Discrete and Continuous Fractional Economic Models Based on Market Equilibrium in the Frame of Different Type Kernels

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This Thesis was submitted in Partial Fulfillment of the Requirements for the Master's Degree of Science in Mathematical Modeling

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Palestinian Technical University- Kadoorie

May 2021

النماذج الاقتصادية الكسرية المنفصلة والمتصلة بناءً على توازن السوق في إطار أنواع مختلفة من التجمعات

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قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في النمذجة الرياضية

عمادة البحث العلمي والدراسات العليا

مايو 2021

جامعة فلسطين التقنية خضوري

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Discrete and Continuous Fractional Economic Models Based on Market Equilibrium in the Frame of Different Type Kernels

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص باستثناء ما تم الإشارة إليه حيثما ورد وأن هذه الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل درجة علمية أو بحث علمي لدى مؤسسة تعليمية أو بحثية أخرى.

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I hereby declare that this thesis is the product of my own efforts, expect what has been referred to, and this thesis as a whole or any part of it has not been submitted as a requirement for attaining a scientific degree to any other educational or research institution.

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COMMITTEE DECISION

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Dedication

To all those who lighten my darkness, surround my heart with their prayers, who have stood by me all my life and have been my mentors, to all those whom I am so grateful to have in my life.

To my dear parents, whom supported my heart and my soul, whom illuminated the darkness of my heart with their love and prayers. Without you, I would not have reached what I have reached today: Hassan and Salma.

To my brothers and my hearts whom were and still are my solid rock in this life: Dr. Amro, Mr. Qais, Eng. Abdullah, Mr. Abd Alrahman and Mohammad. To my lovely sister ever my beautiful sweet: Raghad.

To my little prince, my nephew: Alhassan, the sugar of this bitter life.

To my lovely friends and my soul mates.

To everyone who inspires me by his/her science.

I dedicate my thesis.

Acknowledgment

First and foremost, all praise, loves and thanks be to God, for giving me the patience and strength to complete this a success, who guided my way to this point in my life.

I'm most grateful to Prof. Dr. Saed Malak my supervisor for his patient, support and guidance through the process of writing this thesis. It was really an honor to have been one of his students.

I'm so grateful to Prof. Dr. Thabet Abdeljawad my cosupervisor for his invaluable comments and feedback, his fruitful ideas.

Many special thanks are also due to Dr. Hadi Khalilia who helped me in MATLAB cods and for his comments which contributed to improvement of this work.

I would like to express gratitude to my relatives, friends and stepfamily who supported me in needed moments.

Finally, I would like to thanks unmentioned names who have grate influence, supported me and never let me down.

To each one of the above, I extend my deepest appreciation.

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Abstract

In this thesis we investigate the solutions of continuous and discrete fractional economic models using the non-local discrete fractional differences and derivatives involving Caputo, Caputo – Fabrizio in the sense of Caputo (CFC), Caputo type Atangana – Baleanu (ABC) and ABC with generalized Mittag-Leffler kernel.

These differences and derivatives and fractional factors enable us to solve with equations more generally, as they go beyond dealing with integrals and regular derivatives and enable us to deal with more complex equations and tolerate wider mathematical explanations.

In the case of the economic models that we are dealing with, these differences enable us to better understand the market body and the relationship between

supply and demand and their interactions with the price adjustment on the assumption that the market is in equilibrium.

The main original part is to solve and investigate the economic models by using the discrete counter part of the above, Caputo type for final operators in the power low, exponantial nabla low, nabla Mittag-Leffler low and generalized nabla Mittag-Leffler low and solve the models in the continuous case analytically.

Our research is unique because we review the solutions of these models in the discrete case, in addition to that we are the first to review Generalized ABC with four parameters and the dependence of the rest of the differences on them as special cases.

Finally, we interpret the results we obtain from the model and compare them in more details with the different solutions we obtain.

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الملخص

في هذه الأطروحة نهدف إلى عرض مشكلة اقتصادية وحلها وتحليل نتائجها باستخدام الفروقات والمشتقات الكسرية غير المحلية المنفصلة والمتصلة وذلك يتضمن مشتقة Caputo, Caputo–Fabrizio in the sense of Caputo(CFC), Caputo type Atangana–Baleanu (ABC) and ABC with generalized Mittag-Leffler kernel.

هذه الفروقات و المشتقات والعوامل الجزئية تمكننا من حل المعادلات بشكل أعم بحيث أنها تتجاوز التعامل بالتكاملات والمشتقات العادية وتمكننا من التعامل مع معادلات أصعب وتحتمل تفسيرات رياضية أوسع.

وفي حالة النموذج الإقتصادي الذي نتعامل معه هنا فإن هذه المشتقات تمكننا من تحسين فهم هيئة السوق والعلاقة بين العرض والطلب وتفاعلاتهما مع سعر السلعة على افتراض أن السوق في حالة توازن.

الجزء الأساسي للبحث هو أننا قمنا بحل النماذج الاقتصادية باستخدام الفروق المنفصلة كما أننا استعرضنا حلول هذه النماذج للمشتقات المتصلة يدويًا.

بحثنا يعتبر فريد من نوعه لأننا قمنا بحل هذه النماذج الاقتصادية لأول مرة بإستخدام المشتقات الكسرية المنفصلة إضافةً إلى أننا أول من استخدم مشتقة ABC المعممة بالاعتماد على أربع متغيرات والتي تمكنّا عن طريقها من أخذ بقية المشتقات كحالات خاصة منها.

وبالإضافة الى ذلك قمنا بتفسير النتائج التي حصلنا عليها من النموذج ومقارنتها بشكل أكثر تفصيلًا بإختلاف الحلول التي حصلنا عليها.

Chapter 1

Introduction

1.1. Motivation

The study of discrete fractional calculus has been initiated in nineteenth century.

There has been no noticeable development except very soon, specifically in the past ten years [26].

There has been an increasing interest in developing discrete fractional calculus because its more general than different calculus and provide more efficient numerical tools [30], as the discrete fractional calculus is now being studied to solve equations and models that cannot be solved by regular methods and because they deal with life applications directly [26, 33].

This interest in recent years has focused on the difficulty of solving equations using calculus and partial integration as it has demonstrated a number of unexpected difficulties and technical complications.

Mathematical economics is a theoretical and applied science and its purpose mathematically is a formal description of things, processes, and phenomena [26].

We present the model here by formulating economic concepts in the form of an equation, working on solving it, and interpreting the results using mathematical methods. Having an optimal solution and balance is the main goal of using mathematics in economics.

A revolution occurred in the world of economics at the beginning of the twentieth century, which led to the use of mathematics as a basic method for describing economic phenomena and processes [24, 30].

The market consists of a large number of elements that we have formulated with two models, and each economic model targets the maximum benefit for buyers, the maximum profit for sellers, and the freedom of pricing restricted according to an equilibrium model that has its own controls such as commodities, companies and individuals [11, 24, 30].

Economic theory provides the interactions between price, supply and demand, the dependence of supply and demand on price and how to achieve a point of equilibrium on the supply and demand curves so that the number of supply is equal to the number of demand and the maximum profit of the seller is achieved without regressions [11, 24].

In order to describe the economic process, differentiation and integration, which are the most crucial tools for the construction of the economic phenomena and modelling, have a great importance in economic theory [11, 24].

Since the behavior of the economy may rely on the past fluctuations in the economy, the use of fractional differentiation and integration instead of the integer order ones provide features of memory effect allowing the observation of the history of the economy [30].

The modern stage of the mathematical economy occurs almost simultaneously with the existing of the economic principles contains the expression of economics in terms of the fractional operators, which can be called fractional mathematical economics [38].

Owing to the fact that traditional derivative has some shortcomings causing the deficiency of descriptions of economic concepts, to build an economic model by means of non-local fractional operators, which don’t have any restriction of small neighborhood of the point, which undoubtedly has much more advantages when compared with the models composed by integer-order derivative [11, 38].

Hence, the main original part of this thesis is to investigate the mentioned reviewed economic model in the frame of discrete fractional calculus depending on the discrete versions of fractional differential operators with singular and nonsingular kernels.

To the best of our knowledge no one has previously done a study on these economic models using the discrete fractional calculus operators, we have confirmed this by using keywords for search engines such as: discrete, economic models, solved economic models, solved models using discrete partial and partial disjunction and integration.

1.2. Problem Statement

A competitive market is directly related to competitive equilibrium such that the quantity of goods demanded by buyers equals to the quantity of goods supplied by sellers[30, 38].

The quantity demanded and the quantity supplied can be represented by demand function q_d and supply function q_s, respectively, as follows:

$$q_d(t) = d_0 - d_1 u(t)$$

$$q_s(t) = -s_0 + s_1 u(t) \qquad (1)$$

Where $u(t)$ the price of goods, d_0, s_0, d_1, s_1 are positive constants which are some operators affecting the quantity demand and quantity supplies.

For $q_d\ (t) = q_s(t)$, when the demanded quantity equals the supplied quantity, we obtain the equilibrium price $\mathcal{U} = \left(\frac{d_0+s_0}{d_1+s_1}\right)$.

In a certain cases, the price tends to stay stable and there is no shortage and surplus in economics.

Let us consider the following price adjustment equation as we have it from [30],

$$u'(t) = k(q_d - q_s) \tag{2}$$

Where $k > 0$ is the speed of adjustment constant.

If we substitute q_d and q_s functions in Equation (1) into (2), then we will get

$$u'(t) + k(d_1 + s_1)u(t) = k(d_0 + s_0) \tag{3}$$

Solving Equation (3) as a linear differential equation, one can readily obtain the following solution

$$u(t) = \frac{d_0 + s_0}{d_1 + s_1} + \left(u(0) - \frac{d_0 + s_0}{d_1 + s_1}\right) e^{-k(d_1+s_1)t} \tag{4}$$

Where $u(0)$is the price at the time $t = 0$.

In this solved equation we do not pay attention to the expectation of agents in market. If we consider the expectations of agents, the demand and supply functions involving additional factors d_2 and s_2 change or will change to the following form

$$q_d(t) = d_0 - d_1 u(t) + d_2 u'(t)$$

$$q_s(t) = -s_0 + s_1 u(t) - s_2 u'(t) \qquad (5)$$

For the case $q_d(t) = q_s(t)$ as we obtain,

$$u'(t) - \frac{d_1 + s_1}{d_2 + s_2} u(t) = -\frac{d_0 + s_0}{d_2 + s_2} \qquad (6)$$

And if we solve the Equation above as a linear one, we will have

$$u(t) = -\frac{d_0 + s_0}{d_1 + s_1} + \left(u(0) + \frac{d_0 + s_0}{d_1 + s_1}\right) e^{\frac{d_1+s_1}{d_2+s_2} t} \qquad (7)$$

We note that in the law of the market, if the price of a commodity rises, buyers tend to buy a quantity of that commodity before its price increases further, and as for sellers, they tend to provide a smaller quantity of the product in order to achieve high gains in later times. The models we will study have the condition $q_d\ (t) = q_s(t)$. However, when $u'(t) = 0$ for all $t\ >\ 0$, market is in a changing economy which means dynamic equilibrium.

We have generalized the models above so that they can be solved using discrete fractional differential derivatives, as these fractional derivatives have fewer properties than the traditional properties, which makes them excellent in describing anomalous phenomena. In this thesis we review the solution of models using Caputo, Caputo–Fabrizio (CF) fractional derivative which is the convolution of exponential function and the first order derivative [28] , Atangana–Baleanu (AB) fractional derivative containing the convolution of Mittag-Leffler function and first order derivative, generalized Atangana–Baleanu.

The use of the discrete fractional differences mention in the previous paragraph and other differences that we did not mention is due to the need to have rules for solving time-related equations, they started with Caputo-Fabrizio fractional then were done with the Mittag-Leffler with one parameter instead of the exponential, then the Mittag-Leffler with two parameters, and so on [4].

Herein, we stress that when a constant coefficient linear differential equation is handled by means of Caputo–Fabrizio in the sense of Caputo (CFC) or Caputo type Atangana–Baleanu (ABC), we should put condition in order to have a nontrivial or false solution [5]. But when we solve the problem by using Caputo or generalized ABC, no need to put any condition.

At the end, we do some significant discussions by graphics and proposed them. For more details about fractional operators see [7, 12–17, 23, 25, 31–33, 40, 41]. Also, for some related study on economic see [22, 26].

1.3. Objectives of the Thesis

We summarize our claims by:

- To solve and discuss the new economic models by investigating and analyzing them by the means of non-local fractional operators involving Caputo, Caputo–Fabrizio in the sense of Caputo (CFC), Caputo type Atangana–Baleanu (ABC) and ABC with generalized Mittag-Leffler kernel in the discrete and continuous case.
- To observe the controversial subjects in the market in more detailed manner, by supporting the results with simulation analysis.
- The main original part is to solve and investegate the economic models by using the discrete countar part of the above, Caputo type for final operators in the power low, exponantial nabla low, nabla Mittag-Leffler low and generalized nabla Mittag-Leffler low.

1.4. Research Questions

This thesis will answer two research questions.

1- How can we investigate the mentioned reviewed economic model in the frame of discrete fractional calculus depending on the discrete versions of fractional differential operators with singular and nonsingular kernels?

2- When can we reach a market equilibrium if we take into account the impact of non-local operators?

The purpose of this thesis is to introduce the solutions for economic models by means of Caputo, CF, AB and generalized AB including Mittag-Leffler function with three parameters. In addition to getting the solutions of aforementioned models having non-local properties. Fractional operators refer to those beyond traditional differentiation and integration when applying to the economic models. Thereby, these fractional operators improve the comprehension of supply and demand and their interactions with price of commodity under the assumption that market is in equilibrium [24, 30].

1.6. Literature Review

Atici and Sengül [21], discussed modeling with fractional difference equations. They defined and developed some basics of discrete fractional calculus such as Leibniz rule and summation by parts formula.

Atici, Marshall and Belcher [20], presented new approach for modeling with discrete fractional equations. They defined and introduced a new class of nonlinear discrete fractional equations to model tumor growth rates in mice.

Bas and Oarsman [23], introduced real world applications of fractional models by Atangana-Baleanu fractional derivative. They defined some modeling problems, Newton's law of cooling, population growth, logistic equation, blood alcohol model.

Bas, Acay and Ozarslan [22], considered fractional models with singular and non-singular kernels for energy efficient buildings. They defined and analyzed the fractional version of the heating and cooling model for buildings with energy efficiency. They applied the Caputo fractional derivative, Caputo-Fabrizio, and Atangana-Baleanu in the Caputo sense in the analysis and investigation of the governing model.

Sene [36], addressed the Mittag-Leffler input stability of the fractional differential equations with exogenous inputs.

Abdeljawad [4], discussed the fractional difference operators with discrete generalized Mittag-Leffler kernels. He defined the fractional difference operators with discrete generalized Mittag-Leffler kernels with three parameter for both Riemann type (ABR) and Caputo type (ABC) cases, where AB stands for Atangana-Baleanu [3].

Acay , Bas and Abdeljawad [11], the most recent work about the fractional economic models based on market equilibrium in the frame of different type kernels, investigated and analyzed certain problems in economics by means of non-local fractional operators involving Caputo, Caputo–Fabrizio in the sense of Caputo (CFC), Caputo type Atangana–Baleanu (ABC) and ABC with generalized Mittag-Leffler kernel which they used and generalized.

1.7. Structure of the Thesis

The rest of the thesis is organized as follows:

Chapter 2, presents some of the definitions, lemmas and mathematical theory of discrete fractional derivatives. Chapter 3, the continuous case and discussion of the result obtain.

Chapter 4, In this section we present for the first time the solutions of the models in the discrete fractional derivative Caputo, CFC, ABC and generalized ABC. Chapter 5, the main results by graphs. Finally, the conclusion and comments.

Chapter 2

Mathematical Preliminaries and Fractional Calculus

Here we will present the Definitions, lemmas and remakes that we used in the fourth and fifth chapters to find our solutions we obtain.

2.1. Fractional Differences

Definition 1 [3, 26]**.** (i) Let ν be a natural number, then the ν rising factorial of m is written as

$$m^{\overline{\nu}} = \prod_{i=0}^{\nu-1}(m+i), \quad m^{\overline{0}} = 1, m \in \mathbb{R} \tag{8}$$

(ii)For any real number the ∂ rising function becomes

$$m^{\overline{\partial}} = \frac{\Gamma(m+\partial)}{\Gamma(m)}$$

$$such\ that \quad 0^{\overline{\partial}} = 0, m \in \mathbb{R} \tag{9}$$

In addition, we have

$$\nabla(m^{\overline{\nu}}) = \nu m^{\overline{\nu-1}} \tag{10}$$

Hence $m^{\overline{\nu}}$ is increasing on $\mathbb{N}_0$ ={0,1,2,3,4,..} for $\nu > 0$.

Remark 1 [3]**.** The backward difference operator of a function φ on h$\mathbb{Z}$ is given by

$$\nabla_h \varphi(u) = \frac{\varphi(u) - \varphi(u-h)}{h} \tag{11}$$

For $h = 1$, we get the backward difference

$$\nabla \varphi(u) = \varphi(u) - \varphi(u-1) \tag{12}$$

The backward jumping operator is

$$\varpi(u) = u - h \tag{13}$$

For $a \in \mathbb{R}$ and $h > 0$, $\mathbb{N}_{a,h} = \{a, a+h, a+2h, ..\}$

Definition 2[3, 37]**.** For arbitrary m, $\beta \in \mathbb{R}$ and h >0, the Nabla h-fractional function is defined by

$$m_h^{\overline{\beta}} = h^{\beta} \frac{\Gamma\left(\frac{m}{h} + \beta\right)}{\Gamma\left(\frac{m}{h}\right)} \tag{14}$$

For $h = 1$, we write $m^{\overline{\beta}} = \frac{\Gamma(m+\beta)}{\Gamma(m)}$.

Definition 3[3]**.** The Mittag-Leffler function of one parameter has the following form:

$$E_{\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\beta r + 1)}, \quad (z \in \mathbb{C}; Re(\beta) > 0). \tag{15}$$

And the one of two parameter α and β becomes

$$E_{\beta,\alpha}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\beta r + \alpha)}, \quad (z, \alpha \in \mathbb{C}; Re(\beta) > 0). \tag{16}$$

Where $E_{\beta,1}(z) = E_\beta(z)$ and $E_1(z) = e^z$.

Definition 4[1–3]**.** (Nabla Discrete Mittag-Leffler) For λ∈R, |λ|<1 and α, β, τ ∈C with $Re(\beta) > 0$, the Nabla discrete Mittag-Leffler is

$$E_{\overline{\beta,\alpha}}(\lambda,\tau) = \sum_{r=0}^{\infty} \lambda^r \frac{\tau^{\overline{r\beta+\alpha-1}}}{\Gamma(\beta r + \alpha)} \tag{17}$$

For $\alpha = 1$, we have

$$E_{\overline{\beta}}(\lambda,\tau) \triangleq E_{\overline{\beta,1}}(\lambda,\tau) = \sum_{r=0}^{\infty} \lambda^r \frac{\tau^{\overline{r\beta}}}{\Gamma(\beta r + 1)} \tag{18}$$

Next we have the definition on h$\mathbb{Z}$.

Definition 5 [3]**.** (Nabla h-Discrete Mittag-Leffler) For λ∈R, $|\lambda h^\beta|<1$ and α, β, τ ∈C with $Re(\beta) > 0$, the nabla h-discrete Mittag-Leffler is

$${}_{\mathrm{h}}E_{\overline{\beta,\alpha}}(\lambda,\tau) = \sum_{r=0}^{\infty} \lambda^r \frac{\tau_h{}^{\overline{r\beta+\alpha-1}}}{\Gamma(\beta r + \alpha)} \tag{19}$$

For $\alpha = 1$, we have

$${}_{\mathrm{h}}E_{\overline{\beta}}(\lambda,\tau) \triangleq {}_{\mathrm{h}}E_{\overline{\beta,1}}(\lambda,\tau) = \sum_{r=0}^{\infty} \lambda^r \frac{\tau_h{}^{\overline{r\beta}}}{\Gamma(\beta r + 1)}$$

Remark 2 [3].

1. For h=1, we obtain the nabla discrete Mittag-Leffler function (ML) defined in definition 3.

2. The smaller $0 < h < 1$, the larger the interval of convergence to which λ belongs. For example, when h=1, to guarantee convergence when $\lambda = \frac{-\beta}{1-\beta}$ we must have $0 < \beta < \frac{1}{2}$. When h is small enough we can get convergence for $0 < \beta < 1$.

For more properties of Mittag-Leffler read [8].

The Three parameter case of Mittag-Leffler:

Definition 6 [4]**.** For $\lambda \in \mathbb{R}, h > 0$ and $\beta, \mu, \gamma, \zeta \in \mathbb{C}\ with\ Re(\beta) > 0$, the generalized Nabla h-discrete ML function is defined by

$$ {}_{\mathrm{h}}E^{\gamma}_{\beta,\mu}(\lambda,\zeta) = \sum_{r=0}^{\infty} \lambda^r \frac{{\zeta_h}^{\overline{r\beta+\mu-1}} \gamma_r}{\Gamma(\beta r + \mu) r!}\ ,\ |\ \lambda \mathrm{h}^{\beta}\ |< 1 \tag{20} $$

$$ \gamma_{\mathrm{r}} = \gamma \times (\gamma + 1) \times (\gamma + 2) \ldots \ldots \ldots . (\gamma + \mathrm{k} - 1) $$

Definition 7 [3, 37]**.** (Nabla h-fractional sums): Let $\varpi(u) = u - h, h > 0$ be the backward jump operator. Then for a function $\varphi: \mathbb{N}_{a,h} = \{a, a + h, a + 2h, ..\} \to \mathbb{R}$, the nabla left h- fractional sum of order $\beta\ > 0$ is given by

$$ \left({}_{\mathrm{a}}\nabla_h^{-\beta} \varphi\right)(\mathrm{u}) = \frac{1}{\Gamma(\beta)} \int_a^u \varphi(s) \left(u - \varpi(s)\right)_h^{\overline{\beta-1}} \nabla_h s \tag{21} $$

$$ = \frac{1}{\Gamma(\beta)} \sum_{i=\frac{a}{h}+1}^{\frac{u}{h}} \varphi(ih) \left(u - \varpi(ih)\right)_h^{\overline{\beta-1}} h, u \in N_{a+h,h} \tag{22} $$

Definition 8 [3, 37]**.(** Nabla h-Riemann–Liouville (RL) fractional difference)

Nabla h- (RL) fractional difference of order $\beta > 0$, starting from a has the form

$$\left({}_{a}\nabla_{h}^{\beta}\varphi\right)(u) = \left({}_{a}\nabla_{h}^{n}\ {}_{a}\nabla_{h}^{-(n-\beta)}\ \varphi\right)(u) \tag{23}$$

$$\left({}_{a}\nabla_{h}^{\beta}\varphi\right)(\mathrm{u}) = \frac{\nabla_{h}^{n}}{\Gamma(n-\beta)} \sum_{i=\frac{a}{h}+1}^{\frac{u}{h}} \varphi(ih)\left(u-\varpi(ih)\right)_{h}^{\overline{n-\beta-1}} h \tag{24}$$

$$u \in N_{a+h,h}$$

Definition 9 [4]**.** Assume j, v: $\mathbb{N}_{a} \rightarrow \mathbb{R}$, be discrete functions. Then the Nabla discrete convolution of j *with* v is defined by

$$(j * v)(t) = \sum_{s=a+1}^{t} v(t-\varpi(s)+a)j(s) \tag{25}$$

Definition 10 [3]**.** (Nabla discrete exponential kernel in h$\mathbb{Z}$) the Nabla discrete exponential kernel can be expressed as

$${}_{\mathrm{h}}\varrho_{\lambda}\left(t,\varpi(s)\right) = \left(\frac{1}{1-\lambda h}\right)^{\frac{t-\varpi(s)}{h}} = \left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t-\varpi(s)}{h}},\ where\ \lambda = \frac{-\beta}{1-\beta}$$

Also, using $\lambda s = \frac{1+hs}{s}$, we have

$${}_{\mathrm{h}}\varrho_{\lambda s}(t,a) = (1-hs)^{\frac{t-a}{h}}, \quad t \in \mathbb{N}_{a,h} \tag{26}$$

2.2. Caputo Fractional Difference

Definition 11 [3]. (The h- Caputo fractional difference) Let $\beta > 0$, $l = [\beta] + 1, [\beta]$ is the greatest integer less than or equal to $[\beta]$, $h > 0$, $a \in \mathbb{R}$, $a_h(\beta) = a + (l-1)h$. Assume φ is defined on $\mathbb{N}_{a,h} = \{a, a+h, a+2h, ..\}$. If $0 < \beta < 1$, then $a_h(\beta) = a$. Then

- The left h- Caputo fractional difference of order β starting at $a_h(\beta)$ is defined by

$$({}_{a_h(\beta)}^{C}\nabla^{\beta}\varphi)(t) = (\nabla_{a_h(\beta)}^{-(l-\beta)} \nabla_h^l \varphi)(t),\ t \in \mathbb{N}_{a+lh,h} \tag{27}$$

If we put $h = 1$, we obtain the definitions stated in [1, 2].

Definition 12 [9, 27]**.** The left Caputo fractional derivative is defined by

$${}_a^C D^{\beta}\varphi(t) = \frac{1}{\Gamma(l-\beta)} \int_a^t (t-y)^{(l-\beta-1)} \varphi^{(l)}(y)\, dy \tag{28}$$

Where $\beta \in \mathbb{C},\ Re(\beta) > 0,\ l = [Re(\beta)] + 1$

Lemma 1. For any $\beta > 0$, we have

$$\left({}_a\nabla_h^{-\beta} \nabla_h \varphi\right)(t) = \left(\nabla_h\, {}_a\nabla_h^{-\beta} \varphi\right)(t) - \frac{(t-a)_h^{\overline{\beta-1}}}{\Gamma(\beta)} \varphi(a) \tag{29}$$

Where φ is defined on $\mathbb{N}_{a,h}$.

For the proof see [3, 37].

2.3. CFC Fractional Difference

Definition 13 [3, 19] **.** For $\beta \in (0,1)$ and φ defined on $\mathbb{N}_{a,h}$. Let $H(\beta,h) = B(\beta)\left[\frac{\beta}{h} + (1-\beta)\right]$ such that $B(\beta)$ *is a normalized function such that* $B(0) = B(1) = 1$ *and* $\lambda = \frac{-\beta}{1-\beta}$, Then, we define

- The left (h-nabla) CFC fractional difference by

$$\left({}^{CFC}_{\ a}\nabla_h^{\beta}\varphi\right)(t) = H(\beta,h)\left(\frac{1-\beta+\beta h}{1-\beta}\right) \times$$

$$\sum_{r=\frac{a}{h}+1}^{\frac{t}{h}} h(\nabla_h\varphi)(rh)\left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t-\varpi(rh)}{h}}$$

OR

$$\left({}^{CFC}_{\ a}\nabla_h^{\beta}\varphi\right)(t) = H(\beta,h)\sum_{r=\frac{a}{h}+1}^{\frac{t}{h}} h\nabla_h\varphi(rh)\left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t-rh}{h}} \qquad (30)$$

$$t \in \mathbb{N}_{a+h,h}.$$

- The Left (h-nabla) Caputo–Fabrizio Riemann (CFR) fractional difference

$$\left({}^{CFR}_{\ a}\nabla_h^{\beta}\varphi\right)(t) = H(\beta,h)\left(\frac{1-\beta+\beta h}{1-\beta}\right)\nabla_h \times$$

$$\sum_{r=\frac{a}{h}+1}^{\frac{t}{h}} h\varphi(rh)\left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t-\varpi(rh)}{h}}$$

$$({}^{CFR}_{a}\nabla^{\beta}_{h}\varphi)(t) = H(\beta,h)\nabla_h \sum_{r=\frac{a}{h}+1}^{\frac{t}{h}} h\varphi(rh)\left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t-rh}{h}} \quad (31)$$

$$t \in \mathbb{N}_{a+h,h}.$$

Below we recall the continuous counterparts.

Definition 14 [19]**.** The left side Caputo-Fabrizio fractional derivative in the caputo sense is presented as for $\beta \in (0,1)$ and φ be defined on $\mathbb{N}_a$. Let $B(\beta) is\ a\ normalized\ function\ \ and\ \lambda = \frac{-\beta}{1-\beta}$ then

$$({}^{CFC}_{a}\mathrm{D}^{\beta}\,\varphi)(t) = \frac{B(\beta)}{1-\beta}\int_a^t \dot{\varphi}(n)\exp\big(\lambda(t-n)\big)\,dn \quad (32)$$

Lemma 2[3, 8, 8, 19]**.** (The relation between Riemann and Caputo type h fractional difference with exponential kernels)

$$({}^{CFC}_{a}\nabla^{\beta}\,\varphi)(t) = ({}^{CFR}_{a}\nabla^{\beta}\,\varphi)(t) -$$

$$H(\beta,h)\left(\frac{1-\beta+\beta h}{1-\beta}\right)\varphi(a)\left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t-a}{h}} \quad (33)$$

2.4. ABC Fractional Difference

In this subsection, we recall the continuous counterparts.

Definition 15 [3, 8, 18]**.** Let φ be defined on $\mathbb{N}_{a,h}$, $\beta \in (0,1)$, such that $|\lambda h^{\beta}| < 1$, then the left nabla ABC fractional difference (in the sense of Atangana and Baleanu) is defined by

$$\left({}^{ABC}_{\ a}\nabla_h^{\beta}\varphi\right)(t) = H(\beta,h)\left(\frac{1-\beta+\beta h}{1-\beta}\right)\times$$

$$\sum_{r=\frac{a}{h}+1}^{\frac{t}{h}} h(\nabla_h\varphi)(rh)\,{}_hE_{\overline{\beta}}\big(\lambda, t-\varpi(rh)\big)$$

$$\left({}^{ABC}_{\ a}\nabla_h^{\beta}\varphi\right)(t) = H(\beta,h)\left(\frac{1-\beta+\beta h}{1-\beta}\right)\ \left[\nabla_h\varphi(t) * {}_{\mathrm{h}}E_{\overline{\beta}}(\lambda, t-a)\right] \qquad (34)$$

$$t \in \mathbb{N}_{a+h,h}.$$

And in the left Riemann sense by:

$$\left({}^{ABR}_{\ a}\nabla_h^{\beta}\varphi\right)(t) = H(\beta,h)\left(\frac{1-\beta+\beta h}{1-\beta}\right)\nabla_h \times$$

$$\sum_{r=\frac{a}{h}+1}^{\frac{t}{h}} h\varphi(rh)\,{}_hE_{\overline{\beta}}\big(\lambda, t-\varpi(rh)\big)$$

$$\left({}^{ABR}_{\ a}\nabla_h^{\beta}\varphi\right)(t) = H(\beta,h)\left(\frac{1-\beta+\beta h}{1-\beta}\right)\nabla_h\ \left[\varphi(t)\,{}_{\mathrm{h}}E_{\overline{\beta}}(\lambda, t-a)\right], \qquad (35)$$

$$t \in \mathbb{N}_{a+h,h}.$$

Definition 16 [19]**.** The left side Atangana-Baleanu fractional derivative in the Caputo sense with Mittag-Leffler $B(\beta)\ is\ normalized\ function\ and\ 0 < \beta < 1$, is given by

$$({}^{ABC}_{a}\mathrm{D}^{\beta}\,\varphi)(t) = \frac{B(\beta)}{1-\beta}\int_{a}^{t}\dot{\varphi}(n)\,E_{\beta}\big(\lambda(t-n)^{\beta}\big)\,dn \tag{36}$$

Lemma 3[3]**.** (The relation between the ABC and Riemann fractional with ML kernels)

$$({}^{ABC}_{a}\nabla^{\beta}_{h}\varphi)(t) = ({}^{ABR}_{a}\nabla^{\beta}_{h}\varphi)(t) - H(\beta,h)\left(\frac{1-\beta+\beta h}{1-\beta}\right)\ \varphi(a)_{\mathrm{h}}E_{\overline{\beta}}(\lambda, t-a) \tag{37}$$

2.5. Generalized ABC Fractional Differences with Three Parameters

Definition 17 [4]**.** The discrete generalized ABC and ABR fractional derivatives with kernel $E^{\gamma}_{\overline{\beta,\mu}}(\lambda,t)$, where $0 < \beta < \frac{1}{2}.Re(\mu) > 0$, $\gamma \in \mathbb{R}$, are defined respectively by

$$({}^{ABC}_{a}\nabla^{\beta,\mu,\gamma}\varphi)(t) = \frac{B(\beta)}{1-\beta}\sum_{r=a+1}^{t} E^{\gamma}_{\overline{\beta,\mu}}\big(\lambda, t-\varpi(r)\big)\,\nabla_{r}\varphi(r),\quad t \in \mathbb{N}_{a} \tag{38}$$

And

$$\left({}_{a}^{ABR}\nabla^{\beta,\mu,\gamma}\varphi\right)(t) = \frac{B(\beta)}{1-\beta}\nabla_t \sum_{r=a+1}^{t} E_{\overline{\beta,\mu}}^{\gamma}\big(\lambda, t-\varpi(r)\big)\,\varphi(r),$$

$$t \in \mathbb{N}_a \qquad (39)$$

$$B(\beta) \mathit{is\ a\ normalized\ function\ and}\ \lambda = \frac{-\beta}{1-\beta}$$

Below we recall the continuous counterparts.

Definition 18 [5] **.** The left side ABC operator with generalized Mittag-Leffler function $E_{\beta,\mu}^{\gamma}\left(\lambda t^{\beta}\right)$ such that $\gamma \in \mathbb{R},\ Re(\mu) > 0,\ 0 < \beta < 1$ and $\lambda = \frac{-\beta}{1-\beta}$ is defined by

$$\left({}_{a}^{ABC}\mathrm{D}^{\beta,\mu,\gamma}\varphi\right)(t) = \frac{B(\beta)}{1-\beta}\int_a^t \dot{\varphi}(n)\, E_{\beta,\mu}^{\gamma}\left(\lambda(t-n)^{\beta}\right) dn \qquad (40)$$

Lemma 4 [4]. (The relation between generalized ABC and ABR with three parameters $E_{\overline{\beta,\mu}}^{\gamma}(\lambda,t)$ kernel)

$$\left({}_{a}^{ABC}\nabla^{\beta,\mu,\gamma}\varphi\right)(t) = \left({}_{a}^{ABR}\nabla^{\beta,\mu,\gamma}\varphi\right)(t) - \frac{B(\beta)}{1-\beta}E_{\overline{\beta,\mu}}^{\gamma}(\lambda, t-a)\varphi(a)$$

$$,\ t \in \mathbb{N}_a,\ \lambda = \frac{-\beta}{1-\beta}, 0 < \beta < \frac{1}{2}, Re(\mu) > 0, \gamma \in \mathbb{R}, \varphi \in \mathbb{N}_a \qquad (41)$$

Lemma 5 [4]**.** The solution of nonhomogeneous linear system

$\left({}^{ABC}_{\ a}\mathrm{D}^{\beta,\mu,\gamma}\varphi\right)(t) = r\varphi(t) + g(t)$, with $0 < \beta < 1,\ \gamma, \mu \in \mathbb{C}\,, t \geq a$

is given by

$$\varphi(t) = \varphi(a) \sum_{n=0}^{\infty} r^n \left(\frac{1-\beta}{B(\beta)}\right)^n E_{\beta,n(1-\mu)+1}^{-\gamma n}(\lambda, t-a) + g(t)$$

$$* \sum_{n=0}^{\infty} r^n \left(\frac{1-\beta}{B(\beta)}\right)^{n+1} E_{\beta,(n+1)(1-\mu)}^{-\gamma n}(\lambda, t-a)$$

$$\varphi(t) = \varphi(a) + \varphi(a) \sum_{n=0}^{\infty} r^n \left(\frac{1-\beta}{B(\beta)}\right)^n E_{\beta,n(1-\mu)+1}^{-\gamma n}(\lambda, t-a)$$

$$+g(t) * \sum_{n=0}^{\infty} r^n \left(\frac{1-\beta}{B(\beta)}\right)^{n+1} E_{\beta,(n+1)(1-\mu)}^{-\gamma(n+1)}(\lambda, t-a) \tag{42}$$

2.6. Discrete Laplace Transform

Definition 19 [3, 29]**.** Assume $\varphi(t)$ is defined on $\mathbb{N}_{a,h}$. Then the h- discrete Laplace transform of φ is defined by

$$\mathcal{L}_{a,h}\{\varphi(t)\}(s) = \int_0^{\infty} {}_{\mathrm{h}}\varrho_{\lambda s}^{\varpi}(t,a)\varphi(t)\nabla_h t$$

$$= \int_0^{\infty} \frac{{}_{\mathrm{h}}\varrho_{\lambda s}(t,a)}{1-hs}\varphi(t)\nabla_h t$$

$$= \int_0^{\infty} (1 - hs)^{\frac{t-a-h}{h}} \varphi(t) \nabla_h t, \tag{43}$$

$$= h \sum_{t=\frac{a}{h}+1}^{\infty} (1 - hs)^{t-\frac{a}{h}-1} \varphi(ht) \tag{44}$$

When $a = 0$ we have

$$\mathcal{L}_{0,h}\{\varphi(t)\}(s) = \mathcal{L}_h\{\varphi(t)\}(s)$$

$$= h \sum_{t=1}^{\infty} (1 - hs)^{t-1} \varphi(ht) \tag{45}$$

When $h = 1$ we have

$$\mathcal{L}_{a,1}\{\varphi(t)\}(s) = \mathcal{L}_a\{\varphi(t)\}(s)$$

$$= \int_0^{\infty} (1 - s)^{t-a-1} \varphi(t) \nabla \ t,$$

$$= \sum_{t=a+1}^{\infty} (1 - s)^{(t-a)-1} \varphi(t)$$

Lemma 6 [3]**.** Let φ be defined on $\mathbb{N}_{a,h.}$Then

$$\mathcal{L}_{a,h}\{\nabla_h \varphi(t)\}(s) = s\mathcal{L}_{a,h}\{\varphi(t)\}(s) - \varphi(a) \tag{46}$$

Lemma 7 [3]**.** The Laplace transform of Nabla h-discrete fractional derivative with exponential kernel is given by

$$\mathcal{L}_{a,h}\left\{{}_{\mathrm{h}}\varrho_{\lambda}\ (t, a)\right\}(s) = \frac{1}{s - \lambda} = \frac{(1 - \beta)}{(1 - \beta)s + \beta}\ , \lambda = \frac{-\beta}{1 - \beta} \tag{47}$$

Lemma 8 [3]**.** Let $\beta \in \mathbb{R}\backslash\{\dots\dots\dots,-3,-2,-1,0\}$ and $\mid 1-hs \mid< 1$, then we have

$$\mathcal{L}_{a,h}\left\{(t-a)_h^{\overline{\beta-1}}\right\}(s)=\frac{\Gamma(\beta)}{s^\beta},h>0 \tag{48}$$

Lemma 9 [4]**.**(i) Assume j,v , are discrete functions defined on $\mathbb{N}_a$.

Then, we have (The h-convolution Theorem)

$$(\mathcal{L}_{a,h}(j*v))(s)=\mathcal{L}_{a,h}(j)(s)\mathcal{L}_{a,h}(v)(s)$$

For $h=1$

$$(\mathcal{L}_a(j*v))(s)=\mathcal{L}_a(j)(s)\mathcal{L}_a(v)(s)$$

This is consistent with the continuous counterpart:

(ii) Assume j,v are functions $[0,\infty)\to\mathbb{R}$ Then we have

$$(\mathcal{L}_a(j*v))(s)=\mathcal{L}_a(j)(s)\mathcal{L}_a(v)(s)$$

Lemma 10 [3]**.** For $\beta,\gamma,\lambda\in\mathbb{C}$, $Re(\gamma)>0$, and $s\in\mathbb{C}$ with $Re(s)>0$

$\mid \lambda s^{-\beta} \mid< 1$, we have

$$(\mathcal{L}_{0,h}\{{}_{\mathrm{h}}E_{\overline{\beta,\gamma}}(\lambda,t)\})(s)\cong(\mathcal{L}_h\{{}_{\mathrm{h}}E_{\overline{\beta,\gamma}}(\lambda,t)\})(s)=s^{-\gamma}\left[1-\lambda s^{-\beta}\right]^{-1}$$

In fact

$$(\mathcal{L}_{0,h}\{{}_{\mathrm{h}}E_{\overline{\beta,\gamma}}(\lambda,t)\})\cong(s)(\mathcal{L}_h\{{}_{\mathrm{h}}E_{\overline{\beta,\gamma}}(\lambda,t-a)\})(s)=s^{-\gamma}\left[1-\lambda s^{-\beta}\right]^{-1}$$

Lemma 11[3]**.** For any $\beta > 0$ we have

$$\mathcal{L}_{a,h}\left\{{}_{a}\nabla_{h}^{-\beta}\varphi(t)\right\}(s) = s^{-\beta}\varphi(s)$$

$$where\ \mathcal{L}_{a,h}\{\varphi(t)\}(s) = \varphi(s)$$

Lemma 12 [6, 9, 10, 27, 35]. For $0 < \beta < 1$ the Laplace transform of certain functions hold that

- $(\mathcal{L}_h\{E_\beta(-ct^\beta)\})(s) = \frac{s^\beta}{s(s^\beta+c)}$
- $(\mathcal{L}_h\{1 - E_\beta(-ct^\beta)\})(s) = \frac{c}{s(s^\beta+c)}$
- $(\mathcal{L}_h\{t^{\beta-1}E_{\beta,\beta}(-ct^\beta)\})(s) = \frac{1}{(s^\beta+c)}$
- $(\mathcal{L}_h\{E_{\overline{\beta}}(\lambda, t-a)\})(s) = \frac{s^\beta}{s(s^\beta-\lambda)}$
- $(\mathcal{L}_h\{E_{\overline{\beta,\beta}}(\lambda, t-a)\})(s) = \frac{1}{(s^\beta-\lambda)}$

  $with\ c, \lambda \in \mathbb{C}$

Remark 3 [34]**.** One of Laplace transform rules

- $(\mathcal{L}_h\{\frac{1}{s-c}\})(s) = e^{ct}$

Lemma 13. For $\gamma, \beta, \mu \in \mathbb{C}, Re(\mu) > 0$ and $s \in \mathbb{C}$ with $Re(s) > 0$,

$|\ \lambda s^{-\beta}\ | < 1$,

$$\mathcal{L}_{a,h}\left\{{}_{\mathrm{h}}E_{\overline{\beta,\mu}}^{\gamma}(\lambda, t-a)\right\}(s) = s^{-\mu}\left[1 - \lambda s^{-\beta}\right]^{-\gamma}$$

$For\ \gamma = 1,$

$$\mathcal{L}_{a,h}\left\{{}_{\mathrm{h}}E_{\overline{\beta,\mu}}(\lambda, t-a)\right\}(s) = s^{-\mu}\left[1-\lambda s^{-\beta}\right]^{-1}$$

Proof: In view of Lemma 8, and by using the Definition 8, and that

$$\sum_{k=0}^{\infty}(\lambda s^{-\beta})^{k} = \frac{1}{1-\lambda s^{-\beta}}\ for\ \ |\lambda s^{-\beta}| < 1 \tag{49}$$

We can conclude

$$\mathcal{L}_{a,h}\left\{{}_{\mathrm{h}}E^{\gamma}_{\overline{\beta,\mu}}(\lambda, t-a)\right\}(s) = \sum_{k=0}^{\infty}\frac{\lambda^{k}(\gamma)_{k}}{k!\, s^{\beta k+\mu}}$$

$$= s^{-\mu}\sum_{k=0}^{\infty}\frac{\left(\lambda s^{-\beta}\right)^{k}(\gamma)_{k}}{k!}$$

$$= s^{-\mu}\left[1-\lambda s^{-\beta}\right]^{-\gamma}$$

Where by differentiating Equation (49) $\gamma - times$, we find that

$$\sum_{k=0}^{\infty}\frac{\left(\lambda s^{-\beta}\right)^{k}(\gamma)_{k}}{k!} = \frac{1}{[1-\lambda s^{-\beta}]^{\gamma}}, for\ \ |\lambda s^{-\beta}| < 1$$

since $(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, \gamma \in \mathbb{C}$

Lemma 14 [3] **.**(i) The Laplace transform of h-discrete CFC is given as

$$\mathcal{L}_{a,h}\left\{{}^{CFC}_{\ a}\nabla_h^{\beta}\varphi(t)\right\}(s) = \frac{H(\beta,h)(1-\beta+\beta h)s}{(1-\beta)(s-\lambda)}\varphi(s)$$

$$-\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(s-\lambda)}\varphi(a) \qquad (50)$$

For $h = 1$ we have

$$\mathcal{L}_{a}\left\{{}^{CFC}_{\ a}\nabla^{\beta}\,\varphi(t)\right\}(s) = \frac{B(\beta)}{(1-\beta)(s-\lambda)}[s\varphi(s)-\varphi(a)]$$

(ii) The Laplace transform of CFC fractional derivative

$$\mathcal{L}_{a}\left\{{}^{CFC}_{\ a}\mathrm{D}^{\beta}\,\varphi(t)\right\}(s) = \frac{B(\beta)}{(1-\beta)(s-\lambda)}[s\varphi(s)-\varphi(a)] \qquad (51)$$

Lemma 15[3, 18]**.**(i) The h-discrete Laplace transform of ABC is given as

$$\mathcal{L}_{a,h}\left\{{}^{ABC}_{\ a}\nabla_h^{\beta}\varphi(t)\right\}(s) = \frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)s\left(1-\lambda s^{-\beta}\right)}[s\varphi(s)-\varphi(a)] \qquad (52)$$

For $h = 1$ we have

$$\mathcal{L}_{a}\left\{{}^{ABC}_{\ a}\nabla^{\beta}\,\varphi(t)\right\}(s) = \frac{B(\beta)}{(1-\beta)s(1-\lambda s^{-\beta})}[s\varphi(s)-\varphi(a)]$$

(ii) The Laplace transform of ABC fractional derivative is given by

$$\mathcal{L}_{a}\left\{{}^{ABC}_{\ a}\mathrm{D}^{\beta}\,\varphi(t)\right\}(s) = \frac{B(\beta)}{(1-\beta)s\left(1-\lambda s^{-\beta}\right)}[s\varphi(s)-\varphi(a)] \qquad (53)$$

Lemma 16 [4]**.** (i) The h-discrete Laplace transform of generalized ABC with three parameters is given as

$$\mathcal{L}_a\left\{{}^{ABC}_{\ a}\nabla^{\beta,\mu,\gamma}\varphi(t)\right\}(s) = \frac{B(\beta)s^{1-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\beta)}\varphi(s)$$

$$-\frac{B(\beta)s^{-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\beta)}\varphi(a) \tag{54}$$

(ii) The Laplace transform of generalized ABC fractional derivative is

$$\mathcal{L}_a\left\{{}^{ABC}_{\ a}\mathrm{D}^{\beta,\mu,\gamma}\varphi(t)\right\}(s) = \frac{B(\beta)s^{1-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\beta)}\varphi(s)$$

$$-\frac{B(\beta)s^{-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\beta)}\varphi(a) \tag{55}$$

The definitions presented in the next section are introduced for the first time.

2.7. Generalized ABC Fractional Differences with Four Parameters.

The modified definitions in this section will result in CF definitions as a special case.

Definition 20. The discrete generalized ABC and ABR fractional derivatives with kernel $E^{\gamma}_{\overline{\beta,\mu}}(\lambda,t)$, where $0 < \theta,\beta < \frac{1}{2}, h = 1,\ Re(\mu) > 0$, $\gamma \in \mathbb{R}$, are defined respectively by

$$\left({}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}\varphi\right)(t)=\frac{B(\theta)}{1-\theta}\sum_{r=a+1}^{t}E_{\overline{\beta,\mu}}^{\gamma}\big(\lambda,t-\varpi(r)\big)\,\nabla_{r}\varphi(r),$$

$$t\in\mathbb{N}_a \tag{56}$$

And

$$\left({}_{a}^{ABR}\nabla^{\beta,(\theta,\mu,\gamma)}\varphi\right)(t)=\frac{B(\theta)}{1-\theta}\nabla_t\sum_{r=a+1}^{t}E_{\overline{\beta,\mu}}^{\gamma}\big(\lambda,t-\varpi(r)\big)\,\varphi(r),$$

$$t\in\mathbb{N}_a \tag{57}$$

$$B(\theta) is\ a\ normalized\ function\ and\ \lambda=\frac{-\theta}{1-\theta}$$

Where

If $\beta=\mu=\gamma=1$, we will have

$$\left({}_{a}^{ABC}\nabla^{1,(\theta,1,1)}\varphi\right)(t)=\frac{B(\theta)}{1-\theta}\sum_{r=a+1}^{t}E_{\overline{1,1}}\big(\lambda,t-\varpi(r)\big)\,\nabla_{r}\varphi(r),\ t\in\mathbb{N}_a$$

By using Definition 10 with $h=1$, we have

$$\left({}_{a}^{ABC}\nabla^{\theta}\,\varphi\right)(t)=\left(\frac{H(\theta)}{1-\theta}\right)\sum_{r=a+1}^{t}(\nabla_r\varphi)(r)(1-\beta)^{t-\varpi(r)}\ =\left({}_{a}^{CFC}\nabla_h^{\beta}\varphi\right)(t)$$

Which is the definition of CFC with order θ and $h=1$.

If $\beta=\theta$, we will have

$$\left({}_{a}^{ABC}\nabla^{\beta,\mu,\gamma}\varphi\right)(t)=\frac{B(\beta)}{1-\beta}\sum_{r=a+1}^{t}E_{\overline{\beta,\mu}}^{\gamma}\big(\lambda,t-\varpi(r)\big)\,\nabla_{r}\varphi(r)$$

Which is the definition of ABC with order β and $h=1$.

Lemma 17. (The relation between generalized ABC and ABR with four parameters with $E^{\gamma}_{\beta,\mu}(\lambda,t)$ kernel)

$$({}^{ABC}_{a}\nabla^{\beta,(\theta,\mu,\gamma)}\varphi)(t) = ({}^{ABR}_{a}\nabla^{\beta,(\theta,\mu,\gamma)}\varphi)(t)$$

$$-\frac{B(\theta)}{1-\theta}E^{\gamma}_{\beta,\mu}(\lambda,t-a)\varphi(a)$$

$$t \in \mathbb{N}_a\ \lambda = \frac{-\theta}{1-\theta}, 0 < \theta < \frac{1}{2}, Re(\mu) > 0, \gamma \in \mathbb{R}, \varphi \in \mathbb{N}_a \tag{58}$$

Proof:

From the relation which are deduced from Lemma 13 and by the help of the discrete h-convolution theorem with (h=1) stated in Lemma 9.

$$\mathcal{L}_a\left\{{}^{ABR}_{a}\nabla^{\beta,(\theta\mu,\gamma)}\varphi(t)\right\}(s) = \frac{B(\theta)s^{1-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\theta)}\varphi(s)$$

And

$$\mathcal{L}_a\left\{{}^{ABC}_{a}\nabla^{\beta,(\theta\mu,\gamma)}\varphi(t)\right\}(s) = \frac{B(\theta)s^{1-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\theta)}\varphi(s)$$

$$-\frac{B(\theta)s^{-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\theta)}\varphi(a)$$

So, we conclude that

$$\mathcal{L}_a\left\{{}^{ABC}_{\ a}\nabla^{\beta,(\theta\mu,\gamma)}\varphi(t)\right\}(s) = \mathcal{L}_a\left\{{}^{ABR}_{\ a}\nabla_h^{\beta,(\theta\mu,\gamma)}\varphi(t)\right\}(s)$$

$$-\frac{B(\theta)s^{-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\theta)}\varphi(a) \tag{59}$$

Applying the Laplace inverse transform for (59) and by using Lemma 13, we have

$$({}^{ABC}_{\ a}\nabla^{\beta,(\theta,\mu,\gamma)}\varphi)(t) = ({}^{ABR}_{\ a}\nabla^{\beta,(\theta,\mu,\gamma)}\varphi)(t) - \frac{B(\theta)}{1-\theta}E^{\gamma}_{\beta,\mu}(\lambda, t-a)\varphi(a)$$

Remark 4. From Lemma17 we conclude that the Laplace transform of generalized ABC with four parameters is given as

$$\mathcal{L}_{a,h}\left\{{}^{ABC}_{\ a}\nabla^{\beta,(\theta\mu,\gamma)}\varphi(t)\right\}(s) = \frac{B(\theta)s^{1-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\theta)}\varphi(s)$$

$$-\frac{B(\theta)s^{-\mu}\left(1-\lambda s^{-\beta}\right)^{-\gamma}}{(1-\theta)}\varphi(a)$$

Chapter 3

Continuous Fractional Economic Models

3.1. Continuous Fractional Derivative

Most of the research deals with solving models using Continuous fractional difference because they are easy to handle and simple in terms of interpreting the results [38].

Here, in this Chapter, we will present a simple review for the solutions of the two economic models in Caputo case and detailed explanation for ABC, CFC and Generalized ABC and interpret their results using continuous fractional derivative.

3.2. Economic Model in Caputo Fractional Derivative

Case 1: The price adjustment without considering the expectation by means of Caputo derivative is as follows

$$ {}_{a}^{C}D^{\beta}u(t) + k(d_1 + s_1)u(t) = k(d_0 + s_0),\ \beta \in (0,1) \tag{60} $$

Taking Laplace transform, the solution for equation (60) as shown in [11] is

$$ u(t) = u(0)E_{\beta}\left(-k(d_1 + s_1)t^{\beta}\right) + \frac{(d_0 + s_0)}{(d_1 + s_1)}\left[1 - E_{\beta}\left(-k(d_1 + s_1)t^{\beta}\right)\right] \tag{61} $$

Where $E_\beta(.)$ is the ML function given in Definition 3.

Case 2: If we take the price adjustment with expectation of agents in the Caputo fractional derivative present by

$$ {}_{a}^{C}D^{\beta}u(t) - \frac{(d_1+s_1)}{(d_2+s_2)}u(t) = -\frac{(d_0+s_0)}{(d_2+s_2)}, \beta \in (0,1) \tag{62} $$

Again taking the Laplace transform for the both side in equation (62) we have as seen in [11]. The solution is

$$ u(t) = u(0)E_\beta\left(\frac{(d_1+s_1)}{(d_2+s_2)}t^\beta\right) - \frac{(d_0+s_0)}{(d_1+s_1)}\left[1 - E_\beta\left(\frac{(d_1+s_1)}{(d_2+s_2)}t^\beta\right)\right] \tag{63} $$

3.3. Economic Model in CFC Fractional Derivative

In [11] the authors gave the solution by means of MATLAB, hear we r the solution analytically.

Case1: The price adjustment equation without considering the expectation with CFC is as follows

$$ {}_{a}^{CFC}D^{\beta}u(t) + k(d_1+s_1)u(t) = k(d_0+s_0), \beta \in (0,1) \tag{64} $$

In order to solve the equation we shall solve the following equation.

- Modified equation of the CFC fractional derivative

Consider that

$$ {}_{a}^{CFC}D^{\beta}u(t) = ru(t) + g(t) $$

$$ u(a) = a_0 $$

$$ \text{such that } ru(a) + g(a) = 0 \tag{65} $$

Apply the Laplace transform for Eq. (65) and by using the result in Lemma 14, we have,

$$ \mathcal{L}_a\left\{ {}_{a}^{CFC}D^{\beta}u(t)\right\} = \mathcal{L}_a\{ru(t)\} + \mathcal{L}_a\{g(t)\} $$

$$ \frac{B(\beta)}{1-\beta}\frac{1}{s-\lambda}[sU(s) - u(a)] = rU(s) + G(s) $$

Where,

$$ \mathcal{L}_a\{u(t)\}(s) = U(s), \mathcal{L}_a\{g(t)\}(s) = G(s), \lambda = \frac{-\beta}{1-\beta} \tag{66} $$

$$ \frac{B(\beta)s}{(1-\beta)(s-\lambda)}U(s) - \frac{B(\beta)}{(1-\beta)(s-\lambda)}u(a) $$

$$ = rU(s) + G(s) \tag{67} $$

$$ U(s)\left[\frac{B(\beta)s}{(1-\beta)(s-\lambda)} - r\right] = G(s) + \frac{B(\beta)}{(1-\beta)(s-\lambda)}u(a) \tag{68} $$

$$U(s)\left[\frac{B(\beta)s-r(1-\beta)(s-\lambda)}{(1-\beta)(s-\lambda)}\right]=\frac{G(s)(1-\beta)(s-\lambda)+B(\beta)u(a)}{(1-\beta)(s-\lambda)},$$

$$\beta\neq 1, s\neq\lambda \tag{69}$$

$$U(s)=\frac{G(s)(1-\beta)(s-\lambda)+B(\beta)u(a)}{B(\beta)s-r(1-\beta)(s-\lambda)}$$

$$U(s)=\frac{G(s)(1-\beta)(s-\lambda)+B(\beta)u(a)}{[B(\beta)-r(1-\beta)]s+r(1-\beta)\lambda} \tag{70}$$

The substitution of $\lambda=\frac{-\beta}{1-\beta}$, implies that

$$U(s)=\frac{G(s)(1-\beta)s+\beta G(s)}{[B(\beta)-r(1-\beta)]\left[s+\frac{r(1-\beta)\lambda}{B(\beta)-r(1-\beta)}\right]}$$

$$+\frac{B(\beta)u(a)}{[B(\beta)-r(1-\beta)]\left[s+\frac{r(1-\beta)\lambda}{B(\beta)-r(1-\beta)}\right]} \tag{71}$$

Now substitute in the denominator that $\lambda=\frac{-\beta}{1-\beta}$ in the Eq. (71)

$$U(s)=\frac{G(s)(1-\beta)s+\beta G(s)}{[B(\beta)-r(1-\beta)]\left[s-\frac{r\beta}{B(\beta)-r(1-\beta)}\right]}$$

$$+\frac{B(\beta)u(a)}{[B(\beta)-r(1-\beta)]\left[s-\frac{r\beta}{B(\beta)-r(1-\beta)}\right]} \tag{72}$$

Now let $\nu=\frac{r\beta}{B(\beta)-r(1-\beta)}$, Eq. (72) becomes

$$U(s) = \frac{1}{[B(\beta)-r(1-\beta)]}\left[\frac{G(s)(1-\beta)s}{[s-\nu]} + \frac{\beta G(s)}{[s-\nu]} + \frac{B(\beta)u(a)}{[s-\nu]}\right] \quad (73)$$

Laplace inverse transform for Eq. (73)

$$U(s) = \frac{1}{[B(\beta)-r(1-\beta)]} \times$$

$$\left[\mathcal{L}_a^{-1}\left\{\frac{G(s)(1-\beta)s}{[s-\nu]}\right\} + \beta\mathcal{L}_a^{-1}\left\{\frac{G(s)}{[s-\nu]}\right\} + \mathcal{L}_a^{-1}\left\{\frac{B(\beta)u(a)}{[s-\nu]}\right\}\right] \quad (74)$$

So, by using Laplace rules in Remark 3, (74) becomes

$$u(t) = \frac{1}{[B(\beta)-r(1-\beta)]} \times$$

$$\left[\mathcal{L}_a^{-1}\left\{\frac{G(s)(1-\beta)s}{[s-\nu]}\right\} + \beta\mathcal{L}_a^{-1}\left\{\frac{G(s)}{[s-\nu]}\right\} + B(\beta)u(a)e^{\nu(t-a)}\right] \quad (75)$$

In particular if $g(t) = constant = c$, so $\mathcal{L}_a\{g(t)\} = G(s) = \frac{c}{s}$, Eq. (75) becomes,

$$u(t) = \frac{1}{[B(\beta)-r(1-\beta)]} \times$$

$$\left[\mathcal{L}_a^{-1}\left\{\frac{c(1-\beta)}{[s-\nu]}\right\} + \beta\mathcal{L}_a^{-1}\left\{\frac{c}{s[s-\nu]}\right\} + B(\beta)u(a)e^{\nu(t-a)}\right] \quad (76)$$

By using Remark 3 and Lemma 9, we have

$$u(t)=\frac{c(1-\beta)}{B(\beta)-r(1-\beta)}e^{\nu(t-a)}+\frac{\beta c*e^{\nu(t-a)}}{B(\beta)-r(1-\beta)}$$

$$+\frac{B(\beta)u(a)}{B(\beta)-r(1-\beta)}e^{\nu(t-a)} \tag{77}$$

Now sitting $\nu=\frac{r\beta}{B(\beta)-r(1-\beta)}$ in (77) we get

$$u(t)=\frac{c(1-\beta)e^{\frac{r\beta}{B(\beta)-r(1-\beta)}(t-a)}}{B(\beta)-r(1-\beta)}+\frac{\beta c*e^{\frac{r\beta}{B(\beta)-r(1-\beta)}(t-a)}}{B(\beta)-r(1-\beta)}$$

$$+\frac{B(\beta)u(a)e^{\frac{r\beta}{B(\beta)-r(1-\beta)}(t-a)}}{B(\beta)-r(1-\beta)} \tag{78}$$

We have from (64) that

$$r=-k(d_1+s_1)$$

$$g(t)=c=k(d_0+s_0) \tag{79}$$

So, substituting equation(79) into equation (78) we obtain

$$u(t)=\frac{k(d_0+s_0)(1-\beta)e^{\frac{-k(d_1+s_1)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)}(t-a)}}{B(\beta)+k(d_1+s_1)(1-\beta)}$$

$$+\frac{k(d_0+s_0)\beta e^{\frac{-k(d_1+s_1)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)}(t-a)}}{B(\beta)+k(d_1+s_1)(1-\beta)}$$

$$+\frac{B(\beta)u(a)e^{\frac{-k(d_1+s_1)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)}(t-a)}}{B(\beta)+k(d_1+s_1)(1-\beta)} \tag{80}$$

$$, with\ u(a) = \frac{d_0 + s_0}{d_1 + s_1}$$

Case 2: If we consider the price adjustment with expectation of agents the CFC derivative of the model is given by

$$ {}_{a}^{CFC}D^{\beta}u(t) - \frac{(d_1 + s_1)}{(d_2 + s_2)}u(t) = -\frac{(d_0 + s_0)}{(d_2 + s_2)}, \beta \in (0,1) \quad (81)$$

Comparing(81) with (65) we have

$$r = \frac{(d_1 + s_1)}{(d_2 + s_2)}$$

$$g(t) = c = -\frac{(d_0 + s_0)}{(d_2 + s_2)} \quad (82)$$

By substituting (82) in the result of (78) we will have

$$u(t) = \frac{-(d_0 + s_0)(1 - \beta)e^{\frac{(d_1 + s_1)\beta}{B(\beta)(d_2 + s_2) - (d_1 + s_1)(1 - \beta)}(t-a)}}{B(\beta)(d_2 + s_2) - (d_1 + s_1)(1 - \beta)}$$

$$- \frac{\beta(d_0 + s_0)e^{\frac{(d_1 + s_1)\beta}{B(\beta)(d_2 + s_2) - (d_1 + s_1)(1 - \beta)}(t-a)}}{B(\beta)(d_2 + s_2) - (d_1 + s_1)(1 - \beta)}$$

$$+ \frac{B(\beta)(d_2 + s_2)u(a)e^{\frac{(d_1 + s_1)\beta}{B(\beta)(d_2 + s_2) - (d_1 + s_1)(1 - \beta)}(t-a)}}{B(\beta)(d_2 + s_2) - (d_1 + s_1)(1 - \beta)}, \quad (83)$$

$$where\ u(a) = \frac{d_0 + s_0}{d_1 + s_1}$$

3.4. Economic Model in ABC Fractional Derivative

In [11], the solution was evaluated by means of Laplace and MATLAB. Here, to find the solution analytically we first consider a modified equation below.

- Modified equation of ABC with β parameter

Consider that

$$ {}_{a}^{ABC}D^{\beta}u(t) = ru(t) + g(t) $$

$$ u(a) = a_0 $$

$$ \text{such that} \quad ru(a) + g(a) = 0 \tag{84} $$

Apply the Laplace transform for Eq. (84) and by using Lemma 15, we have,

$$ \mathcal{L}_a\left\{ABCD^{\beta}u(t)\right\} = \mathcal{L}_a\{ru(t)\} + \mathcal{L}_a\{g(t)\} $$

$$ \left[\frac{B(\beta)}{1-\beta}\frac{s^{\beta}}{(s^{\beta}-\lambda)}U(s) - \frac{B(\beta)}{1-\beta}\frac{s^{\beta-1}}{(s^{\beta}-\lambda)}u(a)\right] = rU(s) + G(s) $$

Where,

$$ \mathcal{L}_a\{u(t)\}(s) = U(s), \mathcal{L}_a\{g(t)\}(s) = G(s), \lambda = \frac{-\beta}{1-\beta} \tag{85} $$

After reformulate the Eq. (85) we have

$$ \frac{B(\beta)}{1-\beta}\frac{s^{\beta}}{(s^{\beta}-\lambda)}U(s) - \frac{B(\beta)}{1-\beta}\frac{s^{\beta-1}}{(s^{\beta}-\lambda)}u(a) = rU(s) + G(s) \tag{86} $$

$$ U(s)\left[\frac{B(\beta)s^{\beta}}{(1-\beta)(s^{\beta}-\lambda)} - r\right] = G(s) + \frac{B(\beta)s^{\beta-1}}{(1-\beta)(s^{\beta}-\lambda)}u(a) \tag{87} $$

$$U(s)\left[\frac{B(\beta)s^{\beta}-r(1-\beta)(s^{\beta}-\lambda)}{(1-\beta)(s^{\beta}-\lambda)}\right]=$$

$$\frac{G(s)(1-\beta)(s^{\beta}-\lambda)+B(\beta)s^{\beta-1}u(a)}{(1-\beta)(s^{\beta}-\lambda)}, where\ \beta\neq 1, s^{\beta}\neq\lambda \tag{88}$$

$$U(s)=\frac{G(s)(1-\beta)(s^{\beta}-\lambda)+B(\beta)s^{\beta-1}u(a)}{B(\beta)s^{\beta}-r(1-\beta)(s^{\beta}-\lambda)}$$

$$U(s)=\frac{G(s)(1-\beta)(s^{\beta}-\lambda)+B(\beta)s^{\beta-1}u(a)}{s^{\beta}[B(\beta)-r(1-\beta)]+r(1-\beta)\lambda} \tag{89}$$

Substitute in the numerator in Eq. (89) that $\lambda=\frac{-\beta}{1-\beta}$

$$U(s)=\frac{G(s)(1-\beta)s^{\beta}+\beta G(s)}{[B(\beta)-r(1-\beta)]\left[s^{\beta}+\frac{r(1-\beta)\lambda}{B(\beta)-r(1-\beta)}\right]}+\frac{B(\beta)s^{\beta-1}u(a)}{[B(\beta)-r(1-\beta)]\left[s^{\beta}+\frac{r(1-\beta)\lambda}{B(\beta)-r(1-\beta)}\right]} \tag{90}$$

Now substitute $\lambda=\frac{-\beta}{1-\beta}$ in (90)

$$U(s)=\frac{G(s)(1-\beta)s^{\beta}+\beta G(s)}{[B(\beta)-r(1-\beta)]\left[s^{\beta}-\frac{r\beta}{B(\beta)-r(1-\beta)}\right]}+\frac{B(\beta)s^{\beta-1}u(a)}{[B(\beta)-r(1-\beta)]\left[s^{\beta}-\frac{r\beta}{B(\beta)-r(1-\beta)}\right]} \tag{91}$$

Now return $\nu=\frac{r\beta}{B(\beta)-r(1-\beta)}$ to (91) we have

$$U(s) = \frac{1}{[B(\beta) - r(1-\beta)]}\left[\frac{G(s)(1-\beta)s^{\beta}}{[s^{\beta} - \nu]} + \frac{\beta G(s)}{[s^{\beta} - \nu]} + \frac{B(\beta)s^{\beta-1}u(a)}{[s^{\beta} - \nu]} \right] \quad (92)$$

The Laplace inverse transform for Eq. (92)

$$U(s) = \frac{1}{[B(\beta) - r(1-\beta)]} \times$$

$$\left[\mathcal{L}_a^{-1}\left\{\frac{G(s)(1-\beta)s^{\beta}}{[s^{\beta}-\nu]}\right\} + \beta\mathcal{L}_a^{-1}\left\{\frac{G(s)}{[s^{\beta}-\nu]}\right\} + \mathcal{L}_a^{-1}\left\{\frac{B(\beta)s^{\beta-1}u(a)}{[s^{\beta}-\nu]}\right\}\right] \quad (93)$$

So, by using Laplace rules in Lemma (12), Eq. (93) becomes

$$U(s) = \frac{1}{[B(\beta) - r(1-\beta)]} \times$$

$$\left[\mathcal{L}_a^{-1}\left\{\frac{G(s)(1-\beta)s^{\beta}}{[s^{\beta}-\nu]}\right\} + \beta\mathcal{L}_a^{-1}\left\{\frac{G(s)}{[s^{\beta}-\nu]}\right\} + B(\alpha)u(a)E_{\alpha}(\nu t^{\alpha})\right] \quad (94)$$

In particular if $g(t) = constant = c$, so $\mathcal{L}_a\,\{g(t)\} = G(s) = \frac{c}{s}$, Eq.(94) becomes,

$$u(t) = \frac{1}{[B(\beta) - r(1-\beta)]} \times$$

$$\left[\mathcal{L}_a^{-1}\left\{\frac{c(1-\beta)s^{\beta}}{s[s^{\beta}-\nu]}\right\} + \beta\mathcal{L}_a^{-1}\left\{\frac{c}{s[s^{\beta}-\nu]}\right\} + B(\beta)u(a)E_{\beta}(\nu t^{\beta})\right] \quad (95)$$

$$u(t) = \frac{c(1-\beta)E_\beta\left(\nu t^\beta\right)}{[B(\beta)-r(1-\beta)]} - \frac{c\beta}{\nu[B(\beta)-r(1-\beta)]}\mathcal{L}_a^{-1}\left\{\frac{-\nu}{s[s^\beta-\nu]}\right\} + \tag{96}$$

$$\frac{B(\beta)u(a)E_\beta(\nu t^\beta)}{[B(\beta)-r(1-\beta)]}$$

$$u(t) = \frac{c(1-\beta)E_\beta\left(\nu t^\beta\right)}{[B(\beta)-r(1-\beta)]} - \frac{c\beta}{\nu[B(\beta)-r(1-\beta)]}\left\{1 - E_\beta\left(\nu t^\beta\right)\right\} +$$

$$\frac{B(\beta)u(a)E_\beta(\nu t^\beta)}{[B(\beta)-r(1-\beta)]} \tag{97}$$

Now substitute $\nu = \frac{r\beta}{B(\beta)-r(1-\beta)}$ in (97)

$$u(t) = \frac{c(1-\beta)}{[B(\beta)-r(1-\beta)]}E_\beta\left(\frac{r\beta}{B(\beta)-r(1-\beta)}t^\beta\right) -$$

$$\frac{c\beta}{\nu[B(\beta)-r(1-\beta)]}\left\{1 - E_\beta\left(\frac{r\beta}{B(\beta)-r(1-\beta)}t^\beta\right)\right\} +$$

$$\frac{B(\beta)u(a)}{[B(\beta)-r(1-\beta)]}\ E_\beta\left(\frac{r\beta}{B(\beta)-r(1-\beta)}t^\beta\right) \tag{98}$$

Case 1: The price adjustment equation without considering the expectation of agents in the framework of ABC derivative is

$$ {}_{a}^{ABC}D^\beta u(t) + k(d_1 + s_1)u(t) = k(d_0 + s_0), \beta \in (0,1) \tag{99}$$

We have from Eq.(99) that

$$r = -k(d_1 + s_1)$$

$$g(t) = c = k(d_0 + s_0) \tag{100}$$

By substitute (100) in (98) we have

$$u(t)$$

$$= \frac{k(d_0 + s_0)(1-\beta)}{B(\beta) + k(d_1 + s_1)(1-\beta)} E_\beta \left(\frac{-k(d_1 + s_1)\beta}{B(\beta) + k(d_1 + s_1)(1-\beta)} t^\beta \right)$$

$$+ \frac{(d_0 + s_0)}{(d_1 + s_1)} \left\{ 1 - E_\beta \left(\frac{-k(d_1 + s_1)\beta}{B(\beta) + k(d_1 + s_1)(1-\beta)} t^\beta \right) \right\} +$$

$$\frac{B(\beta)u(a)}{B(\beta) + k(d_1 + s_1)(1-\beta)} E_\beta \left(\frac{-k(d_1 + s_1)\beta}{B(\beta) + k(d_1 + s_1)(1-\beta)} t^\beta \right)$$

$$where\ u(a) = \frac{d_0 + s_0}{d_1 + s_1} \tag{101}$$

Case 2: The price adjustment with expectation of agents in the framework of ABC is

$${}_{a}^{ABC}D^\beta u(t) - \frac{(d_1 + s_1)}{(d_2 + s_2)} u(t) = -\frac{(d_0 + s_0)}{(d_2 + s_2)}, \beta \in (0,1) \tag{102}$$

We have from (102) and (84) that

$$r = \frac{(d_1 + s_1)}{(d_2 + s_2)}$$

$$g(t) = c = -\frac{(d_0 + s_0)}{(d_2 + s_2)} \tag{103}$$

By substituting (103) in (98)

$$u(t) =$$

$$\frac{-(d_0+s_0)(1-\beta)}{[B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)]} E_\beta \left(\frac{(d_1+s_1)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)} t^\beta\right) -$$

$$\frac{(d_0+s_0)}{(d_1+s_1)} \left\{1 - E_\beta \left(\frac{(d_1+s_1)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)} t^\beta\right)\right\} +$$

$$\frac{B(\beta)u(a)(d_2+s_2)}{[B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)]} E_\beta \left(\frac{(d_1+s_1)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)} t^\beta\right)$$

$$where\ u(a) = \frac{d_0 + s_0}{d_1 + s_1} \qquad (104)$$

3.5. Economic Model in Generalized ABC Fractional Derivative with three parameters

- Modified ABC equation

By using Lemma 5, we have that

$$ {}_{a}^{ABC}D^{\beta,\mu,\gamma}u(t) = ru(t) + g(t) \qquad (105)$$

The explicit solution for (105) is

$$u(t) = u(a)\sum_{n=0}^{\infty} r^n \left(\frac{1-\beta}{B(\beta)}\right)^n E_{\beta,n(1-\mu)+1}^{-\gamma n}(\lambda, t-a)$$

$$+ g(t) * \sum_{n=0}^{\infty} r^n \left(\frac{1-\beta}{B(\beta)}\right)^{n+1} E_{\beta,(n+1)(1-\mu)}^{-\gamma(n+1)}(\lambda, t-a))$$

$$= u(a) + u(a)\sum_{n=0}^{\infty} r^n \left(\frac{1-\beta}{B(\beta)}\right)^n E_{\beta,n(1-\mu)+1}^{-\gamma n}(\lambda, t-a)$$

$$+ g(t) * \sum_{n=0}^{\infty} r^n \left(\frac{1-\beta}{B(\beta)}\right)^{n+1} E_{\beta,(n+1)(1-\mu)}^{-\gamma(n+1)}(\lambda, t-a)) \qquad (106)$$

Case 1: The price adjustment equation without considering the expectation of agents in the framework of Generalized ABC is

$$ {}_{a}^{ABC}D^{\beta,\mu,\gamma}u(t) + k(d_1 + s_1)u(t) = k(d_0 + s_0), \beta \in (0,1) \qquad (107)$$

Referring to Lemma 5 that was mentioned before and substitute $r = -k(d_1 + s_1), g(t) = c = k(d_0 + s_0)$ in (106), we have

$$u(t) = u(a)\sum_{n=0}^{\infty}(-K(d_1+s_1))^n \left(\frac{1-\beta}{B(\beta)}\right)^n E_{\beta,n(1-\mu)+1}^{-\gamma n}(\lambda, t-a)$$

$$+ K(d_0+s_0)\sum_{n=0}^{\infty}(-K(d_1+s_1))^n \left(\frac{1-\beta}{B(\beta)}\right)^{n+1} E_{\beta,(n+1)(1-\mu)}^{-\gamma(n+1)}(\lambda, t-a))$$

$$= u(a) + u(a)\sum_{n=0}^{\infty}(-K(d_1+s_1))^n \left(\frac{1-\beta}{B(\beta)}\right)^n E_{\beta,n(1-\mu)+1}^{-\gamma n}(\lambda, t-a)$$

$$+ K(d_0+s_0)\sum_{n=0}^{\infty}(-K(d_1+s_1))^n \left(\frac{1-\beta}{B(\beta)}\right)^{n+1} E_{\beta,(n+1)(1-\mu)}^{-\gamma(n+1)}(\lambda, t-a)) \qquad (108)$$

Case 2: the price adjustment with expectation of agents in terms of generalized ABC can be expressed as

$$ {}_{a}^{ABC}D^{\beta,\mu,\gamma}u(t) - \frac{(d_1+s_1)}{(d_2+s_2)}u(t) = -\frac{(d_0+s_0)}{(d_2+s_2)}, \beta \in (0,1) \tag{109} $$

From (109) we have that $r = \frac{(d_1+s_1)}{(d_2+s_2)}, g(t) = c = -\frac{(d_0+s_0)}{(d_2+s_2)}$,then by substitute them in (106) we have

$$
\begin{aligned}
u(t) &= u(a)\sum_{n=0}^{\infty}\left(\frac{(d_1+s_1)}{(d_2+s_2)}\right)^n\left(\frac{1-\beta}{B(\beta)}\right)^n E_{\beta,n(1-\mu)+1}^{-\gamma n}(\lambda, t-a) \\
&\quad - \frac{(d_0+s_0)}{(d_2+s_2)}\sum_{n=0}^{\infty}\left(\frac{(d_1+s_1)}{(d_2+s_2)}\right)^n\left(\frac{1-\beta}{B(\beta)}\right)^{n+1} E_{\beta,(n+1)(1-\mu)}^{-\gamma(n+1)}(\lambda, t-a)) \\
&= u(a) + u(a)\sum_{n=0}^{\infty}\left(\frac{(d_1+s_1)}{(d_2+s_2)}\right)^n\left(\frac{1-\beta}{B(\beta)}\right)^n E_{\beta,n(1-\mu)+1}^{-\gamma n}(\lambda, t-a) \\
&\quad - \frac{(d_0+s_0)}{(d_2+s_2)}\sum_{n=0}^{\infty}\left(\frac{(d_1+s_1)}{(d_2+s_2)}\right)^n\left(\frac{1-\beta}{B(\beta)}\right)^{n+1} E_{\beta,(n+1)(1-\mu)}^{-\gamma(n+1)}(\lambda, t-a))
\end{aligned}
\tag{110}
$$

3.6. Summary

We can summarize the result as we have in [11] as follows

They have analyzed the economic models in frame of the non-local fractional operators such as Caputo, CFC, ABC and generalized ABC containing Mittag-Leffler function with three parameters.

The price adjustment equation is solved by the underlying fractional derivatives when the expectation of agents are considered and not considered by two models. So, two separate solutions of the models have been obtained for each fractional operator.

In order to obtain accurate solutions which satisfy the initial condition when using the CFC and ABC fractional derivatives, they put a condition such that there is no trivial solution .In addition, when we solve the underlying models in terms of Caputo and generalized ABC fractional derivatives, no need to put any condition and so one can obtain the solution directly.

We here get the solution analytically by considering the nonhomogeneous linear equation

$$Du(t) = ru(t) + g(t)$$

But in [11] they do the calculation using MATLAB.

Chapter 4

Discrete Fractional Economic Models

In this Chapter we will solve the economic models using discrete fractional differences including Caputo, CFC, ABC with one parameter and using the ABC fractional difference with four parameter which no one deal with before. The ABC generalized fractional difference, used to generate successive approximation of the linear system the homogeneous and nonhomogeneous one and produce explicit solution for it.

We review for the first time the solutions of these models using these differences, as the solutions of the models based on an ABC generalized fractional difference with four parameters were not review before.

4.1. Economic Model in the Framework of Caputo Fractional Difference

Case1: The price adjustment equation by means of Caputo fractional derivative without considering the expectations of agents is as follows:

$${}_{a}^{C}\nabla_{h}{}^{\beta}u(t)+k(d_1+s_1)u(t)=k(d_0+s_0),\beta\in(0,1) \qquad (111)$$

By applying the Laplace transform $\mathcal{L}_{a,h}$ to both sides of Eq. (111), we have

$$\mathcal{L}_{a,h}\left\{{}_{a}^{C}\nabla_{h}{}^{\beta}u(t)\right\} + k(d_1 + s_1)\mathcal{L}_{a,h}\left\{u(t)\right\} \tag{112}$$

$$= \mathcal{L}_{a,h}\left\{k(d_0 + s_0)\right\}$$

By Definition 11 and utilizing for a = 0, we have

$$\mathcal{L}_{h}\left\{C\nabla_{h}^{\beta}u(t)\right\} = \mathcal{L}_{h}\left\{(\nabla_{h}^{-(1-\beta)}\nabla_{h})u(t)\right\} \tag{113}$$

By using the Lemma 1 (the relation between Caputo and Riemann), Eq. (113) becomes

$$\mathcal{L}_{h}\left\{(\nabla_{h}^{-(1-\beta)}\nabla_{h})u(t)\right\} = \mathcal{L}_{h}\left\{\nabla_{h}\nabla_{h}^{-(1-\beta)}u(t) - \frac{(t)_{h}^{\overline{-\beta}}}{\Gamma(1-\beta)}u(0)\right\} \tag{114}$$

$$\mathcal{L}_{h}\left\{\nabla_{h}\nabla_{h}^{-(1-\beta)}u(t) - \frac{(t)_{h}^{\overline{-\beta}}}{\Gamma(1-\beta)}u(0)\right\}$$

$$= \mathcal{L}_{h}\left\{\nabla_{h}\nabla_{h}^{-(1-\beta)}u(t)\right\} - \mathcal{L}_{h}\left\{\frac{(t)_{h}^{\overline{-\beta}}}{\Gamma(1-\beta)}u(0)\right\} \tag{115}$$

By using Lemma 6, Eq. (115) becomes

$$s\mathcal{L}_{h}\left\{\nabla_{h}^{-(1-\beta)}u(t)\right\}(s) - \nabla^{-(1-\beta)}u(0) - \mathcal{L}_{h}\left\{\frac{(t)_{h}^{\overline{-\beta}}}{\Gamma(1-\alpha)}u(0)\right\} \tag{116}$$

$$s\mathcal{L}_{h}\left\{\nabla_{h}^{-(1-\beta)}u(t)\right\}(s) - \mathcal{L}_{h}\left\{\frac{(t)_{h}^{\overline{-\beta}}}{\Gamma(1-\beta)}u(0)\right\} \tag{117}$$

By using Lemma 11, Eq. (117) becomes

$$s^{\beta}U(s) - \mathcal{L}_h\left\{\frac{(t)_h^{-\overline{\beta}}}{\Gamma(1-\beta)}u(0)\right\} \quad (118)$$

By using Lemma 8, Eq. (118) becomes

$$s^{\beta}U(s) - u(0)s^{\beta-1} \quad (119)$$

Now, Eq. (112) becomes

$$s^{\beta}U(s) - u(0)s^{\beta-1} + k(d_1+s_1)U(s) = \frac{k(d_0+s_0)}{s} \quad (120)$$

$$U(s) = \frac{K(d_0+s_0)}{s(s^{\beta}+k(d_1+s_1))} + \frac{u(0)s^{\beta-1}}{s^{\beta}+k(d_1+s_1)} \quad (121)$$

By applying the Laplace inverse transform $\mathcal{L}_h^{-1}$ to Eq. (121)

$$\mathcal{L}_h^{-1}\{U(s)\} = \mathcal{L}_h^{-1}\left\{\frac{k(d_0+s_0)}{s(s^{\beta}+k(d_1+s_1))}\right\} + \mathcal{L}_h^{-1}\left\{\frac{u(0)s^{\alpha-1}}{s^{\beta}+k(d_1+s_1)}\right\} \quad (122)$$

$$\mathcal{L}_h^{-1}\{U(s)\} = k(d_0+s_0)\mathcal{L}_h^{-1}\left\{\frac{1}{s(s^{\beta}+k(d_1+s_1))}\right\}$$

$$+ u(0)\mathcal{L}^{-1}\left\{\frac{s^{\beta}}{s(s^{\beta}+k(d_1+s_1))}\right\} \quad (123)$$

By using Lemma 12, we have

$$u(t) = K(d_0+s_0)\mathcal{L}_h^{-1}\left\{\frac{1}{s^{\beta+1}\left(1+s^{-\beta}k(d_1+s_1)\right)}\right\}$$

$$+u(0)E_{\overline{\beta}}(-k(d_1+s_1), t-a) \quad (124)$$

By using Lemma 13 with $\mu = \beta + 1$ and $\gamma = 1$

$$u(t) = k(d_0 + s_0)E_{\overline{\beta,\beta+1}}(-k(d_1 + s_1), t)$$

$$+u(0)E_{\overline{\beta}}(-k(d_1 + s_1), t) \tag{125}$$

$E_{\overline{\beta}}(.)$ is the discrete ML function given in Definition 4.

Case 2: On the other hand, if we consider the expectations of agents, the price adjustment equation with Caputo fractional derivative can be presented by

$$ {}_{a}^{C}\nabla_h^{\beta} u(t) - \frac{(d_1 + s_1)}{(d_2 + s_2)} u(t) = -\frac{(d_0 + s_0)}{(d_2 + s_2)} \tag{126}$$

If we apply the Laplace transform to Eq. (126), we have

$$\mathcal{L}_{a,h}\left\{ \ {}_{a}^{c}\nabla_h^{\beta} u(t)\right\} - \frac{(d_1+s_1)}{(d_2+s_2)}\mathcal{L}_{a,h}\{u(t)\} = \mathcal{L}_{a,h}\left\{-\frac{(d_0+s_0)}{(d_2+s_2)}\right\} \tag{127}$$

By using the result in Eq. (119) and utilizing for $a = 0$, Eq. (127) will be

$$s^{\beta}U(s) - u(0)s^{\beta-1} - \frac{(d_1 + s_1)}{(d_2 + s_2)}U(s) = -\frac{(d_0 + s_0)}{s(d_2 + s_2)} \tag{128}$$

$$U(s) = \frac{-\frac{(d_0+s_0)}{s(d_2+s_2)} + u(0)s^{\beta-1}}{s^{\beta} - \frac{(d_1+s_1)}{(d_2+s_2)}} \tag{129}$$

$$U(s) = -\frac{(d_0 + s_0)}{(d_2 + s_2)} \frac{1}{s\left(s^{\beta} - \frac{(d_1+s_1)}{(d_2+s_2)}\right)} + \frac{u(0)S^{\beta}}{s\left(s^{\beta} - \frac{(d_1+s_1)}{(d_2+s_2)}\right)} \tag{130}$$

By taking the Laplace inverse transform $\mathcal{L}_h^{-1}$ to Eq. (130), then we obtain

$$\mathcal{L}_h^{-1}\{U(s)\} = -\frac{(d_0+s_0)}{(d_2+s_2)}\mathcal{L}_{0,h}^{-1}\left\{\frac{1}{s^{\beta+1}\left(1-s^{-\beta}\frac{(d_1+s_1)}{(d_2+s_2)}\right)}\right\}$$

$$+u(0)\mathcal{L}_{0,h}^{-1}\left\{\frac{s^\beta}{s\left(s^\beta-\frac{(d_1+s_1)}{(d_2+s_2)})\right)}\right\} \tag{131}$$

Using the relations in Lemma.12 and Lemma 13 with $\mu = \beta + 1$ and $\gamma = 1$,

we obtain the solution as follows

$$u(t) = -\frac{(d_0+s_0)}{(d_2+s_2)}\left[E_{\overline{\beta,\beta+1}}\left(\frac{(d_1+s_1)}{(d_2+s_2)},t\right)\right]$$

$$+u(0)E_{\overline{\beta}}\left(\frac{(d_1+s_1)}{(d_2+s_2)},t\right) \tag{132}$$

$E_{\overline{\beta}}(.)$ is the discrete ML function given in Definition 4.

Remark:

We can use the Relation $s^\beta U(s) - u(0)s^{\beta-1}$directly form Lemma 8 in [3].

4.2. Economic Model in the Framework of CFC

4.2.1. CFC Modified Function with Parameter β

Let's make the solution generally, consider that

$$ {}_{a}^{CFC}\nabla_h^{\beta} u(t) = ru(t) + g(t) $$

$$ u(a) = a_0 $$

$$ \text{such that } ru(a) + g(a) = 0 \tag{133} $$

By applying the Laplace transform to Eq. (133) and using Lemma 14, we have,

$$ \mathcal{L}_{a,h}\left\{ {}_{a}^{CFC}\nabla_h^{\beta} u(t)\right\} = \mathcal{L}_{a,h}\{ru(t)\} + \mathcal{L}_{a,h}\{g(t)\}, $$

OR,

$$ \frac{H(\beta,h)(1-\beta+\beta h)}{1-\beta}\frac{1}{s-\lambda}[sU(s)-u(a)] = rU(s) + G(s), $$

where,

$$ \mathcal{L}_{a,h}\{u(t)\}(s) = U(s), \mathcal{L}_{a,h}\{g(t)\}(s) = G(s), \lambda = \frac{-\beta}{1-\beta} \tag{134} $$

After reformulate Eq. (134), we have

$$ \frac{H(\beta,h)(1-\beta+\beta h)s}{(1-\beta)(s-\lambda)}U(s) - \frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(s-\lambda)}u(a) = rU(s) + G(s) \tag{135} $$

Then,

$$U(s)\left[\frac{H(\beta,h)(1-\beta+\beta h)s}{(1-\beta)(s-\lambda)}-r\right]$$

$$=G(s)+\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(s-\lambda)}u(a) \tag{136}$$

$$U(s)\left[\frac{H(\beta,h)(1-\beta+\beta h)s-r(1-\beta)(s-\lambda)}{(1-\beta)(s-\lambda)}\right]$$
$$=\frac{G(s)(1-\beta)(s-\lambda)+H(\beta,h)(1-\beta+\beta h)u(a)}{(1-\beta)(s-\lambda)}$$

Where, (137)

$$\beta\neq 1, s\neq\lambda$$

To make $U(s)$ the subject of equation, we see that

$$U(s)=\frac{G(s)(1-\beta)(s-\lambda)+H(\beta,h)(1-\beta+\beta h)u(a)}{H(\beta,h)(1-\beta+\beta h)s-r(1-\beta)(s-\lambda)}$$

$$U(s)=\frac{G(s)(1-\beta)(s-\lambda)+H(\beta,h)(1-\beta+\beta h)u(a)}{[H(\beta,h)(1-\beta+\beta h)-r(1-\beta)]s+r(1-\beta)\lambda} \tag{138}$$

Substitute in the numerator in Eq. (138) that $\lambda=\frac{-\beta}{1-\beta}$

$$U(s)=\frac{G(s)(1-\beta)s+\beta G(s)}{[H(\beta,h)(1-\beta+\beta h)-r(1-\beta)]\left[s+\frac{r(1-\beta)\lambda}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}\right]}$$

$$+\frac{H(\beta,h)(1-\beta+\beta h)u(a)}{[H(\beta,h)(1-\beta+\beta h)-r(1-\beta)]\left[s+\frac{r(1-\beta)\lambda}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}\right]} \tag{139}$$

Now substitute in the denominator that $\lambda = \frac{-\beta}{1-\beta}$ in Eq. (139)

$$U(s) = \frac{G(s)(1-\beta)s + \beta G(s)}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]\left[s - \frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}\right]}$$

$$+ \frac{H(\beta,h)(1-\beta+\beta h)u(a)}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]\left[s - \frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}\right]} \quad (140)$$

If we let $\nu = \frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}$, Then, Eq. (140) becomes

$$U(s) = \frac{1}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]}\left[\frac{G(s)(1-\beta)s}{[s-\nu]}\right.$$

$$\left. + \frac{\beta G(s)}{[s-\nu]} + \frac{H(\beta,h)(1-\beta+\beta h)u(a)}{[s-\nu]}\right] \quad (141)$$

By taking the Laplace inverse transform to Eq. (141)

$$U(s) = \frac{1}{[H(\beta,h)(1-\beta+\beta h)-r(1-\beta)]}\left[\mathcal{L}_{a,h}^{-1}\left\{\frac{G(s)(1-\beta)s}{[s-\nu]}\right\} + \right.$$

$$\left. \beta\mathcal{L}_{a,h}^{-1}\left\{\frac{G(s)}{[s-\nu]}\right\} + \mathcal{L}_{a,h}^{-1}\left\{\frac{H(\beta,h)(1-\beta+\beta h)u(a)}{[s-\nu]}\right\}\right] \quad (142)$$

So, by using Laplace rules in Lemma 7, Eq. (142) becomes

$$u(t) = \frac{1}{[H(\beta,h)(1-\beta+\beta h)-r(1-\beta)]}\left[\mathcal{L}_{a,h}^{-1}\left\{\frac{G(s)(1-\beta)s}{[s-\nu]}\right\} + \right.$$

$$\left. \beta\mathcal{L}_{a,h}^{-1}\left\{\frac{G(s)}{[s-\nu]}\right\} + H(\beta,h)(1-\beta+\beta h)u(a)_h\varrho_\nu^\beta(t,a)\right] \quad (143)$$

In particular if $g(t) = constant$ for example c, so $\mathcal{L}_{a,h}\{g(t)\} = G(s) = \frac{c}{s}$,

Eq. (143) becomes,

$$u(t) = \frac{1}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]}\Bigg[\mathcal{L}_{a,h}^{-1}\left\{\frac{c(1-\beta)}{[s-\nu]}\right\}$$

$$+ \beta\mathcal{L}_{a,h}^{-1}\left\{\frac{c}{s[s-\nu]}\right\} + H(\beta,h)(1-\beta$$

$$+ \beta h)u(a)_h\varrho_\nu(t,a)\Bigg] \tag{144}$$

By using Lemma 7 and Lemma 9, (144) will be

$$u(t) = \frac{c(1-\beta)_h\varrho_\nu(t,a)}{H(\beta,h)(1-\beta+\beta h) - r(1-\beta)}$$

$$+ \frac{\beta c *_h \varrho_\nu(t,a)}{H(\beta,h)(1-\beta+\beta h) - r(1-\beta)}$$

$$+ \frac{H(\beta,h)(1-\beta+\beta h)u(a)_h\varrho_\nu(t,a)}{H(\beta,h)(1-\beta+\beta h) - r(1-\beta)} \tag{145}$$

Such that $H(\beta,h) = B(\beta)\left[\frac{\beta}{h} + (1-\beta)\right]$, $\nu = \frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}$

By substitute ν in (145), we will have

$$u(t) = \frac{c(1-\beta)_h\varrho_{\frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}}(t,a)}{H(\beta,h)(1-\beta+\beta h) - r(1-\beta)}$$

$$+ \frac{\beta c *_h \varrho_{\frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}}(t,a)}{H(\beta,h)(1-\beta+\beta h) - r(1-\beta)}$$

$$+ \frac{H(\beta,h)(1-\beta+\beta h)u(a)_h\varrho_{\frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}}(t,a)}{H(\beta,h)(1-\beta+\beta h) - r(1-\beta)}$$

(146)

If we take h=1 for (146) we have

$$u(t)=\frac{c(1-\beta)\varrho_{\frac{r\beta}{B(\beta)-r(1-\beta)}}(t,a)}{B(\beta)-r(1-\beta)}+\frac{\beta c*\varrho_{\frac{r\beta}{B(\beta)-r(1-\beta)}}(t,a)}{B(\beta)-r(1-\beta)}+\frac{B(\beta)u(a)\varrho_{\frac{r\beta}{B(\beta)-r(1-\beta)}}(t,a)}{B(\beta)-r(1-\beta)} \tag{147}$$

Such that $B(\beta)$ is normalized function and ϱ is the exponential Nabla defined in Definition 10.

4.2.2. The Model without Considering the Expectation of Agents

The price adjustment equation with CFC without considering the expectations of agents can be written as

$${}_{a}^{CFC}\nabla_h^{\beta}u(t)+k(d_1+s_1)u(t)=k(d_0+s_0),\beta\in(0,1) \tag{148}$$

In reference to (148) with

$$r=-k(d_1+s_1)$$

$$g(t)=c=k(d_0+s_0)$$

Then the solution of equation (148) becomes,

$$u(t)=\frac{k(d_0+s_0)(1-\beta)_h\varrho_{\frac{-k(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}}(t,a)}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}+\frac{\beta k(d_0+s_0)_h\varrho_{\frac{-k(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}}(t,a)}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}$$

$$+\frac{H(\beta,h)(1-\beta+\beta h)u(a)_h\varrho_{\frac{-k(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}}(t,a)}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}$$

where,

$$u(a)=\frac{d_0+s_0}{d_1+s_1} \tag{149}$$

Solving equation (149) with h=1 with

$$r=-k(d_1+s_1)$$

$$g(t)=c=k(d_0+s_0)$$

We obtains,

$$u(t)=\frac{k(d_0+s_0)(1-\beta)\varrho_{\frac{-k(d_1+s_1)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)}}(t,a)}{B(\beta)+k(d_1+s_1)(1-\beta)}$$

$$+\frac{\beta k(d_0+s_0)\varrho_{\frac{-k(d_1+s_1)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)}}(t,a)}{B(\beta)+k(d_1+s_1)(1-\beta)}$$

$$+\frac{B(\beta)u(a)\varrho_{\frac{-k(d_1+s_1)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)}}(t,a)}{B(\beta)+k(d_1+s_1)(1-\beta)}$$

Where,

$$u(a)=\frac{d_0+s_0}{d_1+s_1} \tag{150}$$

4.2.3. The Model with Consider the Expectations of Agents

Moreover, if we consider the expectations of agents, the price adjustment equation with CFC can be given by

$$ {}_{a}^{CFC}\nabla_{h}^{\beta}u(t) - \frac{(d_1 + s_1)}{(d_2 + s_2)}u(t) = -\frac{(d_0 + s_0)}{(d_2 + s_2)} \qquad (151)$$
$$\beta \in (0,1)$$

In reference with (151) with

$$r = \frac{(d_1 + s_1)}{(d_2 + s_2)}$$
$$g(t) = c = -\frac{(d_0 + s_0)}{(d_2 + s_2)}$$

We obtain,

$$u(t) = \frac{(d_0 + s_0)(1-\beta)\, {}_{h}\varrho_{\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}}(t,a)}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)} + \frac{\beta(d_0 + s_0)\, {}_{h}\varrho_{\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}}(t,a)}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)} + \frac{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)u(a)\, {}_{h}\varrho_{\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}}(t,a)}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}$$

Where,

$$u(a) = \frac{d_0 + s_0}{d_1 + s_1} \qquad (152)$$

If we want to solve (151) for $h = 1$ we will substitute $c, g(t)$ then we see that,

$$u(t) = -\frac{(d_0 + s_0)(1-\beta)\varrho_{\frac{(d_1+s_1)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)}}(t,a)}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)} + -\frac{\beta(d_0+s_0)\varrho_{\frac{(d_1+s_1)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)}}(t,a)}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)} + \frac{(d_2+s_2)B(\beta)u(a)\varrho_{\frac{(d_1+s_1)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)}}(t,a)}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)}$$

Where,

$$u(a) = \frac{d_0 + s_0}{d_1 + s_1} \tag{153}$$

4.3. Economic Model in the Framework of ABC

4.3.1. ABC Modified Function with β Parameter

To make the solution more general, consider that

$$ {}^{ABC}_{a}\nabla^{\beta}_{h} u(t) = ru(t) + g(t)$$

$$u(a) = a_0$$

$$\text{such that } ru(a) + g(a) = 0 \tag{154}$$

Apply the Laplace transform for Eq. (154) and by using Lemma 15, we have,

$$\mathcal{L}_{a,h}\left\{ {}^{ABC}_{a}\nabla^{\beta}_{h} u(t)\right\} = \mathcal{L}_{a,h}\{ru(t)\} + \mathcal{L}_{a,h}\{g(t)\}$$

$$\left[\frac{H(\beta,h)(1-\beta+\beta h)}{1-\beta}\frac{s^{\beta}}{(s^{\beta}-\lambda)}U(s)-\frac{H(\beta,h)(1-\beta+\beta h)}{1-\beta}\frac{s^{\beta-1}}{(s^{\beta}-\lambda)}u(a)\right]$$

$$=rU(s)+G(s)$$

$$\mathcal{L}_{a,h}\{u(t)\}(s)=U(s),\mathcal{L}_{a,h}\{g(t)\}(s)=G(s),\lambda=\frac{-\beta}{1-\beta} \tag{155}$$

After reformulate Eq. (155) we have

$$\frac{H(\beta,h)(1-\beta+\beta h)}{1-\beta}\frac{s^{\beta}}{(s^{\beta}-\lambda)}U(s)$$

$$-\frac{H(\beta,h)(1-\beta+\beta h)}{1-\beta}\frac{s^{\beta-1}}{(s^{\beta}-\lambda)}u(a)=rU(s)+G(s) \tag{156}$$

$$U(s)\left[\frac{H(\beta,h)(1-\beta+\beta h)s^{\beta}}{(1-\beta)(s^{\beta}-\lambda)}-r\right]$$

$$=G(s)+\frac{H(\beta,h)(1-\beta+\beta h)s^{\beta-1}}{(1-\alpha)(s^{\beta}-\lambda)}u(a) \tag{157}$$

$$U(s)\left[\frac{H(\beta,h)(1-\beta+\beta h)s^{\beta}-r(1-\beta)(s^{\beta}-\lambda)}{(1-\beta)(s^{\beta}-\lambda)}\right]$$

$$=\frac{G(s)(1-\beta)(s^{\beta}-\lambda)+H(\beta,h)(1-\beta+\beta h)s^{\beta-1}u(a)}{(1-\beta)(s^{\beta}-\lambda)},$$

Where,

$$\beta\neq 1,s^{\alpha}\neq\lambda \tag{158}$$

To make $U(s)$ the subject of equation, we obtains

$$U(s) = \frac{G(s)(1-\beta)(s^{\beta}-\lambda) + H(\beta,h)(1-\beta+\beta h)s^{\beta-1}u(a)}{H(\beta,h)(1-\beta+\beta h)s^{\beta} - r(1-\beta)(s^{\beta}-\lambda)}$$

$$U(s) = \frac{G(s)(1-\beta)(s^{\beta}-\lambda) + H(\beta,h)(1-\beta+\beta h)s^{\beta-1}u(a)}{s^{\beta}[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)] + r(1-\beta)\lambda} \tag{159}$$

Substitute in the numerator in Eq. (159) that $\lambda = \frac{-\beta}{1-\beta}$

$$U(s) = \frac{G(s)(1-\beta)s^{\beta} + \beta G(s)}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]\left[s^{\beta} + \frac{r(1-\beta)\lambda}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}\right]} + \frac{H(\beta,h)(1-\beta+\beta h)s^{\beta-1}u(a)}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]\left[s^{\beta} + \frac{r(1-\beta)\lambda}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}\right]} \tag{160}$$

Now substitute in the denominator that $\lambda = \frac{-\beta}{1-\beta}$ in the Eq. (160)

$$U(s) = \frac{G(s)(1-\beta)s^{\beta} + \beta G(s)}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]\left[s^{\beta} - \frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}\right]} + \frac{H(\beta,h)(1-\beta+\beta h)s^{\beta-1}u(a)}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]\left[s^{\beta} - \frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}\right]} \tag{161}$$

Now let $\nu = \frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}$, Eq. (161) becomes

$$U(s) = \frac{1}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]}$$
$$\left[\frac{G(s)(1-\beta)s^{\beta}}{[s^{\beta}-\nu]} + \frac{\beta G(s)}{[s^{\beta}-\nu]} + \frac{H(\beta,h)(1-\beta+\beta h)s^{\beta-1}u(a)}{[s^{\beta}-\nu]} \right] \tag{162}$$

Taking the Laplace inverse transform to Eq. (162)

$$U(s) = \frac{1}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]}$$
$$\times \left[(1-\beta)\mathcal{L}_{a,h}^{-1}\left\{\frac{G(s)s^{\beta}}{[s^{\beta}-\nu]}\right\} + \beta\mathcal{L}_{a,h}^{-1}\left\{\frac{G(s)}{[s^{\beta}-\nu]}\right\} + H(\beta,h)(1-\beta+\beta h)\mathcal{L}_{a,h}^{-1}\left\{\frac{s^{\beta-1}u(a)}{[s^{\beta}-\nu]}\right\} \right] \tag{163}$$

So, by using Laplace rules in Lemma 12, Eq. (163) becomes

$$u(t) = \frac{1}{[H(\beta,h)(1-\beta+\beta h) - r(1-\beta)]}$$
$$\times \left[\mathcal{L}_{a,h}^{-1}\left\{\frac{G(s)(1-\beta)s^{\beta}}{[s^{\beta}-\nu]}\right\} + \beta\mathcal{L}_{a,h}^{-1}\left\{\frac{G(s)}{[s^{\beta}-\nu]}\right\} + H(\beta,h)(1-\beta+\beta h)u(a)E_{\overline{\beta}}(\nu, t-a) \right] \tag{164}$$

In particular if $g(t) = constant = c$, so $\mathcal{L}_{a,h}\{g(t)\} = G(s) = \frac{c}{s}$, then Eq.(164) becomes,

$$u(t) = \frac{1}{[H(\beta,h)(1-\beta+\beta h)-r(1-\beta)]}\left[c(1-\beta)\mathcal{L}_a^{-1}\left\{\frac{s^\beta}{s[s^\beta-\nu]}\right\} + \beta\mathcal{L}_a^{-1}\left\{\frac{c}{s[s^\beta-\nu]}\right\} + H(\beta,h)(1-\beta+\beta h)u(a)E_{\overline{\beta}}(\nu,t-a)\right] \quad (165)$$

By using Lemma 12 and Lemma 13 with $\mu = \beta + 1$ and $\gamma = 1$

$$u(t) = \frac{c(1-\beta)E_{\overline{\beta}}(\nu,t-a)}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)} + \frac{\beta c}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}\mathcal{L}_{a,h}^{-1}\left\{\frac{1}{s^{\beta+1}[1-s^{-\beta}\nu]}\right\} + \frac{H(\beta,h)(1-\beta+\beta h)u(a)E_{\overline{\beta}}(\nu,t-a)}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)} \quad (166)$$

$$u(t) = \frac{c(1-\beta)E_{\overline{\beta}}(\nu,t-a)}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)} + \frac{\beta c E_{\overline{\beta,\beta+1}}(\nu,t-a)}{(H(\beta,h)(1-\beta+\beta h)-r(1-\beta))} + \frac{H(\beta,h)(1-\beta+\beta h)u(a)E_{\overline{\beta}}(\nu,t-a)}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)} \quad (167)$$

By substituting v, (167) it will be

$$u(t) = \frac{c(1-\beta)}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)} E_{\overline{\beta}}\left(\frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}, t-a\right)$$
$$+ \frac{\beta c}{(H(\beta,h)(1-\beta+\beta h)-r(1-\beta))} E_{\overline{\beta,\beta+1}}\left(\frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}, t-a\right)$$
$$+ \frac{H(\beta,h)(1-\beta+\beta h)u(a)}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)} E_{\overline{\beta}}\left(\frac{r\beta}{H(\beta,h)(1-\beta+\beta h)-r(1-\beta)}, t-a\right)$$

(168)

With $h = 1$ Equation (168) will be

$$u(t) = \frac{c(1-\beta)}{B(\beta)-r(1-\beta)} E_{\overline{\beta}}\left(\frac{r\beta}{B(\beta)-r(1-\beta)}, t-a\right)$$
$$+ \frac{\beta c}{(B(\beta)-r(1-\beta))} E_{\overline{\beta,\beta+1}}\left(\frac{r\beta}{B(\beta)-r(1-\beta)}, t-a\right)$$
$$+ \frac{B(\beta)u(a)}{B(\beta)-r(1-\beta)} E_{\overline{\beta}}\left(\frac{r\beta}{B(\beta)-r(1-\beta)}, t-a\right)$$

(169)

Such that $B(\beta)$ is normalized function

4.3.2. The Model without Considering the Expectation of Agents

The price adjustment equation without considering the expectations of agents with using ABC derivative is given by

$$ {}_{a}^{ABC}\nabla_{h}^{\beta} u(t) + k(d_1 + s_1)u(t) = k(d_0 + s_0), \beta \in (0,1) \qquad (170)$$

By reference to (170) with

$$r = -k(d_1 + s_1)$$

$$g(t) = c = k(d_0 + s_0)$$

We see that,

$$u(t) = \frac{k(d_0 + s_0)(1-\beta)}{H(\beta,h)(1-\beta+\beta h) + k(d_1 + s_1)(1-\beta)} \times E_{\overline{\beta}}\left(\frac{-k(d_1 + s_1)\beta}{H(\beta,h)(1-\beta+\beta h) + k(d_1 + s_1)(1-\beta)}, t-a\right) + \frac{k(d_0 + s_0)\beta}{H(\beta,h)(1-\beta+\beta h) + k(d_1 + s_1)(1-\beta)} \times E_{\overline{\beta,\beta+1}}\left(\frac{-k(d_1 + s_1)\beta}{H(\beta,h)(1-\beta+\beta h) + k(d_1 + s_1)(1-\beta)}, t-a\right) + \frac{H(\beta,h)(1-\beta+\beta h)u(a)}{H(\beta,h)(1-\beta+\beta h) + k(d_1 + s_1)(1-\beta)} \times E_{\overline{\beta}}\left(\frac{-k(d_1 + s_1)\beta}{H(\beta,h)(1-\beta+\beta h) + k(d_1 + s_1)(1-\beta)}, t-a\right)$$

where,

$$u(a) = \frac{d_0 + s_0}{d_1 + s_1} \tag{171}$$

For $h = 1$, by substituting $c, g(t)$we will have

$$u(t) = \frac{k(d_0+s_0)(1-\beta)}{B(\beta)+k(d_1+s_1)(1-\beta)} E_{\overline{\beta}}\left(\frac{-k(d_1+s_1)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)}, t-a\right) + \frac{k(d_0+s_0)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)} E_{\overline{\beta,\beta+1}}\left(\frac{-k(d_1+s_1)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)}, t-a\right) + \frac{B(\beta)u(a)}{B(\beta)+k(d_1+s_1)(1-\beta)} E_{\overline{\beta}}\left(\frac{-k(d_1+s_1)\beta}{B(\beta)+k(d_1+s_1)(1-\beta)}, t-a\right)$$

where,

$$u(a) = \frac{d_0 + s_0}{d_1 + s_1} \tag{172}$$

4.3.3. The Model with Consider the Expectations of Agents

If we consider the expectations of agent in the price adjustment equation with ABC we will have

$${}_{a}^{ABC}\nabla_h^{\beta} u(t) - \frac{(d_1 + s_1)}{(d_2 + s_2)} u(t) = -\frac{(d_0 + s_0)}{(d_2 + s_2)}, \beta \in (0,1) \tag{173}$$

We have r, c from (173) such that

$$r = \frac{(d_1 + s_1)}{(d_2 + s_2)}$$

$$g(t) = c = -\frac{(d_0 + s_0)}{(d_2 + s_2)}$$

By using r, c we have the solution

$$u(t) = \frac{-(d_0 + s_0)(1-\beta)}{H(\beta,h)(1-\beta+\beta h)(d_2 + s_2) - (d_1 + s_1)(1-\beta)}$$

$$E_{\overline{\beta}}\left(\frac{(d_1 + s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2 + s_2) - (d_1 + s_1)(1-\beta)}, t-a\right)$$

$$+ \frac{-(d_0 + s_0)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2 + s_2) - (d_1 + s_1)(1-\beta)}$$

$$E_{\overline{\beta,\beta+1}}\left(\frac{(d_1 + s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2 + s_2) - (d_1 + s_1)(1-\beta)}, t-a\right)$$

$$+ \frac{H(\beta,h)(1-\beta+\beta h)(d_2 + s_2)u(a)}{\big(H(\beta,h)(1-\beta+\beta h)(d_2 + s_2) - (d_1 + s_1)(1-\beta)\big)}$$

$$E_{\overline{\beta}}\left(\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}, t-a\right)$$

where,

$$u(a)=\frac{d_0+s_0}{d_1+s_1} \tag{174}$$

To solve the model in h=1, we substitute , r , we will see

$$u(t)=\frac{-(d_0+s_0)(1-\beta)}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)}E_{\overline{\beta}}\left(\frac{(d_1+s_1)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)}, t-a\right)+$$

$$\frac{-(d_0+s_0)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)}E_{\overline{\beta,\beta+1}}\left(\frac{(d_1+s_1)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)}, t-a\right)+$$

$$\frac{B(\beta)(d_2+s_2)u(a)}{(B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta))}E_{\overline{\beta}}\left(\frac{(d_1+s_1)\beta}{B(\beta)(d_2+s_2)-(d_1+s_1)(1-\beta)}, t-a\right)$$

where,

$$u(a)=\frac{d_0+s_0}{d_1+s_1} \tag{175}$$

4.4. Economic Model in the Framework of ABC with Four Parameters with Generalized Nabla Mittag-Leffler Kernel

4.4.1. The ABC Fractional Linear System with Constant Coefficient with Four Variables

Lets do the solution more general, consider the system

$$\left({}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u\right)(t) = ru(t) + g(t),\ u(a) = a_0$$

$$0 < \theta < \frac{1}{2}, 0 < \beta < \frac{1}{2}, \gamma, \mu \in C, t \in N_{a+1} \tag{176}$$

If we apply the discrete Laplace transform $\mathcal{L}_a$ with $h = 1$ to the system (176) above

$$\mathcal{L}_a\left\{\left({}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u\right)(t)\right\}(s) = \mathcal{L}_a\{ru(t)\} + \mathcal{L}_a\{g(t)\} \tag{177}$$

Now we have from the definition of the ABC with four parameters that

$$\left({}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u\right)(t)$$

$$= \frac{B(\theta)}{1-\theta} \sum_{s=a+1}^{t} E\frac{\gamma}{\beta,\mu}(\lambda, t-s)\nabla_s u(s)$$

$$t \in N_a, 0 \le \theta \le \frac{1}{2}, \gamma \in R,\ Re(\mu) > 0, \lambda = \frac{-\theta}{1-\theta} \tag{178}$$

So Eq. (178) becomes

$$\mathcal{L}_a\left\{\left({}_a^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u\right)(t)\right\}(s)$$

$$= \mathcal{L}_a\left\{\frac{B(\theta)}{1-\theta}\sum_{s=a+1}^{t} E\frac{\gamma}{\beta,\mu}(\lambda,t-s)\nabla_s u(s)\right\}$$

$$= r\breve{u}(s) + \breve{g}(s) \tag{179}$$

$$\breve{u}(s) = \mathcal{L}_a\{u(t)\}, \breve{g}(s) = \mathcal{L}_a\{g(t)\} \tag{180}$$

We have from Remark 4

$$\frac{B(\theta)}{1-\theta}\breve{u}(s)s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma} - \frac{B(\theta)}{1-\theta}u(a)s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma}$$

$$= r\breve{u}(s) + \breve{g}(s) \tag{181}$$

We want to find $u(t)$, so from Eq.(181), we have

$$\frac{B(\theta)}{1-\theta}\breve{u}(s)s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma} - r\breve{u}(s)$$

$$= \breve{g}(s) + \frac{B(\theta)}{1-\theta}u(a)s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma} \tag{182}$$

$$\breve{u}(s) = \frac{u(a)}{s - r\frac{B(\theta)}{1-\theta}s^{\mu}[1-\lambda s^{-\beta}]^{\gamma}}$$

$$+ \frac{\breve{g}(s)}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma} - r} \tag{183}$$

And hence

$$u(t) = \mathcal{L}_a^{-1}\{\breve{u}(s)\}(t) \tag{184}$$

To find the above discrete Laplace inverse, we expand $\breve{u}(s)$ in the form

$$\breve{u}(s) = \frac{u(a)}{s}\sum_{n=0}^{\infty} r^n\left(\frac{1-\theta}{B(\theta)}\right)^n s^{-(1-\mu)}[1-\lambda s^{-\beta}]^{\gamma n}$$

$$+\breve{g}(s)\sum_{n=0}^{\infty} r^n\left(\frac{1-\theta}{B(\theta)}\right)^{n+1} s^{-(1-\mu)(n+1)}[1-\lambda s^{-\beta}]^{\gamma(n+1)}$$

Such that $f = r\frac{1-\theta}{B(\theta)}s^{\mu-1}[1-\lambda s^{-\beta}]^{\gamma}, 0 < \mu < 1$ and

$$|f| < 1 \tag{185}$$

Now by using Lemma 13. And do the Laplace inverse for Eq. (185) we reach the solution

$$u(t) = u(a)\sum_{n=0}^{\infty} r^n\left(\frac{1-\theta}{B(\theta)}\right)^n E\frac{-\gamma n}{\beta, n(1-\mu)+1}(\lambda, t-a)$$

$$+g(t)\sum_{n=0}^{\infty} r^n\left(\frac{1-\theta}{B(\theta)}\right)^{n+1} E\frac{-\gamma(n+1)}{\beta,(n+1)(1-\mu)}(\lambda, t-a)$$

$$= u(a) + u(a)\sum_{n=0}^{\infty} r^n\left(\frac{1-\theta}{B(\theta)}\right)^n$$

(168)

$$E\frac{-\gamma n}{\beta, n(1-\mu)+1}(\lambda, t-a)$$

$$+\,g(t)\sum_{n=0}^{\infty} \quad r^{n}\left(\frac{1-\theta}{B(\theta)}\right)^{n+1} E\frac{-\gamma(n+1)}{\beta,(n+1)(1-\mu)}(\lambda, t-a)$$

4.4.2. The Model without Expectations of Agents

The price adjustment equation by virtue of the ABC with generalized Mittag-Leffler without considering the expectations of agents is given as follows

$$ {}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t)+k(d_1+s_1)u(t)=k(d_0+s_0) \qquad (187)$$

Now by arranging Eq. (187), we have

$$ {}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t)=-k(d_1+s_1)u(t)+k(d_0+s_0) \qquad (188)$$

Now using the result in Eq.(186) with $-k(d_1+s_1)=r$, $k(d_0+s_0)=g(t)$, we conclude

$$u(t)=u(a)+u(a)\sum_{n=0}^{\infty} \quad (-k(d_1+s_1))^{n}$$

$$\left(\frac{1-\theta}{B(\theta)}\right)^{n} E\frac{-\gamma n}{\beta, n(1-\mu)+1}(\lambda, t-a)$$

$$+k(d_0+s_0)*\sum_{n=0}^{\infty} \quad (-k(d_1+s_1))^{n}\left(\frac{1-\theta}{B(\theta)}\right)^{n+1}$$

$$E\frac{-\gamma(n+1)}{\beta,(n+1)(1-\mu)}(\lambda,t-a) \tag{189}$$

Using the result in Eq. (189) for a=0, we can get

$$u(t)=u(0)+u(0)\sum_{n=0}^{\infty}\quad(-k(d_1+s_1))^n$$

$$\left(\frac{1-\theta}{B(\theta)}\right)^n E\frac{-\gamma n}{\beta,n(1-\mu)+1}(\lambda,t)+k(d_0+s_0)$$

$$*\sum_{n=0}^{\infty}\quad(-k(d_1+s_1))^n$$

$$\left(\frac{1-\theta}{B(\theta)}\right)^{n+1} E\frac{-\gamma(n+1)}{\beta,(n+1)(1-\mu)}(\lambda,t) \tag{190}$$

4.4.3. Model with Expectation of Agents

If we take into account the expectations of agents, the price adjustment equation in terms of generalized ABC can be expressed as follows

$${}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t)-\frac{(d_1+s_1)}{(d_2+s_2)}u(t)=-\frac{(d_0+s_0)}{(d_2+s_2)} \tag{191}$$

Now by arranging Eq. (191), we have

$${}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t)=\frac{(d_1+s_1)}{(d_2+s_2)}u(t)-\frac{(d_0+s_0)}{(d_2+s_2)} \tag{192}$$

Now using the result in Eq. (186) with $\frac{(d_1+s_1)}{(d_2+s_2)}=r,\ \ -\frac{(d_0+s_0)}{(d_2+s_2)}=g(t)$ we conclude

$$u(t) = u(a) + u(a)\sum_{n=0}^{\infty} \left(\frac{(d_1 + s_1)}{(d_2 + s_2)}\right)^n$$

$$\left(\frac{1-\theta}{B(\theta)}\right)^n E\frac{-\gamma n}{\beta, n(1-\mu)+1}(\lambda, t-a)$$

$$-\frac{(d_0 + s_0)}{(d_2 + s_2)} * \sum_{n=0}^{\infty} \left(\frac{(d_1 + s_1)}{(d_2 + s_2)}\right)^n \left(\frac{1-\theta}{B(\theta)}\right)^{n+1}$$

$$E\frac{-\gamma(n+1)}{\beta, (n+1)(1-\mu)}(\lambda, t-a) \tag{193}$$

Using the result in Eq.(193) for $a = 0$, we can get

$$u(t) = u(0) + u(0)\sum_{n=0}^{\infty} \left(\frac{(d_1 + s_1)}{(d_2 + s_2)}\right)^n \left(\frac{1-\theta}{B(\theta)}\right)^n$$

$$E\frac{-\gamma n}{\beta, n(1-\mu)+1}(\lambda, t) - \frac{(d_0 + s_0)}{(d_2 + s_2)}$$

$$* \sum_{n=0}^{\infty} \left(\frac{(d_1 + s_1)}{(d_2 + s_2)}\right)^n \left(\frac{1-\theta}{B(\theta)}\right)^{n+1} E\frac{-\gamma(n+1)}{\beta, (n+1)(1-\mu)}(\lambda, t) \tag{194}$$

4.5

4.5.1. Economic Model in Framework of General Definition CFC.

Case 1: The price adjustment equation with CFC without considering the expectations of agents is given by

$$ {}_{a}^{CFC}\nabla_h^{\beta} u(t) + k(d_1 + s_1)u(t) = k(d_0 + s_0) $$
$$ \beta \in (0,1) \tag{195} $$

By applying the Laplace transform to both sides to Eq.(195) , we have

$$ \mathcal{L}_{a,h}\left\{ \ {}_{a}^{CFC}\nabla_h^{\beta} u(t)\right\} + k(d_1 + s_1)\mathcal{L}_{a,h}\{u(t)\} $$
$$ = \mathcal{L}_{a,h}\{k(d_0 + s_0)\} \tag{196} $$

We have from Definition 13, that

$$ {}_{a}^{CFC}\nabla_h^{\beta} u(t) = H(\beta, h) \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} \quad h(\nabla_h u)(sh) \times $$
$$ \left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t}{h}-s} $$
$$ \beta \in (0,1), \text{u be difined on } N_{a,h}, $$
$$ H(\beta, h) = B(\beta)\left[\frac{\beta}{h} + (1-\beta)\right] \tag{197} $$

By take the Laplace transform $\mathcal{L}_{a,h}$ for Eq. (197), we have

$$\mathcal{L}_{a,h}\left\{ \ {}_{a}^{CFC}\nabla_h^{\beta} u(t)\right\}$$

$$= \mathcal{L}_{a,h}\left\{H(\beta,h)\sum_{s=a/h+1}^{\frac{t}{h}} h(\nabla_h u)(sh)\left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t}{h}-s}\right\} \tag{198}$$

By using Lemma 2, (The relation between CFC and CFR) with $a = 0$, we have

$$\mathcal{L}_{0,h}\left\{CFC\nabla_h^{\beta} u(t)\right\}$$

$$= \mathcal{L}_{0,h}\left\{CFR\nabla_h^{\beta} u(t)\right.$$

$$\left. - \frac{H(\beta,h)(1-\beta+\beta h)}{1-\beta} u(0)\left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t}{h}}\right\}$$

$$= \mathcal{L}_{0,h}\left\{CFR\nabla_h^{\beta} u(t)\right\}$$

$$- \mathcal{L}_{0,h}\left\{\frac{H(\beta,h)(1-\beta+\beta h)}{1-\beta} u(0)\left(\frac{1-\beta}{1-\beta+\beta h}\right)^{\frac{t}{h}}\right\}$$

$$= \frac{H(\beta,h)(1-\beta+\beta h)s}{(1-\beta)s+\beta} U(s) - \frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(s-\lambda)} u(0) \tag{199}$$

And hence,

$$\mathcal{L}_{0,h}\{k(d_1+s_1)u(t)\} = k(d_1+s_1)U(s)$$

$$\mathcal{L}_{0,h}\{k(d_0+s_0)\} = \frac{k(d_0+s_0)}{s} \tag{200}$$

So, Eq. (196) becomes

$$\frac{H(\beta,h)(1-\beta+\beta h)s}{(1-\beta)s+\beta}U(s) - \frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)s+\beta}u(0)$$

$$+ k(d_1+s_1)U(s) = \frac{k(d_0+s_0)}{s} \tag{201}$$

From Eq.(201) we have

$$U(s)\left[\frac{H(\beta,h)(1-\beta+\beta h)s}{(1-\beta)s+\beta} + k(d_1+s_1)\right]$$

$$= \frac{k(d_0+s_0)}{s} + \frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(s-\lambda)}u(0) \tag{202}$$

So, By using Lemma 7, $\left(\frac{1}{s-\lambda} = \frac{(1-\beta)}{((1-\beta)s+\beta)}\right)$, Eq. (202) becomes

$$U(s) = \left(\frac{k(d_0+s_0)}{s} + \right.$$

$$\left.\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(s-\lambda)}u(0)\right)\left(\frac{(1-\beta)(s-\lambda)}{H(\beta,h)(1-\beta+\beta h)s+((1-\beta)s+\beta)k(d_1+s_1)}\right) \tag{203}$$

$$U(s) = \frac{s^{-1}k(d_0+s_0)(1-\beta)(s-\lambda)}{H(\beta,h)(1-\beta+\beta h)s+((1-\beta)s+\beta)k(d_1+s_1)}$$

$$+ \frac{H(\beta,h)(1-\beta+\beta h)}{H(\beta,h)(1-\beta+\beta h)s+((1-\beta)s+\beta))k(d_1+s_1)}u(0)$$

$$= \frac{K(d_0 + s_0)(1-\beta)(1-\lambda s^{-1})}{H(\beta,h)(1-\beta+\beta h)s + ((1-\beta)k(d_1+s_1)s + \beta k(d_1+s_1))}$$

$$+ \frac{H(\beta,h)(1-\beta+\beta h)}{H(\beta,h)(1-\beta+\beta h)s + ((1-\beta)s+\beta)k(d_1+s_1)} u(0)$$

$$U(s) = \frac{k(d_0+s_0)(1-\beta)(1-\lambda s^{-1})}{[H(\beta,h)(1-\beta+\beta h) + (1-\beta)k(d_1+s_1)]s + \beta k(d_1+s_1)}$$

$$+ \frac{H(\beta,h)(1-\beta+\beta h)}{[H(\beta,h)(1-\beta+\beta h) + (1-\beta)k(d_1+s_1)]s + \beta k(d_1+s_1)} u(0)$$

(204)

Now by doing the Laplace inverse transform $\mathcal{L}_{a,h}^{-1}$ for Eq.(204) , we have

$$\mathcal{L}_{0,h}^{-1}\{U(s)\} = k(d_0+s_0) \times$$

$$(1-\beta)\mathcal{L}_{0,h}^{-1}\left\{\frac{(1-\lambda s^{-1})}{[H(\beta,h)(1-\beta+\beta h)+(1-\beta)k(d_1+s_1)]s+\beta k(d_1+s_1)}\right\}$$

$$+H(\beta,h)(1-\beta+\beta h)u(0) \times$$

$$\mathcal{L}_{0,h}^{-1}\left\{\frac{1}{[H(\beta,h)(1-\beta+\beta h)+(1-\beta)k(d_1+s_1)]s+\beta k(d_1+s_1)}\right\}$$

(205)

We use the MATLAB code for Laplace inverse with $u(0) = \frac{d_0+s_0}{d_1+s_1}$, $\lambda = \frac{-\beta}{1-\beta}$

and we have

$$u(t)=\frac{k(d_0+s_0)(1-\beta)_h\varrho_{\frac{-k(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}}(t)}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}$$

$$+\frac{\beta k(d_0+s_0)_h\varrho_{\frac{-k(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}}(t)}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}$$

$$+\frac{H(\beta,h)(1-\beta+\beta h)u(a)_h\varrho_{\frac{-k(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}}(t)}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)} \tag{206}$$

Case 2: Moreover, if we consider the expectations of agents, the price adjustment equation with CFC can be given by

$$ {}_{a}^{CFC}\nabla_h^{\beta}u(t)-\frac{(d_1+s_1)}{(d_2+s_2)}u(t)=-\frac{(d_0+s_0)}{(d_2+s_2)},\beta\in(0,1) \tag{207}$$

Applying the Laplace transform to both side of Eq. (207), we have

$$\mathcal{L}_{a,h}\left\{ {}_{a}^{CFC}\nabla_h^{\beta}u(t)\right\}-\frac{(d_1+s_1)}{(d_2+s_2)}\mathcal{L}_{a,h}\{u(t)\}=\mathcal{L}_{a,h}\left\{-\frac{(d_0+s_0)}{(d_2+s_2)}\right\} \tag{208}$$

By using the result in Eq. (202) with $a=0$ and hence,

$$\mathcal{L}_{0,h}\left\{-\frac{(d_1+s_1)}{(d_2+s_2)}\right\}=-\frac{(d_1+s_1)}{(d_2+s_2)}U(s)$$

$$\mathcal{L}_{0,h}\left\{-\frac{(d_0+s_0)}{(d_2+s_2)}\right\}=-\frac{(d_0+s_0)}{s(d_2+s_2)}$$

So, Eq. (208) becomes

$$\frac{H(\beta,h)(1-\beta+\beta h)s}{(1-\beta)s+\beta}U(s)-\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(s-\lambda)}u(0)$$

$$-\frac{(d_1+s_1)}{(d_2+s_2)}U(s)=-\frac{(d_0+s_0)}{s(d_2+s_2)} \tag{209}$$

From Eq. (209) we have

$$U(s)\left[\frac{H(\beta,h)(1-\beta+\beta h)s}{(1-\beta)s+\beta}-\frac{(d_1+s_1)}{(d_2+s_2)}\right]$$

$$=-\frac{(d_0+s_0)}{s(d_2+s_2)}+\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(s-\lambda)}u(0) \tag{210}$$

So, by using Lemma 7, $\left(\frac{1}{s-\lambda}=\frac{(1-\beta)}{((1-\beta)s+\beta)}\right)$, and suppose that $M=-\frac{(d_1+s_1)}{(d_2+s_2)}, V=-\frac{(d_0+s_0)}{(d_2+s_2)}$, Eq. (210) becomes

$$U(s)=\left(\frac{V}{s}+\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(s-\lambda)}u(0)\right)\left(\frac{(1-\beta)(s-\lambda)}{H(\beta,h)(1-\beta+\beta h)s+((1-\beta)s+\beta)M}\right) \tag{211}$$

$$U(s)=\frac{s^{-1}V(1-\beta)(s-\lambda)}{H(\beta,h)(1-\beta+\beta h)s+((1-\beta)s+\beta)K(d_1+s_1)}$$

$$+\frac{H(\beta,h)(1-\beta+\beta h)}{H(\beta,h)(1-\beta+\beta h)s+((1-\beta)s+\beta))K(d_1+s_1)}u(0)$$

$$= \frac{V(1-\beta)(1-\lambda s^{-1})}{H(\beta,h)(1-\beta+\beta h)s + ((1-\beta)Ms+\beta M)}$$

$$+ \frac{H(\beta,h)(1-\beta+\beta h)}{H(\beta,h)(1-\beta+\beta h)s + ((1-\beta)s+\beta)M} u(0)$$

$$U(s) = \frac{V(1-\beta)(1-\lambda s^{-1})}{[H(\beta,h)(1-\beta+\beta h)+(1-\beta)M]s+\beta M}$$

$$+ \frac{H(\beta,h)(1-\beta+\beta h)}{[H(\beta,h)(1-\beta+\beta h)+(1-\beta)M]s+\beta M} u(0) \qquad (212)$$

Now by doing the Laplace inverse transform $\mathcal{L}_{0,h}^{-1}$ for Eq. (212) , we have

$$\mathcal{L}_{0,h}^{-1}\{U(s)\} = V(1-\beta)\mathcal{L}_{0,h}^{-1}\left\{\frac{(1-\lambda s^{-1})}{[H(\beta,h)(1-\beta+\beta h)+(1-\beta)M]s+\beta M}\right\}$$

$$+H(\beta,h)(1-\beta+\beta h)u(0) \times$$

$$\mathcal{L}_{0,h}^{-1}\left\{\frac{1}{[H(\beta,h)(1-\beta+\beta h)+(1-\beta)M]s+\beta M}\right\} \qquad (213)$$

We use the MATLAB code for Laplace inverse with $M = -\frac{(d_1+s_1)}{(d_2+s_2)}$,

$V = -\frac{(d_0+s_0)}{(d_2+s_2)}$, $u(0) = \frac{d_0+s_0}{d_1+s_1}$, $\lambda = \frac{-\beta}{1-\beta}$ we have

$$u(t) = \frac{(d_0+s_0)(1-\beta)_h\varrho_{\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}}(t)}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}$$

$$+ \frac{\beta(d_0+s_0)_h\varrho_{\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}}(t)}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}$$

$$+\frac{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)u(a)_h\varrho_{\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}}(t)}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}$$

(214)

4.5.2. Economic Model in Framework of General Definition of ABC

Case 1: The price adjustment equation with ABC without considering the expectations of agents is given by

$$ {}_{a}^{ABC}\nabla_h^{\beta}u(t)+k(d_1+s_1)u(t)=k(d_0+s_0),\beta\in(0,1) \quad (215)$$

Applying the Laplace transform to both side of Eq. (215), we have

$$\mathcal{L}_{a,h}\left\{\ {}_{a}^{ABC}\nabla_h^{\beta}u(t)\right\}+k(d_1+s_1)\mathcal{L}_{a,h}\{u(t)\} \quad (216)$$

$$=\mathcal{L}_{a,h}\{k(d_0+s_0)\}$$

We have from Definition 15,

$${}_{a}^{ABC}\nabla_h^{\beta}u(t)$$

$$=H(\beta,h)\frac{1-\beta+\beta h}{1-\beta}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} h(\nabla_h u)(sh)*_h E_{\overline{\beta}}(\lambda,t-s)$$

$$=H(\beta,h)\frac{1-\beta+\beta h}{1-\beta}\left(\nabla_h u(t)\right)\times_h E_{\overline{\beta}}(\lambda,t-a)$$

$$\beta\in(0,1), u \text{ be difined on } N_{a,h},$$

$$H(\beta, h) = B(\beta)\left[\frac{\beta}{h} + (1-\beta)\right] \tag{217}$$

By doing the Laplace transform for Eq. (217), we have

$$\mathcal{L}_{a,h}\left\{ {}_{a}^{ABC}\nabla_h^{\beta} u(t)\right\} = \mathcal{L}_{a,h}\left\{\begin{matrix} H(\beta,h)\dfrac{1-\beta+\beta h}{1-\beta}\left(\nabla_h u(t)\right) \\ \times_h E_{\overline{\beta}}(\lambda, t-a) \end{matrix}\right\} \tag{218}$$

By using Lemma 3, (the relation between ABC and ABR) with $a = 0$, we have

$$\mathcal{L}_{0,h}\left\{ {}^{ABC}\nabla_h^{\beta} u(t)\right\}$$

$$= \mathcal{L}_{0,h}\left\{ {}^{ABR}\nabla_h^{\beta} u(t) - H(\beta,h)\frac{1-\beta+\beta h}{(1-\beta)s(1-\lambda s^{-\beta})}u(0)\right\}$$

$$= \mathcal{L}_{0,h}\left\{ {}^{ABR}\nabla_h^{\beta} u(t)\right\}$$

$$- \mathcal{L}_{0,h}\left\{H(\beta,h)\frac{1-\beta+\beta h}{(1-\beta)s(1-\lambda s^{-\beta})}u(0)\right\}$$

$$= H(\beta,h)\frac{1-\beta+\beta h}{(1-\beta)s(1-\lambda s^{-\beta})}\left(sU(s) - u(0)\right) \tag{219}$$

And hence,

$$\mathcal{L}_{0,h}\{k(d_1+s_1)u(t)\} = k(d_1+s_1)U(s)$$

$$\mathcal{L}_{0,h}\{K(d_0+s_0)\} = \frac{k(d_0+s_0)}{s}$$

So, Eq. (216) becomes

$$\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(1-\lambda s^{-\beta})}U(s)-\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)s(1-\lambda s^{-\beta})}u(0)$$

$$+k(d_1+s_1)U(s)=\frac{k(d_0+s_0)}{s} \qquad (220)$$

From Eq. (220) we have

$$U(s)\left[\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(1-\lambda s^{-\beta})}+k(d_1+s_1)\right]$$

$$=\frac{k(d_0+s_0)}{s}+\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)s(1-\lambda s^{-\beta})}u(0) \qquad (221)$$

$$U(s)$$

$$=\left(\frac{k(d_0+s_0)}{s}+\frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)s(1-\lambda s^{-\beta})}u(0)\right)$$

$$\times\left(\frac{(1-\beta)(1-\lambda s^{-\beta})}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))k(d_1+s_1)}\right) \qquad (222)$$

$$U(s)=\frac{s^{-1}k(d_0+s_0)(1-\beta)(1-\lambda s^{-\beta})}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))k(d_1+s_1)}$$

$$+\frac{s^{-1}H(\beta,h)(1-\beta+\beta h)}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))k(d_1+s_1)}u(0)$$

$$U(s)$$

$$=\frac{k(d_0+s_0)(1-\beta)(s^{-1}-\lambda s^{-(\beta+1)})}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))k(d_1+s_1)}$$

$$+\frac{s^{-1}H(\beta,h)(1-\beta+\beta h)}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))k(d_1+s_1)}u(0) \tag{223}$$

Now by doing the Laplace inverse transform $\mathcal{L}_{0,h}^{-1}$ for Eq. (223) , we have

$$\mathcal{L}_{0,h}^{-1}\{U(s)\} = k(d_0+s_0)(1-$$

$$\beta)\mathcal{L}_{0,h}^{-1}\left\{\frac{(s^{-1}-\lambda s^{-(\beta+1)})}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))k(d_1+s_1)}\right\} + H(\beta,h)(1-$$

$$\beta+\beta h)u(0)\mathcal{L}_{0,h}^{-1}\left\{\frac{s^{-1}}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))k(d_1+s_1))}\right\} \tag{224}$$

we use the MATLAB code for Laplace inverse with $\lambda = \frac{-\beta}{1-\beta}$ and

$u(0) = \frac{d_0+s_0}{d_1+s_1}$ we have

$$u(t) = \frac{k(d_0+s_0)(1-\beta)}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}$$

$$E_{\overline{\beta}}\left(\frac{-k(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)},t\right)$$

$$+\frac{k(d_0+s_0)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}$$

$$E_{\overline{\beta,\beta+1}}\left(\frac{-k(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)},t\right)$$

$$+\frac{H(\beta,h)(1-\beta+\beta h)u(a)}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)}\times$$

$$E_{\overline{\beta}}\left(\frac{-k(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)+k(d_1+s_1)(1-\beta)},t\right) \tag{225}$$

Case 2: when considering the expectations of agents, the price adjustment equation in terms of ABC can be presented by

$$ {}_{a}^{ABC}\nabla_h^{\beta} u(t) - \frac{(d_1 + s_1)}{(d_2 + s_2)} u(t) = -\frac{(d_0 + s_0)}{(d_2 + s_2)}, \beta \in (0,1) \tag{226} $$

Applying the Laplace transform to both side of Eq.(226) , we have

$$ \mathcal{L}_{a,h}\left\{ \ {}_{a}^{ABC}\nabla_h^{\beta} u(t)\right\} - \frac{(d_1 + s_1)}{(d_2 + s_2)} \mathcal{L}_{a,h}\{u(t)\} $$

$$ = \mathcal{L}_{a,h}\left\{-\frac{(d_0 + s_0)}{(d_2 + s_2)}\right\} \tag{227} $$

By using the result in Eq. (219) with $a = 0$ and hence,

$$ \mathcal{L}_{0,h}\left\{-\frac{(d_1 + s_1)}{(d_2 + s_2)}\right\} = -\frac{(d_1 + s_1)}{(d_2 + s_2)} U(s) $$

$$ \mathcal{L}_{0,h}\left\{-\frac{(d_0 + s_0)}{(d_2 + s_2)}\right\} = -\frac{(d_0 + s_0)}{s(d_2 + s_2)} $$

Eq. (227) becomes

$$ \frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)(1-\lambda s^{-\beta})} U(s) - \frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)s(1-\lambda s^{-\beta})} u(0) $$

$$ -\frac{(d_1 + s_1)}{(d_2 + s_2)} U(s) = -\frac{(d_0 + s_0)}{s(d_2 + s_2)} \tag{228} $$

Suppose that $M = -\frac{(d_1+s_1)}{(d_2+s_2)}, V = -\frac{(d_0+s_0)}{(d_2+s_2)}$, Eq. (228) becomes

$$ U(s) = \left(\frac{V}{s} + \frac{H(\beta,h)(1-\beta+\beta h)}{(1-\beta)s(1-\lambda s^{-\beta})} u(0)\right) \times $$

$$\left(\frac{(1-\beta)(1-\lambda s^{-\beta})}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))M}\right) \tag{229}$$

$$U(s)=\frac{s^{-1}V(1-\beta)(1-\lambda s^{-\beta})}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))M}$$

$$+\frac{s^{-1}H(\beta,h)(1-\beta+\beta h)}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))M}u(0)$$

$$U(s)=\frac{V(1-\beta)(s^{-1}-\lambda s^{-(\beta+1)})}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))M}$$

$$+\frac{s^{-1}H(\beta,h)(1-\beta+\beta h)}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))M}u(0) \tag{230}$$

Now by doing the Laplace inverse transform $\mathcal{L}_{0,h}^{-1}$ for Eq. (230) , we have

$$\mathcal{L}_{0,h}^{-1}\{U(s)\}=V(1-\beta)\times$$

$$\mathcal{L}_{0,h}^{-1}\left\{\frac{\left(s^{-1}-\lambda s^{-(\beta+1)}\right)}{H(\beta,h)(1-\beta+\beta h)+\left((1-\beta)(1-\lambda s^{-\beta})\right)M}\right\}$$

$$+H(\beta,h)(1-\beta+\beta h)\times$$

$$u(0)\mathcal{L}_{0,h}^{-1}\left\{\frac{s^{-1}}{H(\beta,h)(1-\beta+\beta h)+((1-\beta)(1-\lambda s^{-\beta}))M}\right\} \tag{231}$$

We use the MATLAB code for Laplace inverse with $M=-\frac{(d_1+s_1)}{(d_2+s_2)}$,

$V=-\frac{(d_0+s_0)}{(d_2+s_2)}, \lambda=\frac{-\beta}{1-\beta}$, $u(0)=\frac{d_0+s_0}{d_1+s_1}$ we have

$$u(t)=\frac{-(d_0+s_0)(1-\beta)}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}$$

$$E_{\overline{\beta}}\left(\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)},t\right)$$

$$+\frac{-(d_0+s_0)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)}$$

$$E_{\overline{\beta,\beta+1}}\left(\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)},t\right)$$

$$+\frac{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)u(a)}{(H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta))}$$

$$E_{\overline{\beta}}\left(\frac{(d_1+s_1)\beta}{H(\beta,h)(1-\beta+\beta h)(d_2+s_2)-(d_1+s_1)(1-\beta)},t\right) \quad (232)$$

4.5.3. Economic Model in the Framework of ABC with Generalized Mittag-Leffler Kernel

Case 1: The price adjustment equation by virtue of the ABC with generalized Mittag-Leffler without considering the expectations of agents is given as follows

$$ {}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t)+k(d_1+s_1)u(t)=k(d_0+s_0) \quad (233)$$

Applying the Laplace transform to both side of Eq. (233), we have

$$\mathcal{L}_a\left\{ {}_{a}^{ABC}\nabla_h^{\beta,(\theta,\mu,\gamma)}u(t)\right\}+k(d_1+s_1)\mathcal{L}_a\{u(t)\}$$
$$=\mathcal{L}_a\{k(d_0+s_0)\} \quad (234)$$

We have from Definition of Generalized ABC that

$$ {}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t) = \frac{B(\theta)}{1-\theta}\sum_{s=a+1}^{t} E\frac{\gamma}{\beta,\mu}(\lambda, t-s)\nabla_s u(s) \tag{235} $$

By doing the Laplace transform for Eq. (235), we have

$$ \mathcal{L}_{a,h}\left\{ {}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t)\right\} $$

$$ = \mathcal{L}_{a,h}\left\{\frac{B(\theta)}{1-\theta}\sum_{s=a+1}^{t} E\frac{\gamma}{\beta,\mu}(\lambda, t-s)\nabla_s u(s)\right\} \tag{236} $$

By using Lemma (17). (The relation between Generalized ABC and ABR)

with $a = 0$, $h = 1$, we have

$$ \mathcal{L}\left\{ {}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t)\right\} $$

$$ = \mathcal{L}\left\{ {}^{ABR}\nabla^{\beta,(\theta,\mu,\gamma)}u(t)\right\} - \frac{B(\theta)}{1-\theta}u(0)s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma} $$

$$ = \frac{B(\theta)}{1-\theta}u(s)s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma} $$

$$ - \frac{B(\theta)}{1-\theta}u(0)s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma} \tag{237} $$

And hence,

$$ \mathcal{L}\{k(d_1+s_1)u(t)\} = k(d_1+s_1)U(s) $$

$$ \mathcal{L}\{k(d_0+s_0)\} = \frac{k(d_0+s_0)}{s} $$

So, Eq. (234) becomes

$$\frac{B(\theta)}{1-\theta}U(s)s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}-\frac{B(\theta)}{1-\theta}u(0)s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma}$$

$$+k(d_1+s_1)U(s)=\frac{k(d_0+s_0)}{s} \tag{238}$$

From Eq. (238) we have

$$U(s)\left[\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}+k(d_1+s_1)\right]$$

$$=\frac{k(d_0+s_0)}{s}+\frac{B(\theta)}{1-\theta}u(0)s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma} \tag{239}$$

$$U(s)=\left(\frac{k(d_0+s_0)}{s}+\frac{B(\theta)}{1-\theta}u(0)s^{-\mu}[1\right.$$

$$\left.-\lambda s^{-\beta}]^{-\gamma}\right)\left(\frac{1}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}+k(d_1+s_1)}\right)$$

$$U(s)=\frac{s^{-1}k(d_0+s_0)}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}+k(d_1+s_1)}$$

$$+\frac{\frac{B(\theta)}{1-\theta}s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma}}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}+k(d_1+s_1)}u(0)$$

$$U(s)=\frac{s^{-1}k(d_0+s_0)+\frac{B(\theta)}{1-\theta}s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma}u(0)}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}+k(d_1+s_1)} \tag{240}$$

We use the MATLAB code for Laplace inverse and we have

$$u(t) = u(0) + u(0)\sum_{n=0}^{\infty} \left\{(-k(d_1 + s_1))^n \left(\frac{1-\theta}{B(\theta)}\right)^n \mathcal{L}^{-1}\left\{s^{-n(1-\mu)+1}[1-\lambda s^{-\beta}]^{-\gamma n}\right\}\right\}$$

$$+k(d_0 + s_0)\sum_{n=0}^{\infty} \left\{ \begin{array}{c} (-k(d_1 + s_1))^n \left(\frac{1-\theta}{B(\theta)}\right)^{n+1} \times \\ \mathcal{L}^{-1}\left\{s^{-(n+1)(1-\mu)}[1-\lambda s^{-\beta}]^{-\gamma(n+1)}\right\} \end{array} \right\} \quad (241)$$

By using Lemma 13, we have

$$u(t) = u(0) + u(0)\sum_{n=0}^{\infty} \left\{(-k(d_1 + s_1))^n \left(\frac{1-\theta}{B(\theta)}\right)^n E\,\frac{-\gamma n}{\beta, n(1-\mu)+1}(\lambda, t)\right\}$$

$$+k(d_0 + s_0)\sum_{n=0}^{\infty} \left\{ \begin{array}{c} (-k(d_1 + s_1))^n \left(\frac{1-\theta}{B(\theta)}\right)^{n+1} \times \\ E\,\frac{-\gamma(n+1)}{\beta, (n+1)(1-\mu)}(\lambda, t) \end{array} \right\} \quad (242)$$

Case 2: if we take into account the expectations of agents, the price adjustment equation in terms of generalized ABC can be expressed as follows

$${}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t) - \frac{(d_1 + s_1)}{(d_2 + s_2)}u(t) = -\frac{(d_0 + s_0)}{(d_2 + s_2)} \quad (243)$$

Applying the Laplace transform to both side of Eq. (243), we have

$$\mathcal{L}\left\{ {}_{a}^{ABC}\nabla^{\beta,(\theta,\mu,\gamma)}u(t)\right\} - \frac{(d_1+s_1)}{(d_2+s_2)}\mathcal{L}\{u(t)\}$$

$$= \mathcal{L}\left\{-\frac{(d_0+s_0)}{(d_2+s_2)}\right\} \tag{244}$$

By using the result in Eq. (237) and hence,

$$\mathcal{L}\left\{-\frac{(d_1+s_1)}{(d_2+s_2)}\right\} = -\frac{(d_1+s_1)}{(d_2+s_2)}U(s)$$

$$\mathcal{L}\left\{-\frac{(d_0+s_0)}{(d_2+s_2)}\right\} = -\frac{(d_0+s_0)}{s(d_2+s_2)}$$

So, Eq. (244) becomes

$$\frac{B(\theta)}{1-\theta}U(s)s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma} - \frac{B(\theta)}{1-\theta}u(0)s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma}$$

$$-\frac{(d_1+s_1)}{(d_2+s_2)}U(s) = -\frac{(d_0+s_0)}{s(d_2+s_2)} \tag{245}$$

we have that

$$U(s)\left[\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma} + \frac{(d_1+s_1)}{(d_2+s_2)}\right]$$

$$= -\frac{(d_0+s_0)}{s(d_2+s_2)} + \frac{B(\theta)}{1-\theta}u(0)s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma} \tag{246}$$

Suppose that $M = -\frac{(d_1+s_1)}{(d_2+s_2)}, V = -\frac{(d_0+s_0)}{(d_2+s_2)}$, Eq. (246) becomes

$$U(s) = \left(\frac{V}{s} + \frac{B(\theta)}{1-\theta}u(0)s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma}\right)\left(\frac{1}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma} + M}\right)$$

$$U(s)=\frac{s^{-1}V}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}+M}$$

$$+\frac{\frac{B(\theta)}{1-\theta}s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma}}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}+M}u(0)$$

$$U(s)=\frac{s^{-1}V+\frac{B(\theta)}{1-\theta}s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma}u(0)}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}+M} \quad (247)$$

Now by doing the Laplace inverse transform $\mathcal{L}^{-1}$ for Eq. (247) , we have

$$\mathcal{L}^{-1}\{U(s)\}=\mathcal{L}^{-1}\left\{\frac{s^{-1}V+\frac{B(\theta)}{1-\theta}s^{-\mu}[1-\lambda s^{-\beta}]^{-\gamma}u(0)}{\frac{B(\theta)}{1-\theta}s^{1-\mu}[1-\lambda s^{-\beta}]^{-\gamma}+M}\right\} \quad (248)$$

We use the MATLABE code for Laplace inverse to Eq. (248) and we have

$$u(t)=u(0)+u(0)\sum_{n=0}^{\infty}\left\{(-M)^n\left(\frac{1-\theta}{B(\theta)}\right)^n\mathcal{L}^{-1}\left\{\begin{matrix}s^{-n(1-\mu)+1}\times\\ [1-\lambda s^{-\beta}]^{-\gamma n}\end{matrix}\right\}\right\}$$

$$+V\sum_{n=0}^{\infty}\left\{(-M)^n\left(\frac{1-\theta}{B(\theta)}\right)^{n+1}\mathcal{L}^{-1}\left\{\begin{matrix}s^{-(n+1)(1-\mu)}\times\\ [1-\lambda s^{-\beta}]^{-\gamma(n+1)}\end{matrix}\right\}\right\} \quad (249)$$

Now we have, $M=-\frac{(d_1+s_1)}{(d_2+s_2)}, V=-\frac{(d_0+s_0)}{(d_2+s_2)}$, so Eq.(249) become

$$u(t)=u(0)+u(0)\sum_{n=0}^{\infty}\left\{\left(\frac{(d_1+s_1)}{(d_2+s_2)}\right)^n\left(\frac{1-\theta}{B(\theta)}\right)^n\mathcal{L}^{-1}\left\{\begin{matrix}s^{-n(1-\mu)+1}\times\\ [1-\lambda s^{-\beta}]^{-\gamma n}\end{matrix}\right\}\right\}$$

$$-\frac{(d_0+s_0)}{(d_2+s_2)}\sum_{n=0}^{\infty}\left\{\left(\frac{(d_1+s_1)}{(d_2+s_2)}\right)^n\left(\frac{1-\theta}{B(\theta)}\right)^{n+1}\mathcal{L}^{-1}\left\{\begin{matrix}s^{-(n+1)(1-\mu)}\times\\ [1-\lambda s^{-\beta}]^{-\gamma(n+1)}\end{matrix}\right\}\right\} \quad (250$$

By using Lemma 13, we have

$$u(t)=u(0)+u(0)\sum_{n=0}^{\infty}\left\{\left(\frac{(d_1+s_1)}{(d_2+s_2)}\right)^n\left(\frac{1-\theta}{B(\theta)}\right)^n E\,\frac{-\gamma n}{\beta, n(1-\mu)+1}(\lambda,t)\right\}$$

$$-\frac{(d_0+s_0)}{(d_2+s_2)}\sum_{n=0}^{\infty}\left\{\left(\frac{(d_1+s_1)}{(d_2+s_2)}\right)^n\left(\frac{1-\theta}{B(\theta)}\right)^{n+1} E\,\frac{-\gamma(n+1)}{\beta,(n+1)(1-\mu)}(\lambda,t)\right\}$$

(251)

Chapter 5

Comparative Analysis and Discussion

In this section we will discuss the figures of our solutions (Caputo, CFC and ABC with one Parameter), In all graphics we use an arbitrary value of β (0.3, 0.38,0.4) to make three cases and have a graph for each value then we compared the results in the three cases for the three differences. Also, we use $d_0 = 10, s_0 = 100, d_1 = 14, s_1 = 97, d_2 = 18, s_2 = 94$ and as a special case we use $h = 1$ to make the result more amenable to analysis and interpretation.

5.1. The Graphs of the solution in the Framework of Caputo Fractional Difference

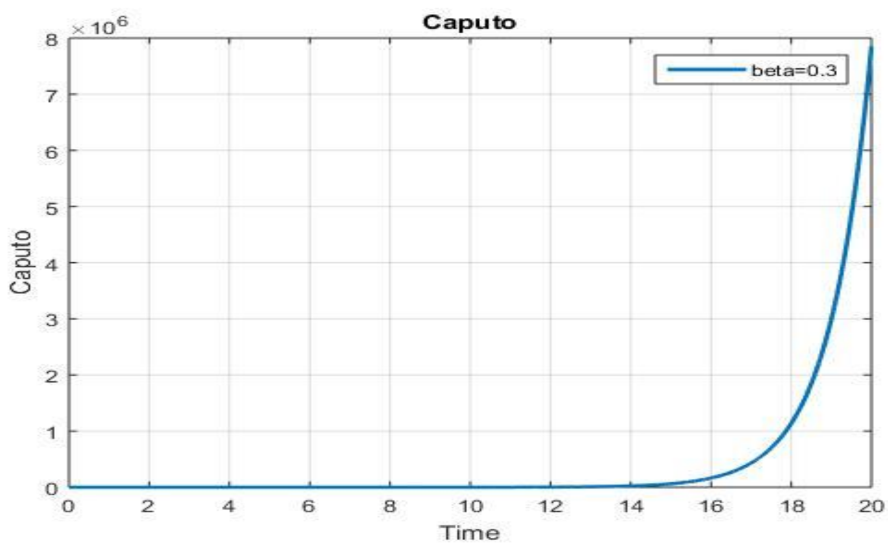

Fig.1 Caputo with $\beta = 0.3$

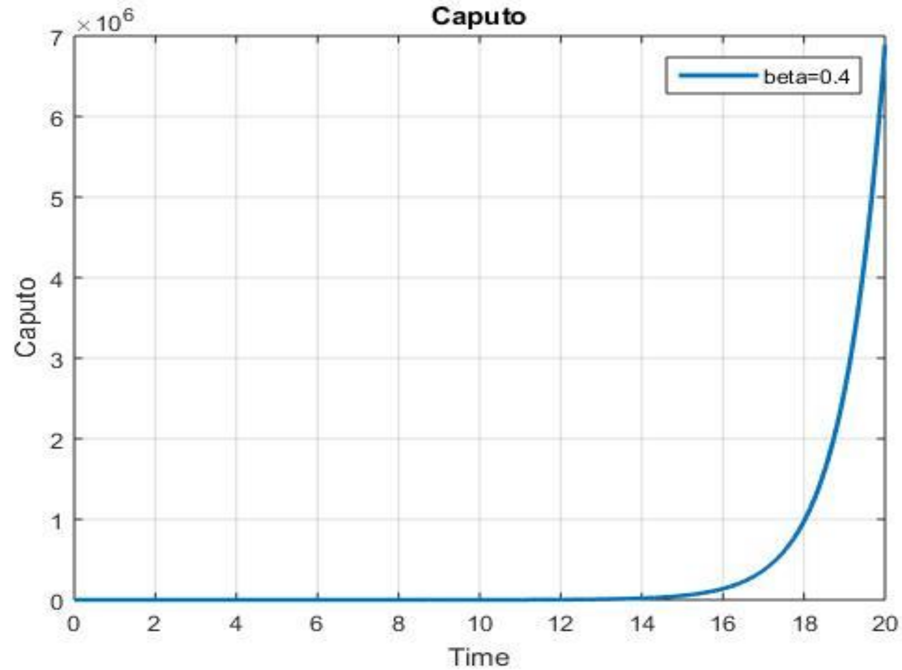

Fig.2 Caputo with $\boldsymbol{\beta = 0.4}$

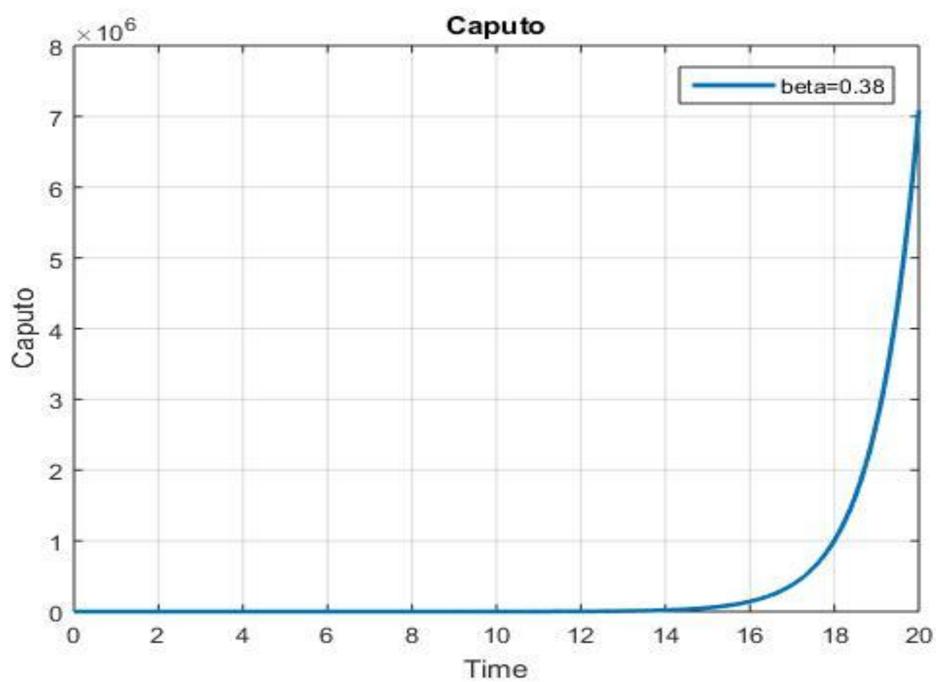

Fig.3 Caputo with $\boldsymbol{\beta = 0.38}$

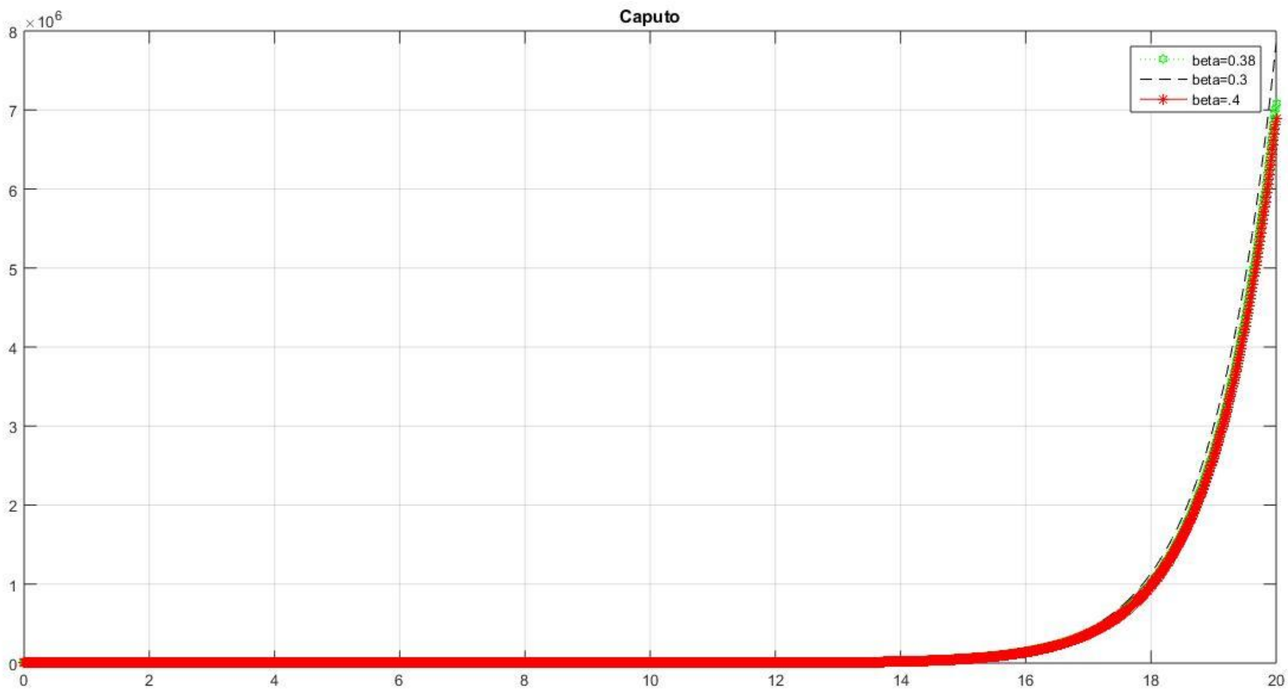

Fig.4 Comparative analysis for all β values

We can see from the figure shown that as time increasing the value of u(t) that solved in Caputo fractional difference is increases with $\beta = 0.3, 0.38, 0.4$, and there is no big difference in graphics with different β values.

5.2. The Graphs of the solution in the Framework of CFC Fractional Difference

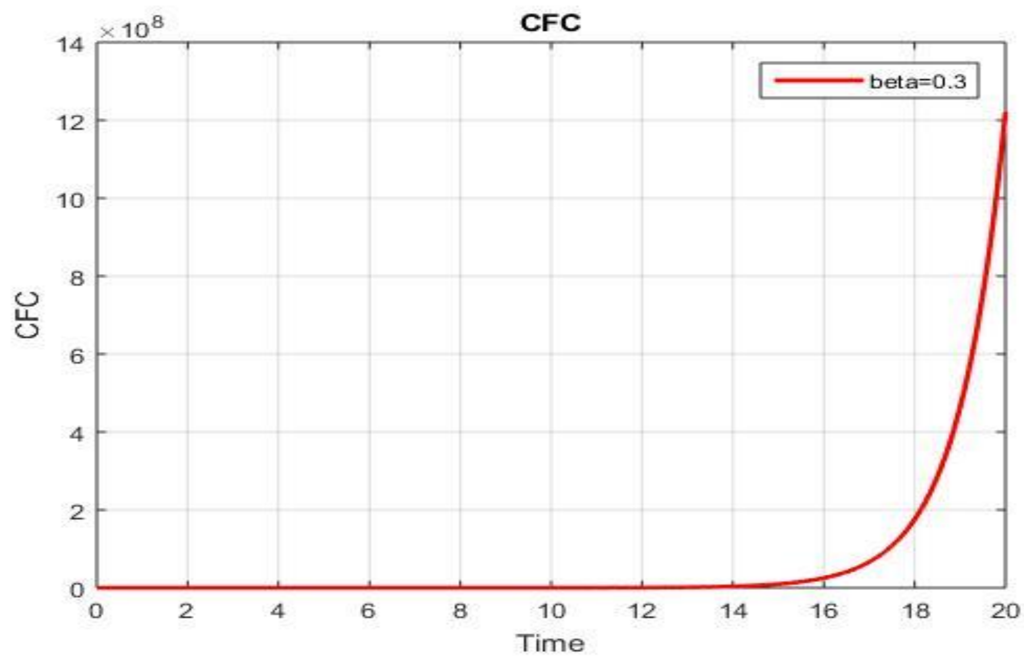

Fig.5 CFC with $\boldsymbol{\beta = 0.3}$

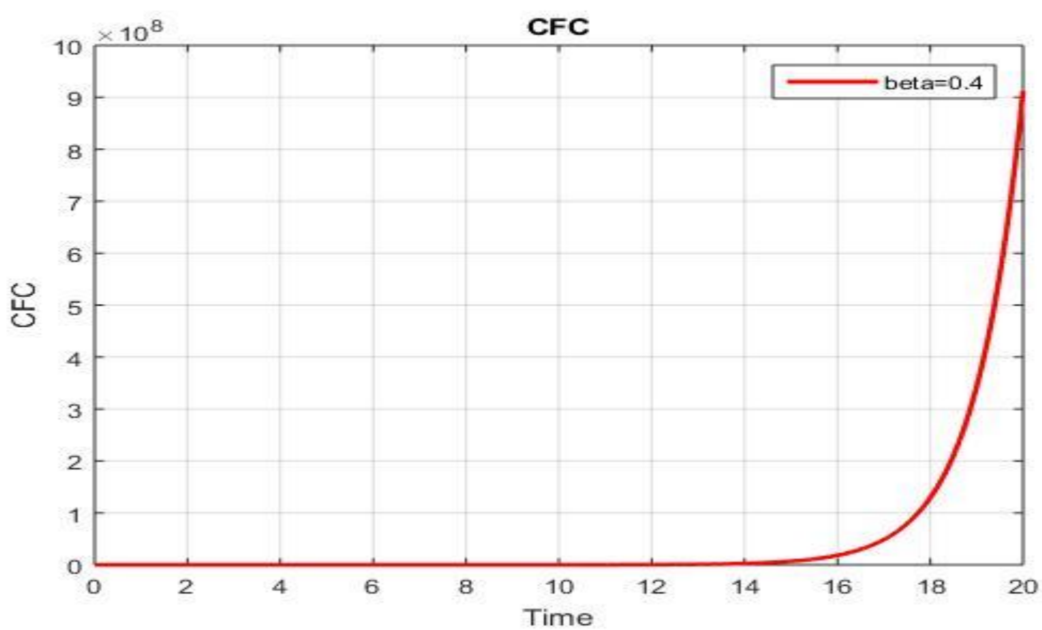

Fig.6 CFC with $\boldsymbol{\beta = 0.4}$

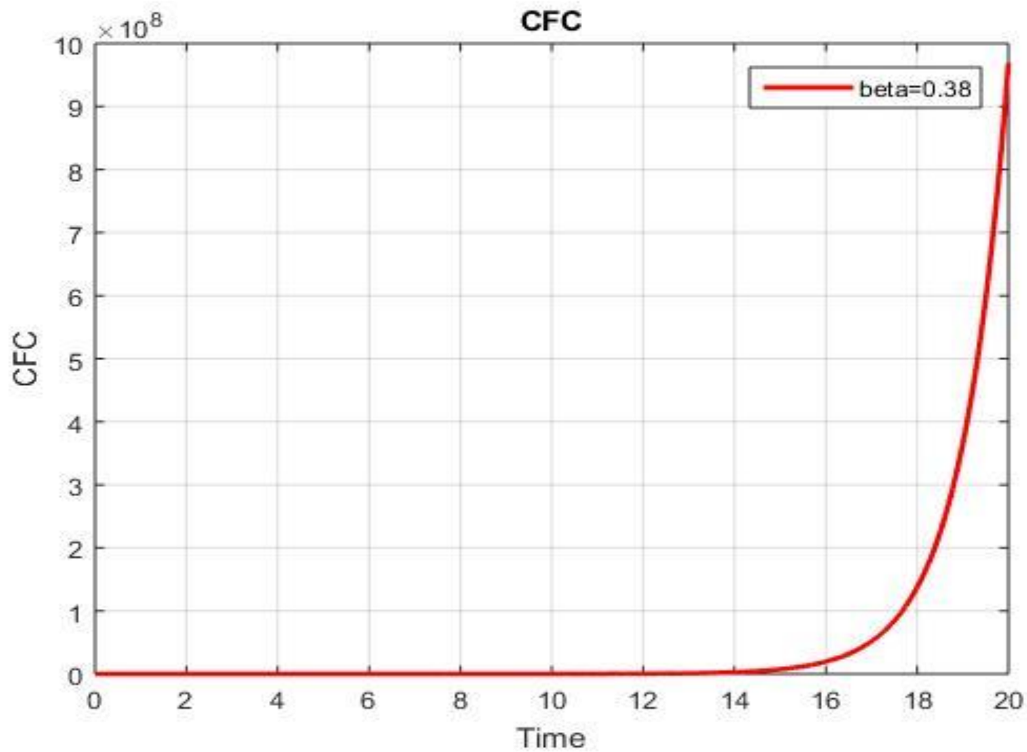

Fig.7 CFC with $\beta = 0.38$

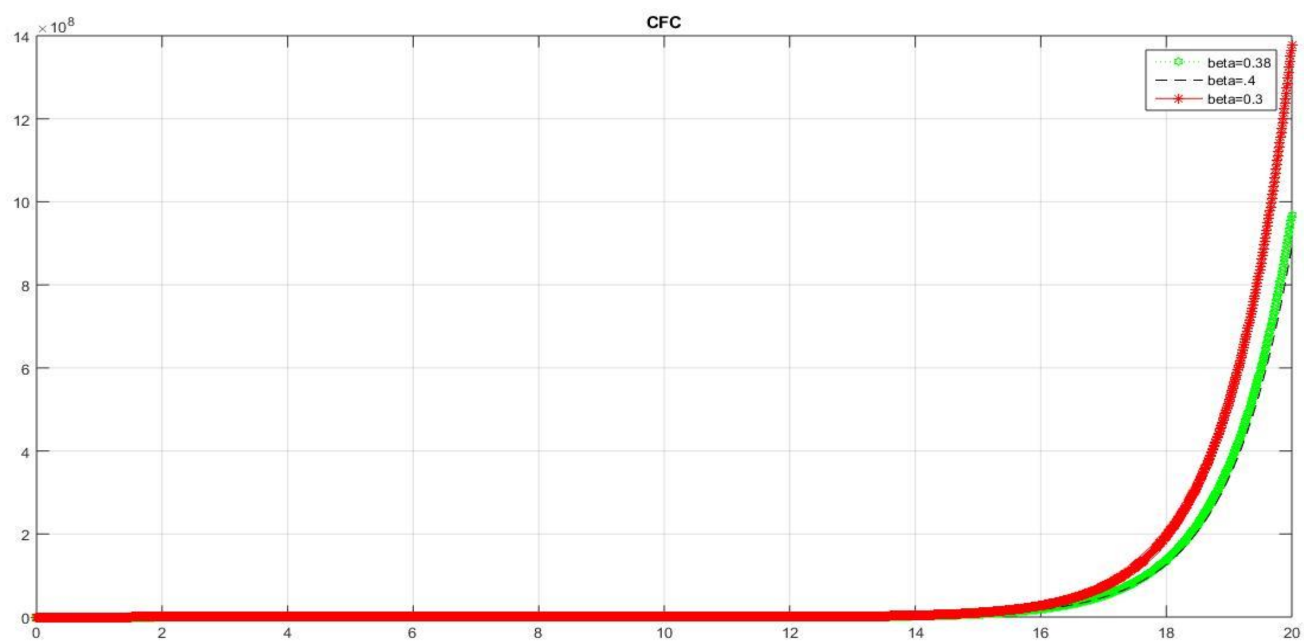

Fig.8 Comparative analysis for all β values

We see from the figure that as time increasing the value of u(t) is increases with $\beta = 0.3, 0.38, 0.4$, and there is no big difference in graphics with different β values, This compatibility is due to the convergence of the $u(t)$ values over time, which results in the graphics being very close to each other

5.3. The Graphs of the solution in the Framework of ABC Fractional Difference with Parameter β

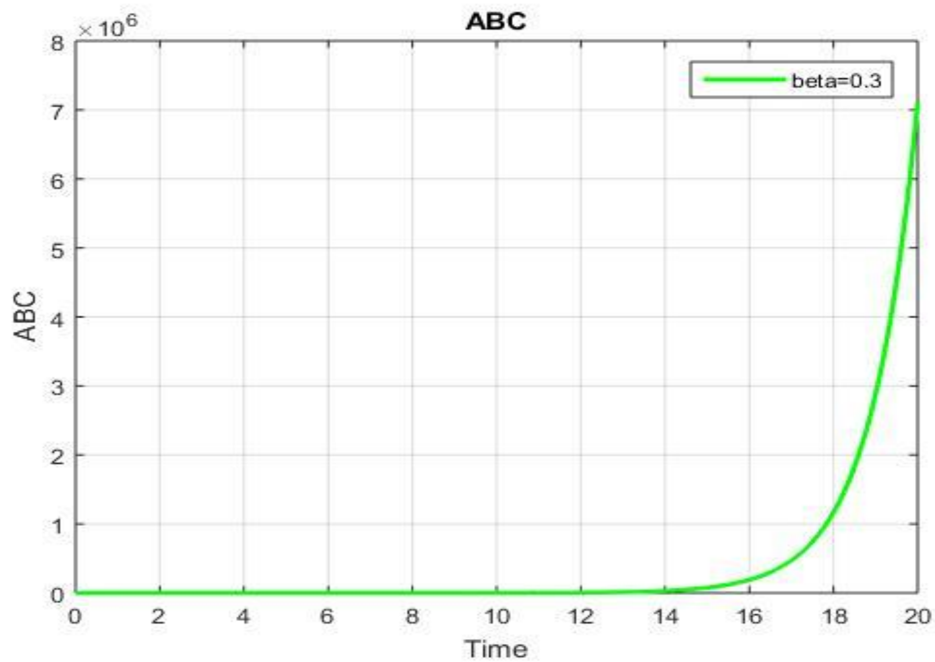

Fig.9 ABC with $\beta = 0.3$

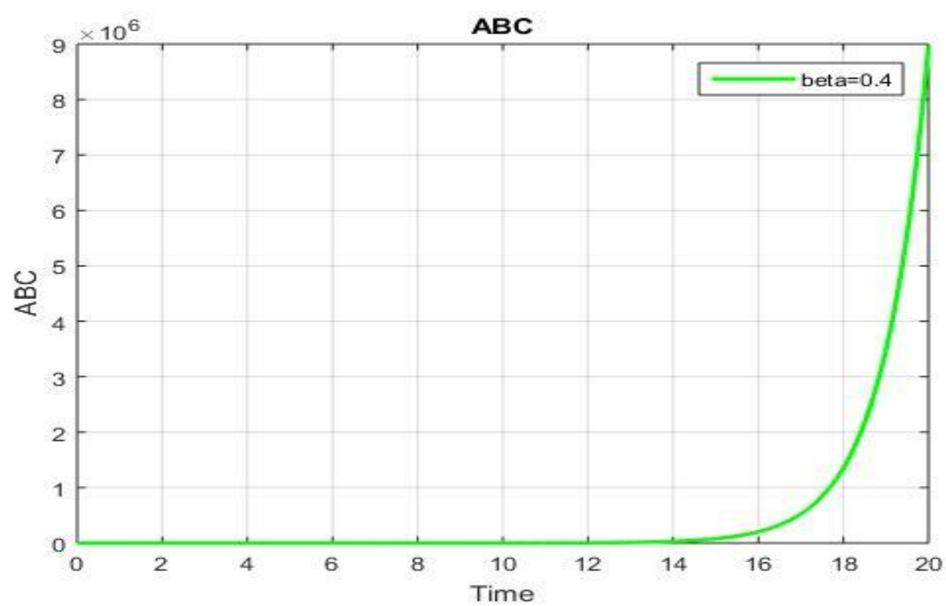

Fig.10 ABC with $\beta = 0.4$

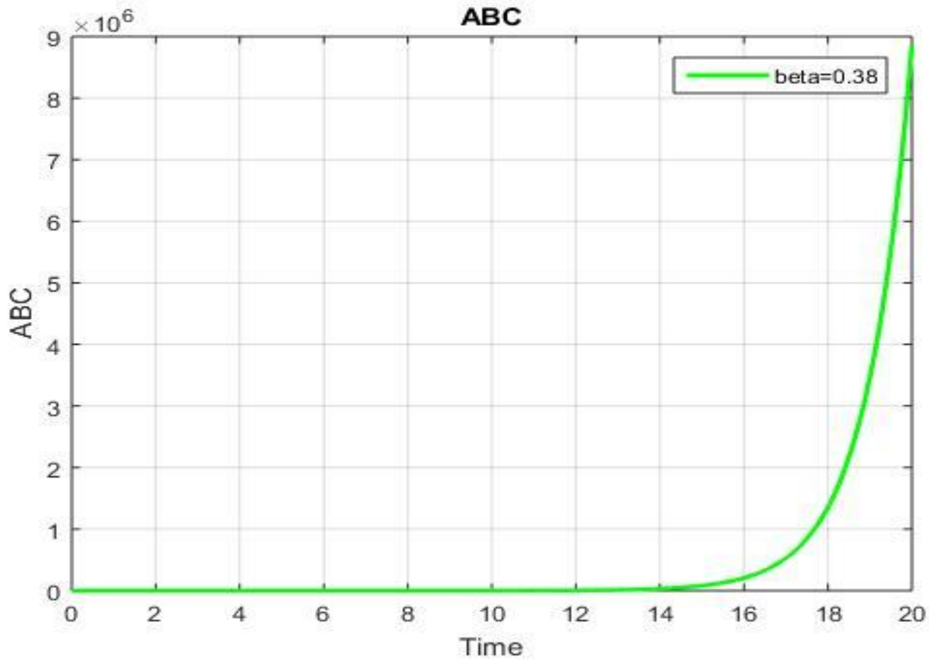

Fig.11 ABC with $\beta = 0.38$

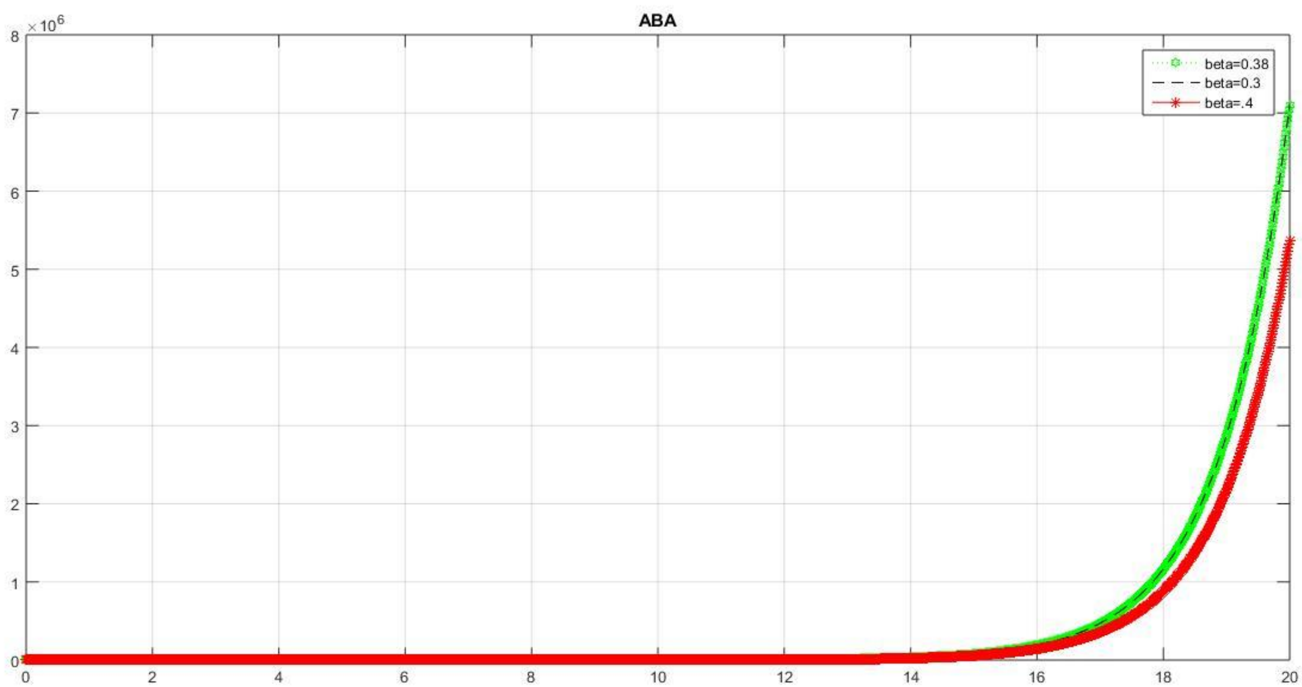

Fig.12 Comparative analysis for all β values

We can see that as time increasing also the value of u(t) with ABC fractional difference is increasing for $\beta = 0.3, 0.38, 0.4$, and there is no big difference in graphics with the original one,

5.4. The Comparative of the solution in CFC, Caputo and ABC with β Parameter

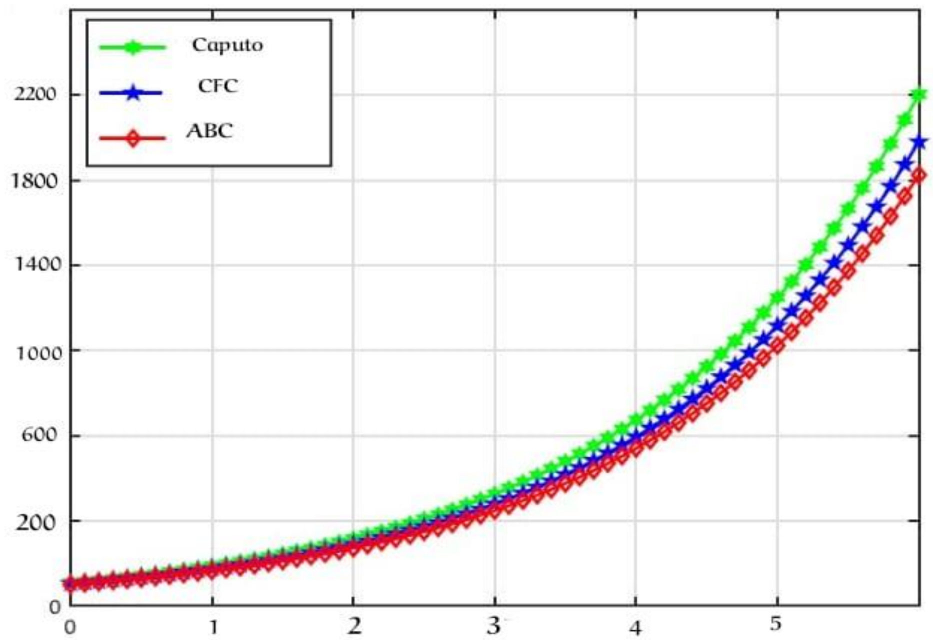

Fig.13 Comparative analysis for β =0.2

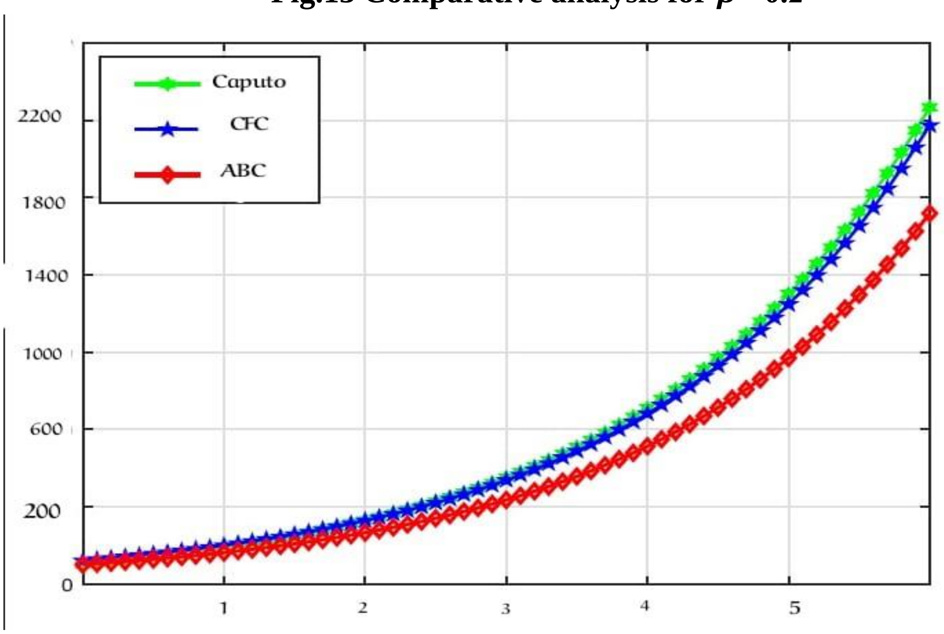

Fig. 14 Comparative analysis for β =1

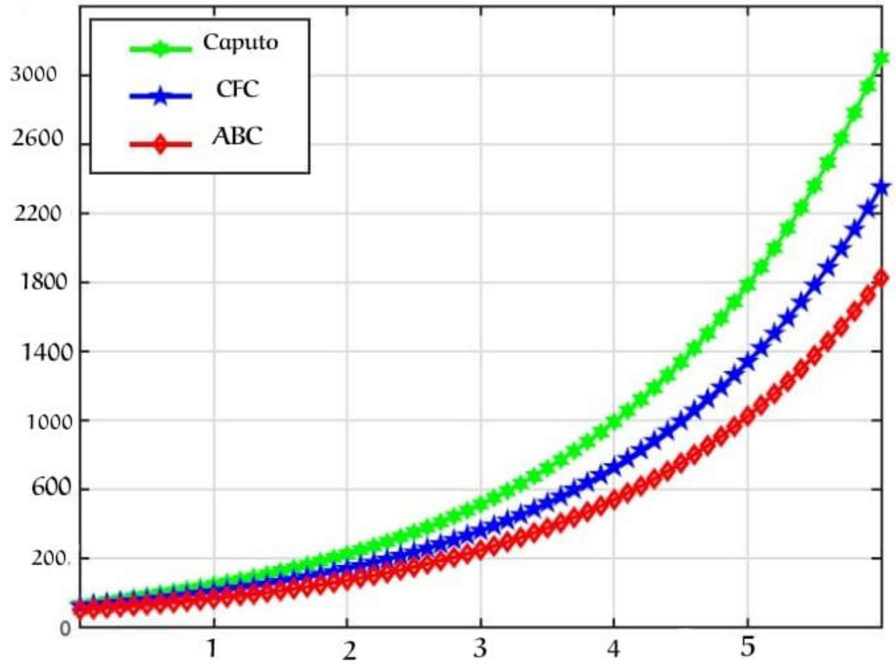

Fig.15 Comparative analysis for β =0.9

We note here that the results in cases for the same solution of differences are almost the same with the difference in the beta value. We relied on close values of beta to see their effect on the same solution. In graphs 4, 8 and 12, we notice that the values are very close here.

In graphs 13, 14 and 15, we took lower values of time to be able to draw the solutions to appear in the same window, as well as we took common outputs for all result with different beta values, and we noticed here that the graphs are very close, and this indicates that The solutions are the same for different differences which indication of the correctness of the solutions.

Chapter 6

Conclusion

1- The price adjustment equation has an important effect on the market and an important value for sellers to get the equilibrium as we solved it using discrete fractional difference and continuous fractional derivative like (Caputo, CFC, ABC, Generalized ABC containing Mittag-Leffler function with three parameter) when expectations of agents are taken and not taken, and by this we get two separate solutions for each derivative of each fractional type.

2- The continuous fractional derivative(CFC, ABC, generalized ABC) that was solved in a previous paper using MATLAB see [11], in this thesis we solved this fractional derivative analytically, and we noticed that the result was similar to what was obtained by programming using MATLAB.

3- The two models were solved by using the discrete fractional differences for (Caputo, CFC, ABC, Generalized ABC) analytically and by MATLAB. The solution was compared in both ways and we found that the solution is identical.

4- In order to avoid obtaining inaccurate and trivial solutions, we have used an initial condition in the case of fractional derivatives (CFC, ABC), since if this initial condition is not used then we arrive at that the solution will be trivial, useless and cannot be explained mathematically and economically. The reason has been clarified in previous research see [4].When we set the initial condition $ru(a) + g(a) = 0$,we concluded that $u(a) = \frac{d_0+s_0}{d_1+s_1}$ such that $u(a)$is the price of goods when $time = a$ and d_0, s_0, d_1, s_1 are defined as some of the factors influencing the market.

5- When we solve the (Caputo and generalized ABC) fractional derivatives, we do not need an initial condition so that we can obtain the solution directly with Mittag-Leffler Kernel with three parameter we can benefit from the properties of Mittag-Leffler that are related to semi groups so that we can make the solutions easier to understand.

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Appendix

1- Caputo fractional difference

For Beta=0.38

```
function G_z= Gama_func(z)
% this funciton to calculate the value of Gama function
% G_z=1;
% for n=1:100
%     G_z_n=(1+1/n)^z / (1+z/n);
%     G_z= G_z_n * G_z;   % accumulate the products
% end
% G_z= G_z / z;   % final value for Gama function
G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);
end
function [E_a,Gama_f]=E_alpha(Lamda,alpha)
% this function to evaluate the E_alpha
E_a=zeros(length(Lamda),1);
Gama_f=[];  % initialize to plot gama func
for k= 0:100
   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func
   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);
   E_a = E_a+ E_ak;    % accumulate the summation
```

```
end

end

% this code to plot Caputo equation

clear, close, clc

%% Initialization

a=0; % inital value

d0=10; %factorial affecting on market

s0=100; %factorial affecting on market

d1=14; %factorial affecting on market

s1=97; %factorial affecting on market

d2=18; %factorial affecting on market

s2=94;     % constan

 % constants

Beta=0.38

alpha=0.38;

c= (-(d0+s0)/(d2+s2));

 r=((d1+s1)/(d2+s2));

B_alpha=sqrt(alpha^2)/alpha;    % normalized function

P_a=(d0+s0)/(d1+s1);

v=r*alpha /(B_alpha -r*(1-alpha));

lamda= -alpha/(1-alpha);

t= 0:.01:20;    % time
```

```
%% Equation 1
Lamda2 =r*t.^alpha;
Lamda2=Lamda2';
[E_a,Gama_f]=E_alpha(Lamda2,alpha);
u2= -c*(1-E_a) + P_a*E_a;
figure(),plot(t,u2,'LineWidth',2);
grid;
xlabel('Time');
ylabel('Caputo');
title('Caputo');
legend( 'beta=0.38');
```

For Beta=0.3

```
function G_z= Gama_func(z)
% this funciton to calculate the value of Gama function
% G_z=1;
% for n=1:100
%     G_z_n=(1+1/n)^z / (1+z/n);
%     G_z= G_z_n * G_z;   % accumulate the products
% end
% G_z= G_z / z;   % final value for Gama function
G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);
```

```
end
function [E_a,Gama_f]=E_alpha(Lamda,alpha)
% this function to evaluate the E_alpha
E_a=zeros(length(Lamda),1);
Gama_f=[];  % initialize to plot gama func
for k= 0:100
   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func
   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);
   E_a = E_a+ E_ak;    % accumulate the summation
end
end
% this code to plot Caputo equation
clear, close, clc
%% Initialization
a=0; % inital value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constan
 % constants
```

```
Beta=0.3;
alpha=0.3;
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));
lamda= -alpha/(1-alpha);
t= 0:.01:20;    % time
%% Equation 2
Lamda2 =r*t.^alpha;
Lamda2=Lamda2';
[E_a,Gama_f]=E_alpha(Lamda2,alpha);
u2= -c*(1-E_a) + P_a*E_a;
figure(),plot(t,u2,'LineWidth',2);
grid;
xlabel('Time');
ylabel('Caputo');
title('Caputo');
legend( 'beta=0.3');
```

For Beta=0.4

```
function G_z= Gama_func(z)
% this funciton to calculate the value of Gama function
% G_z=1;
% for n=1:100
%     G_z_n=(1+1/n)^z / (1+z/n);
%     G_z= G_z_n * G_z;   % accumulate the products
% end
% G_z= G_z / z;   % final value for Gama function
G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);
end
function [E_a,Gama_f]=E_alpha(Lamda,alpha)
% this function to evaluate the E_alpha
E_a=zeros(length(Lamda),1);
Gama_f=[];  % initialize to plot gama func
for k= 0:100
   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func
   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);
   E_a = E_a+ E_ak;    % accumulate the summation
end
end
% this code to plot Caputo equation


clear, close, clc
```

```
%% Initialization
a=0; % initial value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constant
 % constants
Beta=0.4;
alpha=0.4;
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));
lamda= -alpha/(1-alpha);
t= 0:.01:20;    % time
%% Equation 3
Lamda2 =r*t.^alpha;
Lamda2=Lamda2';
[E_a,Gama_f]=E_alpha(Lamda2,alpha);
```

```
u2= -c*(1-E_a) + P_a*E_a;

figure(),plot(t,u2,'LineWidth',2);

grid;

xlabel('Time');

ylabel('Caputo');

title('Caputo');

legend( 'beta=0.4');
```

For all Beta values

```
function G_z= Gama_func(z)

% this funciton to calculate the value of Gama function

% G_z=1;

% for n=1:100

%     G_z_n=(1+1/n)^z / (1+z/n);

%     G_z= G_z_n * G_z;   % accumulate the products

% end

% G_z= G_z / z;   % final value for Gama function

G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);

end

function [E_a,Gama_f]=E_alpha(Lamda,alpha)

% this function to evaluate the E_alpha

E_a=zeros(length(Lamda),1);

Gama_f=[];  % initialize to plot gama func
```

```
for k= 0:100
   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func
   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);
   E_a = E_a+ E_ak;    % accumulate the summation
end
end
% this code to plot Caputo equation
clear, close, clc
%% Initialization
a=0; % initial value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constant
 % constants
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));
 t= 0:.01:20;    % time
```

```
%%For Beta =0.38
Beta=0.38;
alpha=0.38;
lamda= -alpha/(1-alpha);
Lamda2 =r*t.^alpha;
Lamda2=Lamda2';
[E_a,Gama_f]=E_alpha(Lamda2,alpha);
u2_1= -c*(1-E_a) + P_a*E_a;
figure(),plot(t,u2_1,'LineWidth',2);
grid;
xlabel('Time');
ylabel('Caputo');
title('Caputo');
legend( 'beta=0.38');
%% For Beta=0.3
alpha=0.3
Lamda2 =r*t.^alpha;
Lamda2=Lamda2';
[E_a,Gama_f]=E_alpha(Lamda2,alpha);
u2_2= -c*(1-E_a) + P_a*E_a;
figure(),plot(t,u2_2,'LineWidth',2);
grid;
```

```
xlabel('Time');

ylabel('Caputo');

title('Caputo');

legend( 'beta=0.3');

%% For Beta=0.4

alpha=0.4;

Lamda2 =r*t.^alpha;

Lamda2=Lamda2';

[E_a,Gama_f]=E_alpha(Lamda2,alpha);

u2_3= -c*(1-E_a) + P_a*E_a;

figure(),plot(t,u2_3,'LineWidth',2);

grid;

xlabel('Time');

ylabel('Caputo');

title('Caputo');

legend( 'beta=0.4');

%%

plot(t,u2_1,':hg',t, u2_2,'--k',t,u2_3,'-*r');

grid;

legend('beta=0.38' , 'beta=0.3','beta=.4');

title('Caputo')
```

2- CFC fractional difference

For Beta=0.38

```
function G_z= Gama_func(z)
% this funciton to calculate the value of Gama function
% G_z=1;
% for n=1:100
%     G_z_n=(1+1/n)^z / (1+z/n);
%     G_z= G_z_n * G_z;   % accumulate the products
% end
% G_z= G_z / z;   % final value for Gama function
G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);


end
function [E_a,Gama_f]=E_alpha(Lamda,alpha)
% this function to evaluate the E_alpha
E_a=zeros(length(Lamda),1);
Gama_f=[];  % initialize to plot gama func
for k= 0:100
   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func
   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);
   E_a = E_a+ E_ak;    % accumulate the summation
end
```

```
end
% this code to plot CFC equation
clear, close, clc

%% Initialization
a=0; % inital value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constan
 % constants
Beta=0.38;
alpha=0.38;
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));
lamda= -alpha/(1-alpha);
t= 0:.01:20;    % time
```

```
%% Equation 1

P1_1= B_alpha *(1-alpha)* P_a/(B_alpha - r*(1-alpha)) *exp(v*(t-a));   % part 1 of the eq1

P1_2= alpha*c*(exp(v*(t-a))-1) / ((B_alpha - r*(1-alpha)));   %part 2 of the eq1

P1_3= c / (B_alpha - r*(1-alpha)) *exp(v*(t-a));            % Part 3 of the eq1

u1= P1_1 + P1_2 - P1_3; % equation 1

figure(),plot(t,u1,'r','LineWidth',2);

grid;

xlabel('Time');

ylabel('CFC');

title('CFC');

legend('beta=0.38')
```

For Beta=0.3

```
function G_z= Gama_func(z)

% this funciton to calculate the value of Gama function

% G_z=1;

% for n=1:100

%     G_z_n=(1+1/n)^z / (1+z/n);

%     G_z= G_z_n * G_z;   % accumulate the products

% end

% G_z= G_z / z;   % final value for Gama function

G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);
```

```
end

function [E_a,Gama_f]=E_alpha(Lamda,alpha)

% this function to evaluate the E_alpha

E_a=zeros(length(Lamda),1);

Gama_f=[];  % initialize to plot gama func

for k= 0:100

   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func

   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);

   E_a = E_a+ E_ak;    % accumulate the summation

end


end

% this code to plot CFC equation

clear, close, clc

%% Initialization

a=0; % inital value

d0=10; %factorial affecting on market

s0=100; %factorial affecting on market

d1=14; %factorial affecting on market

s1=97; %factorial affecting on market

d2=18; %factorial affecting on market

s2=94;     % constan

 % constants
```

```
Beta=0.3;

alpha=0.3;

c= (-(d0+s0)/(d2+s2));

 r=((d1+s1)/(d2+s2));

B_alpha=sqrt(alpha^2)/alpha;    % normalized function

P_a=(d0+s0)/(d1+s1);

v=r*alpha /(B_alpha -r*(1-alpha));


lamda= -alpha/(1-alpha);

t= 0:.01:20;    % time

%% Equation 2

P1_1= B_alpha *(1-alpha)* P_a/(B_alpha - r*(1-alpha)) *exp(v*(t-a));   % part 1 of the eq2

P1_2= alpha*c*(exp(v*(t-a))-1) / ((B_alpha - r*(1-alpha)));   %part 2 of the eq2

P1_3= c / (B_alpha - r*(1-alpha)) *exp(v*(t-a));            % Part 3 of the eq2

u1= P1_1 + P1_2 - P1_3; % equation 2

figure(),plot(t,u1,'r','LineWidth',2);

grid;

xlabel('Time');

ylabel('CFC');

title('CFC');

legend('beta=0.3')
```

For Beta=0.4

```
function G_z= Gama_func(z)

% this funciton to calculate the value of Gama function

% G_z=1;

% for n=1:100

%     G_z_n=(1+1/n)^z / (1+z/n);

%     G_z= G_z_n * G_z;   % accumulate the products

% end

% G_z= G_z / z;   % final value for Gama function

G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);


end

function [E_a,Gama_f]=E_alpha(Lamda,alpha)

% this function to evaluate the E_alpha

E_a=zeros(length(Lamda),1);

Gama_f=[];  % initialize to plot gama func

for k= 0:100

   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func

   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);

   E_a = E_a+ E_ak;    % accumulate the summation

end
```

```
end
% this code to plot CFC equation


clear, close, clc


%% Initialization
a=0; % inital value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constan
 % constants
Beta=0.4;
alpha=0.4;
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));


lamda= -alpha/(1-alpha);
```

```
t= 0:.01:20;    % time

%% Equation 3

P1_1= B_alpha *(1-alpha)* P_a/(B_alpha - r*(1-alpha)) *exp(v*(t-a));   % part 1 of the eq3

P1_2= alpha*c*(exp(v*(t-a))-1) / ((B_alpha - r*(1-alpha)));   %part 2 of the eq3

P1_3= c / (B_alpha - r*(1-alpha)) *exp(v*(t-a));            % Part 3 of the eq3

u1= P1_1 + P1_2 - P1_3; % equation 3

figure(),plot(t,u1,'r','LineWidth',2);

grid;

xlabel('Time');

ylabel('CFC');

title('CFC');

legend('beta=0.4')
```

For all Beta values

```
function G_z= Gama_func(z)

% this funciton to calculate the value of Gama function

% G_z=1;

% for n=1:100

%     G_z_n=(1+1/n)^z / (1+z/n);

%     G_z= G_z_n * G_z;   % accumulate the products

% end

% G_z= G_z / z;   % final value for Gama function

G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);
```

```
end

function [E_a,Gama_f]=E_alpha(Lamda,alpha)

% this function to evaluate the E_alpha

E_a=zeros(length(Lamda),1);

Gama_f=[];  % initialize to plot gama func

for k= 0:100

   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func

   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);

   E_a = E_a+ E_ak;    % accumulate the summation

end


end

% this code to plot CFC equation


clear, close, clc


%% Initialization

a=0; % inital value

d0=10; %factorial affecting on market

s0=100; %factorial affecting on market

d1=14; %factorial affecting on market

s1=97; %factorial affecting on market
```

```
d2=18; %factorial affecting on market

s2=94;     % constan

 % constants

c= (-(d0+s0)/(d2+s2));

 r=((d1+s1)/(d2+s2));

B_alpha=sqrt(alpha^2)/alpha;    % normalized function

P_a=(d0+s0)/(d1+s1);

v=r*alpha /(B_alpha -r*(1-alpha));


lamda= -alpha/(1-alpha);

t= 0:.01:20;    % time


%% Equation 5

Beta=0.38;

alpha=0.38;

P1_1_2= B_alpha *(1-alpha)* P_a/(B_alpha - r*(1-alpha)) *exp(v*(t-a));   % part 1 of the
eq5

P1_2_2= alpha*c*(exp(v*(t-a))-1) / ((B_alpha - r*(1-alpha)));   %part 2 of the eq5

P1_3_2= c / (B_alpha - r*(1-alpha)) *exp(v*(t-a));            % Part 3 of the eq5


u1_1= P1_1_2 + P1_2_2 - P1_3_2; % equation 5

figure(),plot(t,u1_1,'r','LineWidth',2);

grid;
```

```
xlabel('Time');
ylabel('CFC');
title('CFC');
legend('beta=0.38')
%% Equation 6
alpha=0.4;
P1_1_1= B_alpha *(1-alpha)* P_a/(B_alpha - r*(1-alpha)) *exp(v*(t-a));   % part 1 of the eq6
P1_2_1= alpha*c*(exp(v*(t-a))-1) / ((B_alpha - r*(1-alpha)));   %part 2 of the eq6
P1_3_1= c / (B_alpha - r*(1-alpha)) *exp(v*(t-a));            % Part 3 of the eq6

u1_2= P1_1_1 + P1_2_1 - P1_3_1; % equation 6
figure(),plot(t,u1_2,'r','LineWidth',2);
grid;
xlabel('Time');
ylabel('CFC');
title('CFC');
legend('beta=0.4')
%% Equation 7
alpha=0.3;
P1_1_3= B_alpha *(1-alpha)* P_a/(B_alpha - r*(1-alpha)) *exp(v*(t-a));   % part 1 of the eq7
P1_2_3= alpha*c*(exp(v*(t-a))-1) / ((B_alpha - r*(1-alpha)));   %part 2 of the eq7
```

```
P1_3_3= c / (B_alpha - r*(1-alpha)) *exp(v*(t-a));            % Part 3 of the eq7

u1_3= P1_1_3 + P1_2_3 - P1_3_3; % equation 7

figure(),plot(t,u1_3,'r','LineWidth',2);

grid;

xlabel('Time');

ylabel('CFC');

title('CFC');

legend('beta=0.3')

%% %% all figure

plot(t,u1_1,':hg',t, u1_2,'--k',t,u1_3,'-*r');

grid;

legend('beta=0.38', 'beta=.4', 'beta=0.3');

title('CFC')
```

3- ABC fractional difference

For Beta=0.38

```
function G_z= Gama_func(z)

% this funciton to calculate the value of Gama function

% G_z=1;

% for n=1:100

%     G_z_n=(1+1/n)^z / (1+z/n);

%     G_z= G_z_n * G_z;   % accumulate the products

% end

% G_z= G_z / z;   % final value for Gama function

G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);


end

function [E_a,Gama_f]=E_alpha(Lamda,alpha)

% this function to evaluate the E_alpha

E_a=zeros(length(Lamda),1);

Gama_f=[];  % initialize to plot gama func

for k= 0:100

   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func

   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);

   E_a = E_a+ E_ak;     % accumulate the summation

end
```

```
end
% this code to plot ABC equation

clear, close, clc

%% Initialization
a=0; % inital value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constan
 % constants
Beta=0.38
alpha=0.38;
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));
```

```
lamda= -alpha/(1-alpha);

t= 0:.01:20;    % time


Lamda3 =v*t.^alpha; Lamda3=Lamda3';

[E_a,Gama_f]=E_alpha(Lamda3,alpha);

 %%Equation 1

P3_1= (c*(1-alpha)*E_a)./(B_alpha - r*(1-alpha));      % part 1 of eq1

P3_2= (alpha*c/((B_alpha-r*(1-alpha)))) .*(1-E_a) ;   % part 2 of eq1

P3_3= (B_alpha * P_a*E_a) ./(B_alpha-r*(1-alpha));      % part 3 of eq1


u3= P3_1 - P3_2 + P3_3; %equation 1

figure(),plot(t,u3,'g','LineWidth',2)

grid;

xlabel('Time')

ylabel('ABC')

title('Figure of u3 Vs t');

title('ABC');

legend( 'beta=0.38')
```

For Beta=0.3

```
function G_z= Gama_func(z)

% this funciton to calculate the value of Gama function

% G_z=1;

% for n=1:100
```

```
%     G_z_n=(1+1/n)^z / (1+z/n);
%     G_z= G_z_n * G_z;   % accumulate the products
% end
% G_z= G_z / z;   % final value for Gama function
G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);


end
function [E_a,Gama_f]=E_alpha(Lamda,alpha)
% this function to evaluate the E_alpha
E_a=zeros(length(Lamda),1);
Gama_f=[];  % initialize to plot gama func
for k= 0:100
    Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func
    E_ak=Lamda .^k ./ Gama_func(alpha*k +1);
    E_a = E_a+ E_ak;    % accumulate the summation
end


end
% this code to plot ABC equation


clear, close, clc


%% Initialization
```

```
a=0; % inital value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constan
 % constants
Beta=0.3 ;
alpha=0.3;
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));

lamda= -alpha/(1-alpha);
t= 0:.01:20;    % time

Lamda3 =v*t.^alpha; Lamda3=Lamda3';
[E_a,Gama_f]=E_alpha(Lamda3,alpha);
 %%Equation 2
P3_1= (c*(1-alpha)*E_a)./(B_alpha - r*(1-alpha));       % part 1 of eq2
```

```
P3_2= (alpha*c/((B_alpha-r*(1-alpha)))) .*(1-E_a) ;   % part 2 of eq2

P3_3= (B_alpha * P_a*E_a) ./(B_alpha-r*(1-alpha));      % part 3 of eq2


u3= P3_1 - P3_2 + P3_3; %equation 2

figure(),plot(t,u3,'g','LineWidth',2)

grid;

xlabel('Time')

ylabel('ABC')

title('Figure of u3 Vs t');

title('ABC');

legend( 'beta=0.3')
```

For Beta=0.4

```
function G_z= Gama_func(z)

% this funciton to calculate the value of Gama function

% G_z=1;

% for n=1:100

%     G_z_n=(1+1/n)^z / (1+z/n);

%     G_z= G_z_n * G_z;   % accumulate the products

% end

% G_z= G_z / z;   % final value for Gama function

G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);


end
```

```
function [E_a,Gama_f]=E_alpha(Lamda,alpha)
% this function to evaluate the E_alpha
E_a=zeros(length(Lamda),1);
Gama_f=[];  % initialize to plot gama func
for k= 0:100
   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func
   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);
   E_a = E_a+ E_ak;    % accumulate the summation
end

end
% this code to plot ABC equation

clear, close, clc

%% Initialization
a=0; % inital value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constan
```

```
 % constants
Beta=0.4;
alpha=0.4;
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));

lamda= -alpha/(1-alpha);
t= 0:.01:20;    % time

Lamda3 =v*t.^alpha; Lamda3=Lamda3';
[E_a,Gama_f]=E_alpha(Lamda3,alpha);
 %%Equation 3
P3_1= (c*(1-alpha)*E_a)./(B_alpha - r*(1-alpha));      % part 1 of eq3
P3_2= (alpha*c/((B_alpha-r*(1-alpha)))) .*(1-E_a) ;   % part 2 of eq3
P3_3= (B_alpha * P_a*E_a) ./(B_alpha-r*(1-alpha));     % part 3 of eq3

u3= P3_1 - P3_2 + P3_3; %equation 3
figure(),plot(t,u3,'g','LineWidth',2)
grid;
xlabel('Time')
```

```
ylabel('ABC')
title('Figure of u3 Vs t');
title('ABC');
legend( 'beta=0.4')
```

For all Beta values

```
function G_z= Gama_func(z)
% this funciton to calculate the value of Gama function
% G_z=1;
% for n=1:100
%     G_z_n=(1+1/n)^z / (1+z/n);
%     G_z= G_z_n * G_z;   % accumulate the products
% end
% G_z= G_z / z;   % final value for Gama function
G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);


end
function [E_a,Gama_f]=E_alpha(Lamda,alpha)
% this function to evaluate the E_alpha
E_a=zeros(length(Lamda),1);
Gama_f=[];  % initialize to plot gama func
for k= 0:100
   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func
   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);
```

```
    E_a = E_a+ E_ak;    % accumulate the summation
end

end
% this code to plot ABC equation

clear, close, clc

%% Initialization
a=0; % inital value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constan
 % constants
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));
```

```
lamda= -alpha/(1-alpha);

t= 0:.01:20;    % time


Lamda3 =v*t.^alpha; Lamda3=Lamda3';

[E_a,Gama_f]=E_alpha(Lamda3,alpha);

%%Equation4

alpha=0.3;

beta=0.3;

Lamda3 =v*t.^alpha; Lamda3=Lamda3';

[E_a,Gama_f]=E_alpha(Lamda3,alpha);

P3_1_1= (c*(1-alpha)*E_a)./(B_alpha - r*(1-alpha));      % part 1 of eq4

P3_2_1= (alpha*c/((B_alpha-r*(1-alpha)))) .*(1-E_a) ;   % part 2 of eq4

P3_3_1= (B_alpha * P_a*E_a) ./(B_alpha-r*(1-alpha));     % part 3 of eq4


u3_1= P3_1_1 - P3_2_1 + P3_3_1;

figure(),plot(t,u3_1,'g','LineWidth',2)

grid;

xlabel('Time')

ylabel('ABC')


title('ABC');

legend( 'beta=0.38')

%%Equation5
```

```
alpha=0.3;
P3_1_2= (c*(1-alpha)*E_a)./(B_alpha - r*(1-alpha));      % part 1 of eq5
P3_2_2= (alpha*c/((B_alpha-r*(1-alpha)))) .*(1-E_a) ;   % part 2 of eq5
P3_3_2= (B_alpha * P_a*E_a) ./(B_alpha-r*(1-alpha));     % part 3 of eq5

u3_2= P3_1_2 - P3_2_2 + P3_3_2;
figure(),plot(t,u3_2,'g','LineWidth',2)
grid;
xlabel('Time')
ylabel('ABC')

title('ABC');
legend( 'beta=0.3')
%%Equation6
alpha=0.4;
P3_1_3= (c*(1-alpha)*E_a)./(B_alpha - r*(1-alpha));      % part 1 of eq6
P3_2_3= (alpha*c/((B_alpha-r*(1-alpha)))) .*(1-E_a) ;   % part 2 of eq6
P3_3_3= (B_alpha * P_a*E_a) ./(B_alpha-r*(1-alpha));     % part 3 of eq6

u3_3= P3_1_3 - P3_2_3 + P3_3_3;
figure(),plot(t,u3_3,'g','LineWidth',2)
grid;
xlabel('Time')
```

```
ylabel('ABC')

title('ABC');
legend( 'beta=0.4')

%% %% all figure
plot(t,u3_1,':hg',t, u3_2,'--k',t,u3_3,'-*r');
grid;
legend('beta=0.38',  'beta=0.3','beta=.4');
title('ABC')
```

4- For all Differences

For Beta=0.2

```
 function G_z= Gama_func(z)
% this funciton to calculate the value of Gama function
% G_z=1;
% for n=1:100
%     G_z_n=(1+1/n)^z / (1+z/n);
%     G_z= G_z_n * G_z;   % accumulate the products
% end
% G_z= G_z / z;   % final value for Gama function
G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);
```

```
end

function [E_a,Gama_f]=E_alpha(Lamda,alpha)

% this function to evaluate the E_alpha

E_a=zeros(length(Lamda),1);

Gama_f=[];  % initialize to plot gama func

for k= 0:100

   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func

   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);

   E_a = E_a+ E_ak;    % accumulate the summation

end


end

% this code to plot Caputo equation


clear, close, clc


%% Initialization

a=0; % inital value

d0=10; %factorial affecting on market

s0=100; %factorial affecting on market

d1=14; %factorial affecting on market

s1=97; %factorial affecting on market
```

```
d2=18; %factorial affecting on market
s2=94;    % constan
 % constants
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));

lamda= -alpha/(1-alpha);
t= 0:.01:03;    % time

Lamda3 =v*t.^alpha; Lamda3=Lamda3';
[E_a,Gama_f]=E_alpha(Lamda3,alpha);
%%Equation4
alpha=0.3;
beta=0.2;
Lamda3 =v*t.^alpha; Lamda3=Lamda3';
[E_a,Gama_f]=E_alpha(Lamda3,alpha);
P3_1_1= (c*(1-alpha)*E_a)./(B_alpha - r*(1-alpha));      % part 1 of eq1
P3_2_1= (alpha*c/((B_alpha-r*(1-alpha)))) .*(1-E_a) ;   % part 2 of eq1
P3_3_1= (B_alpha * P_a*E_a) ./(B_alpha-r*(1-alpha));      % part 3 of eq1
```

```
u3_1= P3_1_1 - P3_2_1 + P3_3_1;
figure(),plot(t,u3_1,'g','LineWidth',2)
grid;
xlabel('Time')
ylabel('Caputo')


title('Caputo');
legend( 'beta=0.2')
%%Equation5
alpha=0.3;
P3_1_2= (c*(1-alpha)*E_a)./(B_alpha - r*(1-alpha));      % part 1 of eq5
P3_2_2= (alpha*c/((B_alpha-r*(1-alpha)))) .*(1-E_a) ;   % part 2 of eq5
P3_3_2= (B_alpha * P_a*E_a) ./(B_alpha-r*(1-alpha));     % part 3 of eq5


u3_2= P3_1_2 - P3_2_2 + P3_3_2;
figure(),plot(t,u3_2,'g','LineWidth',2)
grid;
xlabel('Time')
ylabel('CFC')


title('CFC');
legend( 'beta=0.2')
%%Equation3
```

```
alpha=0.4;
P3_1_3= (c*(1-alpha)*E_a)./(B_alpha - r*(1-alpha));      % part 1 of eq6
P3_2_3= (alpha*c/((B_alpha-r*(1-alpha)))) .*(1-E_a) ;   % part 2 of eq6
P3_3_3= (B_alpha * P_a*E_a) ./(B_alpha-r*(1-alpha));      % part 3 of eq6

u3_3= P3_1_3 - P3_2_3 + P3_3_3;
figure(),plot(t,u3_3,'g','LineWidth',2)
grid;
xlabel('Time')
ylabel('ABC')

title('ABC');
legend( 'beta=0.2')

%% %% all figure
plot(t,u3_1,':hg',t, u3_2,'--k',t,u3_3,'-*r');
grid;
legend('Caputo','CFC','ABC');
title('beta=0.2')
```

For Beta=0.9

```
function G_z= Gama_func(z)
```

```
% this funciton to calculate the value of Gama function
% G_z=1;
% for n=1:100
%     G_z_n=(1+1/n)^z / (1+z/n);
%     G_z= G_z_n * G_z;   % accumulate the products
% end
% G_z= G_z / z;   % final value for Gama function
G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);


end
function [E_a,Gama_f]=E_alpha(Lamda,alpha)
% this function to evaluate the E_alpha
E_a=zeros(length(Lamda),1);
Gama_f=[];  % initialize to plot gama func
for k= 0:100
    Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func
    E_ak=Lamda .^k ./ Gama_func(alpha*k +1);
    E_a = E_a+ E_ak;    % accumulate the summation
end


end
% this code to plot Caputo equation
```

```
clear, close, clc


%% Initialization

a=0; % inital value

d0=10; %factorial affecting on market

s0=100; %factorial affecting on market

d1=14; %factorial affecting on market

s1=97; %factorial affecting on market

d2=18; %factorial affecting on market

s2=94;     % constan

 % constants

c= (-(d0+s0)/(d2+s2));

 r=((d1+s1)/(d2+s2));

B_alpha=sqrt(alpha^2)/alpha;    % normalized function

P_a=(d0+s0)/(d1+s1);

v=r*alpha /(B_alpha -r*(1-alpha));


lamda= -alpha/(1-alpha);

t= 0:.01:03;    % time


%% Equation 1

Beta=0.9;

alpha=0.38;
```

```
P1_1_2= B_alpha *(1-alpha)* P_a/(B_alpha - r*(1-alpha)) *exp(v*(t-a));   % part 1 of the
eq1
P1_2_2= alpha*c*(exp(v*(t-a))-1) / ((B_alpha - r*(1-alpha)));   %part 2 of the eq1
P1_3_2= c / (B_alpha - r*(1-alpha)) *exp(v*(t-a));            % Part 3 of the eq1

u1_1= P1_1_2 + P1_2_2 - P1_3_2; % equation 1
figure(),plot(t,u1_1,'r','LineWidth',2);
grid;
xlabel('Time');
ylabel('Caputo');
title('beta=0.9');
legend('Caputo')

%% Equation 2
Beta=0.9;
alpha=0.4;
P1_1_1= B_alpha *(1-alpha)* P_a/(B_alpha - r*(1-alpha)) *exp(v*(t-a));   % part 1 of the
eq2
P1_2_1= alpha*c*(exp(v*(t-a))-1) / ((B_alpha - r*(1-alpha)));   %part 2 of the eq2
P1_3_1= c / (B_alpha - r*(1-alpha)) *exp(v*(t-a));            % Part 3 of the eq2

u1_2= P1_1_1 + P1_2_1 - P1_3_1; % equation 2
figure(),plot(t,u1_2,'r','LineWidth',2);
```

```
grid;
xlabel('Time');
ylabel('CFC');
title('beta=0.9');
legend('CFC')
%% Equation 3
Beta=0.9;
alpha=0.3;
P1_1_3= B_alpha *(1-alpha)* P_a/(B_alpha - r*(1-alpha)) *exp(v*(t-a));   % part 1 of the
eq3
P1_2_3= alpha*c*(exp(v*(t-a))-1) / ((B_alpha - r*(1-alpha)));   %part 2 of the eq3
P1_3_3= c / (B_alpha - r*(1-alpha)) *exp(v*(t-a));           % Part 3 of the eq3

u1_3= P1_1_3 + P1_2_3 - P1_3_3; % equation 3
figure(),plot(t,u1_3,'r','LineWidth',2);
grid;
xlabel('Time');
ylabel('ABC');
title('bet=0.9');
legend('ABC')
%% %% all figure
plot(t,u1_1,':hg',t, u1_2,'--k',t,u1_3,'-*r');
grid;
```

```
legend('Caputo', 'CFC', 'ABC');

title('beta=0.9')
```

For Beta=1

```
function G_z= Gama_func(z)

% this funciton to calculate the value of Gama function

% G_z=1;

% for n=1:100

%     G_z_n=(1+1/n)^z / (1+z/n);

%     G_z= G_z_n * G_z;   % accumulate the products

% end

% G_z= G_z / z;   % final value for Gama function

G_z=integral(@(t)(exp(-t).*t.^(z-1)),0,Inf);


end

function [E_a,Gama_f]=E_alpha(Lamda,alpha)

% this function to evaluate the E_alpha

E_a=zeros(length(Lamda),1);

Gama_f=[];  % initialize to plot gama func

for k= 0:100

   Gama_f(k+1)=Gama_func(alpha*k +1);  % to plot gama func

   E_ak=Lamda .^k ./ Gama_func(alpha*k +1);

   E_a = E_a+ E_ak;    % accumulate the summation

end
```

```
end
% this code to plot Caputo equation

clear, close, clc

%% Initialization
a=0; % initial value
d0=10; %factorial affecting on market
s0=100; %factorial affecting on market
d1=14; %factorial affecting on market
s1=97; %factorial affecting on market
d2=18; %factorial affecting on market
s2=94;     % constant
 % constants
c= (-(d0+s0)/(d2+s2));
 r=((d1+s1)/(d2+s2));
B_alpha=sqrt(alpha^2)/alpha;    % normalized function
P_a=(d0+s0)/(d1+s1);
v=r*alpha /(B_alpha -r*(1-alpha));
 t= 0:.01:20;    % time

%%For Beta =0.9
```

```
Beta=0.9;

alpha=0.38;

lamda= -alpha/(1-alpha);

Lamda2 =r*t.^alpha;

Lamda2=Lamda2';

[E_a,Gama_f]=E_alpha(Lamda2,alpha);

u2_1= -c*(1-E_a) + P_a*E_a;


figure(),plot(t,u2_1,'LineWidth',2);

grid;

xlabel('Time');

ylabel('beta=1');

title('beta=1');

legend( 'Caputo');

%% For Beta=0.9

Beta=0.9

alpha=0.3

Lamda2 =r*t.^alpha;

Lamda2=Lamda2';

[E_a,Gama_f]=E_alpha(Lamda2,alpha);

u2_2= -c*(1-E_a) + P_a*E_a;


figure(),plot(t,u2_2,'LineWidth',2);
```

```
grid;
xlabel('Time');
ylabel('beta=1');
title('beta=1');
legend( 'ABC');
%% For Beta=0.4
beta=1;
alpha=0.4;
Lamda2 =r*t.^alpha;
Lamda2=Lamda2';
[E_a,Gama_f]=E_alpha(Lamda2,alpha);
u2_3= -c*(1-E_a) + P_a*E_a;


figure(),plot(t,u2_3,'LineWidth',2);
grid;
xlabel('Time');
ylabel('beta=1');
title('beta=1');
legend( 'Caputo');
%% all differences
plot(t,u2_1,':hg',t, u2_2,'--k',t,u2_3,'-*r');
grid;
legend('Caputo' , 'CFC','ABC');title('beta=1')
```