Forecasting the Behavior of the Global Market Price Using Markov Chain of the Fuzzy States - Case study: gold price

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This Thesis was submitted in partial fulfillment of the requirements for the Master's Degree of Science in Mathematical Modeling

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Forecasting the Behavior of the Global Market Price Using Markov Chain of the Fuzzy States - Case study: gold price

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص، باستثناء ما تم الإشارة إليه حيثما ورد، و أن هذه الرسالة ككل ، أو أي جزء منها لم يقدم من قبل لنيل درجة علمية أو بحث علمي لدى مؤسسة تعليمية او بحثية أخرى.

Declaration

I hereby declare that this thesis is the product of my own efforts, except what has been referred to, and this thesis as a whole or any part of it has not been submitted as a requirement for attaining a scientific degree to any other educational or research institution.

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COMMITTEE DECISION

This thesis/dissertation (Forecasting the Behavior of the Global Market Price Using Markov Chain of the Fuzzy States-Case Study: Gold Price)

Examination committee	**Signature**
Prof. Dr. Saed Mallak (Supervisor)	
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Asst. Prof. Dr. Mohammad Jarrar (Internal Examiner)	

DEDICATION

To all those who lighten my dark nights with their prayers, stand by my side and encourage me until the end of time, to those I feel unable to express my great gratitude for their precious support,

To both of whom I am indebted to forever and I

Owe everything that made it all possible, to my parents

Raba'a and Ismail,

To the sweetest heart ever my love sister; Sara,

To the solid rock of my life, my brothers; Mohannad and Mohammad

To everyone who inspires me by his/her science

I dedicate my work…

Acknowledgment

First and foremost, all praise and thanks be to God, for giving me the patience and strength to make this a success, who guided my way to this point in my life.

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Contents

Forecasting the Behavior of the Global Market Price Using Markov Chain of the Fuzzy States:

Case Study Gold Price

By

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Prof. Dr. Saed Mallak

Abstract

The estimation regarding to the return rate of global market prices has been one of the most popular topics for the financial area due to its high return. In this work we consider gold prices as case study.

Closing retraction R_t is studied as a fuzzy concept and several types of fuzzy numbers are applied to R_t: triangle, trapezoidal, parabolic, and K-Trapezoidal-Triangular fuzzy numbers. We create a Markov chain model (MC) and a Markov model with fuzzy states (MCFS) and compare between them. The two models MC and MCFS are used to predict for long run time. At the end, we estimate the expected return price in specific months. Whereas, the MCFS model has more accurate results than the MC model.

In this work Excel is used to process the data and the MATLAB is used for other calculations.

التنبؤ بسلوك سعر السوق العالمي باستخدام سلاسل ماركوف بالحالات الضبابية

دراسة حالة : سعر الذهب

إعداد : ديمة عبده

بإشراف: أ.د / سائد ملاك

الملخص

تقدير معدل العائد لأسعار السوق العالمية من أكثر الموضوعات شيوعًا في المنطقة المالية نظرًا لارتفاع عائدها. في هذا العمل نعتبر أسعار الذهب دراسة حالة.

تم دراسة تراجع الإغلاق R_t كمفهوم غامض وتم تطبيق عدة أنواع من الأرقام الغامضة على R_t: مثلث ، شبه منحرف ،قطع مكافئ ، K- شبه منحرف- مثلث أرقام غامضة. نقوم بإنشاء نموذج سلسلة ماركوف (MC) ونموذج ماركوف بحالات ضبابية (MCFS) ونقارن بينهما. يتم استخدام النموذجين (MC) و (MCFS) للتنبؤ بوقت المدى الطويل، وفي النهاية ، نقوم بتقدير سعر العائد المتوقع في أشهر محددة ، في حين أن نموذج (MCFS) له نتائج أكثر دقة من نموذج (MC).

في هذا العمل ، يتم استخدام Excel لمعالجة البيانات ويتم استخدام MATLAB للعمليات الحسابية الأخرى.

Chapter 1
Introduction

1.1 Motivation

Due to uncertainty either in the parameters, which define the problem, or in the situations in which the problem occurs, the classical approaches for uncertainty analysis are very important as they help for making effective decision-making.

It is very intractable to make statistical involvement while the available data is inadequate. Probability Theory has been applied successfully for many real world problems but still there are some restrictions to the probabilistic method. For example, probabilistic methods are based on enormous collection of data, which is random in nature, to achieve the necessary level of confidence. But in wide-ranging, it is difficult to get the exact probability of the events on account of the complicated systems. Thus, results based on probability theory do not always provide helpful information to the practitioners due to the restriction of being able to handle only quantitative information. Moreover, in real world applications, sometimes there is a lack of data to precisely deal with the statistics of parameters [2].

To conquer these complications, methodologies based on fuzzy set theory are being used in the risk analysis for spreading the basic event uncertainty [2-3].

The probabilistic approaches deal with uncertainty, which is random in nature, while the fuzzy approach deals with the uncertainty, which is due to inaccuracy associated with the complexity of the system in addition to haziness of human judgement difficulties. Fuzzy set theory has been considered as a useful tool, especially for dealing with the complicated systems, in which the interactions of the systems' variables may be too sophisticated to be accurately identified [2-3].

One of these complex and difficult systems is the economic system and the global market price, where the quick change of market price in the global market has attracted several investors into examining price fluctuations. The estimation concerning to the exact daily, monthly or yearly price of the market price has always been a difficult task in the business sector. Markov process is a stochastic model that can be used to model a random system [8].

Markov decision processes (MDPs) are mathematical models that give mathematical outline for modelling the situations where outcomes are partially random and partially under the supervision of a decision maker. The Markov property means that evolution of the Markov process in the future depends only on the present state and does not depend on past history. The

Markov process does not remember the past if the present state is given. Hence, the Markov process is called the process with memory less property This model relies on crisp states [10] and [19].

The idea of fuzzy logic was first advanced by Zadeh (1965) where he defined fuzzy sets in order to describe unclear situations mathematically [30].

Fuzzy matrices were introduced for the first time by Thomason (1977), who discussed the convergence of powers of fuzzy matrix [25].

Kruce et al. (1987) introduced the fuzzy Markov chain as a classical Markov chain based on fuzzy probabilities where he used a fuzzy set to denote the transition matrix with the uncertain data in the Markov chains [14].

Fuzzy Markov chain is demonstrated as the concept of fuzzy relation and its compositions (Sanchez, 1976) [21]. It can be used while the decision maker prefers subjective probabilities to model the uncertainties (Vajargah and Gharehdaghi, 2012) [27].

Yoshida (1994) constructed a Markov fuzzy process, with a transition possibility measure [29]. Slowinski (1998) showed that we can use a fuzzy set representation in order to deal with uncertain data and flexible requirements [23].

Fuzzy Markov chains approaches are given by Avrachenkov and Sanchez (2000), they analyzed fuzzy Markov chains and its properties in detail [28].

Kuranoa et al. (2006) used fuzzy states to show fuzzy transition probabilities [15]. Pardo and Fuente (2010) used Markovian decision processes with fuzzy states to calculate the best policy to be implemented regarding publicity decisions in a queueing system [20]. Zhou et al. (2013) used fuzzy probability-based Markov chain model to estimate regional long-term electric power demand [31]. Ky and Fuente (2016) used combination of Markov model and fuzzy time series model for forecasting stock market data [16]. Kıral and Uzun (2017) used Markov chain of the fuzzy states to estimate stock market index [12].

1.2 Crisp VS Fuzzy

A classical (crisp) set is a collection of well-defined objects called elements (terms or members) and the set is denoted by capital letters and the elements of a set are denoted by small letters [9]. An example of a crisp set:

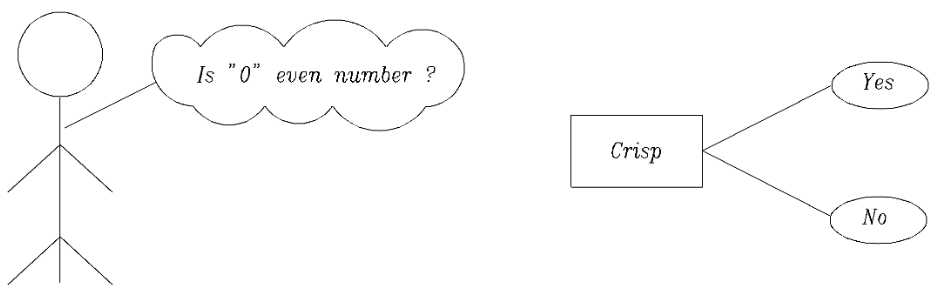

Figure 1-1: Crisp set example

Therefore, the membership function or characteristic function is defined by [5]:

$$\mu_{A(x)} = \begin{cases} 1, & x \in A \\ 0, & Otherwise \end{cases}$$

In this case there are two possible options member belongs to set, member does not belongs to set then the membership function is denoted by ‘1’otherwise ‘0’.

A Fuzzy set is a super set of a crisp set, in a crisp set we only discuss whether the element is in the set or not. While a fuzzy set includes all the elements having degrees of membership. The degree lies between 1 and 0 [2], [4] and [9].

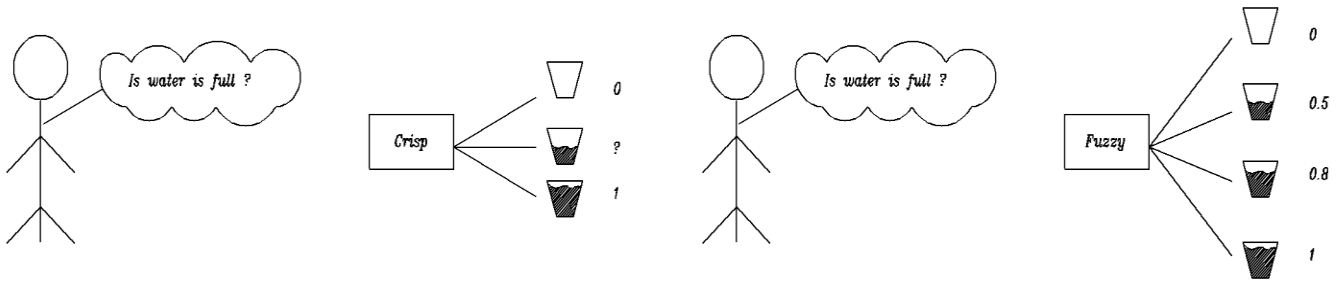

Figure 1-2: Crisp & Fuzzy Set Example

1.3 Objectives of the Thesis

In order to deal with uncertain and complex nature of the market price movements, we will use MCFS technique. For this aim, we will categorize the closing returns of the global market prices return data as fuzzy states. Then we will determine the Markov transition matrix of the fuzzy states. Furthermore, we will estimate the preceding expected market price return.

Most financial instruments and other global market price are traded after hours, although in far smaller volumes. Therefore, the closing return price of any financial instruments is often different from its after-hours price. So, the estimation regarding to the return rate of price market has been one of the most popular topic for the financial area due to its high return. Therefore, several studies were carried out under the rapid fluctuation of the global market price to predict the direction of the price movement [8].

The aim of this research is to determine the probabilistic transition matrix of the closing returns of market prices using Markov Chain Model of Fuzzy States (MCFS). We use this method in order to consider the information while system moves between the extreme values of the states which are more realistic and flexible than the classical Markov Chain techniques. With this study, we can say that the use of MCFS can give valuable information to the investors about the market price movements [11].

In order to achieve the desired goal, this study was conducted on a sample of gold return prices.

1.4 Obstacles Encountered

The problem arose when substituting some values of R_t in the relationship mentioned in the references [11], [12]and [26]

$$\tilde{S}_i = \frac{(i+1)(0.225 - R_t)}{0.225}$$

The result of the compensation is greater than one, and this contradicts the definition of the membership function, and after reviewing the relationship and using the mathematical sequence in finding the value of $\tilde{S}_i$, it became clear that the attached relationship in Reference [13] is false, as the correct relationship is as follows:

$$\tilde{S}_i = \frac{(i+1)(0.225) - R_t}{0.225}$$

What confirmed the validity of our conclusion, that the error in the reference [13] has been amended.

1.5 Structure of the Thesis

The present thesis is organized into five chapters including the present one: chapter one that includes motivation, objectives and literature review with clarifying the differences between the classic concept and the fuzzy concept.

Chapter 2 presents some of the fundamental definitions and mathematical theory of Markov chains and fuzzy set theory.

In Chapter 3, Closing retraction R_t was studied as a fuzzy concept, and several types of fuzzy numbers were applied to R_t: triangle, trapezoidal, parabolic, and K-Trapezoidal-Triangular fuzzy numbers. We create a Markov model with classic and fuzzy states and compare between them. At the end, we estimate the expected return price in specific months as an example.

Chapter 4 deals with the main results.

In Chapter 5, we give our conclusions and comments.

Chapter 2

Basic Concepts

2.1 Stochastic Processes

Stochastic modelling is an interesting and challenging area of probability and statistics.

We begin with a formal definition. A stochastic process is a family of random variables $\{X_\theta\}$, indexed by a parameter θ, where θ belongs to some index setΘ, where Θ represents time.

If Θ is representing specific time points, we have a discrete time stochastic process and we replace the general subscript θ by n. Hence, we talk about the discrete time process $\{X_n\}$

In general, for a discrete time process, the random variable X_n depends on earlier values of the process, $X_{n-1}, X_{n-2}, \ldots$ similarly, in continuous time, $X(t)$ generally depends on values $X(u)$ for $u < t$. Therefore, we are often interested in conditional distributions of the form $P_r(X_{t_k} | X_{t_{k-1}} X_{t_{k-2}}, \ldots, X_{t_1})$for some set of times $t_k > t_{k-1} > \cdots > t_1$ In general; this conditional distribution will depend upon values of $X_{t_{k-1}} X_{t_{k-2}}, \ldots, X_{t_1}$[24].

2.2 Markov Processes and Markov Chains

"A Markov process $\{X_t\}$ is a stochastic process with the property that, given the value of X_t, the values of X_s for $s > t$ are not influenced by the values of X_u for $u < t$. In words, the probability of any particular future behavior of the process, when its current state is known exactly, is not altered by additional knowledge concerning its past behavior. A discrete-time Markov chain is a Markov process whose state space is a finite or countable set, and whose (time) index set is $T = \{0, 1, 2, \ldots\}$

In formal terms, the Markov property is that

$$P_r\{X_{n+1} = j \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i\}$$

$$= P_r\left\{X_{n+1} = j \mid X_n = i\right\}$$

For all time points n and all states $i_0, \ldots, i_{n-1}, i, j$.

It is frequently convenient to label the set of all possible values of the random variable of the Markov chain by the nonnegative integers $(0, 1, 2, \ldots)$ which we will do unless the contrary is explicitly stated, and it is customary to say X_n starts from state i if $X_n = i$.

The probability of X_{n+1} in state j given that X_n is in state i is called the one-step transition probability and is denoted by $P_{ij}^{n,n+1}$. That is,

$$P_{ij}^{n,n+1} = P_r\{X_{n+1} = j \mid X_n = i\}.$$

The notation emphasizes that in general the transition probabilities are functions not only of the initial and final states, but also of the time of transition as well. When the one-step transition probabilities are independent of the time variable n, we say that the Markov chain has stationary transition probabilities (homogeneous); we limit our discussion to this case. Thus $P_{ij}^{n,n+1} = P_{ij}$ is independent of n, and P_{ij} is the conditional probability that the state value undergoes a transition from i to j

In one trial. It is customary to arrange these numbers in a matrix P the countable square array

$$P = \left\| \begin{matrix} P_{00} & P_{01} & P_{02} & P_{03} & \cdots \\ P_{10} & P_{11} & P_{12} & P_{13} & \cdots \\ P_{20} & P_{21} & P_{22} & P_{23} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & P_{i2} & P_{i3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} \right\|$$

And refer to $P = \| P_{ij} \|$ as the Markov matrix or *transition probability matrix* of the process.

The i^{th} row of P, for $i = 0, 1, ...$, is the probability distribution of the values of X_{n+1} under the condition that $X_n = i$. If the number of states is finite, then P is a finite square matrix whose order (the number of rows) is equal to the number of states. Clearly, the quantities P_{ij} satisfy the conditions" [24].

$P_{ij} \geq 0$ For $i, j = 0, 1, 2, ...$

$$\sum_{j=0}^{\infty} P_{ij} = 1 \text{ for } i = 0, 1, 2, \ldots$$

2.2 The Long Run Behavior of Markov Chains

2.2.1 Regular Transition Probability Matrices

"Suppose that a transition probability matrix P = ‖ P_{ij} ‖ on a finite number of states labeled 0, 1, …., N has the property that when raised to some power k, the matrix P^k has all of its elements strictly positive, Such a transition probability matrix, or the corresponding Markov chain, is called regular. The most important fact concerning a regular Markov chain is the existence of a limiting probability distribution $\pi = (\pi_0,\ \pi_1, \dots, \pi_N)$, where $\pi_j > 0$ for j = 0, 1, …., N and $\sum_j \pi_j = 1$, and this distribution is independent of the initial state. Formally, for a regular transition probability matrix P = ‖ P_{ij} ‖ we have the convergence

$\lim_{n \to x} P_{ij}^{(n)} = \pi_j > 0$ For j = 0, 1, …., N,

Or, in terms of the Markov chain$\{X_n\}$,

$\lim_{n \to \infty} \Pr\left\{X_n = j \,\middle|\, X_0 = i\right\} = \pi_j > 0$ for j = 0, 1, …., N.

The convergence means that, in the long run ($n \to \infty$), the probability of finding the Markov chain in state j is approximately π_j no matter in which state the chain began at time 0 ″[24].

Definition: ergodic chain A Markov chain is called an ergodic chain if it is possible to go from every state to every state (not necessarily in one move) [24].

2.2.2 Estimating of Transition Probability

The transition probabilities represent parameters that have to be estimated, and one of the suitable forms for time series data is Markov method, when data is available for each time from the chain. These data represent cases that the studied phenomenon lies within, and there is a movement or transition of this phenomenon between the cases. For example, let it be 21 cases that we can estimate the transition matrix through [32].

The Maximum Likelihood Method (M.L.E) will be used, and it's one of the important methods in statistical theory when estimating parameters because it gives the estimation that have the maximum probability [32].

The transition of this type of data is from i_t to j_{t+1} suppose that there is a sample consists of samples of observations as Markov chain and suppose that the number $n_i(0)$ represents the observed elements in case (i) at the time $(t = 0)$. That the observed elements points towards a chain of cases at time $(t = 0, 1, 2, \dots, T)$, so,the Markov process in stability is [32]:

$$P_r(x_0, x_1, \dots, x_r) = P_r(x_o) \prod_t p_r\,(X_t | X_{t-1}) \qquad (1)$$

Let $n_{ij}(t)$ represents the number of observed elements for each $(X_t = j, X_{t-1} = i)$.

$$n_{ij} = \sum_t n_{ij}\,(t) \qquad (2)$$

The probability in the previous equation can be written in a proportional form as:

$$P_r(x_0, x_1, \dots, x_T | n) = P_r(x_o) \prod_{i.j} (P_{ij}^{nij}) \qquad (3)$$

As Codman and Anderson explained that the form n_{ij}represents a set of sufficient statistics and the distribution of $n_{ij}(t)$ can be obtained considering $n_i(t-1) = \sum_j n_{ij}(t)$ of observations that are Multinomial distributed by a probability (P_{ij}). The probability density function (pdf) for $[n_{ij}(t)]$ is:

$$p_r\left(n_{11(t)}, n_{12(t)}, \dots \middle| n'(0) p_{11}, \dots\right) =$$

$$\prod_t \left[\prod_i \left\{n_i(t-1)! \,/\, \prod_j \{n_{ij}(t)!\} \prod_j (P_{ij}^{nij(t)})\right\}\right] = \left[\prod_{t.i} \big(n_i(t-1)! \,/\, \prod_j n_{ij}(t)\,!\big)\right] \left[\prod_{i.j} P_{ij}^{nij}\right] \qquad (4)$$

So that:

$n'(0) = [n_1(0), n_2(0), \dots, n_r(0)]$ Representsvector element that represents the numbers

In cases at time *(t=0)*

If $n_{ij}(t)$ for each $(t,\ i,\ j)$ is known, we can get an estimation of the stable transition probabilities with confirmation of the condition:

$$\sum P_{ij} = 1,\ j = 1, 2, \dots, n \qquad (5)$$

By taking the logarithm for each sides of the previous equation (4), and by taking the above condition (5) into consideration by using Langrangian Multipliers, the Langrang function will be obtained as follows:

$$\log P_r(X_1,\ \ X_2, \dots X_T | n) = \sum_i \lambda_i \left(\sum_j P_{ij} - 1\right) \qquad (6)$$

$$= \log\Big(\prod_{i.j} P_{ij}^{n_{ij}}\Big) - \sum_i \lambda_i \Big(\sum_j P_{ij} - 1\Big) + c$$

Where c is a constant that represents:

$$c = \log \prod_{i.j} \{n_i(t-1)! \Big/ \prod_j n_{ij}\,(t)!\}$$

To obtain the maximum likelihood function for the previous equation (6), the partial derivative is taken according to P_{ij}, λ_i respectively:

$$\partial \,/\, \partial P_{ij}[\sum_i \sum_j n_{ij} \log P_{ij} - \sum_i \lambda_i (\sum_j P_{ij} - 1)] = n_{ij}/P_{ij} - \lambda_i = 0 \qquad (7)$$

$$\partial \,/\, \partial \lambda_j[\sum_i \sum_j n_{ij} \log P_{ij} - \sum_i \lambda_i (\sum_j P_{ij} - 1)] = \sum_j P_{ij} - 1 = 0 \qquad (8)$$

And from equation (6) equal to zero we get:

$$n_{ij} = \widehat{\lambda}\ \widehat{P_{ij}} \qquad (9)$$

And so,

$$\sum_j n_{ij} = \widehat{\lambda_i} \sum_j \widehat{P_{ij}}$$

As,

$$\sum_j \hat{P}_{ij} = 1$$

Then,

$$\sum_j n_{ij} = \widehat{\lambda_i} \qquad (10)$$

By substitution (10) in the above equation (9), we obtain:

$$\widehat{P_{ij}} = n_{ij} \,/\, \sum_j n_{ij} \geq 0 \qquad (11)$$

2.3 Fuzzy Logic

The beginning of fuzzy logic is to allow truth values to be any number in the interval [0, 1]. If p is an atomic proposition, then we let tv(p) denote the truth of p, tv(p) ∈ [0, 1] for any proposition in fuzzy logic. If tv (p) = 1, then p is absolutely true, if tv (p) = 0 then p is absolutely false and if, for example, tv (p) = 0.60 then the truth of p is 0.60 [2].

Basically, a set is defined as a collection of objects, which share certain characteristics. A classical set is a collection of distinct objects. The classical set is defined in such a way that the universe of discourse is split into two groups: members and nonmembers. Consider an object x in a crisp set A. This object is either a member or a nonmember of the given set A. In case of crisp sets, no partial membership exists. This binary issue of membership can be represented mathematically by the indicator function,

$$\mu_{A(x)} = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

Where $\mu_{A(x)}$ is the membership in set A for element x in the universe. The membership concept represents mapping from an element x in universe X to one of the two elements in universe Y (either to element 0 or 1). There exist a function-theoretic set called value set for any set A defined on universe X, based on the mapping of characteristic function, the whole set is assigned a membership value 1, and the null set is assigned a membership value 0 [2], [4] and [9].

A fuzzy set may be viewed as an extension and generalization of the basic concepts of crisp sets. An important property of a fuzzy set is that it allows partial membership, i.e. between 0 and 1. Zadeh [30] extended the notion of valuation set {0,1} (definitely in / definitely out) to the interval of real values (degree of membership) between 1 and 0 denoted as [0,1]. 0.0 represents absolutely false and 1.0 represents absolutely truth.

The fuzzy set $\tilde{A}$ in the universe of discourse Ω is defined as a set of ordered pairs $(x,\mu_{\tilde{A}(x)})$, i.e. $\tilde{A}=\{(x,\mu_{\tilde{A}(x)}) \mid x\in \Omega\}$ where $\mu_{\tilde{A}(x)}$ is the degree of membership of x in fuzzy $\tilde{A}$ and it indicates the degree that x belongs to $\tilde{A}$. Clearly $\mu\, \tilde{A}(x) \in [0,1]$.

The height of a fuzzy set $\tilde{A}$ is defined as [4]:

$$\boldsymbol{ht}(\tilde{A}) = \text{Sup}\ \{\tilde{A}(x) \mid x \text{ in } X\} \qquad (12).$$

A membership function defines the fuzziness in a fuzzy set irrespective of the elements in the set, which are discrete or continuous. The membership functions are generally represented in graphical form. There exist certain limitations for the shapes used to represent graphical form of membership function. The rule that describe fuzziness graphically are also fuzzy.

Standard shapes of the membership functions are maintained over the years. The membership function defines all the information contained in a fuzzy set; hence it is important to discuss the various features of the membership functions. For a fuzzy set A a membership function, denoted by $\mu_{\tilde{A}(x)}$ maps R to the membership space N, i.e. $\mu_{\tilde{A}(x)}$:R→N. The membership value ranges in the interval [0,1], i.e. the range of the membership function is a subset of the non-negative real numbers whose supremum is finite. The main basic features involved in characterizing membership function are the following:

(i) The core of a membership function for some fuzzy set A is defined as that region of universe that is characterized by complete membership in the set A. The core has elements x of the universe such that $\mu_{\tilde{A}(x)}$= 1. The core of a fuzzy set may be an empty set.

(ii) The support of a membership function for a fuzzy set A is defined as that region of universe that is characterized by a nonzero membership in the set A. The support comprises elements x of the universe such that $\mu_{\tilde{A}(x)}$>0.

(iii) The boundary of a membership function for a fuzzy set A is defined as the region of universe that contains a nonzero but not a complete membership. In other words the boundary comprises those elements x of the universe such that 0< $\mu_{A(x)}$<1.

(iv) The boundary elements are those which possess partial membership in the fuzzy set A [2],[4]and [9].

2.4 Fuzzy Numbers

A fuzzy number is a convex (the lines connecting them lie completely inside the shape) and normal fuzzy set of the real line ℝ that is,

(i) $\exists x_0 \in \mathbb{R}$ With $\mu_{\breve{A}(x_0)} = 1$

(ii) $\mu_{\tilde{A}(x)}$ Is a piecewise continuous, and its membership function is defined as:

$$\mu_{\tilde{A}(x)} = \begin{cases} f_A(x), & a_1 \leq x \leq a_2 \\ 1 \quad , & x = a_2 \\ g_A(x) \ , & a_2 \leq x \leq a_3 \\ 0 \quad , & otherwise \end{cases}$$

Where $0 \leq \mu_{\tilde{A}(x)} \leq 1$ and $a_1, a_2\ and\ a_3 \in \mathbb{R}$ such that, $a_1 \leq a_2 \leq a_3$ and the two functions $f_A, g_A: \mathbb{R} \longrightarrow [0,1]$ are called the sides of the fuzzy numbers [4].

2.5 Special Type of Fuzzy Numbers

2.5.1 Triangle Fuzzy Numbers

"A triangular fuzzy number $\widetilde{N}$, (Figure 2.1), is defined by three numbers $a_1 < a_2 < a_3$ where:

(1) $\widetilde{N}(x) = 1\ at\ x = a_2$

(2) The graph of $\widetilde{N}(x)$ on $[a_1, a_2]$ is straight line from $(a_1, 0)$ to$(a_2, 1)$ also; on $[a_2, a_3]$ the graph is a straight line from $(a_2, 1)$ to $(a_3, 0)$.

(3) $\widetilde{N}(x) = 0$ For $x \leq a_1\ or\ x \geq a_3$.

We write $\widetilde{N} = (a_1/a_2/a_3)$ for triangular fuzzy number$\widetilde{N}$. If at least of the graphs described above is not a straight line (curve), then $\widetilde{N}$ is called triangular shaped fuzzy number and we write$\widetilde{N} \approx (a_1/a_2/a_3)$.

A fuzzy number $\widetilde{N} = (a_1, a_2, a_3)$ said to be triangle fuzzy number (TFN) if its membership function is defined as below "[4] and [18]:

$$\mu_{\tilde{A}(x)} = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} , & a_1 \le x \le a_2 \\ 1 , & x = a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} , & a_2 \le x \le a_3 \\ 0 , & otherwise \end{cases}$$

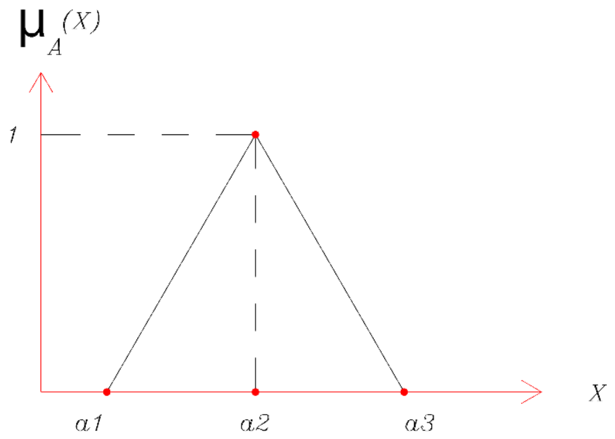

Figure 2-1: Triangular fuzzy number A= (a_1, a_2, a_3)

2.5.2 Trapezoidal Fuzzy Numbers

"A trapezoidal fuzzy number $\tilde{N}$ is defined by four numbers $a_1 < a_2 < a_3 < a_4$ where:

(1) $\tilde{N}$(x)=1 on $[a_2 , a_3]$

(2) The graph of $\tilde{N}$(x) on $[a_1 , a_2]$ is a straight line from $(a_1 , 0)$ to $(a_2, 1)$ and on $[a_3 , a_4]$ the graph is also a straight line from $(a_3 , 1)$ to $(a_4 , 0)$.

(3) $\widetilde{N}(x)=0$ for $x \leq a_1$ or $x \geq a_4$

We write $\widetilde{N} = (a_1, a_2, a_3, a_4)$ for trapezoidal fuzzy number $\widetilde{N}$. Its membership function is defined as below (Figure 2-2) "[4] and [18]:

$$\mu_{\tilde{A}(x)} = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \dfrac{a_3-x}{a_3-a_2}, & a_3 \leq x \leq a_4 \\ 0, & otherwise \end{cases}$$

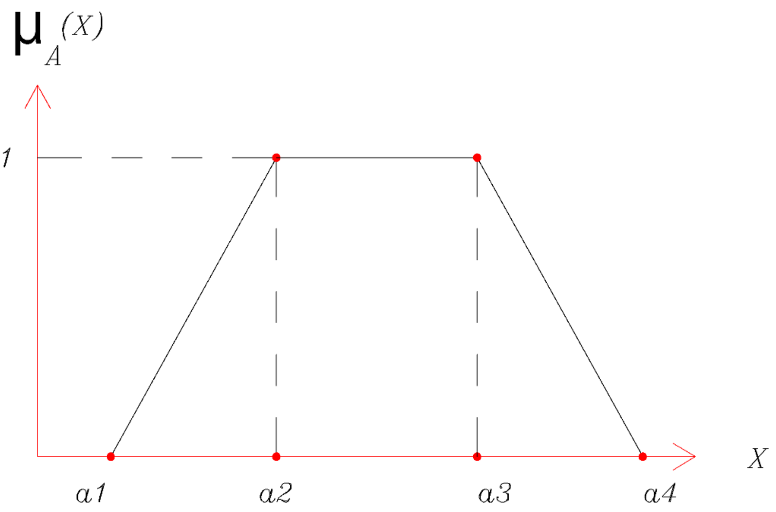

Figure 2-2: Trapezoidal fuzzy number A= (a_1, a_2, a_3, a_4)

2.5.3 Parabolic Fuzzy Numbers

A fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ said to be parabolic fuzzy number (PFN) (Figure 2.3) if its membership function is defined as below [2]:

$$\mu_{\tilde{A}(x)} = \begin{cases} (\frac{x-a_1}{a_2-a_1})^2, & a_1 \leq x \leq a_2 \\ 1, & x = a_2 \\ (\frac{a_3-x}{a_3-a_2})^2, & a_2 \leq x \leq a_3 \\ 0, & otherwise \end{cases}$$

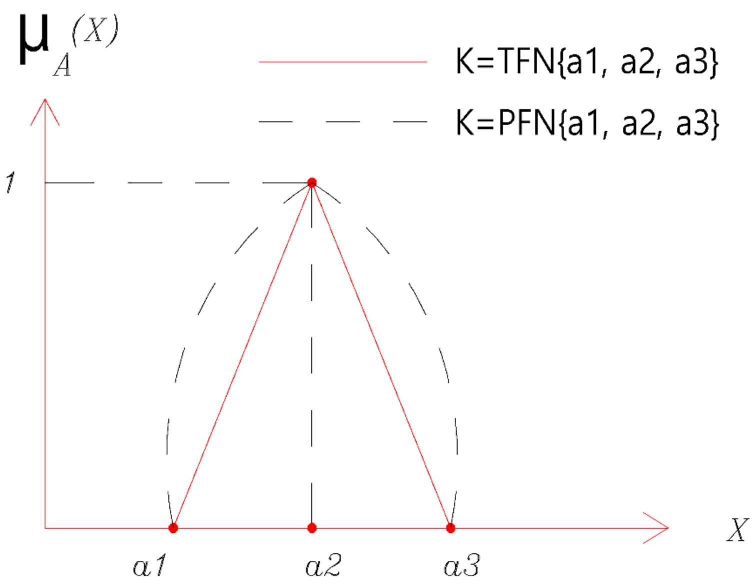

Figure 2-3: Parabolic fuzzy number A= (a_1, a_2, a_3)

2.5.4 K-Trapezoidal-Triangular Fuzzy Numbers

"Fuzzy number that is determined by n real numbers $a_1, a_2, \dots, a_{2k+3}$ and c_i,$0 \le c_i \le 1, i = 1, 2, \dots, k$, where n is a multiple of 3, K is the number of trapezoidal and $a_1 < a_2 < \dots < a_{2k+3}$ (Figure 2-4). Denote it by $\bar{A}_{(c_1,\dots,c_k)} = (a_1, a_2, \dots, a_{2k+3})$ and whose membership function is given by "[17]:

$$\mu_{\bar{A}_{(c_1,\dots,c_k)}}(x) = \begin{cases} 0 \quad , & x \le a_1 \\ \dfrac{c_1(x - a_1)}{(a_2 - a_1)}, & a_1 \le x \le a_2 \\ \cdot & \\ \cdot & \\ \dfrac{-c_1(x - a_{2k+2})}{a_{2k+2} - a_{2k+1}}, & a_{2k+1} \le x \le a_{2k+2} \\ 0 \quad , & x \ge a_{2k+2} \end{cases}$$

1- $\mu_{\bar{A}_{(c_1,\dots,c_k)}}(x)$ is a line from $(a_1, 0)$ to (a_2, c_1) and from (a_{2k+1}, c_1) to $(a_{2k+2}, 0)$

2- $\mu_{\bar{A}_{(c_1,\dots,c_k)}}(x)$ Is a line from (a_{i+1}, c_i) to $(a_{i+2}, c_{i+1}) i = 1, 2, \dots k - 1$.

3- $\mu_{\bar{A}_{(c_1,\dots,c_k)}}(x)$ Is a line from (a_{k+3+i}, c_{k-i}) to $(a_{k+4+i}, c_{k-i-1}) i = 0, 1, 2, \dots k - 2$.

4- $\mu_{\bar{A}_{(c_1,\dots,c_k)}}(x)$ is a line from (a_{k+1}, c_k) to $(a_{k+2}, 1)$ and from $(a_{k+2}, 1)$ to (a_{k+3}, c_k)

5- $\mu_{\bar{A}_{(c_1,\ldots,c_k)}}(x)$=0 for $x \leq a_1, x \geq a_{2k+3}$

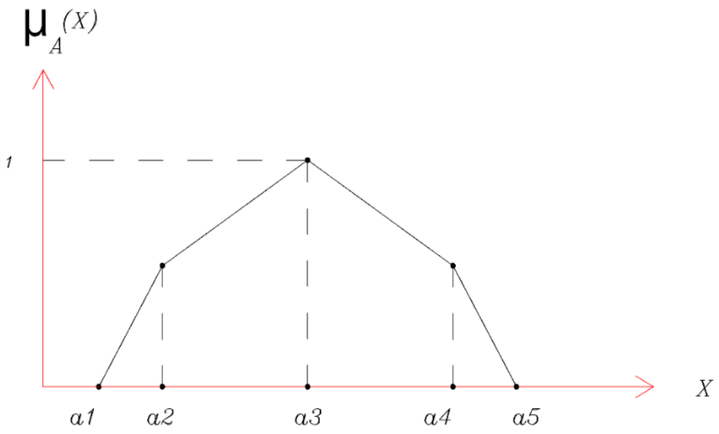

Figure 2-4: K-Trapezoidal-Triangular fuzzy number A= $(a_1, a_2, a_3, a_4, a_5)$

We called such a fuzzy number a k-trapezoidal-triangular fuzzy number. If k=0, a k-trapezoidal-triangular fuzzy number is a triangular fuzzy number.

2.6 Statistical and Mathematical Concepts

1) The midpoint of an interval AB is the point that divides AB in the ratio 1:1.

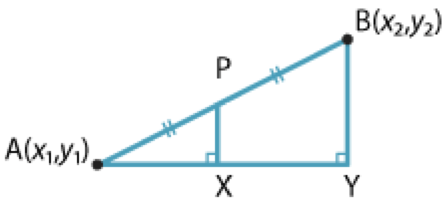

Figure 2-5: Midpoint of an interval.

Assume that the point A has coordinates (x_1, y_1) and the point B has coordinates (x_2, y_2). It is easy to see, using either congruence or similarity that the midpoint P of AB is [22]:

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \quad (13)$$

2) Let X be a discrete random value with set of possible values D and pmf (Probability mass function) $P(x)$. The expected value or mean value of X, denoted by $E(X)$ or μ_X or just μ, is [7]:

$$E(X) = \mu_X = \sum_{x \in D} x.P(x) \quad (14)$$

3) The Mean Absolute Error (MAE) is the average of all absolute errors. The formula is [1]:

$$MAE = \frac{1}{n}\sum_{i=1}^{n}|x_i - x| \quad (15)$$

Where n = the number of errors, i.e. $|x_i - x| > 0$

$|x_i - x|$= the absolute error (the difference between the actual value and the expected value).

4) The Greatest Integer Function is denoted by y =$\lfloor x \rfloor$. For all real numbers, x, the greatest integer function returns the largest integer less than or equal to x. In essence, it rounds down a real number to the nearest integer [4].

Chapter 3

Markov Chains with Fuzzy States and Methodology

3.1. Basics

3.1.1. Probability of Fuzzy Events

Let (Ω, Λ, P) be the standard probabilistic space, where Ω denotes the sample space, Λ the σ – algebra on Ω and P a probability measure. A fuzzy set $\tilde{A}$ on Ω is called a fuzzy event. Let $\mu_{\tilde{A}}(\omega),\ \omega \in \Omega, \mu_{\tilde{A}}(\omega): \Omega \to [0,1]$ be the membership function of the fuzzy event $\widetilde{A}$. Then the probability of the fuzzy event$\tilde{A}$ is defined using the integral of Lebesgue–Stieltjesby Zadeh [30] as the expected value of the membership function of the fuzzy event with respect to the probability distribution P (uncountable set):

$$\mathrm{P}(\tilde{A}) = \int_{\Omega} \mu_{\tilde{A}}(\omega)\, dP = E(\mu_{\tilde{A}}) \qquad (16)$$

And in a countable set is:

$$P(\tilde{A}) = \sum_{\Omega} \mu_{\tilde{A}}(\omega) P_{\omega}. \qquad (17)$$

The conditional probability of the fuzzy event $\tilde{A}$ given the fuzzy event $\tilde{B}$ is [20]:

$$\mathrm{P}(\tilde{A} \mid \tilde{B}) = \frac{P(\tilde{A}, \widetilde{B})}{P(\tilde{B})}, \quad P(\tilde{B}) > 0 \qquad (18)$$

Additionally, the product of two fuzzy events $\tilde{A}$and $\tilde{B}$is [20]:

$$\tilde{A}.\tilde{B} \leftrightarrow \mu_{\tilde{A}.\tilde{B}} = \tilde{A}.\tilde{B}. \qquad (19)$$

3.1.2. Markov Chains Probabilities

Let X_t be the state of the system at time t. we consider a finite state Markov chain in which the transition probability matrix is

$\boldsymbol{P} = [P_{ij}] =$

$\forall\, i, j \in \{0, 1, \dots, N\}$

State	0	1	...	N
0	P_{00}	P_{01}	...	P_{0N}
1	P_{10}	P_{11}	...	P_{1N}
⋮	⋮	⋮	⋱	⋮
N	P_{N0}	P_{N1}	...	P_{NN}

(20)

Where P_{ij} represents the transition probability from state i to state j of one step,

$P_{ij} \geq 0, \forall i, j$:

$$P_{ij} = P\,\{X_{t+1} = j | X_t = i\} = P\,\{X_1 = j | X_0 = i\} \qquad (21)$$

With $\sum_{j=0}^{N} P_{ij} = 1$. Moreover, the probability transition matrix of k steps (where P_{ij}^{k} is the transition probability from state i to state j, in k steps) is:

$$\boldsymbol{P^{k}} = \left[P_{ij}^{k}\right] P_{ij}^{k} \geq 0, \forall i, j \in \{0, 1, \dots, N\} \quad (22)$$

$$P_{\boldsymbol{ij}}^{\boldsymbol{k}} = P\{X_{t+k} = j | X_t = i\} = P\{X_k = j | X_0 = i\} (23)$$

The matrix $\boldsymbol{P}^{k}$ is calculated by the Chapman-Kolmogorov equation with:

$$\boldsymbol{P}^{k} = (P)^{k} . \quad (24)$$

Finally, let $P_i = P\{X_0 = i\}, \forall i \in \{0, 1, \dots, N\}$, i.e. P_i are the initial state probabilities of the Markov chain, with $\sum_{i=0}^{N} P_i = 1$.

3.2. Markov Chains Probabilities with Fuzzy States

On the state of the system, we define a fuzzy partition, i.e. a set of fuzzy states $\{\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n\}$, such that each fuzzy subset $\tilde{A}_i, i \in \{1, 2, \dots, n\}$ represents a fuzzy state or event in the initial Markov chain.

Definition 1. *The probability of fuzzy initial state* $P\left(\tilde{A}_i\right) = P\left(X_0 = \tilde{A}_i\right)$is defined by using the probability of fuzzy event (in a finite situation) calculated by (17):

$$P\left(\tilde{A}_i\right) = P\left\{\tilde{X}_0 = \tilde{A}_i\right\} = \sum_{s=0}^{N} P\{X_0 = s\} \mu_{\tilde{A}_i}(s) = \sum_{s=0}^{N} P_s \mu_{\tilde{A}_i}(s). \quad (25)$$

Proposition 1. *The conditional probability of the fuzzy state* $\tilde{A}_i$, given the initial state *m*, with $j \in \{1, \dots, n\}$ and $m \in \{0, \dots, N\}$, is:

$$P(\tilde{A}_j | m) = P\{\tilde{X}_1 = \tilde{A}_j | X_0 = m\} = \sum_{s=0}^{N} P_{ms} \mu_{\tilde{A}_j} \qquad (26)$$

and it represents the transition probability of fuzzy state (of one step)

Definition 2. *The Markov chain of the fuzzy final state* is defined by the matrix:

$\bar{\boldsymbol{P}} =$

State	$\tilde{A}_1$	$\tilde{A}_2$	...	$\tilde{A}_n$
0	$P(\tilde{A}_1 \| 0)$	$P(\tilde{A}_2 \| 0)$	...	$P(\tilde{A}_n \| 0)$
1	$P(\tilde{A}_1 \| 1)$	$P(\tilde{A}_2 \| 1)$	...	$P(\tilde{A}_n \| 1)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
N	$P(\tilde{A}_1 \| N)$	$P(\tilde{A}_2 \| N)$	...	$P(\tilde{A}_n \| N)$

(27)

This matrix gives the transition probability of the initial state $m, (m \in \{0, \dots, N\})$ to the fuzzy final state $\tilde{A}_j, (j \in \{1, \dots, n\})$.

Proposition 2. The conditional probability of the fuzzy event $\widetilde{A}_j$ given the fuzzy event $\widetilde{A}_j, i, j \in \{1, \dots, n\}$ is a function of a linear combination of probabilities $P(\widetilde{A}_j|m)$, of the form:

$$P(\tilde{A}_j|\tilde{A}_i) = P\{\tilde{X}_1 = \tilde{A}_j|\tilde{X}_0 = \tilde{A}_i\} = \sum_{m=0}^{N} P(\tilde{A}_j|m)\frac{P_m\mu_{\tilde{A}_i}(m)}{P(\tilde{A}_i)} \quad (28)$$

and represents the probability of transition from the fuzzy initial state (of one step).

Definition 3. *The Markov chain with fuzzy initial state and fuzzy final state* is defined by the matrix:

$\tilde{P} =$

State	$\tilde{A}_1$	$\tilde{A}_2$	...	$\tilde{A}_n$
$\tilde{A}_1$	$P(\tilde{A}_1\|\tilde{A}_1)$	$P(\tilde{A}_2\|\tilde{A}_1)$	...	$P(\tilde{A}_n\|\tilde{A}_1)$
$\tilde{A}_2$	$P(\tilde{A}_1\|\tilde{A}_2)$	$P(\tilde{A}_2\|\tilde{A}_2)$	...	$P(\tilde{A}_n\|\tilde{A}_2)$
⋮	⋮	⋮	⋱	⋮
$\tilde{A}_n$	$P(\tilde{A}_1\|\tilde{A}_n)$	$P(\tilde{A}_2\|\tilde{A}_n)$	...	$P(\tilde{A}_n\|\tilde{A}_n)$

(29)

The matrix gives the transition probabilities from fuzzy initial state $\tilde{A}_i$ to fuzzy final state $\widetilde{A}_j, i, j \in \{1, \dots, n\}$.

The matrices $\overline{\boldsymbol{P}}$and $\widetilde{\boldsymbol{P}}$ are stochastic, given that the sum of each of their rows is 1:

1. The m^th row of the matrix $\bar{P}$ are the terms: $P\ (\tilde{A}_1|m)$, $P\ (\tilde{A}_2|m)\ldots$, $P\ (\tilde{A}_n|m)$so:

$$\sum_{j=1}^{n} P\ (\tilde{A}_j|m) = \sum_{j=1}^{n}\sum_{s=0}^{N} P_m\mu_{\tilde{A}_j}(s) = \sum_{s=0}^{N} P_{ms}\left(\sum_{j=1}^{N}\mu_{\tilde{A}_j}(s)\right) = 1$$

(30)

2. The i^th row of the matrix $\widetilde{\boldsymbol{P}}$ are the terms: $P(\tilde{A}_1|\tilde{A}_i)P(\tilde{A}_2|\tilde{A}_i),\ldots,P(\tilde{A}_n|\tilde{A}_i)$. So,

$$\sum_{j=1}^{N} P(\tilde{A}_j\left|\tilde{A}_i\right.) = \sum_{j=1}^{n}\sum_{m=0}^{N} P(\tilde{A}_j|m)\frac{P_m\mu_{\tilde{A}_i}(m)}{P(\tilde{A}_i)}$$

$$= \frac{1}{P(\tilde{A}_i)}\left(\sum_{m=0}^{N} P_m\mu_{\tilde{A}_i}(m)\left(\sum_{j=1}^{n} PP(\tilde{A}_j|m)\right)\right) = \frac{\sum_{m=0}^{N} P_m\mu_{\tilde{A}_i}(m)}{P(\tilde{A}_i)} = 1 \quad (31)$$

Lastly, we desire to indicate that the computations of the matrices $\overline{\boldsymbol{P}}$and $\widetilde{\boldsymbol{P}}$ are significantly simplified by using matrix calculus. To this end, we define matrices $\boldsymbol{Q}$ and $\boldsymbol{S}$

$$
\boldsymbol{Q} = \begin{bmatrix} \mu_{\tilde{A}_1}(0) & \mu_{\tilde{A}_2}(0) & \cdots & \mu_{\tilde{A}_n}(0) \\ \mu_{\tilde{A}_1}(1) & \mu_{\tilde{A}_2}(1) & \cdots & \mu_{\tilde{A}_n}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{\tilde{A}_1}(N) & \mu_{\tilde{A}_2}(N) & \cdots & \mu_{\tilde{A}_n}(N) \end{bmatrix} \qquad (32)
$$

$$
\boldsymbol{S} = \begin{bmatrix} \frac{P_0\mu_{\tilde{A}_1}(0)}{P(\tilde{A}_1)} & \frac{P_1\mu_{\tilde{A}_1}(1)}{P(\tilde{A}_1)} & \cdots & \frac{P_N\mu_{\tilde{A}_1}(N)}{P(\tilde{A}_1)} \\ \frac{P_0\mu_{\tilde{A}_2}(0)}{P(\tilde{A}_2)} & \frac{P_1\mu_{\tilde{A}_2}(1)}{P(\tilde{A}_2)} & \cdots & \frac{P_N\mu_{\tilde{A}_2}(N)}{P(\tilde{A}_2)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{P_0\mu_{\tilde{A}_n}(0)}{P(\tilde{A}_n)} & \frac{P_1\mu_{\tilde{A}_n}(1)}{P(\tilde{A}_n)} & \cdots & \frac{P_N\mu_n(N)}{P(\tilde{A}_n)} \end{bmatrix} \qquad (33)
$$

The matrix $\boldsymbol{Q}$ contains the membership function values of the fuzzy partition which determines the fuzzy states of the system $\{\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n\}$.

With matrices $\boldsymbol{Q}$ and $\boldsymbol{S}$, we derive [20]:

$$\bar{P} = \mathrm{PQ} \qquad (34)$$

$$\tilde{P} = \mathrm{S}\bar{P} = SPQ \qquad (35)$$

3.3 Fuzzy States

In this part we will try to get the fuzzy concept for the Gold return prices:

First, return prices for Gold closing prices will be transformed into 21 states, from the high loss (S_{-10}) to the positive high return (S_{10}), that each state have the same length (k) that is, all of them have the same chance of occurrence.

Then, suppose that the return price for a month of a certain year ($\mathrm{R_t} = 0.765$).

So, the position of that ($\mathrm{R_t}$) is tried to be obtained from 21 states by using the following relation:($\mathrm{I} = \frac{\mathrm{R_t}}{\mathrm{k}}$) (36)

where, I is the position and k is the length of each state.

Assume k=0.225, then: $\mathrm{I} = \frac{0.765}{0.225} = 3.4$.

According to the previous result, the position of ($\mathrm{R_t}$) is difficult to be determined certainly and clearly.

But, we can conclude that ($\mathrm{R_t}$) is lying between the third and the forth state, which moves away from the third state to be closer to the forth state by 0.4. (i.e. it gets closer to the third state by 0.6).

So, the membership degree of ($\mathrm{R_t}$) in $\tilde{\mathrm{S}}_3 = 4 - 3.4 = 0.6$

And the membership degree of ($\mathrm{R_t}$) in $\tilde{\mathrm{S}}_4 = 1 - 0.6 = 0.4$

In general, $i = \left\lfloor \frac{R_t}{k} \right\rfloor$, (i= the greatest integer of $(\frac{R_t}{k})$) (37)

$\tilde{S}_i = (i + 1) - \frac{R_t}{k}$ (38)

$\tilde{S}_{i+1} = 1 - \tilde{S}_i$ (39)

Let, k = 0.225 as a special case. Then, $\tilde{S}_i$ will be:

$$\tilde{S}_i = (i + 1) - \frac{R_t}{0.225}$$

$$\tilde{S}_i = \frac{0.225(i+1) - R_t}{0.225} \quad (40)$$

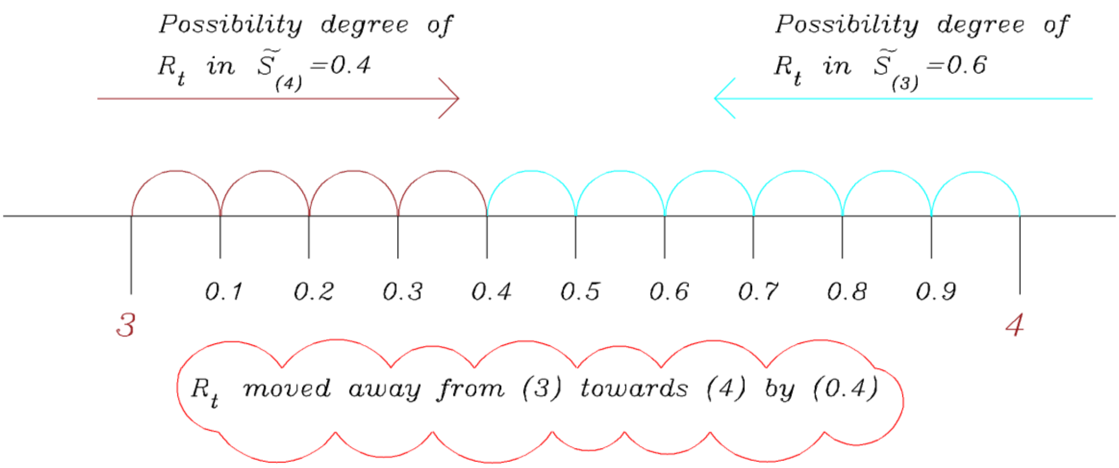

Figure 3-1: R_t as a fuzzy concept

After monthly percentage changes of the market price are transformed into 21 fuzzy states from the high loss S_{-10} to high return S_{10}

Next, we will apply different types of fuzzy numbers to the return price R_t:

1) **Triangle Fuzzy Numbers**:

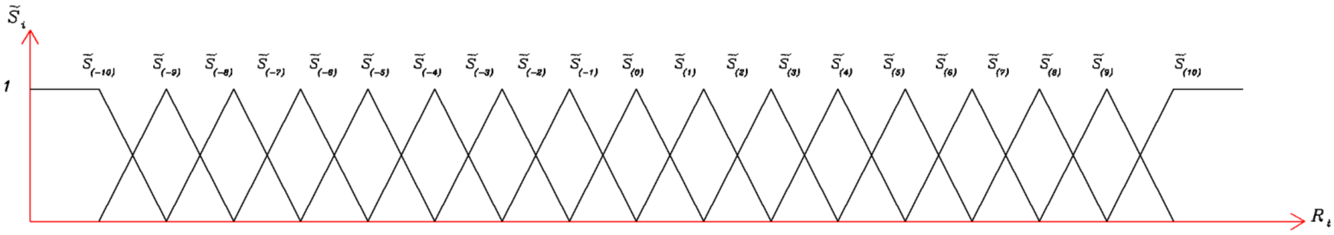

Figure 3-2Triangle fuzzy states for R_t

By using the equation of a straight line given two points on it

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \qquad (41)$$

the points are :

(x_1, y_1)= $\big((i+1)(0.225), 0\big)$

(x_2, y_2) = $(\,(i)(0.225), 1)$

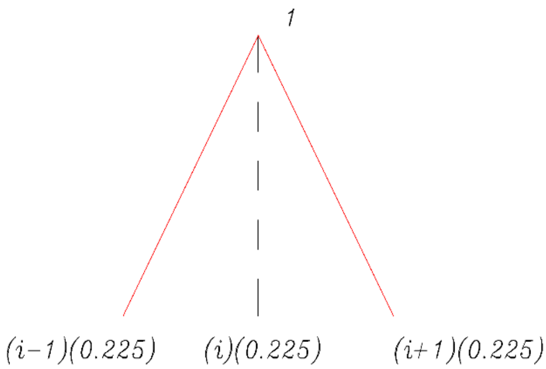

Figure 3-3: Coordinates triangle fuzzy state

$$\frac{y-0}{x-(i+1)(0.225)} = \frac{1}{-0.225}$$

$$y = \frac{x-(i+1)(0.225)}{-0.225} = \frac{(i+1)(0.225)-R_t}{0.225},$$

“Triangle relationship”

which is similar to Eq (40)

Thus, the correct modification of the mathematical relationship that was presented incorrectly in the references [11], [12]and [26] as:

$\frac{(i+1)(0.225-R_t)}{0.225}$ (Before modification)

Was reached, and this was verified from the reference [13] that modified it exactly as we reached it.

$\frac{(i+1)(0.225)-R_t}{0.225}$ (After modification)

2) Trapezoidal Fuzzy Numbers:

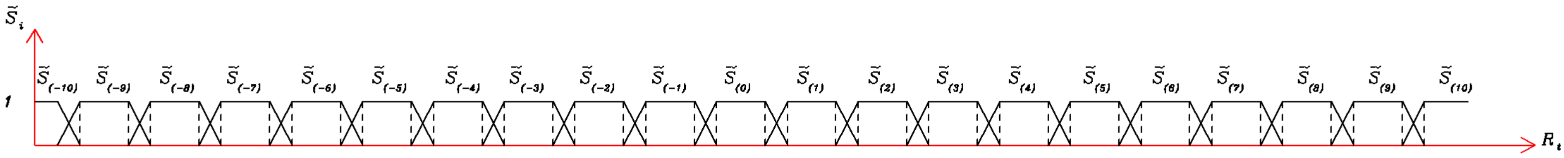

Figure 3-4 Trapezoidal fuzzy states for R_t

Which means → if $R_t \in (a_2\,, a_3) \rightarrow \tilde{S}_i = 1$

But → $i = \left\lfloor \frac{R_t}{0.15} \right\rfloor$, where $\mathbf{0.15} = \frac{(\,a_4 - a_1\,)}{3}$.

Since, $R_t \in (a_2, a_3) \rightarrow$ there is more than one value of R_t gives integer number (i)

$(i = \frac{R_t}{0.15})$

$\Longrightarrow R_{1t}, R_{2t}, \dots, R_{nt}, \dots = i * 0.15$

"For the same i" contraction

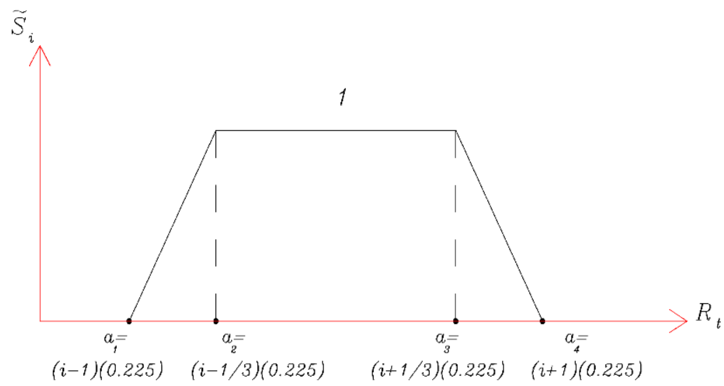

Figure 3-5 : Coordinates trapezoidal fuzzy state

So, we will deal with single value of (a_2, a_3)

Take: <u>Midpoint of an interval</u>.

The midpoint of an interval AB, is the point that divides AB in the ratio 1:1.

Assume that the point A has coordinates $(x_1, y_1)$and the point B has coordinates (x_2, y_2), the midpoint P of AB is [22]:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The midpoint of $(a_2 \ , \ a_3)$ "average" $\rightarrow \frac{a_2+a_3}{2}$

$$\frac{\left(i - \frac{1}{3}\right)(0.225) + (i + \frac{1}{3})(0.225)}{2}$$

$$= \frac{(0.225)i - \frac{1}{3}(0.225) + (0.225)i + \frac{1}{3}(0.225)}{2}$$

$$= \frac{2(0.225)i}{2}$$

$$= (0.225)i$$

$= a_2$ In the triangle fuzzy number.

3) Parabola "Triangle Shape" Fuzzy Numbers:

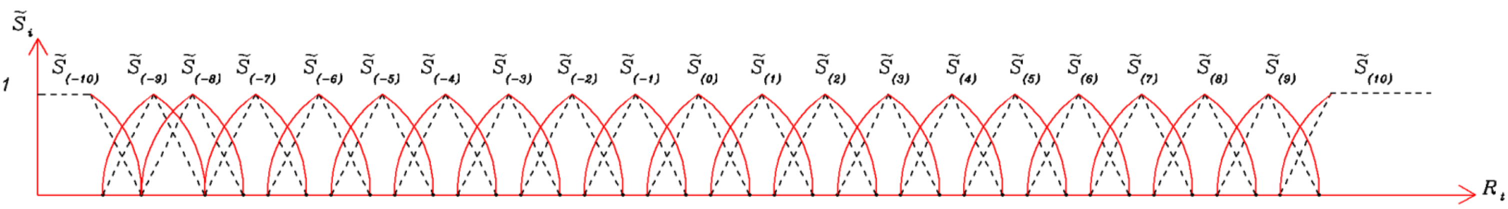

Figure 3-6 : parabola fuzzy states for R_t

By referring to the parabolic membership function [2], we note that the relationship is the square of the triangle membership function

$$\tilde{S}_i = [\frac{(i+1)(0.225)-R_t}{0.225}]^2 \qquad (42)$$

And since through the first paragraph in Section 3.2 we have concluded that the principle of the fuzziness in the estimation of the states for the return in prices is identical to the relationship of the triangle, this means that the use of this type of fuzzy number gives less accurate results.

Example:

Take $R_t = 0.5175$

$$i = \frac{0.5175}{0.225} = 2.3 \rightarrow i = \lfloor 2.3 \rfloor$$

$$= 2$$

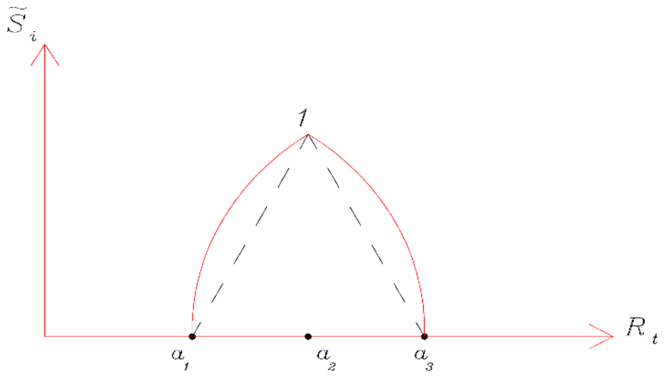

Figure 3-7: Coordinates parabola fuzzy state

In general using Eq (40)

$$\tilde{S}_i = (i+1) - \frac{R_t}{0.225} = "in\ triangle"$$

$$R_t = 0.5175,\ \tilde{S}_2 = 3 - \frac{0.5175}{0.225} = 0.7$$

$$error = 0.7 - 0.7$$

In parabola:

For $R_t = 0.5175$

$S_2 = (0.7)^2 = 0.49$

$error = 0.7 - 0.49 = 0.21$

By comparing the result between the fuzzy concepts that was explained in Section 3.3 (using Eq (40)) with the result that emerged from the application of the Parabola relationship (using Eq (42)).

we note that the Parabola does not give an accurate result as the error rate is equal to 0. 21 in the example that was taken.

4) **K-Triangle-Trapezoidal Fuzzy Numbers:**

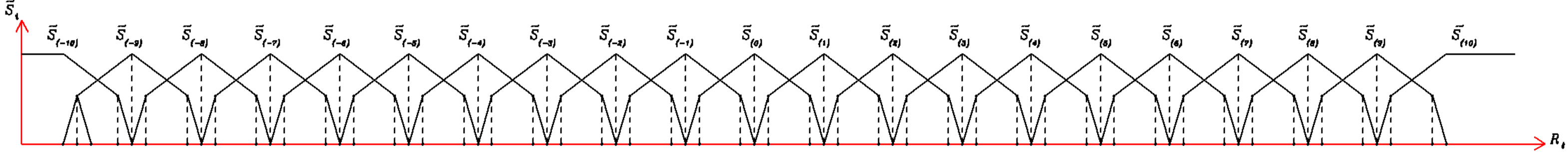

Figure 3-8 K-triangle -trapezoidal fuzzy states for R_t

For line L_1:

$(x_1 , y_1) = ((i + 1)(0.225) , 0)$

$(x_2 , y_2) = (\left(i + \frac{1}{2}\right)(0.225) , 0.501)$

Using (41):

$$\frac{y - 0}{x - a_5} = \frac{0.501 - 0}{a_4 - a_5}$$

$$y = (0.501)\left(\frac{a_5 - x}{a_5 - a_4}\right) \qquad , \qquad a_4 \le R_t \le a_5$$

$$\rightarrow \tilde{S}_i = (0.501)\left(\frac{(i + 1)(0.225) - R_t}{0.1125}\right)$$

For example:

Let $R_t = 0.5175$

$$i = \left\lfloor\frac{0.5175}{0.225}\right\rfloor = 2$$

$$\frac{R_t}{0.225} = 2.3 \rightarrow \tilde{S}_2 = 3 - 2.3 = 0.7$$

$$R_t \in (\left(i + \tfrac{1}{2}\right)(0.225), (i + 1)(0.225))$$

So, $\tilde{S}_2 \mid_{R_t=0.5175} = \left(\frac{(3)(0.225) - 0.5175}{0.1125}\right)(0.501) = 0.7014 \approx 0.7$

Error = 0.014

Take $C_1 = 0.5001$

$\rightarrow \tilde{S}_2 \mid_{R_t=0.5175} = 0.70014$, Closer to 0.7

$error = 0.0014$

So, the closer the C_1 value to 0.5, the more accurate the result.

The best result when $C_1 = 0.5$

$$\tilde{S}_i = \left(\frac{(i+1)(0.225) - R_t}{0.1125}\right)(0.5)$$

$$\tilde{S}_i = \left(\frac{(i+1)(0.225) - R_t}{0.225}\right) \rightarrow \quad triangle\ fuzzy\ number.$$

For line L_2 :

$(x_1, y_1) = (a_3, 1)$

$(x_2, y_2) = (a_4, 0.501)$

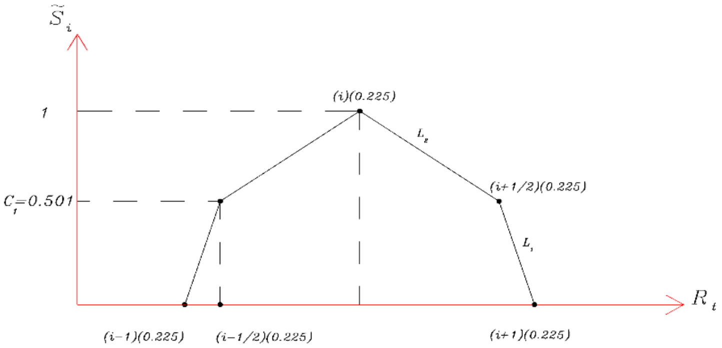

Figure 3-9: Coordinates K-triangle-trapezoidal fuzzy state

Using Eq (41):

$$\rightarrow \frac{y-1}{x-a_3} = \frac{-0.499}{a_4 - a_3} \quad , \qquad a_3 < R_t < a_4$$

$$y - 1 = (x - a_3)\left(\frac{-0.499}{0.1125}\right)$$

$$\tilde{S}_i = 1 - (0.499).\left(\frac{R_t - i(0.225)}{0.1125}\right)$$

Let $R_t = 0.51 \in (a_3, a_4)$

$$i = 2 \rightarrow \frac{0.51}{0.225} = 2.27 \Longrightarrow \tilde{S}_2 = 3 - 2.27 = 0.733$$

$$\tilde{S}_2 = 1 - (0.499).\left(\frac{0.51 - 2(0.225)}{0.1125}\right) = 0.734$$

$$error = 0.001$$

3.4 The Sample and Methodology

3.4.1 The Sample

The study includes monthly data between January 2010 and July 2020. The monthly weighted average of the gold price received from Istanbul.

The return R_t were calculated as monthly percentage of the gold price

$R_t = ((P_t - P_{t-1})/P_{t-1}) * 100\%$, [26] and [6] where t denotes the sessions $(t = 2, 3, \ldots, 127)$.

The average return μ_R is approximately 0.46%, when the standard deviation is 3.69%.

For the given period which is 8 times higher than expected return.

3.4.2 Building the Markov Chain Model (MC)

Closing returns of the gold price are transformed into 21discrete categorical states from high loss S_{-10} to the positive high return S_{10} according to functions below. For this aim, we defined the k integer numbers which is based on R_t as $k-1 < \frac{R_t+0.12\%}{0.24\%} \leq k$ where: $-2.28\% < R_t \leq 2.28\%$

And the k-th Markov states for $k \in \{-9, \ldots, 9\}$ as following [12]:

$$S_{10} = \begin{cases} 1, & R_t > 2.28\% \\ 0, & Otherwise \end{cases}$$

$$S_k = \begin{cases} 1, & (2k-1)0.12\% < R_t \leq (2k+1)0.12\% \\ 0, & Otherwise \end{cases}$$

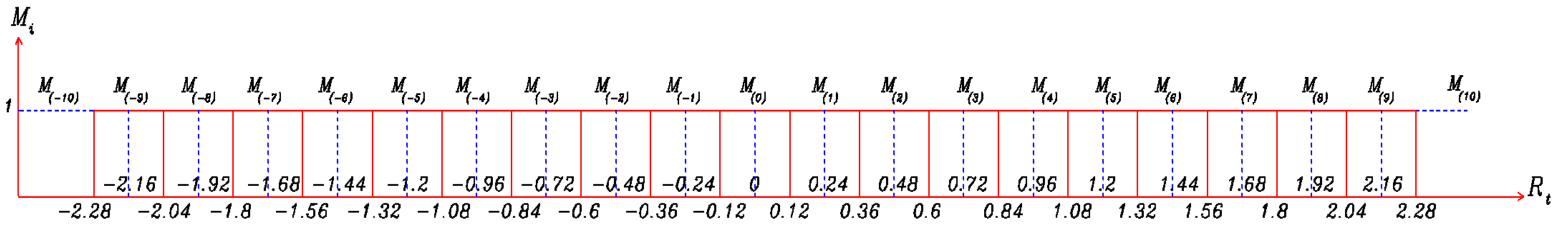

Figure 3-10: Discrete categorical states

Table 3-1: Transformed States of the monthly Closing Returns for Some months

Date	R_t	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
01-2010	-2.1236%	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
02-2010	1.8027%	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***
03-2010	2.9511%	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***
07-2010	1.4602%	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***
10-2010	2.1288%	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***
11-2010	1.5839%	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***
12-2010	-2.3708%	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
01-2011	0.7966%	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
05-2011	1.0708%	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***
08-2011	1.2019%	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***

As it is shown in Table (3-1), we transformed 126 closing returns of Gold price to the defined 21 discrete states. Then, we calculated all transitions numbers of states from the present session to the next session for the period considered. We also used conditional probabilities of the Markov chain to obtain one-step

Transition probability matrix, which is shown in Table (3.4).

If we apply the

previous

relation:

$\overline{\mathrm{P}} = \mathrm{PQ}$

$\widetilde{\mathrm{P}} = \mathrm{S}\overline{\mathrm{P}} = \mathrm{SPQ}$ [20] With the aim of linking the two models: MC & MCFC

The following results will be obtained:

Table 3-2: Matrix S

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 3-3: Matrix Q

1	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***	***0***
0	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***1***

P= one step transition probability matrix Estimated by Maximum Likelihood Method (M.L.E)

$$\widehat{P_{ij}} = n_{ij} \,/\, \sum_j n_{ij} \geq 0 \qquad (11)$$

P=One Step Conditional Probability Matrix

Table 3-4: One-step conditional probability matrix (P)

	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
S_{-10}	0.2581	0.0644	0.0644	0.0323	0.0644	0.0323	0.0323	0.0000	0.0000	0.0323	0.0644	0.0000	0.0323	0.0323	0.0000	0.0000	0.0968	0.0323	0.0323	0.0323	0.0968
S_{-9}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
S_{-8}	0.6667	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000
S_{-7}	0.6000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2000
S_{-6}	0.5714	0.0000	0.0000	0.0000	0.0000	0.1429	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2857	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_{-5}	0.3333	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.3334	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_{-4}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000
S_{-3}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_{-2}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
S_{-1}	0.0000	0.0000	0.0000	0.0000	0.4000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2000	0.0000	0.0000	0.0000	0.2000	0.2000
S_0	0.0000	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3334
S_1	0.5000	0.0000	0.0000	0.0000	0.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.8000
S_3	0.3333	0.0000	0.0000	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3334
S_4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
S_5	0.6667	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.6667	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3333
S_7	0.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000

S_8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
S_9	0.0000	0.0000	0.0000	0.0000	0.2000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2000	0.0000	0.0000	0.2000	0.0000	0.0000	0.4000
S_{10}	0.2000	0.0000	0.0286	0.0571	0.0286	0.0000	0.0000	0.0000	0.0286	0.0286	0.0286	0.0571	0.0000	0.0286	0.0286	0.0286	0.0000	0.0000	0.0000	0.0857	0.3713

Based on the results of S&Q matrices, which appeared in the form of the identity matrix, this is an indication of the validity of the relationship used to connect the two models

$$\widetilde{P} = S\overline{P} = SPQ$$

Lastly, we calculated the probability of next month closing return state $P(x_i)$ by multiplying present return state and the conditional transition matrix. We calculated the expected closing return $\widehat{R}_t$ for next day [12] by using Eq (14):

$$\widehat{R}_t = \sum_i x_i \, P(x_i)$$

Where x_i denotes the middle points of the states for $i = -9, \ldots, 9$ and the boundaries of the states for $i = -10, 10$.

3.4.3 Building the Markov Chain model with Fuzzy States (MCFS)

Closing returns of the Gold price are transformed into 21 fuzzy states from the high loss $\tilde{S}_{-10}$ to high return $\tilde{S}_{10}$, which is shown in Figure (3-11).

We use triangular fuzzy numbers to obtain the membership degree of R_t to the fuzzy states.

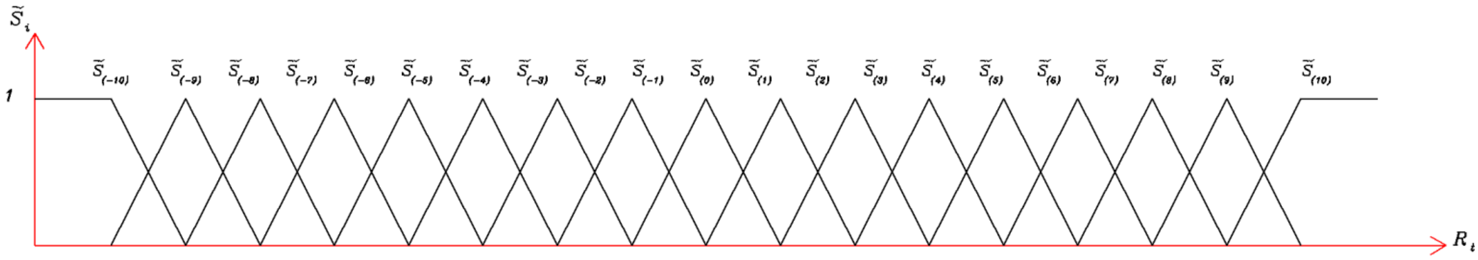

Figure 3-11: fuzzy state for gold return price

Let $[\![x]\!]$ be the greatest integer function and R_t is the closing return of Gold price. We define fuzzy state components of the returns $\tilde{S}_i$ and $\tilde{S}_{i+1}$ as follows:

If $-2.25\% < R_t < 2.25\%$ then $i = \left[\!\left[\frac{R_t}{0.225}\right]\!\right]$ and $\tilde{S}_i = \frac{(i+1)(0.225)-R_t}{0.225}$,

$$\tilde{S}_{i+1} = 1 - \tilde{S}_i$$

If $R_t \leq -2.25\%$ or $R_t \geq 2.25\%$ then $\tilde{S}_{-10} = 1$ or $\tilde{S}_{10} = 1$respectively [13].

Table 3-5: Transformed fuzzy states of the closing returns for some months

Date	R_t	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
01-2010	-2.12%	.44	.56	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
02-2010	1.80%	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.99	.01	0
03-2010	2.95%	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
07-2010	1.46%	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.51	.49	0	0	0
10-2010	2.13%	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.54	.46
11-2010	1.58%	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.96	.04	0	0
12-2010	-2.37%	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
01-2011	0.80%	0	0	0	0	0	0	0	0	0	0	0	0	0	.46	.54	0	0	0	0	0	0
05-2011	1.07%	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.24	.76	0	0	0	0	0
08-2011	1.2%	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.33	.67	0	0	0	0	0

Therefore, R_t numbers are transformed to the triangular fuzzy numbers for the time considered

Within this framework, MCFS that depend on the fuzzy set theory gives more precision and realistic description to the problems than the MC.

Then we calculated the fuzzy transition probability matrix by using the conditional probability of the fuzzy state $\tilde{S}_j$ given the fuzzy state $\widetilde{S}_i$. With this way, we replaced fuzzy transition probability with crisp transition probability.

Then we obtained the probabilistic transition matrix of the fuzzy states by using

$$\widetilde{\mathrm{P}} = \mathrm{S}\overline{\mathrm{P}} = \mathrm{SPQ}$$

First, monthly percentage changes of the gold price are transformed into 21 fuzzy states from the high loss S_{-10} to high return S_{10}.

We classified the states as triangular fuzzy number.

If $-2.25 < R_t < 2.25\%$ then $i = \left\lfloor \frac{R_t}{0.225} \right\rfloor$

$$\tilde{S}_i = \frac{(i+1)(0.225) - R_t}{0.225}, \qquad \tilde{S}_{i+1} = 1 - \tilde{S}_i$$

If $R_t \leq -2.25$ or $R_t \geq 2.25$ then: $\tilde{S}_{-10} = 1\ , \tilde{S}_{10} = 1$

To obtain the Markov chain of the fuzzy states:

$$\tilde{P} = SPQ$$

P = transition probability matrix

Table 3-6: Transition probability matrix.

	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
S_{-10}	0.2581	0.0644	0.0644	0.0323	0.0644	0.0323	0.0323	0.0000	0.0000	0.0323	0.0644	0.0000	0.0323	0.0323	0.0000	0.0000	0.0968	0.0323	0.0323	0.0323	0.0968
S_{-9}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
S_{-8}	0.6667	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000
S_{-7}	0.6000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2000
S_{-6}	0.5714	0.0000	0.0000	0.0000	0.0000	0.1429	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2857	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_{-5}	0.3333	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.3334	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_{-4}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000
S_{-3}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_{-2}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
S_{-1}	0.0000	0.0000	0.0000	0.0000	0.4000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2000	0.0000	0.0000	0.0000	0.2000	0.2000
S_0	0.0000	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3334
S_1	0.5000	0.0000	0.0000	0.0000	0.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.8000
S_3	0.3333	0.0000	0.0000	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3334
S_4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

S_5	**0.6667**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.3333**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**
S_6	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.6667**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.3333**
S_7	**0.5000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.5000**
S_8	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**1.0000**
S_9	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.2000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.2000**	**0.0000**	**0.0000**	**0.2000**	**0.0000**	**0.0000**	**0.4000**
S_{10}	**0.2000**	**0.0000**	**0.0286**	**0.0571**	**0.0286**	**0.0000**	**0.0000**	**0.0000**	**0.0286**	**0.0286**	**0.0286**	**0.0571**	**0.0000**	**0.0286**	**0.0286**	**0.0286**	**0.0000**	**0.0000**	**0.0000**	**0.0857**	**0.3713**

The decision maker associates the decisions with 21 fuzzy states denoted $\{\tilde{S}_{-10}, \tilde{S}_{-9}, \dots \tilde{S}_{10}\}$ which correspond to ‘‘membership degree of closing return price R_t’’

These fuzzy values or fuzzy states $\{\tilde{S}_{-10}, \tilde{S}_{-9}, \dots \tilde{S}_{10}\}$ make up a fuzzy partition of *X*, where *X* is the set of status of the system i.e. $\{\tilde{S}_{-10}, \tilde{S}_{-9}, \dots \tilde{S}_{10}\}$

(*X*0 indicates the status: “state position “i=0” relative to the value of closing return priceR_t” . . .).

We use the standard notation to denote a fuzzy number of the discrete set:

$$\tilde{S} = \{(i, \mu_{\tilde{S}_i})/i = -10, \dots, 10\}$$

So, the fuzzy values $\{\tilde{S}_{-10}, \tilde{S}_{-9}, \dots \tilde{S}_{10}\}$ are written:

$$\tilde{S}_{-10} = \{(-10, \mu_{\tilde{S}_{-10}}(-10)), (-9, \mu_{\tilde{S}_{-10}}(-9)), \dots, (10, \mu_{\tilde{S}_{-10}}(10))\}$$

$$\tilde{S}_{-9} = \{(-10, \mu_{\tilde{S}_{-9}}(-10)), (-9, \mu_{\tilde{S}_{-9}}(-9)), \dots, (10, \mu_{\tilde{S}_{-9}}(10))\}$$

$$\vdots$$

$$\tilde{S}_{10} = \{(-10, \mu_{\tilde{S}_{10}}(-10)), (-9, \mu_{\tilde{S}_{10}}(-9)), \dots, (10, \mu_{\tilde{S}_{10}}(10))\}$$

Where $\mu_{\tilde{S}_{-10}}, \mu_{\tilde{S}_{-9}}, \dots, \mu_{\tilde{S}_{10}}$denote the membership functions of fuzzy states of $\{\tilde{S}_{-10}, \tilde{S}_{-9}, \dots \tilde{S}_{10}\}$ so, $\mu_{\tilde{S}_{-10}}(-10)$ is the degree of possibility that the i[th] position of closing return price R_t with $i = -10$ has of belonging to the fuzzy state $\tilde{S}_{-10}$

$\mu_{\tilde{S}_{-10}}(-9)$ Is the degree of possibility that the i[th] position of closing return price R_t with $i = -9$ has of belonging to the fuzzy state $\tilde{S}_{-10}$

$$\boldsymbol{Q} = \begin{bmatrix} \mu_{\tilde{S}_{-10}}(-10) & \mu_{\tilde{S}_{-9}}(-10) & \cdots & \mu_{\tilde{S}_{10}}(-10) \\ \mu_{\tilde{S}_{-10}}(-9) & \mu_{\tilde{S}_{-9}}(-9) & \cdots & \mu_{\tilde{S}_{10}}(-9) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{\tilde{S}_{-10}}(10) & \mu_{\tilde{S}_{-9}}(10) & \cdots & \mu_{\tilde{S}_{10}}(10) \end{bmatrix}$$

If there are more than one value of R_t that have the same position i, hence, we have a group of finite numbers (n) for the same $\mu_{\tilde{S}_i}(i)$. Where, $\mu_{\tilde{S}_i}(i)$ is a fuzzy number that if we define that values in a group$\bar{A}_i$, the value of:

$$ht\ (\bar{A}) = \bar{A}_i$$

$$ht\ (\bar{A}_i) = Max\left\{\mu_{\tilde{S}_{i1}}(i), \mu_{\tilde{S}_{i2}}(i), \dots, \mu_{\tilde{S}_{in}}(i)\right\} = \mu_{\tilde{S}_i}(i)$$

And since, $\tilde{S}_{i+1} = 1 - \tilde{S}_i$ so, $\mu_{\tilde{S}_{i+1}}(i) = 1 - \mu_{\tilde{S}_i}(i)$

And other values in the same row = 0. “For the same$i$.”

Table 3-7: Matrix Q

	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
S_{-10}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-9}	0	0.1960	0.8040	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-8}	0	0	0.6813	0.3187	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-7}	0	0	0	0.7923	0.2077	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-6}	0	0	0	0	0.8829	0.1171	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-5}	0	0	0	0	0	0.9950	0.0050	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-4}	0	0	0	0	0	0	0.8123	0.1877	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-3}	0	0	0	0	0	0	0	0.7368	0.2632	0	0	0	0	0	0	0	0	0	0	0	0
S_{-2}	0	0	0	0	0	0	0	0	0.7914	0.2086	0	0	0	0	0	0	0	0	0	0	0
S_{-1}	0	0	0	0	0	0	0	0	0	0.9840	0.0160	0	0	0	0	0	0	0	0	0	0
S_0	0	0	0	0	0	0	0	0	0	0	0.6779	0.3221	0	0	0	0	0	0	0	0	0
S_1	0	0	0	0	0	0	0	0	0	0	0	0.8715	0.1285	0	0	0	0	0	0	0	0
S_2	0	0	0	0	0	0	0	0	0	0	0	0	0.6038	0.3962	0	0	0	0	0	0	0
S_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0.7242	0.2758	0	0	0	0	0	0
S_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.2504	0.7496	0	0	0	0	0

S_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.6581	0.3419	0	0	0	0
S_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.5102	0.4898	0	0	0
S_7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.9603	0.0397	0	0
S_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.9880	0.0120	0
S_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.9227	0.0773
S_{10}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

$$
\boldsymbol{S} = \begin{bmatrix} \frac{P_{-10}\mu_{\tilde{S}_{-10}}(-10)}{P(\tilde{S}_{-10})} & \frac{P_{-9}\mu_{\tilde{S}_{-10}}(-9)}{P(\tilde{S}_{-10})} & \cdots & \frac{P_{10}\mu_{\tilde{S}_{-10}}(10)}{P(\tilde{S}_{-10})} \\ \frac{P_{-10}\mu_{\tilde{S}_{-9}}(-10)}{P(\tilde{S}_{-9})} & \frac{P_{-9}\mu_{\tilde{S}_{-9}}(-9)}{P(\tilde{S}_{-9})} & \cdots & \frac{P_{10}\mu_{\tilde{S}_{-9}}(10)}{P(\tilde{S}_{-9})} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{P_{-10}\mu_{\tilde{S}_{10}}(-10)}{P(\tilde{S}_{10})} & \frac{P_{-9}\mu_{\tilde{S}_{10}}(-9)}{P(\tilde{S}_{10})} & \cdots & \frac{P_{10}\mu_{\tilde{S}_{10}}(10)}{P(\tilde{S}_{10})} \end{bmatrix}
$$

We assume that the initial state probabilities P_i are all equal, so pi = 1/21.

With P_i we calculate the fuzzy initial stateprobabilities using Eq. (25)

$$
\Longrightarrow P(\tilde{S}_i) = \sum_{S=-10}^{N} P_s \mu_{\tilde{S}_i}(s).
$$

$P_i =$ The probability of beginning of state (i).

Table 3-8: the probability of fuzzy initial state$P(\tilde{S}_i)$ =P ($X_0 = \tilde{S}_i$)

	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
$P(\tilde{S}_i)$	0.0476	0.0093	0.0707	0.0529	0.0519	0.0530	0.0389	0.0440	0.0502	0.0568	0.0330	0.0568	0.0349	0.0534	0.0251	0.0670	0.0406	0.0691	0.0489	0.0445	0.0513

Table 3-9: Matrix S

	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
S_{-10}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-9}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-8}	0	0.5413	0.4587	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-7}	0	0	0.2869	0.7131	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-6}	0	0	0	0.1904	0.8096	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-5}	0	0	0	0	0.1053	0.8947	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-4}	0	0	0	0	0	0.0061	0.9939	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-3}	0	0	0	0	0	0	0.2030	0.7970	0	0	0	0	0	0	0	0	0	0	0	0	0
S_{-2}	0	0	0	0	0	0	0	0.2496	0.7504	0	0	0	0	0	0	0	0	0	0	0	0
S_{-1}	0	0	0	0	0	0	0	0	0.1749	0.8251	0	0	0	0	0	0	0	0	0	0	0
S_0	0	0	0	0	0	0	0	0	0	0.0230	0.9770	0	0	0	0	0	0	0	0	0	0
S_1	0	0	0	0	0	0	0	0	0	0	0.2699	0.7301	0	0	0	0	0	0	0	0	0
S_2	0	0	0	0	0	0	0	0	0	0	0	0.1755	0.8245	0	0	0	0	0	0	0	0
S_3	0	0	0	0	0	0	0	0	0	0	0	0	0.3536	0.6464	0	0	0	0	0	0	0
S_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0.5241	0.4759	0	0	0	0	0	0
S_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.5325	0.4675	0	0	0	0	0
S_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.4013	0.5987	0	0	0	0

S_7	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0.3378**	**0.6622**	**0**	**0**	**0**
S_8	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0.0386**	**0.9614**	**0**	**0**
S_9	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0.0128**	**0.9872**	**0**
S_{10}	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0.0718**	**0.9282**

The transition probability matrix corresponding to the fuzzy states $\{\tilde{S}_{-10}, \tilde{S}_{-9}, \ldots \tilde{S}_{10}\}$

$\tilde{P} = \left[P\left(\tilde{A}_j / \tilde{A}_i\right)\right]$ Obtained with: $\tilde{P}$ =SPQ, and is:

Table 3-10: Probabilistic transition matrix of the fuzzy states

	$\tilde{S}_{-10}$	$\tilde{S}_{-9}$	$\tilde{S}_{-8}$	$\tilde{S}_{-7}$	$\tilde{S}_{-6}$	$\tilde{S}_{-5}$	$\tilde{S}_{-4}$	$\tilde{S}_{-3}$	$\tilde{S}_{-2}$	$\tilde{S}_{-1}$	$\tilde{S}_{0}$	$\tilde{S}_{1}$	$\tilde{S}_{2}$	$\tilde{S}_{3}$	$\tilde{S}_{4}$	$\tilde{S}_{5}$	$\tilde{S}_{6}$	$\tilde{S}_{7}$	$\tilde{S}_{8}$	$\tilde{S}_{9}$	$\tilde{S}_{10}$
$\tilde{S}_{-10}$	0.2581	0.0126	0.0958	0.0461	0.0637	0.0397	0.0264	0.0061	0	0.0317	0.0442	0.0208	0.0195	0.0361	0.0089	0	0.0494	0.0784	0.0331	0.0301	0.0993
$\tilde{S}_{-9}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\tilde{S}_{-8}$	0.3058	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.1006	0.0523	0	0	0	0.5413
$\tilde{S}_{-7}$	0.6192	0	0	0	0	0	0	0	0	0	0	0	0.0861	0.0565	0	0.0629	0.0327	0	0	0	0.1426
$\tilde{S}_{-6}$	0.5769	0	0	0	0	0.1150	0.0006	0	0	0	0	0	0.1627	0.1067	0	0	0	0	0	0	0.0381
$\tilde{S}_{-5}$	0.3584	0	0	0.2363	0.0619	0.015	0.0001	0	0.236	0.0622	0	0	0.0182	0.0119	0	0	0	0	0	0	0
$\tilde{S}_{-4}$	0.0021	0	0	0.0016	0.0004	0	0.4037	0.0933	0.0016	0.0004	0	0	0	0	0	0	0	0	0	0	0.4969
$\tilde{S}_{-3}$	0	0	0	0	0	0	0.0825	0.019	0	0	0	0	0	0.5772	0.2198	0	0	0	0	0	0.1015
$\tilde{S}_{-2}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0.1808	0.0688	0	0	0	0	0	0.7504
$\tilde{S}_{-1}$	0	0	0	0	0.2914	0.0386	0	0	0	0	0	0	0	0	0	0.1086	0.0564	0	0	0.1523	0.3527
$\tilde{S}_{0}$	0	0	0	0.2581	0.0758	0.0011	0	0.2399	0.0857	0	0	0	0	0	0	0.003	0.0016	0	0	0.0042	0.3306
$\tilde{S}_{1}$	0.3650	0	0	0.0713	0.341	0.0427	0	0.0663	0.0237	0	0	0	0	0	0	0	0	0	0	0	0.09
$\tilde{S}_{2}$	0.0877	0	0	0	0.0775	0.0103	0	0	0	0.1623	0.0026	0	0	0	0	0	0	0	0	0	0.6596
$\tilde{S}_{3}$	0.2155	0	0	0	0	0.2144	0.0011	0	0	0.0696	0.0011	0	0	0	0	0	0	0	0	0	0.4983
$\tilde{S}_{4}$	0.1747	0	0	0	0	0.1738	0.0009	0	0	0	0	0	0	0	0	0	0	0	0	0	0.6506

$\tilde{S}_5$	0.3117	0	0	0	0	0	0	0	0	0	0	0	0.0941	0.0617	0	0	0	0	0	0	0.5325
$\tilde{S}_6$	0.2675	0	0	0	0	0	0	0	0	0.3928	0.0063	0	0.0808	0.0530	0	0	0	0	0	0	0.1996
$\tilde{S}_7$	0.3311	0	0	0	0	0	0	0	0	0.2216	0.0036	0	0	0	0	0	0	0	0	0	0.4437
$\tilde{S}_8$	0.0193	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.9807
$\tilde{S}_9$	0	0	0	0	0.1743	0.0232	0	0	0	0	0	0	0	0	0.0494	0.148	0	0.1896	0.0078	0	0.4077
$\tilde{S}_{10}$	0.1856	0	0.0181	0.0505	0.0471	0.0048	0	0	0.0210	0.0316	0.0184	0.0548	0.0068	0.0192	0.0175	0.0481	0.0091	0.0138	0.0006	0.0734	0.3796

In Table (3-10), the closing returns of gold price are considered as a stochastic process with 21 fuzzy state $\{\tilde{S}_{-10}, \tilde{S}_{-9}, \ldots \tilde{S}_{10}\}$ space with Markovchain structure. The conditional partition degree of passing from state $\tilde{S}_0$ to $\tilde{S}_6$ is $\tilde{P}(\tilde{S}_6|\tilde{S}_0)$ = 0.16%.

Lastly, we calculated next month closing returns partition degrees to the states $P(x_i)$, by multiplying present return partitions and the conditional transition matrix of the fuzzy states. And we calculated the expected closing return $\hat{R}_t$for next month by using Eq (9).

$$\hat{R}_t = \sum_i x_i \, \tilde{P}(x_i).$$

Chapter 4

Main Results

4.1. Explanation

The MCFS model gives information about risky months. When a monthly return basically increased (2.25% or greater), the next months' return also basically increased or decreased with the probability of 37.96 % or 18.56% respectively.

On the other hand, when a monthly return substantially decreased (2.25% or greater), the next month's return also increased or decreased with the probability of 9.93% or 25.81% respectively. This result is not significantly different from that of the MC model in Table 2.As in the MC model gives datum about risky months also. When a monthly return increased (2.25% or greater), the next months' return also increased or decreased with the probability of 37.14 % or 20% respectively. On the other hand, when a monthly return substantially decreased (2.25% or greater), the next month's return also increased or decreased with the probability of 9.63% or 25.81% respectively. This is not sufficient to rely on it to obtain accurate results, so we predicted a closing return price of a particular month and compared the results of both models: MC & MCFS models.

4.2 Estimated closing return $\hat{R}_t$ Using MC model (Matrix P):

If we want to predict the closing return price of month October 2012, then the present month will be September 2012.

First, we calculated the probability of next months' closing return state $P(x_i)$ by multiplying present return state and the conditional transition matrix$\boldsymbol{P}$.

Where the present return price$R_t\ |_{9-2012} = 0.2539$

Table 4-1:Present return state

So, after the present return multiplied by probability transition matrix with MC (P), the next month closing return state $P(x_i)$ will be obtained:

Date	$\boldsymbol{R_t}$	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
09-2012	0.2539	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*

Table 4-2: Probability of next month closing return state

	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
$P(x_i) =$	0.5	0	0	0	0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 4-3: Estimated closing return ($\hat{R}_t$) with MC model for October, 2012

	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
$P(x_i)$	0.5	0	0	0	0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_i	-2.40	-2.16	-1.92	-1.68	-1.44	-1.20	-0.96	-0.72	-0.48	-0.24	0.00	0.24	0.48	0.72	0.96	1.20	1.44	1.68	1.92	2.16	2.40

Then the Eq (9) is applied:

$$\hat{R}_t = \sum x_i \, P(x_i)$$

Where $x_i$denotes the middle points of the discrete categorical states for i=−9,.., 9and the boundaries of the states for i=−10,10.

We get $\hat{R}_t = -1.92$ %

In table (4-3) we have shown the probability (Prediction) of closing return $P(x_i)$ on October, 2012 with MC model and we calculated the expected return $(\hat{R}_t)$

4.3 Estimated closing return $\hat{\mathrm{R}}_{\mathrm{t}}$ Using MCFS model (Matrix $\tilde{\mathbf{p}}$):

First, we calculated the probability of next months' closing return state $\tilde{\mathrm{p}}(\mathrm{x_i})$ by multiplying present return state and the conditional transition matrix $\tilde{\mathbf{p}}$

Where the present return price $R_t \mid_{9-2021} = 0.2539$

Table 4-4: Present return state

Date	R_t	$\tilde{S}_{-10}$	$\tilde{S}_{-9}$	$\tilde{S}_{-8}$	$\tilde{S}_{-7}$	$\tilde{S}_{-6}$	$\tilde{S}_{-5}$	$\tilde{S}_{-4}$	$\tilde{S}_{-3}$	$\tilde{S}_{-2}$	$\tilde{S}_{-1}$	$\tilde{S}_{0}$	$\tilde{S}_{1}$	$\tilde{S}_{2}$	$\tilde{S}_{3}$	$\tilde{S}_{4}$	$\tilde{S}_{5}$	$\tilde{S}_{6}$	$\tilde{S}_{7}$	$\tilde{S}_{8}$	$\tilde{S}_{9}$	$\tilde{S}_{10}$
09-2012	0.2539	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***.87***	***.13***	***0***	***0***	***0***	***0***	***0***	***0***	***0***	***0***

So, after the present return multiplied by probability transition matrix with MCFS ($\tilde{P}$), the next month closing return state $\tilde{P}(x_i)$ will be obtained:

Table 4-5: Probability of next month closing return state

	$\tilde{S}_{-10}$	$\tilde{S}_{-9}$	$\tilde{S}_{-8}$	$\tilde{S}_{-7}$	$\tilde{S}_{-6}$	$\tilde{S}_{-5}$	$\tilde{S}_{-4}$	$\tilde{S}_{-3}$	$\tilde{S}_{-2}$	$\tilde{S}_{-1}$	$\tilde{S}_{0}$	$\tilde{S}_{1}$	$\tilde{S}_{2}$	$\tilde{S}_{3}$	$\tilde{S}_{4}$	$\tilde{S}_{5}$	$\tilde{S}_{6}$	$\tilde{S}_{7}$	$\tilde{S}_{8}$	$\tilde{S}_{9}$	$\tilde{S}_{10}$
$\tilde{P}(x_i) =$	***0.587***	***0.013***	***0.096***	***0.108***	***0.370***	***0.078***	***0.026***	***0.064***	***0.021***	***0.053***	***0.045***	***0.021***	***0.020***	***0.036***	***0.009***	***0***	***0.049***	***0.078***	***0.033***	***0.030***	***0.263***

Table 4-6: Estimated closing return with MCFS model for October 2012

	$\tilde{S}_{-10}$	$\tilde{S}_{-9}$	$\tilde{S}_{-8}$	$\tilde{S}_{-7}$	$\tilde{S}_{-6}$	$\tilde{S}_{-5}$	$\tilde{S}_{-4}$	$\tilde{S}_{-3}$	$\tilde{S}_{-2}$	$\tilde{S}_{-1}$	$\tilde{S}_{0}$	$\tilde{S}_{1}$	$\tilde{S}_{2}$	$\tilde{S}_{3}$	$\tilde{S}_{4}$	$\tilde{S}_{5}$	$\tilde{S}_{6}$	$\tilde{S}_{7}$	$\tilde{S}_{8}$	$\tilde{S}_{9}$	$\tilde{S}_{10}$
$P(x_i)$	0.587	0.013	0.096	0.108	0.370	0.078	0.026	0.064	0.021	0.053	0.045	0.021	0.020	0.036	0.009	0	0.049	0.078	0.033	0.030	0.263
x_i	-2.25	-2.03	-1.80	-1.58	-1.35	-1.13	-0.90	-0.86	-0.45	-0.23	0.00	0.23	0.45	0.68	0.90	1.13	1.35	1.58	1.80	2.03	2.25

Then, the equation (9) is applied:

$$\hat{R}_t = \sum x_i P(x_i)$$

Where $x_i$denotes the middle points of the fuzzy states for $i = -9, \dots, 9$ and the boundaries of the states for $= -10, \dots, 10$

We get $\hat{R}_t = -0.9802\%$ and the actual value of closing return $\boldsymbol{R_t}$**=-1,259% on October, 2012**

In table (4-6) we have shown the probability (Prediction) of closing return $\tilde{P}(x_i)$ on October, 2012 with MCFS model and we calculated the expected return $(\hat{R}_t)$

Table 4-7: Estimated R_t for some months

Date	$\hat{R}_t$ ***with MC model***	$\hat{R}_t$ ***with MCFS model***	**Actual** (R_t)	$\lvert e_i \rvert$**with MC model**	$\lvert e_i \rvert$**with MCFS model**
October ,2012	-1.92%	-0.98%	-1.26%	0.66%	0.28%
October , 2019	0.72%	-0.52%	-1.53%	2.25%	1.01%
November,2010	1.20%	0.88%	1.58%	0.38%	0.70%
May ,2020	1.20%	0.89%	1.06%	0.16%	0.17%
February,2018	0%	-0.38%	-0.62%	0.62%	0.24%
MAE	0.81%	0.48%			

In Table 4-7, we have shown some estimation results for some months, which are chosen randomly. And we used the mean absolute error MAE to measure how our prediction is close to the eventual outcomes.

$MAE = \frac{1}{n}\sum_{i=1}^{n}|x_i - x|$ Where e_i denotes the error. From Table 4-7, one can see that MCFS model is better than MC model for forecasting the gold return price

4.4 Long Run Behavior

4.4.1 Long Run Behavior for MCFS

As the MCFS model can be used to predict the returns for smaller time (one month), can be used to predict for long run time where:

If a fuzzy Markov chain is an Ergodic ($\tilde{P}$) [5]:

$$\exists \pi = (\pi_1, \pi_2, \dots)$$

Such that:

1) π is a probability vector.

2) $\pi_j > 0, \forall j$

3) $\pi P = \pi$

4) $\lim_{n \to \infty} (P)^n = \begin{bmatrix} \pi_1 & \cdots & \pi_n \\ \vdots & \ddots & \vdots \\ \pi_1 & \cdots & \pi_n \end{bmatrix}$

5) $\sum_j \pi_j = 1$

By applying the third and fourth conditions on the matrix obtained from the data of this thesis, we get:

Table 4-8: Long run estimation MCFS

	$\tilde{S}_{-10}$	$\tilde{S}_{-9}$	$\tilde{S}_{-8}$	$\tilde{S}_{-7}$	$\tilde{S}_{-6}$	$\tilde{S}_{-5}$	$\tilde{S}_{-4}$	$\tilde{S}_{-3}$	$\tilde{S}_{-2}$	$\tilde{S}_{-1}$	$\tilde{S}_{0}$	$\tilde{S}_{1}$	$\tilde{S}_{2}$	$\tilde{S}_{3}$	$\tilde{S}_{4}$	$\tilde{S}_{5}$	$\tilde{S}_{6}$	$\tilde{S}_{7}$	$\tilde{S}_{8}$	$\tilde{S}_{9}$	$\tilde{S}_{10}$
$\tilde{S}_{-10}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{-9}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{-8}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{-7}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{-6}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{-5}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{-4}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{-3}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{-2}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{-1}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{0}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{1}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{2}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**

$(P)^{12}=$

$\tilde{S}_3$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_4$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_5$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_6$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_7$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_8$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_9$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**
$\tilde{S}_{10}$	**0.2382**	**0.0030**	**0.0282**	**0.0391**	**0.0591**	**0.0309**	**0.0118**	**0.0080**	**0.0155**	**0.0394**	**0.0164**	**0.0213**	**0.0245**	**0.0335**	**0.0119**	**0.0291**	**0.0195**	**0.0294**	**0.0083**	**0.0350**	**0.2979**

Indicating that

$\pi =$

(0.2382, 0.0030, 0.0282, 0.0391, 0.0591, 0.0309, 0.0118, 0.0080, 0.0155, 0.0394, 0.0164, 0.0213, 0.0245, 0.0335, 0.0119, 0.0291, 0.0195, 0.0249, 0.0083, 0.0350, 0.2979).

For the previous results, the higher the ratio was for the highest return $\tilde{S}_{10} \approx 30\%$

And then for the highest loss $\tilde{S}_{-10} \approx 24\%$

This gives us probability distribution for gold price return. In the long run, no matter the state of gold price in a month, the number of gain months will be approximately equal to the number of loss. Categories of the states and the conditional transition probability matrices are calculated in MATLAB program. If we apply the law of expectation using the equation:

$$\hat{R}_{t\ t\to\infty} = \sum_{i=1}^{21} \pi_i x_i$$

We get,$\hat{R}_t = 0.1104\%$. And its present return state as:

Table 4-9: Present state for long run $\hat{R}_t$ with MCFS

R_t	$\tilde{S}_{-10}$	$\tilde{S}_{-9}$	$\tilde{S}_{-8}$	$\tilde{S}_{-7}$	$\tilde{S}_{-6}$	$\tilde{S}_{-5}$	$\tilde{S}_{-4}$	$\tilde{S}_{-3}$	$\tilde{S}_{-2}$	$\tilde{S}_{-1}$	$\tilde{S}_{0}$	$\tilde{S}_{1}$	$\tilde{S}_{2}$	$\tilde{S}_{3}$	$\tilde{S}_{4}$	$\tilde{S}_{5}$	$\tilde{S}_{6}$	$\tilde{S}_{7}$	$\tilde{S}_{8}$	$\tilde{S}_{9}$	$\tilde{S}_{10}$
0.1104	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0.51*	*0.49*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*

4.4.2 Long Run Behavior for MC

As the MC model can be used to predict the returns for smaller time (one month), can be used to predict for long run time where:

If a Markov chain is an ergodic (P) [5]:

$$\exists \pi = (\pi_1, \pi_2, \dots)$$

Such that:

1) π is a probability vector.

2) $\pi_j > 0, \forall j$

3) $\pi P = \pi$

4) $\lim_{n \to \infty} (P)^n = \begin{bmatrix} \pi_1 & \cdots & \pi_n \\ \vdots & \ddots & \vdots \\ \pi_1 & \cdots & \pi_n \end{bmatrix}$

5) $\sum_j \pi_j = 1$

By applying the third and fourth conditions on the matrix obtained from the data of this thesis, we get:

Table 4-10: Long run estimation MC

$(P)^{n}_{n\to\infty}$:

	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_{0}	S_{1}	S_{2}	S_{3}	S_{4}	S_{5}	S_{6}	S_{7}	S_{8}	S_{9}	S_{10}
S_{-10}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{-9}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{-8}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{-7}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{-6}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{-5}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{-4}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{-3}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{-2}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{-1}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{0}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{1}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{2}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{3}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{4}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{5}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{6}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{7}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{8}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{9}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900
S_{10}	0.2400	0.0155	0.0238	0.0401	0.0559	0.0237	0.0155	0.0079	0.0162	0.0395	0.0238	0.0166	0.0398	0.0240	0.0164	0.0241	0.0232	0.0158	0.0077	0.0405	0.2900

Indicating that

$$\pi =$$

(0.2400, 0.0155, 0.0238, 0.0401, 0.0559, 0.0237, 0.0155, 0.0079, 0.0162, 0.0395, 0.0238, 0.0166, 0.0398, 0.0240, 0.0164, 0.0241, 0.0232, 0.0158, 0.0077, 0.0405, 0.2900).

For the previous results, the higher the ratio was for the highest return $\tilde{S}_{10} \approx 29\%$

And then for the highest loss $\tilde{S}_{-10} = 24\%$

This gives us probability distribution for gold price return. In the long run, no matter the state of gold price in a month, the number of gain months will be approximately equal to the number of loss. Categories of the states and the conditional transition probability matrices are calculated in MATLAB program.

If we apply the law of expectation using the equation:

$$\hat{R}_{t\ t\to\infty} = \sum_{i=1}^{21} \pi_i x_i$$

We get, $\hat{R}_t = 0.0740\%$. And its present return state as:

Table 4-11: Present state for long run $\hat{R}_t$ with MC

R_t	S_{-10}	S_{-9}	S_{-8}	S_{-7}	S_{-6}	S_{-5}	S_{-4}	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
0.0740	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*

Chapter 5

Conclusions

The fruits of this thesis will facilitate the researchers' effort by abbreviating the steps in finding the degree of belonging to the return price, and where the researchers can only rely on the concept of Fuzzy to find the R_t site of the relationship $i = \frac{R_t}{k}$.

It was also through this work that the error in the relationship was discovered [11], [12] and [26]

$$\tilde{S}_i = \frac{(i+1)(0.225-R_t)}{0.225}$$

Where it was modified and proven using mathematical steps, and what confirmed the validity of our conclusion, the amendment that occurred in one of the attached references was exactly as we reached [13].

$$\tilde{S}_i = \frac{(i+1)(0.225)-R_t}{0.225}$$

After applying several types of fuzzy number, Triangle Fuzzy Number was the best to get the most accurate results.

We have predicted behavior of the Gold return price for next month. Using the MC and MCFS models, forecasts of the monthly Gold price returns were compared. The results give sensitive and significant information to the investors about investment opportunities of the Gold return price for the monthly buying and selling strategies when the present return is known. In risky months, when a monthly return substantially increased or decreased, the next month's return also substantially increased or decreased for both models. The transition probabilities of monthly returns in non-risky months would be significantly lower than those in risky months for both models. The MCFS model can be used to forecast the returns for smaller time (less than one month) intervals which may give more investment opportunities. Investors can earn higher than the average return in risky months in a short run. Besides the MCFS model can be used to predict the returns for smaller time (one day) and also different classifications and fuzzy sets which may give more investment opportunities. The probability distribution of gold showed that, the investors can gain higher return in a long run.

All the results obtained in this thesis were calculated using the MATLAB program.

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Appendix

```
% matrix II with i,s in column1 & si in column2
II=[-10 0.438105233
8   0.987994565
13  0.884045456
21  0.388345898
10  0.644326093
-14 0.12075134
6   0.510219249
21  0.757965259
25  0.926831766
9   0.538609992
7   0.960342963
-11 0.536766896
3   0.459406539
16  0.297435054
16  0.887402043
11  0.61873571
4   0.241038519
11  0.325765407
54  0.887200255
5   0.658078368
```

-29 0.143312622

18 0.857615833

-22 0.314158476

0 0.093752574

23 0.651903021

-18 0.347942086

-8 0.036045805

-18 0.099315751

2 0.050895603

-1 0.793276757

9 0.820266067

31 0.220380397

1 0.87149115

-6 0.597967874

-10 0.539171172

-5 0.071097225

-11 0.951836562

-11 0.813431713

-30 0.361587031

-21 0.866295849

-24 0.049461342

-20 0.31497894

21 0.994962598

1 0.871250526

-12 0.226600537

-13 0.505898331

-20 0.424267008

7 0.185035653

19 0.10478018

12 0.048032253

-13 0.431582238

-4 0.510603952

-4 0.812342476

12 0.780931495

-6 0.41055561

-20 0.078141136

-7 0.360561298

-18 0.127365072

9 0.922688999

18 0.898118838

-7 0.486305537

-19 0.218214966

6 0.368169548

-1 0.211047991

-6 0.728092255

-20 0.047704975

-6 0.520304942

2 0.064942619

12 0.270826327

-27 0.400727398

-9 0.196015065

75 0.693180437

-7 0.486305537

-19 0.218214966

6 0.368169548

-1 0.211047991

-6 0.728092255

-20 0.047704975

-6 0.520304942

2 0.064942619

12 0.270826327

-27 0.400727398

-9 0.196015065

51 0.275444253

15 0.817351343

-1 0.838520032

13	0.971693027
-8	0.681279512
5	0.279774253
-10	0.224161086
16	0.224030443
12	0.750834816
-13	0.272591715
0	0.128238858
-7	0.066483148
23	0.425384595
0	0.677873324
-3	0.736760115
3	0.724240569
-11	0.555510623
-8	0.327299869
-16	0.273936502
-13	0.996994591
-1	0.984031127
5	0.245042659
2	0.603849208
10	0.587566728
14	0.15670261

```
9   0.322305924
-6  0.882910086
-5  0.994977208
-2  0.791377056
26  0.652026401
18  0.638736923
25  0.082863298
3   0.07122442
-5  0.582056208
-7  0.792297712
2   0.55300877
24  0.783291949
10  0.135706277
-2  0.404926779
24  0.943461865
9   0.564454206
4   0.250391597
27  0.632825948]
% data processing
for k=1:126
if II(k,1)>10
II(k,1)=10
```

```
II(k,2)=1
elseif II(k,1)<-10
II(k,1)=-10
II(k,2)=1
end
end
% hight of the fuzzy set
A=zeros(21,2)
for i=1:126
a=II(i,1)
mx=II(i,2)
for j=1:126
if a==II(j,1)
O=[mx II(j,2)]
mx=max(O)
x=a+11
A(x,1)=a
A(x,2)=mx
end
end
end
% initial probability vector
```

```
pi=[ 1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21
1/21]
% matrix Q
```

```
UU=transpose(pi)
Q=zeros(21)
for j=1:21
if A(j,2)==0
Q(j,j)=0
Q(j,j+1)=0
else
Q(j,j)=A(j,2)
Q(j,j+1)=1-A(j,2)


end
end
% matrix S
Q=Q(:,1:21)
tee=transpose(pi)
H=(tee*Q)
TeT=(H)'
QT=transpose(Q)

tte=[tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;te
e;tee]
```

```
TTe=[TeT TeT TeT TeT TeT TeT TeT TeT TeT TeT TeT TeT TeT TeT
TeT TeT TeT TeT TeT TeT TeT]
Se=zeros(21)
for i=1:21
for j=1:21
Se(i,j)=(tte(i,j)*QT(i,j))/TTe(i,j)
end
end
% vector I cotains i,s position
I=[-10
8
13
21
10
-14
6
21
25
9
7
-11
3
```

16

16

11

4

11

54

5

-29

18

-22

0

23

-18

-8

-18

2

-1

9

31

1

-6

-10

-5

-11

-11

-30

-21

-24

-20

21

1

-12

-13

-20

7

19

12

-13

-4

-4

12

-6

-20

-7

-18

9

18

-7

-19

6

-1

-6

-20

-6

2

12

-27

-9

75

-7

-19

6

-1

-6

-20

-6

2

12

-27

-9

51

15

-1

13

-8

5

-10

16

12

-13

0

-7

23

0

-3

3

-11

-8

-16

-13

-1

5

2

10

14

9

-6

-5

-2

26

18

25

3

-5

-7

2

24

10

-2

24

```
9

4

27]

% data processing

for k=1:126

if I(k,1)>10

I(k,1)=10

elseif I(k,1)<-10

I(k,1)=-10

end

end

% Transfer counter between states

po=zeros(21)

for i=1:125

a=I(i)+11

b=I(i+1)+11

po(a,b)=po(a,b)+1

end

DD=po

s=0;

for i=1:21

j=1:21
```

```
s=s+DD(i,j)

s(1,1)=31

s(1,21)=35

end

% probability transition matrix MC

sum=s

PP=zeros(21)

for i=1:21

for j=1:21

PP(i,j)=po (i,j)/sum(i)

end

end

% MCFS

F=Se*PP*Q

% long run estimation


DDD=transpose(F)

for j=1:21

DDD(j,j)=DDD(j,j)-1

end

TTT=DDD(1:20,:)
```

W=[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]

DM=[TTT;W]

V=[0

0

0

0

0

0

0

0

0

0

0

0

0

0

0

0

0

0

```
0
1]
B=inv(DM)*V
s=0
for i=1:21
s=s+B(i,1)
end
% xi (midd point of fuzzy states)
c=[
-2.25
-2.03
-1.8
-1.58
-1.35
-1.13
-0.9
-0.68
-0.45
-0.23
0
0.23
0.45
```

```
0.68

0.9

1.13

1.35

1.58

1.8

2.03

2.25]

% estimation law R^t for lon run vector


Z=zeros(21,1)

sem=0

ToT=F(1,1:21)


UoU=ToT'

for i=1:21

z(i,1)=B(i,1)*c(i,1)

sem=sem+z(i,1)




end

% we set kjmomomommi
```

```
E=0

for i=1:21


E=E+F(14,i)

end

W=B'*F

% Rt vector

Rtm=[-2.123573678

1.802701223

2.951089772

4.862622173

2.330026629

-2.952169051

1.460200669

4.779457817

5.641462853

2.128812752

1.583922833

-2.370772552

0.796633529

3.758077113

3.62533454
```

2.560784465

1.070766333

2.626702783

12.17537994

1.201932367

-6.33224534

4.082036438

-4.795685657

0.203905671

5.25332182

-3.903286969

-1.583110306

-3.847346044

0.663548489

-0.17848727

2.065440135

7.150414411

0.253914491

-1.259542772

-2.146313514

-0.915996876

-2.464163226

-2.433022135

-6.606357082

-4.694916566

-5.186128802

-4.345870261

4.726133416

0.253968632

-2.525985121

-2.813827124

-4.370460077

1.758366978

4.476424459

2.914192743

-2.797106003

-0.789885889

-0.857777057

2.749290414

-1.217375012

-4.292581756

-1.431126292

-3.853657141

2.042394975

4.072923261

-1.459418746

-4.099098367

1.492161852

-0.047485798

-1.288820757

-4.285733619

-1.242068612

0.660387911

2.864064077

-5.940163665

-1.84410339

16.9440344

-1.459418746

-4.099098367

1.492161852

-0.047485798

-1.288820757

-4.285733619

-1.242068612

0.660387911

2.864064077

-5.940163665

-1.84410339

11.63802504

3.416095948

-0.188667007

2.931369069

-1.72828789

1.287050793

-2.075436244

3.77459315

2.756062166

-2.761333136

0.196146257

-1.364958708

5.304288466

0.072478502

-0.615771026

0.737045872

-2.37498989

-1.64864247

-3.436635713

-2.924323783

-0.221407004

1.294865402

0.539133928

2.342797486

3.339741913

2.177481167

-1.323654769

-1.123869872

-0.403059838

5.92829406

4.131284192

5.831355758

0.883974505

-1.030962647

-1.528266985

0.550573027

5.448759311

2.444466088

-0.316108525

5.41272108

2.122997804

1.068661891

```
6.157614162]
% Q matrix in MC
Qw=zeros(126,2)
Qm=zeros(21)
for i=1:126
for j=1:21
if Rtm(i)>0.12*(2*(j-11)-1) && Rtm(i)<=0.12*(2*(j-11)+1)
Qm(j,j)=1
Qw(i,1)=Rtm(i)
Qw(i,2)=j-11
elseif Rtm(i)>=2.28
Qm(21,21)=1
Qw(i,1)=Rtm(i)
Qw(i,2)=10

elseif Rtm(i)<=-2.28
Qm(1,1)=1
Qw(i,1)=Rtm(i)
Qw(i,2)=-10
end
end
end
```

```
end
end
% S matrix in MCFS
h=pi'*Qm
Te=h'
QTe=Qm'
tee=pi'
tte=[tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;tee;te
e;tee]
TTee=[Te Te Te Te Te Te Te Te Te Te Te Te Te Te Te Te Te Te Te Te Te]
SSe=zeros(21)
for i=1:21
for j=1:21
SSe(i,j)=(tte(i,j)*QTe(i,j))/TTee(i,j)
end
end
% example for estimation R^t for a specific month
Dad=[0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
Ded=[0 0 0 0 0 0 0 0 0 0 0 0 .06 .94 0 0 0 0 0 0 0]
pap=Dad*PP
pep=Ded*F
cat=zeros(21,1)
```

```
ca=0

caa=-2.4

for i=1:21

cat(i,1)=caa+ca

caa=caa+.24

end

peep=pep'

sees=0

for i=1:21

sees=sees+(peep(i,1)*c(i,1))

end

paap=pap'

saas=0

for i=1:21

saas=saas+(paap(i,1)*cat(i,1))

end
```