under investigation, the classical Lagrangian, which specifies the balance among no dissipative energy, is in the form below

$$ L_c = K_e - V = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} L \dot{q}^2 - \frac{1}{2} K x^2 - \frac{1}{2\varepsilon A}(x_0 - x) q^2. \tag{10} $$

Given that the capacitor microphone model has two degrees of freedom, which are the electrical charge q and mechanical displacement x, we have two classical Euler-Lagrange equations (CELEs) $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L_c}{\partial \dot{q}}) - \frac{\partial L_c}{\partial q} + \frac{\partial P}{\partial \dot{q}} = v$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L_c}{\partial \dot{x}}) - \frac{\partial L_c}{\partial x} + \frac{\partial P}{\partial \dot{x}} = f$. Substituting P and L_c from eqs. (9)–(10) in the CELEs yields

$$L\ddot{q} + R\dot{q} + \frac{1}{\varepsilon A}(x_0 - x)q = v, \tag{11}$$

$$m\ddot{x} + B\dot{x} + Kx - \frac{q^2}{2\varepsilon A} = f. \tag{12}$$

As was noted in [37], many laws of the real-world phenomena cannot be captured by the theory of calculus of variations; for instance, the dynamical equations obtained based on the traditional energy approach cannot describe the behavior of nonconservative systems. On the other hand, the modeling of physical systems via the fractional calculus can provide new features of their complex dynamics with memory effects. Hence, in the following, we introduce a fractional form of the classical Lagrangian (10), which exposes new features of the physical system under study. Then, we derive the FELEs for the system under investigation. The proof of these equations has also been given in the appendix. By fractionalizing eq. (10), we define

$$L_f = \frac{1}{2}m({}_aD_t^\alpha x)^2 + \frac{1}{2}L({}_aD_t^\alpha q)^2 - \frac{1}{2}Kx^2 - \frac{1}{2\varepsilon A}(x_0 - x)q^2, \tag{13}$$

where ${}_aD_t^\alpha$ can be one of those differential operators (3) or (5). Also, the power function P can be fractionalized as follows:

$$P = \frac{1}{2}B({}_aD_t^\alpha x)^2 + \frac{1}{2}R({}_aD_t^\alpha q)^2. \tag{14}$$

Then, we can obtain the FELEs from

$$-{}_tD_b^\alpha \frac{\partial L_f}{\partial_a D_t^\alpha q} - {}_aD_t^\alpha \frac{\partial L_f}{\partial_t D_b^\alpha q} - \frac{\partial L_f}{\partial q} + \frac{\partial P}{\partial_a D_t^\alpha q} = v, \tag{15}$$

$$-{}_tD_b^\alpha \frac{\partial L_f}{\partial_a D_t^\alpha x} - {}_aD_t^\alpha \frac{\partial L_f}{\partial_t D_b^\alpha x} - \frac{\partial L_f}{\partial x} + \frac{\partial P}{\partial_a D_t^\alpha x} = f, \tag{16}$$

where ${}_tD_b^\alpha$ is the right fractional operator in eq. (4) or (6). Considering eqs. (13)–(16), the FELEs read

$$-L {}_tD_b^\alpha {}_aD_t^\alpha q + R {}_aD_t^\alpha q + \frac{1}{\varepsilon A}(x_0 - x)q = v, \tag{17}$$

$$-m {}_tD_b^\alpha {}_aD_t^\alpha x + B {}_aD_t^\alpha x + Kx - \frac{1}{2\varepsilon A}q^2 = f. \tag{18}$$

Notice that, as the fractional order α tends to 1, the FELEs (17)–(18) are reduced to the CELEs previously defined in eqs. (11)–(12).

Now, we want to achieve the fractional Hamilton equations (FHEs). In order to this, we obtain the fractional Hamiltonian function from

$$H_f = L_{\alpha,q}\, {}_aD_t^\alpha q + L_{\beta,q}\, {}_tD_b^\alpha q + L_{\alpha,x}\, {}_aD_t^\alpha x + L_{\beta,x}\, {}_tD_b^\alpha x - L_f, \tag{19}$$

where the generalized momenta are introduced as

$$L_{\alpha,q} = \frac{\partial L_f}{\partial_a D_t^\alpha q} = L {}_aD_t^\alpha q, \qquad L_{\beta,q} = \frac{\partial L_f}{\partial_t D_b^\alpha q} = 0, \qquad L_{\alpha,x} = \frac{\partial L_f}{\partial_a D_t^\alpha x} = m {}_aD_t^\alpha x, \qquad L_{\beta,x} = \frac{\partial L_f}{\partial_t D_b^\alpha x} = 0. \tag{20}$$

Substituting eqs. (13) and (20) into eq. (19), the fractional Hamiltonian function is obtained

$$H_f = L({}_aD_t^\alpha q)^2 + m({}_aD_t^\alpha x)^2 - \frac{1}{2}m({}_aD_t^\alpha x)^2 - \frac{1}{2}L({}_aD_t^\alpha q)^2 + \frac{1}{2}Kx^2 + \frac{1}{2\varepsilon A}(x_0 - x)q^2. \tag{21}$$

Then, the FHEs of motion are derived from

$$\frac{\partial H_f}{\partial q} - {}_tD_b^\alpha\, L_{\alpha,q} - {}_aD_t^\alpha\, L_{\beta,q} + \frac{\partial P}{\partial_a D_t^\alpha q} = v, \tag{22}$$

$$\frac{\partial H_f}{\partial x} - {}_tD_b^\alpha\, L_{\alpha,x} - {}_aD_t^\alpha\, L_{\beta,x} + \frac{\partial P}{\partial_a D_t^\alpha x} = f, \tag{23}$$

which result the corresponding FELEs (17)–(18). Again, as $\alpha \to 1$, the FHEs are reduced to the CELEs (11)–(12).

4 Numerical method

In this section, an efficient numerical technique is suggested for solving the FELEs (17)–(18) considering the Caputo and ABC differential operators. To this aim, we first begin with the ABC derivative and reformulate eqs. (17)–(18) via defining the new state variables $q_1 = q$, $q_2 = {}_aD_t^\alpha q$, $x_1 = x$, and $x_2 = {}_aD_t^\alpha x$. Thus, we derive

$$\begin{cases} {}^{ABC}_{\ a}D_t^\alpha q_1 = q_2, \\ {}^{ABC}_{\ t}D_b^\alpha q_2 = \dfrac{R}{L}q_2 + \dfrac{1}{L\varepsilon A}(x_0 - x_1)q_1 - \dfrac{1}{L}v, \\ {}^{ABC}_{\ a}D_t^\alpha x_1 = x_2, \\ {}^{ABC}_{\ t}D_b^\alpha x_2 = \dfrac{B}{m}x_2 + \dfrac{k}{m}x_1 - \dfrac{1}{2m\varepsilon A}{q_1}^2 - \dfrac{1}{m}f. \end{cases} \tag{24}$$

Using the fractional integral operator in the sense of ABC defined in [27] and supposing that $q_1(a) = q(a)$ and $x_1(a) = x(a)$, as the initial values of charge and displacement, respectively, and $q_2(b) = x_2(b) = 0$, eq. (24) is converted into the following system of fractional integral equations

$$\begin{cases} q_1(t) = q_1(a) + \dfrac{1-\alpha}{N(\alpha)}q_2(t) + \dfrac{\alpha}{\Gamma(\alpha)N(\alpha)}\displaystyle\int_a^t q_2(\nu)(t-\nu)^{\alpha-1}\mathrm{d}\nu, \\ q_2(t) = \dfrac{1-\alpha}{N(\alpha)}\left(\dfrac{R}{L}q_2(t) + \dfrac{1}{L\varepsilon A}(x_0 - x_1(t))q_1(t) - \dfrac{1}{L}v(t)\right) \\ \qquad + \dfrac{\alpha}{\Gamma(\alpha)N(\alpha)}\displaystyle\int_t^b \left(\dfrac{R}{L}q_2(\nu) + \dfrac{1}{L\varepsilon A}(x_0 - x_1(\nu))q_1(\nu) - \dfrac{1}{L}v(\nu)\right)(\nu - t)^{\alpha-1}\mathrm{d}\nu, \\ x_1(t) = x_1(a) + \dfrac{1-\alpha}{N(\alpha)}x_2(t) + \dfrac{\alpha}{\Gamma(\alpha)N(\alpha)}\displaystyle\int_a^t x_2(\nu)(t-\nu)^{\alpha-1}\mathrm{d}\nu, \\ x_2(t) = \dfrac{1-\alpha}{N(\alpha)}\left(\dfrac{B}{m}x_2(t) + \dfrac{k}{m}x_1(t) - \dfrac{1}{2m\varepsilon A}q_1^2(t) - \dfrac{1}{m}f(t)\right) \\ \qquad + \dfrac{\alpha}{\Gamma(\alpha)N(\alpha)}\displaystyle\int_t^b \left(\dfrac{B}{m}x_2(\nu) + \dfrac{k}{m}x_1(\nu) - \dfrac{1}{2m\varepsilon A}q_1^2(\nu) - \dfrac{1}{m}f(\nu)\right)(\nu - t)^{\alpha-1}\mathrm{d}\nu. \end{cases} \tag{25}$$

Now, a uniform partition is considered on $[a, b]$ with $h_N = \frac{b-a}{N}$ as the length of the time step, in which N is an arbitrary positive integer. Furthermore, the numerical approximations of $q_i(t_j)$ and $x_i(t_j)$ are, respectively, denoted by $q_{i,j}$ and $x_{i,j}$, where $i = 1, 2$ and t_j is the time at node j, *i.e.*, $t_j = a + jh_N$ for $0 \le j \le N$. Then, by applying the fractional Euler method in [38], the convolution integrals in eq. (25) are discretized and a system of linear algebraic equations is obtained

$$\begin{cases} Q_1 - \dfrac{1-\alpha}{N(\alpha)}Q_2 - \dfrac{\alpha}{N(\alpha)}B_{N,\alpha}Q_2 = Q_{1,0}, \\ Q_2 - \dfrac{1-\alpha}{N(\alpha)}G_{Q_2}(Q_1, Q_2, X_1, V) - \dfrac{\alpha}{N(\alpha)}F_{N,\alpha}G_{Q_2}(Q_1, Q_2, X_1, V) = 0, \\ X_1 - \dfrac{1-\alpha}{N(\alpha)}X_2 - \dfrac{\alpha}{N(\alpha)}B_{N,\alpha}X_2 = X_{1,0}, \\ X_2 - \dfrac{1-\alpha}{N(\alpha)}G_{X_2}(Q_1, X_1, X_2, F) - \dfrac{\alpha}{N(\alpha)}F_{N,\alpha}G_{X_2}(Q_1, X_1, X_2, F) = 0, \end{cases} \tag{26}$$

where

$$B_{N,\alpha} = (F_{N,\alpha})^T = h_N \begin{bmatrix} \omega_{0,\alpha} & 0 & \dots & 0 \\ \omega_{1,\alpha} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \omega_{N,\alpha} & \dots & \omega_{1,\alpha} & \omega_{0,\alpha} \end{bmatrix}, \quad \omega_{0,\alpha} = 1, \quad \omega_{j,\alpha} = \left(1 + \frac{\alpha-1}{j}\right)\omega_{j-1,\alpha}, \quad j = 1, 2, \dots, \tag{27}$$

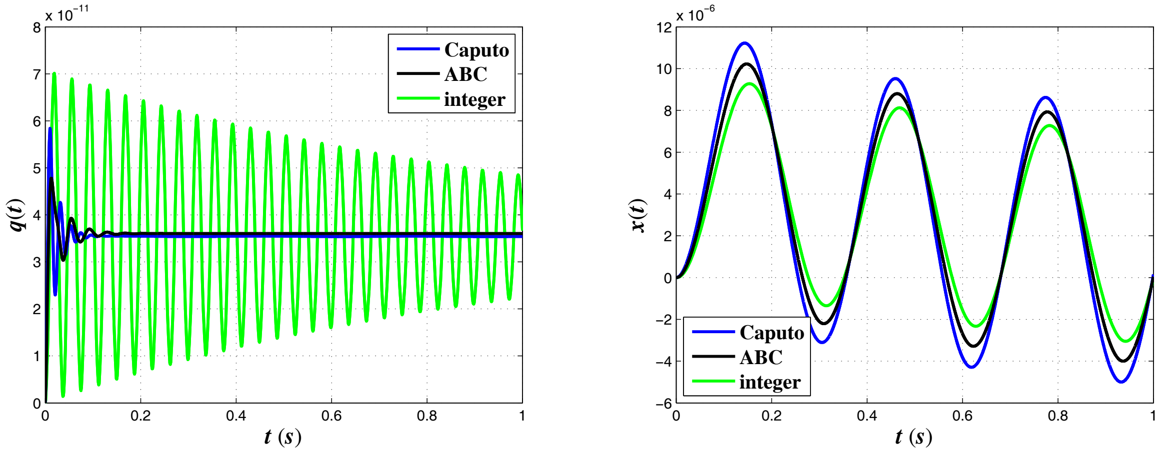

Fig. 2. The dynamics of the electrical charge $q(t)$ and the displacement $x(t)$ for the capacitor microphone within the Caputo and ABC fractional operators with $\alpha = 0.9$ and the integer-order classical solution.

$$Q_i = \begin{bmatrix} q_{i,0} \\ \vdots \\ q_{i,N} \end{bmatrix}, \quad X_i = \begin{bmatrix} x_{i,0} \\ \vdots \\ x_{i,N} \end{bmatrix}, \quad Q_{i,0} = \begin{bmatrix} q_{i,0} \\ \vdots \\ q_{i,0} \end{bmatrix}, \quad X_{i,0} = \begin{bmatrix} x_{i,0} \\ \vdots \\ x_{i,0} \end{bmatrix}, \quad i = 1, 2, \tag{28}$$

$$V = \begin{bmatrix} v(t_0) \\ \vdots \\ v(t_N) \end{bmatrix}, \qquad F = \begin{bmatrix} f(t_0) \\ \vdots \\ f(t_N) \end{bmatrix}, \tag{29}$$

$$G_{Q_2}(Q_1, Q_2, X_1, V) = \begin{bmatrix} \frac{R}{L} q_{2,0} + \frac{1}{L\varepsilon A}(x_0 - x_{1,0}) q_{1,0} - \frac{1}{L} v(t_0) \\ \vdots \\ \frac{R}{L} q_{2,N} + \frac{1}{L\varepsilon A}(x_0 - x_{1,N}) q_{1,N} - \frac{1}{L} v(t_N) \end{bmatrix}, \tag{30}$$

$$G_{X_2}(Q_1, X_1, X_2, F) = \begin{bmatrix} \frac{B}{m} x_{2,0} + \frac{k}{m} x_{1,0} - \frac{1}{2m\varepsilon A} q_{1,0}^2 - \frac{1}{m} f(t_0) \\ \vdots \\ \frac{B}{m} x_{2,N} + \frac{k}{m} x_{1,N} - \frac{1}{2m\varepsilon A} q_{1,N}^2 - \frac{1}{m} f(t_N) \end{bmatrix}. \tag{31}$$

It is noticeable that, the aforesaid results can also be applied to the classic Caputo case by replacing the ABC integral in eq. (25) with its Caputo counterpart defined in [22], and then, repeating the above-mentioned discretization algorithm.

5 Simulation results

In this section, the dynamical behaviors of $q(t)$ and $x(t)$ are investigated considering the Caputo and ABC fractional operators with different values α. The system parameters are taken as $L = 10^6\,\mathrm{H}$, $R = 2 \times 10^6\,\Omega$, $A = 2 \times 10^{-2}\,\mathrm{m}^2$, $\varepsilon = 8.854 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}}$, $v = 12$ volts, $m = 0.01\,\mathrm{kg}$, $B = 10\,\frac{\mathrm{Ns}}{\mathrm{m}}$, $K = 10\,\frac{\mathrm{N}}{\mathrm{m}}$, $f = 0.001\sin(5t)$ and $x_0 = 0.005\,\mathrm{m}$. Furthermore, the initial values are assumed to be $q(0) = 0$ and $x(0) = 0$. As it is depicted in figs. 2–7, the numerical solution of the FELEs not only represents various behaviours for different values of α but also tends to the integer-order classical solution as α tends to 1. Moreover, the ABC derivative provides quite different properties than the classical Caputo, for example, in the transient state. Therefore, more flexible models are provided considering the new fractional operators, which is advantageous in better understanding the complex behaviours of the real-world dynamical phenomena.

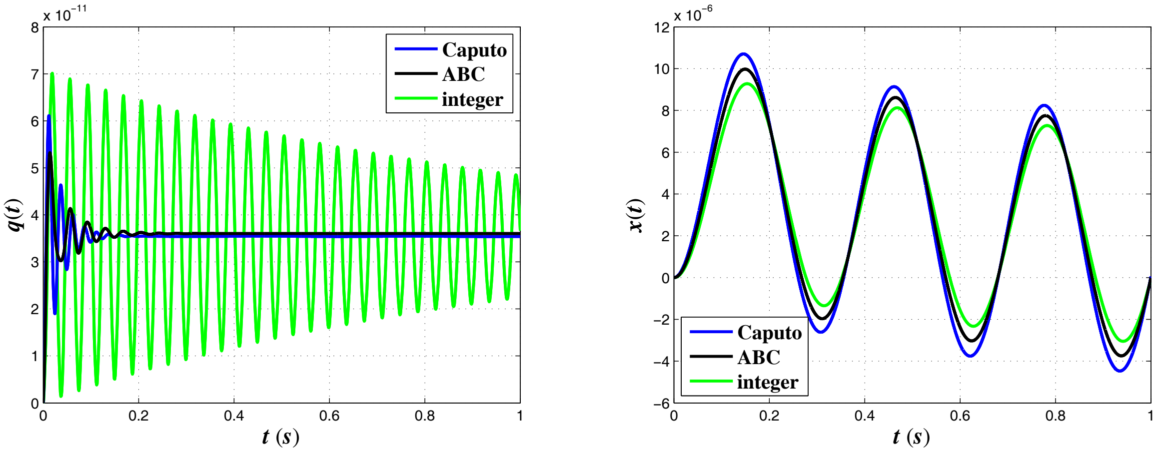

Fig. 3. The dynamics of the electrical charge $q(t)$ and the displacement $x(t)$ for the capacitor microphone within the Caputo and ABC fractional operators with $\alpha = 0.925$ and the integer-order classical solution.

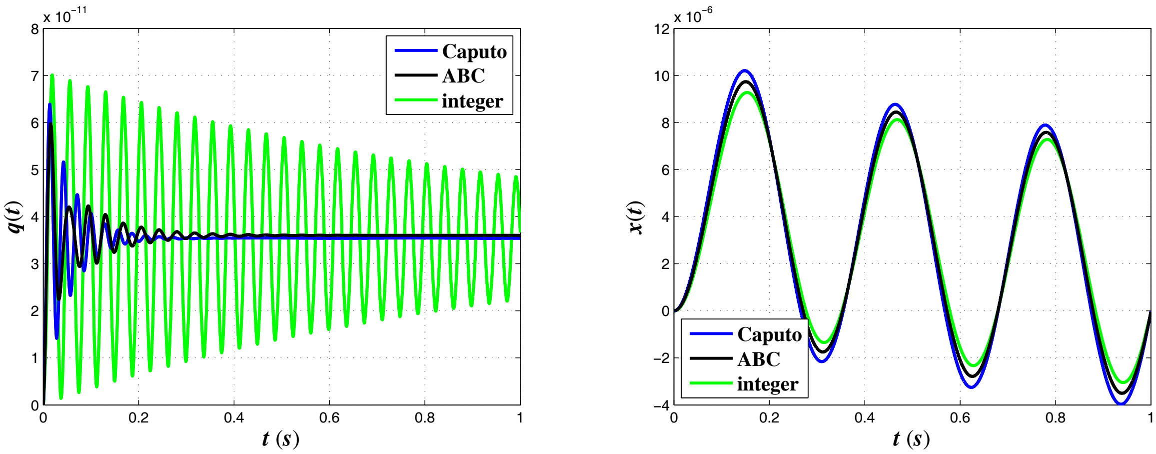

Fig. 4. The dynamics of the electrical charge $q(t)$ and the displacement $x(t)$ for the capacitor microphone within the Caputo and ABC fractional operators with $\alpha = 0.95$ and the integer-order classical solution.

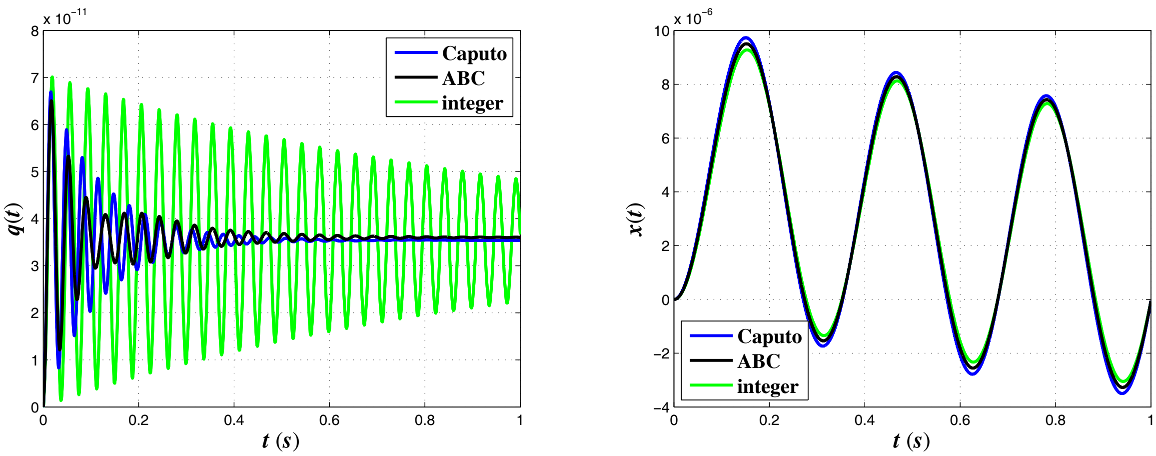

Fig. 5. The dynamics of the electrical charge $q(t)$ and the displacement $x(t)$ for the capacitor microphone within the Caputo and ABC fractional operators with $\alpha = 0.975$ and the integer-order classical solution.

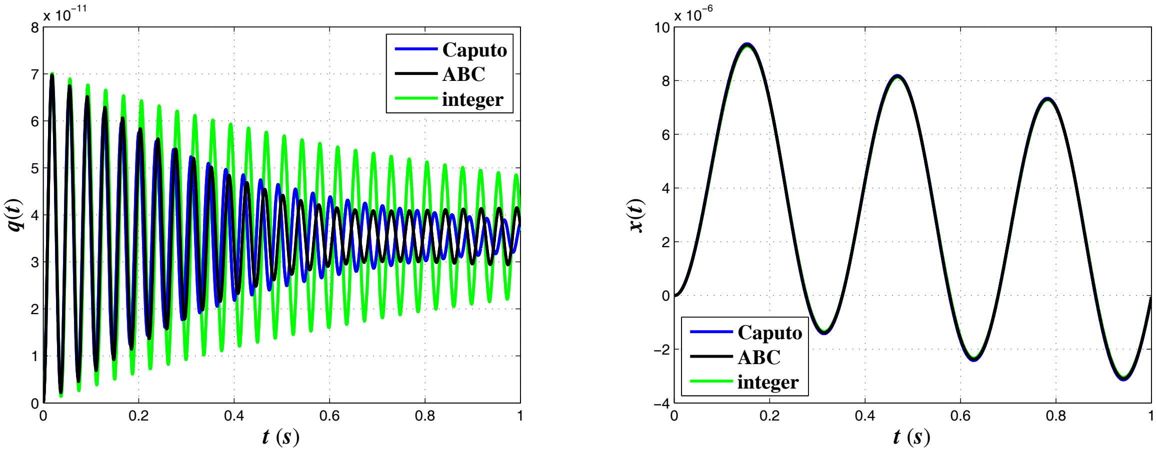

Fig. 6. The dynamics of the electrical charge $q(t)$ and the displacement $x(t)$ for the capacitor microphone within the Caputo and ABC fractional operators with $\alpha = 0.995$ and the integer-order classical solution.

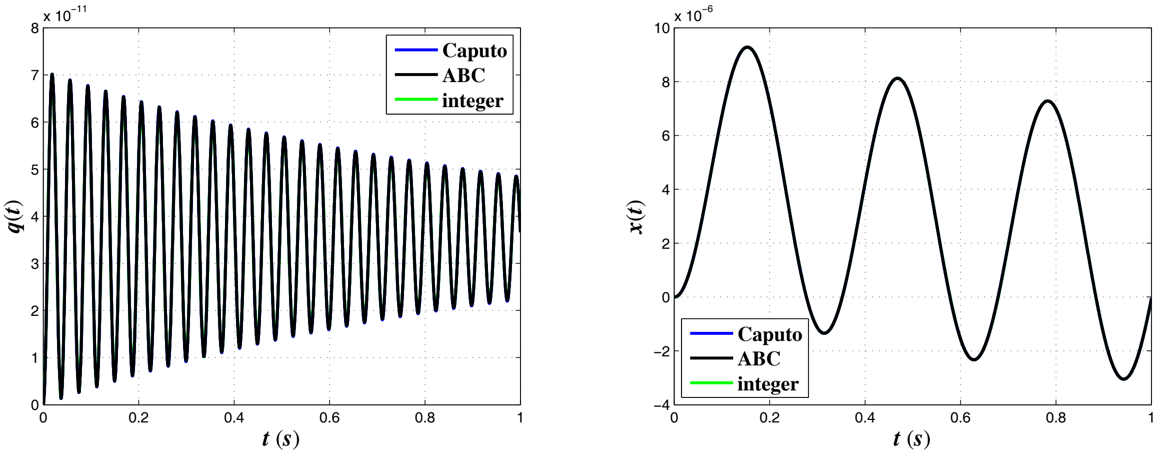

Fig. 7. The dynamics of the electrical charge $q(t)$ and the displacement $x(t)$ for the capacitor microphone within the Caputo and ABC fractional operators with $\alpha = 1$ and the integer-order classical solution.

6 Conclusion

In this work, we used the concept of the fractional calculus in order to study the motion equation of a capacitor microphone. We established the classical and fractional Lagrangian as well as the fractional Hamiltonian of the motion and derived the FELEs and FHEs in the ABC and Caputo sense. Then, we solved the derived fractional equations via a numerical approach, which applied the quadrature rule of Euler convolution in order to discretize the convolution integral. Simulation results showed that the behaviours of the FELEs depend on the fractional derivative order as well as the differential operators. Indeed, various behaviours were exhibited by using different values of α and various derivative operators, which converged to the classical solution as α goes to 1. Consequently, the fractional calculus has new notable features, which help us to have more realistic and flexible models of the real-world dynamics.

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Appendix A. (The proof of the FELEs in the ABC sense)

Base on the fractional calculus of variations presented in [37], we define the fractional Lagrangian function as $L_f = L(t, z, {}_aD_t^\alpha z, {}_tD_b^\alpha z)$, where $z = [\begin{smallmatrix} q \\ x \end{smallmatrix}]$, ${}_aD_t^\alpha z, {}_tD_b^\alpha z$ denote, respectively, the left and right ABC derivatives, and L has continuous first and second (partial) derivatives with respect to all its arguments. As a starting point to achieve the FELEs, the action function of the classical field is considered such that it contains the fractional partial derivatives, *i.e.*, $S[z] = \int_a^b L(t, z, {}_aD_t^\alpha z, {}_tD_b^\alpha z)\mathrm{d}t$. From the Agrawal's fractional Lagrangian mechanics [37], finding the extremum point of S and employing the formula for the fractional integration by parts yield the FELEs. In the following, inspired by the Agrawal's formulation, we suppose that $z(t)$ has continuous left and right fractional derivatives of order α for $a \leq t \leq b$ and satisfies the boundary conditions $z(a) = z_a$ and $z(b) = z_b$. For finding the extremum point $z^*(t)$ of the functional $S[z]$, we take $\varepsilon \in \mathbb{R}$ and consider the following family of curve:

$$z(t) = z^*(t) + \varepsilon\zeta(t), \tag{A.1}$$

where $\zeta(a) = \zeta(b) = 0$. Since the fractional derivatives are linear operators, then it follows

$${}_aD_t^\alpha z(t) = {}_aD_t^\alpha z^*(t) + \varepsilon\, {}_aD_t^\alpha \zeta(t), \tag{A.2}$$

$${}_tD_b^\alpha z(t) = {}_tD_b^\alpha z^*(t) + \varepsilon\, {}_tD_b^\alpha \zeta(t). \tag{A.3}$$

Substituting eqs. (A.1)–(A.3) into the action function $S[z]$, for each ζ we find

$$S = S(\varepsilon) = \int_a^b L(t, z^* + \varepsilon\zeta, {}_aD_t^\alpha z^* + \varepsilon\, {}_aD_t^\alpha \zeta, {}_tD_b^\alpha z^* + \varepsilon\, {}_tD_b^\alpha \zeta)\mathrm{d}t. \tag{A.4}$$

A necessary condition for $S(\varepsilon)$ to have an extremum $z = z^*(t)$ is that for all admissible $\zeta(t)$, we should have

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon} = \int_a^b \left[\frac{\partial L}{\partial z}\zeta + \frac{\partial L}{\partial_a D_t^\alpha z}{}_aD_t^\alpha \zeta + \frac{\partial L}{\partial_t D_b^\alpha z}{}_tD_b^\alpha \zeta\right]\mathrm{d}t = 0. \tag{A.5}$$

As has been shown in [39], the formulation of the fractional integration by parts can be hold considering the ABC fractional derivative. Applying the formulation, we have the following equations for the second and third integrals in eq. (A.5) by considering the boundary conditions $\zeta(a) = \zeta(b) = 0$

$$\int_a^b \frac{\partial L}{\partial_a D_t^\alpha z}{}_aD_t^\alpha \zeta\,\mathrm{d}t = \int_a^b {}_tD_b^\alpha\left(\frac{\partial L}{\partial_a D_t^\alpha z}\right)\zeta\,\mathrm{d}t, \tag{A.6}$$

$$\int_a^b \frac{\partial L}{\partial_t D_b^\alpha z}{}_tD_b^\alpha \zeta\,\mathrm{d}t = \int_a^b {}_aD_t^\alpha\left(\frac{\partial L}{\partial_t D_b^\alpha z}\right)\zeta\,\mathrm{d}t. \tag{A.7}$$

Substituting eqs. (A.6)–(A.7) into eq. (A.5) yields

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon} = \int_a^b \left[\frac{\partial L}{\partial z} + {}_tD_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha z} + {}_aD_t^\alpha \frac{\partial L}{\partial_t D_b^\alpha z}\right]\zeta\,\mathrm{d}t = 0. \tag{A.8}$$

Since $\zeta(t)$ is arbitrary, a well-established concept in the calculus of variations results the FELEs

$$\frac{\partial L}{\partial z} + {}_tD_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha z} + {}_aD_t^\alpha \frac{\partial L}{\partial_t D_b^\alpha z} = 0. \tag{A.9}$$

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