Mathematical Models Described by Gibbs Measures and Phase Transitions on Cayley Tree

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نماذج رياضية موصوفة باستخدام مقاييس جيبس و انتقالات المرحلة على شجرة كايلي

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Declaration

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DEDICATION

I dedicate this thesis to my wonderful parents, brothers, sisters and friends for their love, steadfast support and permanent encouragement throughout my life, whether in my bachelor's or master's degree academic study.

To my esteemed Prof. Dr. Saed Mallak, who has not hesitated to help, support and encourage me.

To the sleeping angel, my grandmother

To all my teachers who inspired me with their knowledge.

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Table of Contents

List of Abbreviations

Abbreviation	Explanation
NN	Nearest-Neighbor
NNN	Next-Nearest-Neighbor
PNNN	Prolonged Next-Nearest- Neighbor
TPNNN	Ternary-Prolonged Next-Nearest- Neighbor
OLNNN	One-Level Next-Nearest- Neighbor
NNNI	Next-Nearest-Neighbor Ising
MRFM	Markov Random Field Method
PFM	Partition Function Method

Mathematical Models Described by Gibbs Measures and Phase Transitions on Cayley Tree

By: Sanabel Abu Oun

Supervised by Prof. Dr. Saed Mallak

Abstract

In this thesis, five models are studied on lattice spin systems which are considered in statistical mechanics. A probability measure that's called Gibbs Measure is defined for these models on Cayley tree (or Bethe lattice) and the existence of phase transition is proved by using two approaches: Markov Random field method and Partition Function method. The first and second models are related to Ising-Vannimenus Model with three different competing interactions on semi-infinite Cayley tree, the analysis of these models is done by using Markov Random field method making use of the Kolmogorov consistency conditions. To achieve this, we constructed the set of recurrence equations that corresponds to the mentioned models and satisfied consistency condition, then we analyzed these equations and determined the conditions on the temperature and the coupling constants in which the phase transition exists. The third set of models (3 models) are called Potts Models, which are a generalization for Ising model, we have constructed the recurrence equations for these models by using Partition function method. In the same way as the previous models, the phase transition conditions are determined. Since we got a high order polynomials with complicated factors, we used Wolfarm Mathematica for equations analysis.

Keywords: Gibbs measure, Phase transition, Cayley tree, Competing interactions, Ising-Vannimenus model, Potts model.

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الملخص

في هذه الأطروحة ، تم دراسة خمسة نماذج على أنظمة الدوران الشبكية والتي تعتبر فرع مهم في الميكانيكا الإحصائية. تم تعريف مقياس احتمالي يسمى مقياس جيبس على هذه النماذج المعرفة على شجرة كايلي وتم إثبات وجود انتقال الطور باستخدام طريقتين : طريقة حقول ماركوف العشوائية وطريقة اقترانات التقسيم. النموذجين الأول و الثاني مرتبطان بنموذج آيزنج فانيمينوس على شجرة كايلي شبه اللانهائية مع ثلاثة تفاعلات متنافسة مختلفة ، تم تحليل هذه النماذج باستخدام طريقة حقول ماركوف العشوائية . لتحقيق ذلك ، قمنا ببناء مجموعة من معادلات التكرار التي تتوافق مع النماذج المذكورة وحالة الاتساق المستوفاة ، ثم قمنا بعمل تحليل لهذه المعادلات وحددنا الشروط على درجة الحرارة وثوابت الاقتران التي ينتج عنها انتقالات مرحلية للنماذج السابقة .مجموعة النماذج الثالثة تسمى نماذج بوتس وهو تعميم لنموذج آيزنج . لقد فسرنا معادلات التكرار لهذا النموذج باستخدام طريقة اقترانات التقسيم . كما هو الحال في النماذج السابقة ، قمنا بتحديد ظروف انتقال الطور. لأننا حصلنا على كثيرات حدود ذات درجات عالية ، فقد قمنا باستخدام برنامج الماثماتيكا للمساعدة بتحليلها.

الكلمات المفتاحية : مقياس جيبس ، الانتقالات المرحلية ، شجرة كايلي، التفاعلات المتنافسة ، آيزنج فانيمينوس مودل ، بوتس مودل.

Chapter One

Introduction

As it is well known, Statistical Mechanics is a branch of physics interested in explaining the macroscopic behavior of matter based on its microscopic structure**.** This structure is represented as a system of an infinite number of random variables attached to vertices of a lattice system (one dimensional or more) such that these elements interact with each other according to their position**.** This interpretation was obtained by using statistical methods and probability theory. The role of probability theory is to represent this system of random variables by using a well- known probability measure called Gibbs measure [28].

Although the basics of Statistical Mechanics have already been laid in 19^{th} century, but the basics of Gibbs measure and the study of infinite random systems began in 1960's by R.L. Dobrushin $(1965-1970)$ [21], D. Ruelle and O .E. Lanford (1969) who proposed the theory of Gibbs measure as a mathematical description of a huge number of interacting components as an equilibrium state of a physical system [32]. As a result, from probabilistic point of view, a Gibbs measure is nothing other than the distribution of a countably infinite family of random variables that achieve some specific conditional probabilities. This concept has received interests from both mathematical physicists and probabilists, which turns many of physical problems and questions to probabilistic problems.

Cayley tree (or Bethe lattice) is a non-realistic lattice which was introduced in to the physical literature in 1935 by the physicist Hans Bethe [14]. The operations and calculations on this lattice are easier and more accurate to calculate and understand than the d dimensional Z^d lattice. As a result, many of the topics in statistical mechanics have recently been taken into account on the Cayley tree [1-

10]. The results obtained on the Cayley tree are very helpful on studying and analyzing the d-dimensional Z^d lattice. As a result, we see that many researchers have employed the well-known models: The Ising and Potts models in conjunction with the Cayley tree [36-38, 1-10].

Ising model is a model that was invented by W. Lenz (1920) [34] and it was investigated by his student E. Ising (1924) in his PhD thesis, this model is a representation of spins for a substance by $+1$ "up orientation " and -1 "down orientation" such that the set $S = \{-1,1\}$ of possible orientation is called the state space . The spins form an infinite linear chain, i.e., they are located at the sites of Z such that these spins interact with each other based on a Hamiltonian (energy function) of the form [28]

$$H(\sigma) = -J \sum_{|x-y|=1} \sigma(x)\sigma(y) - h \sum_{x \in z} \sigma(x)$$

where the first sum is over nearest-neighbor vertices, i.e., $\{x, y \in z \colon |x - y| = 1\}$ and the spins $\sigma(x)$ and $\sigma(y)$ take values in the set of state space $S = \{+1, -1\}$, h represents the action of an external magnetic field and J is a real number. This model was developed for more dimensions and interactions and became an important step towards a mathematical theory of phase transitions even though Ising failed to get phase transition for his model [28]. This model has a relevance to physical, biological and chemical systems, see [23, 28, 33]. In (1981), Ising-Vannimenus model [55] investigated by J. Vannimenus consists of Ising spins ($\sigma = \pm 1$) on a cayley tree of order 2, in which two competing interactions are presented: nearest-neighbor (NN) interactions and next-nearest-neighbor (NNN) interactions. Since the appearance of Vannimenus model, the Ising model on a Cayley tree (or Bethe lattice) with competing interactions has received great attention, see [15, 18, 30, 43, 47, 53], these studies' investigation of Gibbs measures for a developed Ising-Vannimenus model on trees were based on recurrent equations analysis. In 2016, H. Akin used a new rigorous measure-theoretical approach to describe a specifically set of Gibbs measures with a memory of length 2 that corresponds to the Ising-Vannimenus Model on the Cayley tree of order two [5]. In 2017, he dealt with a new Gibbs measures of Ising-Vannimenus model with competing NN and prolonged NNN interactions on a Cayley tree of order three, such that he found the set of translation-invariant Gibbs states for this model using Markov Random field approach [4]. Moreover, in (2018), H. Akin generalized his results in [4] to an arbitrary order Cayley tree [3].

The Potts Model is a model Introduced by R. Potts in 1952 [49] as a generalization of the Ising model, i.e., with q state space such that $q \geq 2$. M. Miyamoto (1982) generalized the Spitzer's results of investigation of Gibbs measure as a Markov chain on Cayley tree whose state space $\{-1, 1\}$ to the case when the state space is a compact set [44]. Zachary (1983) also generalized Spitzer's results to a countable state space on a regular infinite tree [56], F. Peruggi (1984) studied the properties of one-step Markov and m-step ($m = 1$ or $m = 2$) for translationally-invariant probability measures on q state model on Bethe lattices [35]. F. Peruggi, F. d. Liberto, and G. Monroy (1983, 1985, 1987) also generalized specific results on Potts model and phase diagram describing this model [36-38]. In addition, this model has taken a great interest in the recent years. The researchers studied the recurrence equations and obtain some exact results: critical temperatures, partition function, number of phases and curves. For more details, see [6-8,19]. This model has many biological, physical and social applications, for more details see [13].

In this thesis, we are interested in constructing lattice models on Cayley tree which are described by Gibbs measure in order to determine if there is a phase transition or not. Phase transitions can be physically defined as a transformation of a thermodynamic system from one phase to another. In the study of statistical physics, the phenomenon of phase transition commonly describes the transitions between solid, liquid, and gaseous states of matter. We cannot observe the mixture between these three states to approach to another phase. In fact, the transition can only occur from one phase to another phase. In the study of theory of phase transition, the description of Gibbs measures of a given Hamiltonian has brought us a fundamental problem in this equilibrium statistical mechanics. In fact, the main problem of equilibrium statistical physics is to describe all limiting Gibbs measures of a given Hamiltonian on a lattice, this attention because these distributions describe the equilibrium states of a physical system. This is known to be a difficult task. Mathematically, we can say that the phase transition occurs with non-uniqueness of Gibbs measure (more than one limiting Gibbs distribution exist) [31].

In this thesis, five lattice models that are related to Ising-Vannimenus and Potts Models with different competing interactions on Cayley tree will be studied and solved. The existence of phase transition for the models will be proved by using one approach from the following: Markov Random Field Method (MRFM) or Partition Function Method (PFM).

In both chapters 1 and 2, an introduction and some basic definitions about Gibbs measures, Cayley trees, Ising model and Potts model will be presented.

In chapter 3, an Ising-Vannimenus Model with three competing interactions, NN, prolonged NNN and ternary-prolonged NNN interactions on Cayley tree of order three will be studied, and the conditions for phase transition will be determined by using MRFM.

In chapter 4, an Ising-Vannimenus Model on Cayley tree of order four with one-level NNN interaction will be studied. As the previous models, MRFM will be used to analyze the model and prove the existence of phase transition on the translation-invariant Gibbs measures.

In chapter 5, three Potts models will be studied. The first one is a model with q state space on Cayley tree of order three with two competing interactions: NN and ternary-prolonged NNN interactions such that the PFM will be used to prove the existence of phase transition.The second model is a Potts model with $q = 3$ on Cayley tree of order two and with the same competing interactions in the first model in addition to one-level NNN interaction. The last model is a development for the second model on Cayley tree of order three with one-level 3-tuple interactions. The PFM will be used to solve these models.

Finally, in chapter 6, basic results and conclusions will be determined.

Chapter Two

Preliminaries

This Chapter consists of three sections. In section 2.1 basic definitions and properties about Gibbs measures and Phase Transition are presented and in section 2.2 definitions and remarks about Cayley trees and Competing interactions are explained. In section 2.3, basic definitions for Ising model, Potts model and Ising-Vannimenus model on Cayley tree are presented.

2.1 Gibbs Measures and Phase Transition

Gibbs measure is related with Markov chains. As a result, we start with Markov chains definition.

Definition 2.1.1. [41] Let S be a discrete set, finite or countably infinite. Suppose to each pair $i, j \in S$ there is assigned a non-negative number p_{ij} such that these numbers satisfy the constraint $\sum_{j\in S} p_{ij} = 1, \forall i \in S$. Let $X_0, X_1, ...$ be a sequence of random variables whose range is in S.

The sequence (X_n) is a Markov chain if $P(X_{n+1} = j|X_0 = i_0, \dots, X_n = i_n) = P(X_{n+1} = j|X_n = i_n) = p_{i_n j}^{n+1}$ for all n and every sequence $i_0, \dots, i_n \in S$ for which $P(X_0 = i_0, \dots, X_n = i_n) > 0$.

It is known that a Markov chain is stationary (homogeneous) if $p_{i_n j}^{n+1}$ does not depend on n, that is $p_{i_n j}^{n+1} = p_{i_n j}$. Otherwise, it is non-stationary (non-homogeneous or inhomogeneous). See Figure 2.1.1.

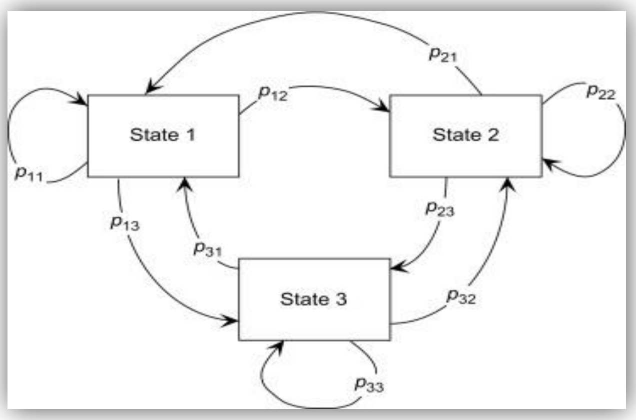

Figure 2.2.1. Schematic illustration of a three-state, first-order Markov chain.

Definition 2.1.2. [16] A Markov chain with memory m is a process satisfying $P\ (X_{n+1} = s_{n+1} \mid X_n = s_n, \ldots\ldots, X_1 = s_1) = P(X_{n+1} = s_{n+1} \mid X_n = s_n, \ldots\ldots,$

$X_{n-m+1} = s_{n-m+1})$ for all $n \geq m$.

Definition 2.1.3. [41] Let S be a countably infinite set and (Φ, E) any measurable space. A family $(\varphi(x))_{x \in S}$ of random variables which are defined on some probability space and take values in Φ is called a random field , or a spin system. S is called the parameter set, Φ is called the state space, or spin space, and $\varphi(x)$ is called the spin at state x.

Definition 2.1.4. (Configurations) [41] Let $\Omega = \Phi^S = \{(\varphi(x))_{x \in S} : \varphi(x) \in \Phi\}$. Then $\sigma \in \Omega$ is called a configuration and Ω is called the set of all possible configurations.

Definition 2.1.5. (The Potential and Hamiltonian) [28] Let $\Gamma = \{A \subset S: A \neq \varphi, |A| < \infty\}$. An interaction potential (or simply a potential) is a family $U = (U_A), A \in \Gamma$ of functions where $U_A : \Omega \to R$ with the following properties:

(i) For each $A \in \Gamma$, U_A is a measurable function with respect to the product sigma algebra.

(ii) For all $\Lambda \in \Gamma$ and $\sigma \in \Omega$, the series

$$H_{\Lambda}^{U}(\sigma) = \sum_{A \in \Gamma, A \cap \Lambda \neq \varphi} U_A(\sigma) \qquad \text{exists.}$$

$H_{\Lambda}^{U}(\sigma)$ is called the (total) energy of σ in Λ for U, and H_{Λ}^{U} called the Hamiltonian in Λ for U.

Definition 2.1.6.(Gibbs Probability distribution) [41] On the space $\Omega^{\Lambda} = \Phi^{\Lambda} = \{(\varphi(x))_{x \in \Lambda} : \varphi(x) \in \Phi\}$ we introduce a probability distribution defining the probability of a configuration by $\mu_{\Lambda}(w^{\Lambda}) = Z_{\Lambda}^{-1}\, exp[-\beta H_{\Lambda}^{\Phi}(w^{\Lambda})]$ where Z_{Λ} is a normalizing factor (Partition function) defined by the condition $\sum_{w^{\Lambda} \in \Omega^{\Lambda}} \mu_{\Lambda}(w^{\Lambda}) = 1$, that is $Z_{\Lambda} = \sum_{w^{\Lambda} \in \Omega^{\Lambda}} exp[-\beta H_{\Lambda}^{\Phi}(w^{\Lambda})]$. $\beta = (kT)^{-1}$, where k is a constant we consider it to be 1 and T is the temperature. This Probability distribution is called a Gibbs probability distribution in Λ corresponding to the given Hamiltonian**.**

Definition 2.1.7. (Convex Set) [28] A subset B of any real valued vector space is called a convex set if for all $\mu, v \in B$ and $0 < s < 1$ then $s\,\mu + (1 - s)v \in B$. See Figure 2.1.2.

Definition 2.1.8. (Extreme elements) [28] An element μ of a convex subset B of any real vector space is said to be extreme in B if $\mu \neq s\, v + (1 - s)\, v'$ for all $0 < s < 1$ and $v, v' \in B$.

Definition 2.1.9. (Extreme Limiting Gibbs Measures) [28] Let $\{\mu_{\Lambda_n}^{H}\}_{n=1}^{\infty}$ be a sequence of Gibbs probability measures that corresponds to the Hamiltonian H. Then the set of extreme limiting Gibbs measures (extreme Gibbs states) for a given Hamiltonian H is a set of infinite limits μ^H such that $\mu^H(A) = \lim_{\Lambda_n \to \infty} \mu_{\Lambda_n}^{H}(A)$ for each cylinder event A.

Theorem 2.1.1. [28] The set of all limiting Gibbs measures (Gibbs states or DLR states) G_H corresponding to the Hamiltonian H represents a convex set , i.e., if $\mu, v \in G_H$ and $0 < s < 1$ then $s\,\mu + (1 - s)v \in G_H$. The most interesting elements of this convex set are its extreme Gibbs states.

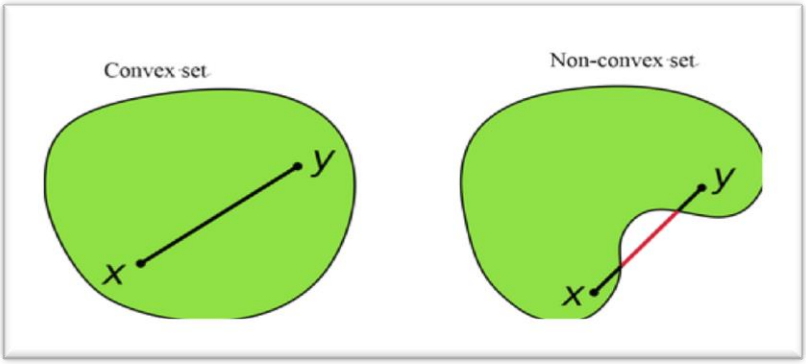

Figure 2.1.2 Convex and Non convex set.

Definition 2.1.10. (Phase Transition) [28] The phenomenon of non-uniqueness of limiting Gibbs measures is interpreted as phase transition. It's known that if $|G_H| > 1$, then we have phase transition.

Definition 2.1.11. (Critical Temperature) [4] If it is possible to find an exact value of temperature T_{cr} such that a phase transition occurs for all $T < T_{cr}$, then T_{cr} is called the critical temperature of the model.

It's known that phase transitions usually exist at low temperatures.

2.2 Cayley Tree and Competing Interactions

In this section, basic definitions about Cayley tree and different types of competing interactions are mentioned.

Remark 2.2.1. [20] Most physical systems are frustrated in the sense that there are usually different interactions, each favor a different type of order state, such competition often be revealed by changing a parameter of the system (such as temperature, pressure or magnetic field) which serves to enhance the effect of particular interaction and drive the system into a different ordered state. These interactions are called (**Competing Interactions**).

Definition 2.2.1. (Cayley Tree) [24] A Regular Cayley tree (Bethe lattice Γ^k) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles with exactly $k + 1$

edges issuing from each vertex. It is denoted as $\Gamma^{\mathrm{k}} = (V, L)$,such that V is the set of vertices of Γ^{k} and L is the set of edges of Γ^{k}.

Definition 2.2.2. (Semi-Infinite Cayley Tree) [24] A Semi-Infinite Cayley tree Γ_{+}^{k} of k-th order is an infinite graph without cycles with $k + 1$ edges issuing from each vertex except $x^{(0)}$ which has only k edges , $x^{(0)}$ represents the root vertex.

See Figure 2.2.1.

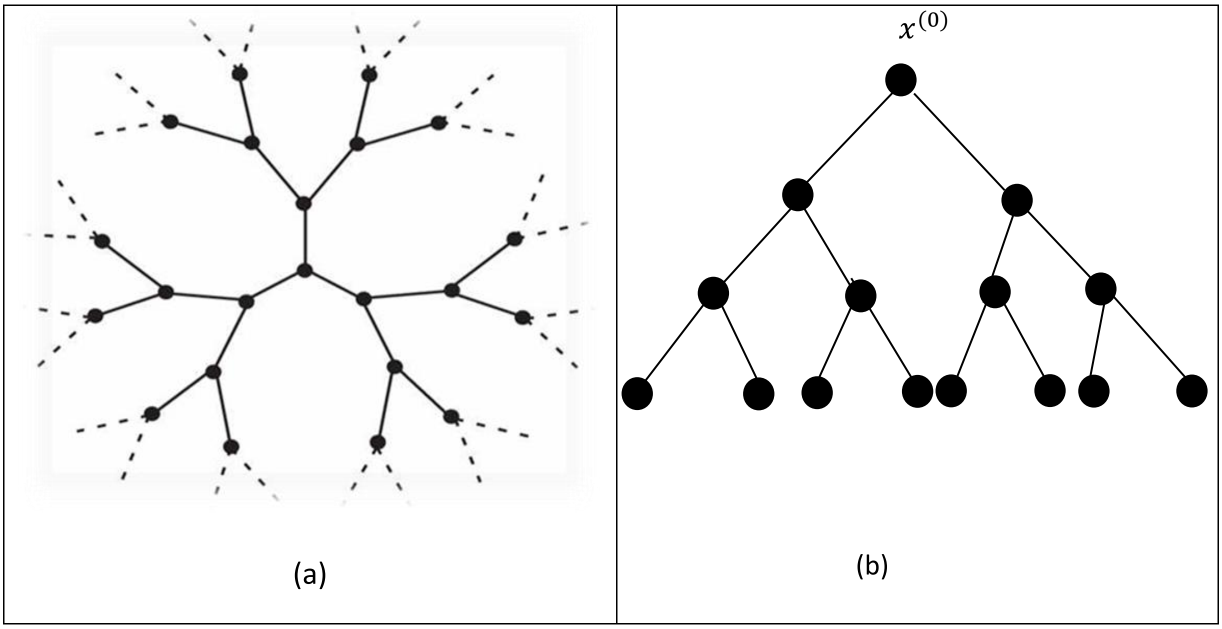

Figure 2.2.1. (a) Bethe lattice (Cayley tree) of order 2 (b) Semi-infinite Cayley tree of order 2

Definition 2.2.3. (Single-Trunk Cayley Tree) [24] The Semi-infinite Cayley tree $\Gamma_{+}^{\mathrm{k}}(l) = (V^{l},\ L^{l})$ is called a single-trunk Cayley tree if from vertex $x^{(0)}$ a single edge l emanates and from any other vertex $x \in V^{l}$, $x \neq x^{(0)}$ exactly $k + 1$ edges emanate. See Figure 2.2.2.

Definition 2.2.4. (Nearest-Neighbor Vertices) [24] Two vertices $x, y \in V$ are called nearest-neighbors if there exists an edge $l \in L$ connecting them, which is denoted by $l = < x, y >$.

Definition 2.2.5. [24] The distance $d(x,y), x,y \in V$ on the Cayley tree Γ^k is the number of edges in the shortest (minimal) path from x to y.

Definition 2.2.6. [1] The sphere of radius n on V is denoted by W_n where :

$$W_n = \{ x \in V : d(x, x^{(0)}) = n\}$$

Such that $x^{(0)}$ represents the root vertex, and the vertices on W_n are called the n^{th} level. For simplicity, we put $|x| = d(x, x^{(0)}), x \in V$.

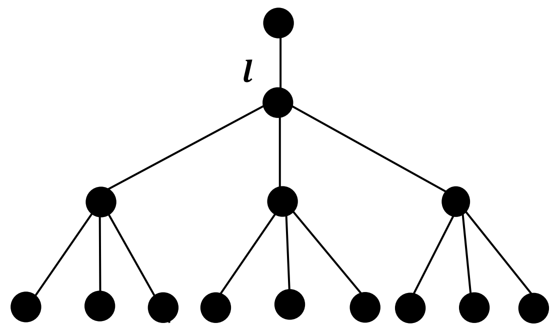

Figure 2.2.2 Single-Trunk Cayley Tree of order three (Γ_3^+ (l))

Definition 2.2.7. [24] The ball of radius n on V is denoted by V_n where : $V_n = \{ x \in V : d(x, x^{(0)}) \leq n\}$, and L_n denote the set of edges in V_n.

Definition 2.2.8. (Next-Nearest-Neighbor Vertices) [1] The two vertices $x,y \in V$ are called next-nearest-neighbors if $d(x,y) = 2$. The next-nearest neighbor vertices x, y are called prolonged iff $|x| \neq |y|$ and denoted by $> x,y <$.

Definition 2.2.9. (One-Level Next-Nearest-Neighbor Vertices) [2] The next-nearest-neighbor vertices x,y are called one-level next-nearest neighbor if $|x| = |y|$ and are denoted by $< \widehat{x,y} >$.

Definition 2.2.10. (The Set of Direct Successors of Vertex x) [24] if $x \in W_n$, then the set of direct successors of this vertex is the set $S(x) = \{y_i \in W_{n+1} : d(x, y_i) = 1,\ i = 1, \dots, k\}$.

Note that for a semi-infinite Cayley tree of order k, $|S(x)| = k$ for any $x \in V$.

Definition 2.2.11. (Ternary-Prolonged Next-Nearest-Neighbor Vertices) [1] The triple of vertices x, y, z are called ternary-prolonged next-nearest-neighbors if $x \in W_n$, $y \in S(x)$ and $z \in S(y)$ for some nonnegative integer n and this denoted by $> x, y, z <$.

See Figure 2.2.3.

Definition 2.2.12. (One-Level k-Tuple Interaction) [25] For a semi-infinite Cayley tree Γ_+^k, consider the set $S(x) = \{y_1, \cdots, y_k\}$ of direct successors for any $x \in V$. Then the jointly interaction of all sites in $S(x)$ is called one-level k-tuple interaction.

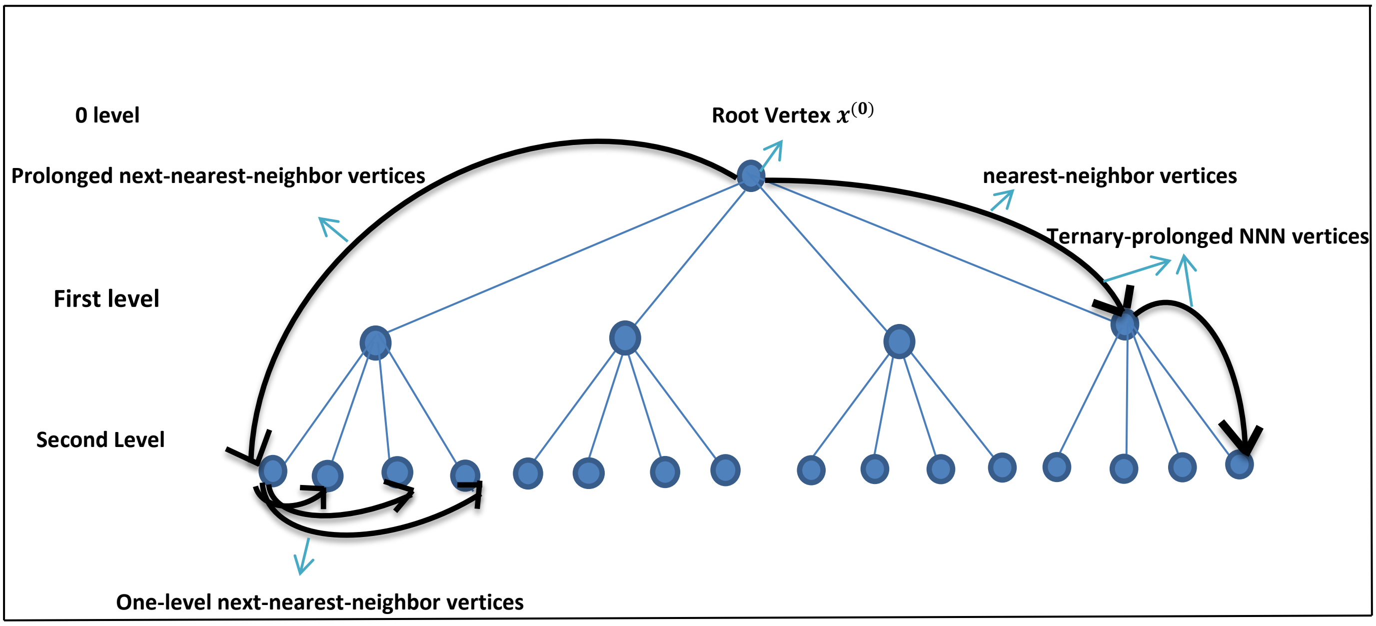

Figure 2.2.3. Nearest-neighbor, Prolonged next-nearest-neighbor, Ternary-prolonged next-nearest-neighbor and One-level next-nearest-neighbor vertices on semi-infinite Cayley tree of order 4.

Remark 2.2.2. [51] The point of difference between Cayley tree and finite dimensional lattices z^d, is that the ratio of the number of boundary vertices to the number of interior vertices in a large finite subset of the tree does not vanish in the thermodynamic limit. If $k \geq 2$ is the order of the tree and $\partial V_n = V_{n+1} \setminus V_n$ (the boundaries for the ball n), then:

$\lim\limits_{n \to \infty} \frac{|\partial \mathrm{Vn}|}{|\mathrm{Vn}|} = \lim_{n \to \infty} \frac{(k+1)k^n}{1+(\frac{(k+1)(k^n-1)}{k-1})} = \mathrm{k} - 1 \geq 1$. So we conclude that the remote boundary expected to have a very strong influence on spins is located deep inside the tree, which is a rich point for phase transition.

Definition 2.2.13. (Descartes's Rule of Signs) [17] Let $f(x)$ be a polynomial all of whose coefficients are real numbers with the terms are arranged in order of decreasing powers of x and consider the equation $f(x) = 0$, then

1) The number of positive roots either is equal to the number of variations in sign of $f(x)$ or is less than that by an even integer.
2) The number of negative roots either is equal to the number of variations in sign of $f(-x)$ or is less than that by an even integer.

Definition 2.2.14. (Limit of Infinity) [48] We say $\lim_{x \to c} f(x) = \infty$ if for every $M > 0$ there exists a $\delta > 0$ such that for all $x \neq c$, if $|x - c| < \delta$ then $f(x) \geq M$.

Remark 2.2.3. [39] If a function $f: R \to R$ is a continuously differentiable function with a fixed point a, i.e., $f(a) = a$.Then the dynamical system obtained by iterating the function f is:

$$x_{n+1} = f(x_n), n = 0, 1, 2, 3, ...$$

is stable at $x = a$ if $|f'(a)| < 1$ and it's unstable if $|f'(a)| > 1$.

Theorem 2.2.1. [29] Let a be a positive equilibrium point for the dynamical system $x_{n+1} = f(x_n), n = 0,1,2,3, \ldots .$ Assume that f' is continuous on an open interval I containing a. Then a is locally asymptotically stable if $|f'(a)| < 1$ and unstable if $|f'(a)| > 1$.

Definition 2.2.15. (Invariant set) [54] A set A is called invariant under the dynamic operator F if $F(A) \subseteq A$.

2.3 Ising and Potts Models on Cayley Tree

2.3.1 Ising Model on Cayley Tree

Ising model is a model with spins that take values in the state space $\Phi = \{-1,1\}$ assigned to the vertices of the tree . Assume a configuration σ on V is defined as a function such that $\sigma : x \in V \to \sigma(x) \in \Phi$ and the set of all configurations denoted by Φ^V. Then the formal Hamiltonian for this model is

$$H(\sigma) = -J \sum_{<x,y>\subset L} \sigma(x)\sigma(y)$$

where the sum is over NN vertices $< x, y >$ [4].

From the point of view of probability theory, solving the Ising model on the Cayley tree amounts to find a probability space characterized by an infinite number of random variables, i.e., the spins are indices in V in our case, and a measure μ. The marginal probabilities can be calculated from an infinite set of given finite dimensional marginal distributions $\mu_n = \mu_{|Vn}$. Note that, when we consider μ_n, we are not dealing with a finite disconnected subgraph of our given cayley tree ; however the spins of V_n that are located on the shell n are in fact connected with the spins of V_{n+1}. The essence to be used is that, given the values of the spins on $V \setminus V_n$ (the values of spins outside the ball V_n). In the nearest-neighbor case, the spins on V_n are conditioned only by the neighboring spins, i.e., the spins of the $n+1^{th}$shell. In turn, given a spin $\sigma(x)$ with $x \in W_n$, then the effects of all its neighbor spins that are on $V \setminus V_n$ can be encoded via an effective external field h_x acting only on $\sigma(x)$ [42]. Thus, a finite-dimensional distribution of a measure μ in the volume (ball) V_n has been defined by the following formula:

$$\mu_n(\sigma_n) = \frac{1}{Z_n} \; exp\left[- \beta \, H_n(\sigma_n) + \sum_{x \in W_n} \sigma(x) h_x \; \right] \tag{1}$$

with the associated partition function Z_n defined as:

$$Z_n = \sum_{\sigma_n \in \Phi^{V_n}} exp[- \beta \;\; H_n(\sigma_n) + \sum_{x \in W_n} \sigma(x) h_x \;] \tag{2}$$

where the spin configurations σ_n belongs to the set of configurations Φ^{V_n} , $\beta = \frac{1}{T}$ where $T > 0$ is the temperature $h = \{h_x \in R, x \in V\}$ is a collection of real functions that define boundary condition and

$$H(\sigma_n) = -J \sum_{<x,y> \subset L_n} \sigma(x)\sigma(y) \tag{3}$$

2.3.2 Potts Model on Cayley Tree [52]

A Potts model on Cayley tree is a generalization of Ising model , i.e., the spin takes values in the set $\Phi = \{1, 2, 3, .., q\}$ assigned to the vertices of the tree. In the same way as Ising model, the set of all configurations on the tree denoted by Φ^V. Let $\sigma \in \Phi^V$, The formal Hamiltonian is:

$$H(\sigma) = -J \sum_{<x,y>\subset L} \delta_{\sigma(x)\sigma(y)} - h \sum_{x\in V} \delta_{1\sigma(x)} \qquad (4)$$

where $J \in R$ is a coupling constant and $< x, y >$ are NN vertices, h is an external field and $\delta_{i\,j}$ is called the Kroneker's delta such that

$$\delta_{i\,j} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$$

The finite-dimensional distribution of a probability measure μ in the ball V_n is the measure:

$$\mu_n = \frac{1}{Z_n} \; exp\left[- \beta \; H_n(\sigma_n) + \sum_{x\in W_n} \sigma(x) h_x \;\right] \qquad (5)$$

where the spin configurations σ_n belongs to the set of configurations $\Phi^{V_n}, \beta = \frac{1}{T}$ where $T > 0$ is the temperature , Z_n is the partition function, $h = \{h_x \in R, x \in V\}$ is a collection of real numbers function that define boundary condition and

$$H_n(\sigma_n) = -J \sum_{<x,y>\subset L_n} \delta_{\sigma(x)\sigma(y)} - h \sum_{x\in V_n} \delta_{1\sigma(x)}$$

2.3.3 Ising-Vannimenus Model on Cayley Tree [4]

Ising-Vannimenus model is a model investigated by J. Vannimenus which with a spins take values in the state space $\Phi = \{-1, 1\}$, which assigned to the vertices of the tree. The formal Hamiltonian is

$$H(\sigma) = -J \sum_{<x,y>} \sigma(x)\sigma(y) \; - J_p \sum_{>x,y<} \sigma(x)\sigma(y) \qquad (6)$$

Which defines the Ising-Vannimenus model with two competing interactions : NN and Prolonged NNN (PNNN), where the sum in the first term ranges all NN and the sum in the second term ranges all PNNN and the spins $\sigma(x)$ and $\sigma(y)$ take it's values in the set Φ. Note that $J,\ J_p\ \in R$ are coupling constants corresponding to NN and PNNN potentials, respectively.

In [9, 10], the authors have studied Gibbs measures with memory of length 2 for generalized Ising-Vannimenus models on a Cayley tree of second order related to the Hamiltonian (6) by means of the vector valued function

$$h :< x,y >\to h_{xy} = (h_{xy,++}, h_{xy,+-}, h_{xy,-+}, h_{xy,--}) \in R^4 ,$$

where $h_{xy,\sigma(x)\sigma(y)} \in$ R and $x \in W_{n-1},\ y \in S(x)$.

The finite-dimensional distribution of a probability measure μ in the ball V_n is the measure

$$\mu_n = \frac{1}{Z_n}\ exp\left[-\beta\ H_n(\sigma_n) + \sum_{x\in W_{n-1}}\sum_{y\in W_n}\sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}\ \right] \qquad (7)$$

where the spin configurations σ_n belongs to the set of configurations Φ^{V_n}, $\beta = \frac{1}{T}$ where $T > 0$ is the temperature, Z_n is the partition function such that

$$Z_n = \sum_{\sigma_n\in\Phi^{V_n}}\exp[-\beta\ H_n(\sigma_n) + \sum_{x\in W_{n-1}}\sum_{y\in W_n}\sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}] \qquad (8)$$

Consider the subsets of the set of states for one dimensional lattices. The best known approximation is the so-called mean field one. It corresponds to restricting the variation to the set of product states (states without correlations between the lattice sites). The second approximation would be to take into account the nearest -neighbour correlations. In physics literature this is called the Bethe-Peierls method. For one dimensional lattices we consider increasing subsets of the set of states: $\vartheta_1 \subset \vartheta_2 \subset \cdots \subset \vartheta_n \subset \cdots$ where ϑ_n is the set of states taking into account non-trivial correlations between n-successive lattice points; ϑ_1 is the set of mean field states, ϑ_2 is the set of Bethe-Peierls states; the latter is extended to the so-called Bethe lattices. All these states correspond in probability theory to so-called Markov chains with memory of length n [22].

Chapter Three

The First Model:

Gibbs Measures of an Ising-Vannimenus Model with Ternary Competing Interactions on Cayley Tree of Order Three

In this chapter, we are going to focus on the phase Transition of Gibbs measures with memory of length 2 associated to Ising-Vannimenus model on a Cayley tree of order 3 with ternary interactions. This model was studied by H. Akin [1] where he analyzed the paramagnetic and ferromagnetic Gibbs measures that correspond to this model. In brief, H. Akin obtained the recurrence equations associated with that model and studied them analytically such that the solutions are correspond to the invariant set $A = \{(v_1, v_4, v_5, v_8) \in R^{+4} : v_1 = v_8,\ v_4 = v_5\}$. He solved this model by using PFM considering two cases : the first case when $c = 1$ and the second case when $b = 1$ where $c = e^{\beta J_t}$ and $b = e^{\beta J_p}$. In addition, he found numerical solutions corresponding to the same set A.

In this chapter, we solve the same model analytically using MRFM instead of PFM where the solutions are given in term of a recurrence equations that define on a non-translation invariant set A_1. The development is that we assume all the parameters of the model a $(e^{\beta J}), b$ and c do not always equal to 1 (general case).

3.1 Model Construction

Consider the Hamiltonian for Ising-Vannimenus Model (6) with prolonged-ternary NNN interaction (see definition 2.2.11). Therefore, the Hamiltonian is [1]

$$H(\sigma) = -J_t \sum_{>x,y,z<} \sigma(x)\sigma(y)\sigma(z) - J_p \sum_{>x,y<} \sigma(x)\sigma(y) - J \sum_{<x,y>} \sigma(x)\sigma(y) \quad (9)$$

This defines the Ising-Vannimenus model with competing NN , PNNN and ternary PNNN (TPNNN) interactions. Where the sum in the first term ranges all TPNNN, and the sum in the second term ranges all the PNNN , and the third term ranges the NN such that the spins $\sigma(x)$, $\sigma(y)$ and $\sigma(z)$ take values in the set $\Phi = \{-1,1\}$, where $x, y, z \in V$. Here J_t, J_P and $J \in R$ are coupling constants corresponding to TPNNN, PNNN and NN potentials, respectively. In brief, our Hamiltonian is the same as Vannimenus one (6) in addition to TPNNN interaction on Cayley tree of order Three.

Now let us Consider a Cayley tree of order 3, with $x \in W_n$, for some $n \in N$ and $S(x) = \{y, z, l\}$ where $y, z, l \in W_{n+1}$ are the direct successors of x. Denote $B_1(x) = \{x, y, z, l\}$ a unite semi-ball with a center x, such that $S(x) = \{y, z, l\}$.

Remark 3.1.1. [4] Φ^{Vn} denotes the set of all spin configurations on V_n and $\Phi^{B_1(x)}$ denotes the set of all configurations on unite semi-ball $B_1(x)$. In our case, it's clear that the set $\Phi^{B_1(x)}$ consists of 16 configurations

$$\Phi^{B_1(x)} = \{\begin{pmatrix} & i & \\ j & k & g \end{pmatrix} : \ i,\ j, k, g \in \Phi\} \tag{10}$$

In table 3.1.1 below, we denote the spin configurations belonging to $\Phi^{B_1(x)}$:

Table 3.1.1 The set of possible configurations ($\Phi^{B_1(x)}$) on Cayley tree of order 3

$\boldsymbol{\sigma_1}^{(1)} = \begin{pmatrix} & + & \\ + & + & + \end{pmatrix}$	$\boldsymbol{\sigma_2}^{(1)} = \begin{pmatrix} & + & \\ - & + & + \end{pmatrix}$	$\boldsymbol{\sigma_3}^{(1)} = \begin{pmatrix} & + & \\ + & - & + \end{pmatrix}$	$\boldsymbol{\sigma_4}^{(1)} = \begin{pmatrix} & + & \\ + & + & - \end{pmatrix}$
$\boldsymbol{\sigma_5}^{(1)} = \begin{pmatrix} & + & \\ - & - & + \end{pmatrix}$	$\boldsymbol{\sigma_6}^{(1)} = \begin{pmatrix} & + & \\ - & + & - \end{pmatrix}$	$\boldsymbol{\sigma_7}^{(1)} = \begin{pmatrix} & + & \\ + & - & - \end{pmatrix}$	$\boldsymbol{\sigma_8}^{(1)} = \begin{pmatrix} & + & \\ - & - & - \end{pmatrix}$
$\boldsymbol{\sigma_9}^{(1)} = \begin{pmatrix} & - & \\ + & + & + \end{pmatrix}$	$\boldsymbol{\sigma_{10}}^{(1)} = \begin{pmatrix} & - & \\ - & + & + \end{pmatrix}$	$\boldsymbol{\sigma_{11}}^{(1)} = \begin{pmatrix} & - & \\ + & - & + \end{pmatrix}$	$\boldsymbol{\sigma_{12}}^{(1)} = \begin{pmatrix} & - & \\ + & + & - \end{pmatrix}$
$\boldsymbol{\sigma_{13}}^{(1)} = \begin{pmatrix} & - & \\ - & - & + \end{pmatrix}$	$\boldsymbol{\sigma_{14}}^{(1)} = \begin{pmatrix} & - & \\ - & + & - \end{pmatrix}$	$\boldsymbol{\sigma_{15}}^{(1)} = \begin{pmatrix} & - & \\ + & - & - \end{pmatrix}$	$\boldsymbol{\sigma_{16}}^{(1)} = \begin{pmatrix} & - & \\ - & - & - \end{pmatrix}$

In the same way as H. Akin (2017) [4], we take a natural definition for the quantities $h\begin{pmatrix} & x & \\ y & z & l \end{pmatrix}$ (the real valued boundary function at the ball $B_1(x)$) as $h_{B_1(x)}$

. In this thesis, we assume the following definition for the vector valued function $h : V \rightarrow R^{16}$ which defined by

$$h : B_1(x) =< x, y, z, l > \rightarrow h_{B_1(x)} = \left(h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(l)} : \sigma(x), \sigma(y), \sigma(z), \sigma(l) \in \Phi\right) \quad (11)$$

Where $h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(l)} \in \mathrm{R}$, $x \in W_n$, $y, z, l \in S(x)$ and $\sigma(x)\sigma(y)\sigma(z)\sigma(l)$ represents the order of the spins in the ball $B_1(x)$, we use the function $h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(l)}$ to describe the Gibbs measure of any configuration $\left({}_{\sigma(y)\ \sigma(z)\ \sigma(l)}^{\quad\ \sigma(x)}\right)$ that belongs to $\Phi^{B_1(x)}$.

The finite-dimensional Gibbs probability measures (distributions) on the configuration space $\Omega^{V_n} = \{\sigma_n = \{\sigma(x) = \pm 1,\ x \in V_n\}\}$ that corresponds to the Hamiltonian (9) at inverse temperature $\beta = \frac{1}{T}$ is defined by the formula [4]

$$\mu_{\boldsymbol{h}}{}^{(\boldsymbol{n})}(\sigma_n) = \frac{1}{Z_n} exp[-\beta\, H_n(\sigma_n) + \sum_{x \in w_{n-1}} \sum_{y,z,l \in s(x)} \sigma(x)\sigma(y)\sigma(z)\sigma(l)\, h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(l)}] \quad (12)$$

with the corresponding partition function defined by

$$Z_n = \sum_{\sigma_n \in \Omega^{V_n}} \exp[-\beta\, H_n(\sigma_n) + \sum_{x \in w_{n-1}} \sum_{y,z,l \in S(x)} \sigma(x)\sigma(y)\sigma(z)\sigma(l)\, h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(l)}] \quad (13)$$

Now, we obtain a new set of Gibbs measures by constructing of an infinite volume distribution (limiting Gibbs measure) with given finite-dimensional distributions. To start with this, consider the following definition

Definition3.1.1. [3, 4, 5] A finite dimensional measures (probability distributions) $\mu_h{}^{(n)}$ are compatible (consistent) if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{w \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee w) = \mu_{n-1}(\sigma_{n-1}) \quad (14)$$

where $\sigma_{n-1} \vee w$ is the concatenation of the configurations.

It is known that if this condition is satisfied , then there exists a unique limiting Gibbs measure μ_h Such that

$$\mu_h\big(\sigma \in \Omega : \sigma|_{V_n} = \sigma_n\big) = \mu_h^{(n)}(\sigma_n) \quad \text{for all} \quad \sigma_n \in \Omega^{V_n}, n \in N .$$

Remark 3.1.1.[2] The limiting Gibbs distribution μ_h satisfying the compatibility condition (14) is called splitting Gibbs measure, and the icon h denotes that this measure corresponds to the boundary function h .

Remark 3.1.2.[2] The splitting Gibbs measure μ_h that corresponds for the Hamiltonian (9) and the boundary function h (11) is called a splitting Gibbs measure with memory of length 2.

Remark 3.1.3.[2, 3, 4] The Method of finding a limiting measure μ_h that satisfied consistency conditions in definition 3.1.1 is called Markov Random field method making use of the Kolmogorov consistency conditions.

More exactly, in the next step we find the set of limiting Gibbs measures μ_h for a sequence of finite dimensional compatible measures $\{\mu_h{}^{(n)} : n \geq 1\}$ in the same way as [4]. The Hamiltonian (interaction energy) on the set of vertices V with the inner configuration $\sigma_{n-1} \in \Omega^{V_{n-1}}$ and with the boundary condition $\eta \in \Omega^{W_n}$ is defined by :

$$H_n(\sigma_{n-1} \vee \eta) = H(\sigma_{n-1}) - J_t \sum_{x \in w_{n-2}} \sum_{z \in S(x)} \sum_{y \in S(z)} \sigma(x)\sigma(z)\eta(y)$$
$$- J_p \sum_{x \in w_{n-2}} \sum_{y \in S^2(x)} \sigma(x)\eta(y) - J \sum_{x \in w_{n-1}} \sum_{y \in S(x)} \sigma(x)\eta(y) \qquad (15)$$

$$= -J_t \sum_{>x,y,z<,x,y,z\in V_{n-1}} \sigma(x)\sigma(y)\sigma(z) - J_t \sum_{x\in w_{n-2}} \sum_{z\in S(x)} \sum_{y\in S(z)} \sigma(x)\sigma(z)\eta(y)$$

$$- J_p \sum_{\substack{>x,y< \\ x,y\in V_{n-1}}} \sigma(x)\sigma(y) - J_p \sum_{x\in w_{n-2}} \sum_{y\in S^2(x)} \sigma(x)\eta(y) - J \sum_{\substack{<x,y>' \\ x,y\in V_{n-1}}} \sigma(x)\sigma(y)$$

$$- J \sum_{x\in w_{n-1}} \sum\nolimits_{y\in S(x)} \sigma(x)\eta(y) \qquad (16)$$

Thus, the compatibility condition (14) satisfied for the sequence of finite-dimensional probability measures $\mu_h{}^{(n)},\ n \geq 1$ (12) for our model if

$$\sum_{\eta\in\Phi^{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \eta) = \mu_h^{(n-1)}(\sigma_{n-1})$$

Then,

$$L_n \sum\nolimits_{\eta\in\Omega^{W_n}} exp[-\beta\, H_n(\sigma_{n-1} \vee\ \eta)$$

$$+ \sum_{y,z,l\,\in w_{n-1}} \sum_{y_i\in S(y)} \sum_{z_i\in S(z)} \sum_{l_i\in S(l)} (\sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)\,h_{\,B_1(y),\sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)}$$

$$+ \sigma(z)\,\eta(z_1)\,\eta(z_2)\,\eta(z_3)\ \ h_{\,B_1(z),\sigma(z)\,\eta(z_1)\,\eta(z_2)\,\eta(z_3)}$$

$$+ \sigma(l)\,\eta(l_1)\eta(l_2)\,\eta(l_3)\ \ h_{\,B_1(l),\sigma(l)\,\eta(l_1)\,\eta(l_2)\,\eta(l_3)}\ )]$$

$$= exp[-\beta\, H_n(\sigma_{n-1}) + \sum_{x\in w_{n-2}} \sum_{y,z,l\,\in S(x)} [\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(l) h_{\,B_1(x),\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(l)}]]$$

where $L_n = \frac{Z_{n-1}}{Z_n}$. Using The Hamiltonian in (15), we obtain:

$$L_n \sum_{\eta\in\Omega^{W_n}} exp[-\beta\, H_n(\sigma_{n-1}) + \beta\, J_t \sum_{x\in w_{n-2}} \sum_{z\in S(x)} \sum_{y\in S(z)} \sigma(x)\sigma(z)\eta(y)$$

$$+ \beta\, J_p \sum_{x\in w_{n-2}} \sum_{y\, \in S^2(x)} \sigma(x)\eta(y) \ + \beta\, J \sum_{x\in w_{n-1}} \sum_{y\, \in S(x)} \sigma(x)\eta(y)$$

$$+ \sum_{y,z,l\in w_{n-1}} \sum_{y_i\in S(y)} \sum_{z_i\in S(z)} \sum_{l_i\in S(l)} (\sigma(y)\, \eta(y_1)\, \eta(y_2)\, \eta(y_3)\, h_{\,B_1(y),\sigma(y)\, \eta(y_1)\, \eta(y_2)\, \eta(y_3)}$$

$$+\ \sigma(z)\, \eta(z_1)\, \eta(z_2)\, \eta(z_3)\ \ h_{\,B_1(z),\sigma(z)\, \eta(z_1)\, \eta(z_2)\, \eta(z_3)}$$

$$+\ \sigma(l)\, \eta(l_1)\, \eta(l_2)\ \ \eta(l_3)\ \ \ h_{\,B_1(l),\sigma(l)\, \eta(l_1)\, \eta(l_2)\, \eta(l_3)}\ \ )]$$

$$=\ \ exp[-\beta\, H_n(\sigma_{n-1})\ +\ \sum_{x\in w_{n-2}} \sum_{y,z,l\in S(x)} (\sigma(x)\ \sigma(y)\ \sigma(z)\ \ \sigma(l) h_{\,B_1(x),\sigma(x)\ \sigma(y)\ \sigma(z)\ \sigma(l)}\,)]$$

$$\longrightarrow$$

$$L_n\, e^{-\beta\, \boldsymbol{H_n}(\sigma_{n-1})} \sum_{\eta\in\Omega^{W_n}} exp[\beta\, J_t \sum_{x\in w_{n-2}} \sum_{z\in S(x)} \sum_{y\in S(z)} \sigma(x)\sigma(z)\eta(y)$$

$$+ \beta\, J_p \sum_{x\in w_{n-2}} \sum_{y\, \in S^2(x)} \sigma(x)\eta(y) \ + \beta\, J \sum_{x\in w_{n-1}} \sum_{y\, \in S(x)} \sigma(x)\eta(y)$$

$$+ \sum_{y,z,w,t\, \in w_{n-1}} \sum_{y_i\in S(y)} \sum_{z_i\in S(z)} \sum_{l_i\in S(l)} (\sigma(y)\, \eta(y_1)\, \eta(y_2)\, \eta(y_3)\, h_{\,B_1(y),\sigma(y)\, \eta(y_1)\, \eta(y_2)\, \eta(y_3)}$$

$$+\ \sigma(z)\, \eta(z_1)\, \eta(z_2)\, \eta(z_3)\ \ h_{\,B_1(z),\sigma(z)\, \eta(z_1)\, \eta(z_2)\, \eta(z_3)}$$

$$+\ \sigma(l)\, \eta(l_1)\, \eta(l_2)\ \ \eta(l_3)\ \ \ h_{\,B_1(l),\sigma(l)\, \eta(l_1)\, \eta(l_2)\, \eta(l_3)}\ \ )]$$

$$=\ \ e^{-\beta\, \boldsymbol{H_n}(\sigma_{n-1})}\, exp[\sum_{x\in w_{n-2}} \sum_{y,z,l\in S(x)} (\sigma(x)\ \sigma(y)\ \sigma(z)\ \ \sigma(l) h_{\,B_1(x),\sigma(x)\ \sigma(y)\ \sigma(z)\ \sigma(l)}\,)]$$

For all $i = 1, 2, 3.$

Now after simplifying we get the following formula

$$\prod_{x\in w_{n-2}} \prod_{y,z,l\, \in\, S(x)} e^{\sigma(x)\ \sigma(y)\ \sigma(z)\ \sigma(l)\ h_{\,B_1(x),\sigma(x)\ \sigma(y)\ \sigma(z)\ \sigma(l)}}$$

$$=\ L_n \prod_{x\, \in\, w_{n-2}} \prod_{y,z,l\, \in\, S(x)} \prod_{y_i\in S(y)} \prod_{z_i\in S(z)} \prod_{l_i\in S(l)} \sum_{\eta\left(y_i\right),\eta(z_i),\eta(l_i)\in\{-1,+1\}} e^{[B\left(h,J,J_p,J_t\right)]} \quad (71)$$

Where

$$B(h,J,J_p,J_t) = \sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)\,h_{B_1(y),\sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)}$$

$$+\sigma(z)\,\eta(z_1)\,\eta(z_2)\,\eta(z_3)\;h_{B_1(z),\sigma(z)\,\eta(z_1)\,\eta(z_2)\,\eta(z_3)}$$

$$+\sigma(l)\,\eta(l_1)\,\eta(l_2)\;\eta(l_3)\,h_{B_1(l),\sigma(l)\,\eta(l_1)\,\eta(l_2)\,\eta(l_3)}$$

$$+\beta J\left[\begin{matrix}\sigma(y)\big(\eta(y_1)+\eta(y_2)+\eta(y_3)\big) + \sigma(z)\big(\eta(z_1)+\eta(z_2)+\eta(z_3)\big) + \\ \sigma(l)\big(\eta(l_1)+\eta(l_2)+\eta(l_3)\big)\end{matrix}\right]$$

$$+\beta J_t\left[\sigma(x)\left(\sigma(y)\left(\sum_{i=1}^{3}\eta(y_i)\right)+\sigma(z)\left(\sum_{i=1}^{3}\eta(z_i)\right)+\sigma(l)\left(\sum_{i=1}^{3}\eta(l_i)\right)\right)\right]$$

$$+\beta J_p\left[\sigma(x)\sum_{i=1}^{3}(\eta(y_i)+\eta(z_i)+\eta(l_i)\big)\right]$$

Next step, we rewrite equation (17) for all possible values of $\sigma(x),\sigma(y)$, $\sigma(z),\sigma(l) \in \Phi$. For doing this, we assume $\sigma(x)$, $\sigma(y),\sigma(z)$ and $\sigma(l)$ are fixed such that $x \in W_{n-2}$ and $y,z,l \in W_{n-1}$. Then for each case we consider all possible values of configurations for the set of boundaries $\{y_1, y_2, y_3, z_1, z_2, z_3, l_1, l_2, l_3\}$.

To simplify, assume:

$\sigma(x) = i, \sigma(y) = j, \sigma(z) = k, \sigma(l) = g, \eta(y_1) = u,\ \eta(y_2) = v,\ \eta(y_3) = p, \eta(z_1) = r,\ \eta(z_2) = s,\ \eta(z_3) = q,\ \eta(l_1) = o,\ \eta(l_2) = n,\ \eta(l_3) = m$ where $i,j,k,g,u,v,p,r,s,q,o,n,m \in \Phi$ (See Figure 3.1.1).

Thus:

$$e^{ijkg\,h_{B_1(x),i;j,k,g}} = L_2 \sum_{u,v,p,q,,\ldots} e^{\beta J_p i(u+v+p+r+s+q+o+n+m)}\ x\ \ e^{\beta J\left(j(u+v+p)+k(r+s+q)+g(o+n+m)\right)}$$

$$x\ e^{\beta J_t\left(i\,j\,(u+v+p)+i\,k\,(r+s+q)+i\,g(o+n+m)\right)}\ x\ e^{juvp\,h_{B_1(y),j;u,v,p}}$$

$$x\ e^{krsq\,h_{B_1(z),k;r,s,q}}\ \ x\ \ e^{gonm\,h_{B_1(l),g;o,n,m}} \qquad (18)$$

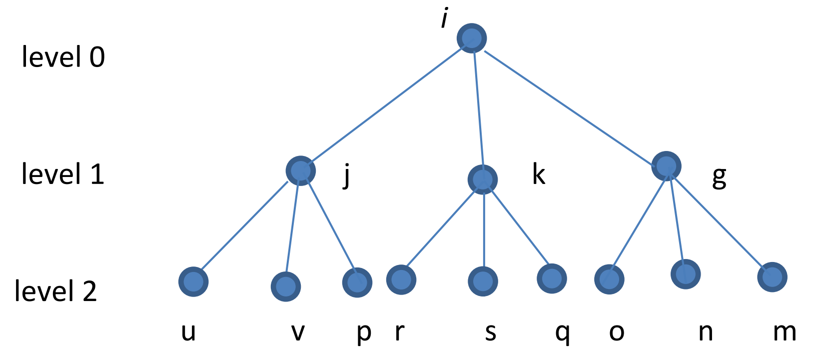

Figure 3.1.1. The set of Configurations on semi-finite Cayley tree of order three with 2 levels**.**

As H. Akin [3, 4], we define the real vector-valued function h (11) in the following way:

Let $h = (h_1 , \; h_2 , h_3 , h_4 , h_5 , h_6 , h_7 , h_8) \in R^8$ such that for $x \; \in W_n$

$$h_1 = h_{B_1(x),\, \sigma_1{}^{(1)}} = h_{xyzl,\sigma_1{}^{(1)}} \tag{19}$$

$$h_2 = h_{xyzl,\, \sigma_2{}^{(1)}} = h_{xyzl,\, \sigma_3{}^{(1)}} = h_{xyzl,\, \sigma_4{}^{(1)}} \tag{20}$$

$$h_3 = h_{xyzl,\, \sigma_5{}^{(1)}} = h_{xyzl,\, \sigma_6{}^{(1)}} = h_{xyzl,\, \sigma_7{}^{(1)}} \tag{21}$$

$$h_4 = h_{xyzl,\, \sigma_8{}^{(1)}} \tag{22}$$

$$h_5 = h_{xyzl,\, \sigma_9{}^{(1)}} \tag{23}$$

$$h_6 = h_{xyzl,\, \sigma_{10}{}^{(1)}} = h_{xyzl,\, \sigma_{11}{}^{(1)}} = h_{xyzl,\, \sigma_{12}{}^{(1)}} \tag{24}$$

$$h_7 = h_{xyzl,\, \sigma_{13}{}^{(1)}} = h_{xyzl,\, \sigma_{14}{}^{(1)}} = h_{xyzl,\, \sigma_{15}{}^{(1)}} \tag{25}$$

$$h_8 = h_{xyzl,\, \sigma_{16}{}^{(1)}} \tag{26}$$

3.2 Construction of Recurrence Equations

In this section, we construct the recurrence equations that give an explicit formula for Gibbs measures with memory of length two that corresponds to the Hamiltonian (9) and satisfies consistency conditions (14) by means of equation (18). Assume that $a = e^{\beta J}$, $b = e^{\beta J_p}$ and $c = e^{\beta J_t}$. By using the equations (19)-(26) and for simplifying, we can assume new variables

$$v_i{}' = e^{h_{B_1(x),\sigma_j^{(1)}}} \quad \text{for } x \in W_{n-1} \text{ and } v_i = e^{h_{B_1(y),\sigma_j^{(1)}}} \text{ for } y \in S(x).$$

Where $i = 1,2,\dots,8$ and $j \in \{1,2 \dots\dots,16\}$.

To simplify, let $x \in W_0$. Then From (18), through direct enumeration, we get the following eight equations:

First equation : We consider the case corresponding to h_1 (19) (See Figure 3.2.1).

In this case, $i = j = k = g = +1$. Then from (18), $e^{ijkg\, h_{B_1(X),i;j,k,g}} = e^{h_{B_1(x),+;+++}}$ $= e^{h_{B_1(x),\sigma_1^{(1)}}}$

$$e^{h_{B_1(x),\sigma_1^{(1)}}} = v_1{}' = L_2\left(a^3b^3c^3v_1 + \frac{3\,a\,b\,c}{v_2} + \frac{3\,v_3}{a\,b\,c} + \frac{v_4}{a^3b^3c^3}\right)^3 \tag{27}$$

Such that $v_1 = e^{h_{B_1(w),\sigma_1^{(1)}}}$, $v_2 = e^{h_{B_1(w),\sigma_2^{(1)}}} = e^{h_{B_1(w),\sigma_3^{(1)}}} = e^{h_{B_1(w),\sigma_4^{(1)}}}$,

$$v_3 = e^{h_{B_1(w),\sigma_5^{(1)}}} = e^{h_{B_1(w),\sigma_6^{(1)}}} = e^{h_{B_1(w),\sigma_7^{(1)}}} \;\; and \;\; v_4 = e^{h_{B_1(w),\sigma_8^{(1)}}},\ \forall\, w \in S(x).$$

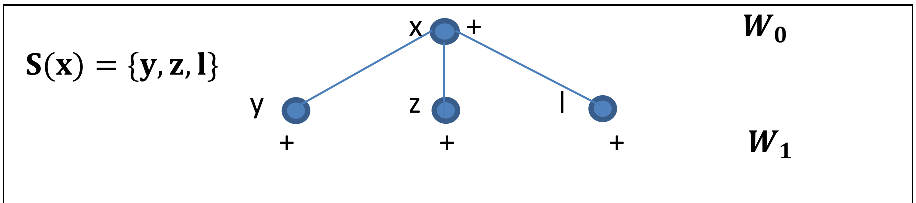

Figure 3.2.1. The Semi-ball $B_1(x)$ with $\sigma_1{}^{(1)}$ configuration**,** the Boundary Function h that corresponds for this model is $h_1(19)$.

Similarly, we find the other 7 equations :

$$v_2{}'^{(-1)} = L_2 \left(a^3b^3c^3v_1 + \frac{3\,a\,b\,c}{v_2} + \frac{3\,v_3}{a\,b\,c} + \frac{v_4}{a^3b^3c^3}\right)^2 \left(\frac{b^3}{a^3c^3v_5} + \frac{3\,b\,v_6}{a\,c} + \frac{3\,a\,c}{b\,v_7} + \frac{a^3\,c^3\,v_8}{b^3}\right) \tag{28}$$

$$v_3{}' = L_2 \left(a^3b^3c^3v_1 + \frac{3\,a\,b\,c}{v_2} + \frac{3\,v_3}{a\,b\,c} + \frac{v_4}{a^3b^3c^3}\right) \left(\frac{b^3}{a^3c^3v_5} + \frac{3\,b\,v_6}{a\,c} + \frac{3\,a\,c}{b\,v_7} + \frac{a^3\,c^3\,v_8}{b^3}\right)^2 \tag{29}$$

$$(v_4{}')^{(-1)} = L_2 \left(\frac{b^3}{a^3c^3v_5} + \frac{3\,b\,v_6}{a\,c} + \frac{3\,a\,c}{b\,v_7} + \frac{a^3\,c^3\,v_8}{b^3}\right)^3 \tag{30}$$

$$(v_5{}')^{(-1)} = L_2 \left(\frac{a^3\,v_1}{c^3b^3} + \frac{3\,a\,v_6}{b\,c\,v_2} + \frac{3\,b\,c\,v_3}{a} + \frac{b^3\,c^3}{a^3v_4}\right)^3 \tag{31}$$

$$v_6{}' = L_2 \left(\frac{a^3\,v_1}{c^3b^3} + \frac{3\,a\,v_6}{b\,c\,v_2} + \frac{3\,b\,c\,v_3}{a} + \frac{b^3\,c^3}{a^3v_4}\right)^2 \left(\frac{1}{a^3c^3b^3\,v_5} + \frac{3\,c\,v_6}{a\,b} + \frac{3\,a\,b\,c}{v_7} + \frac{a^3\,b^3v_8}{c^3}\right) \tag{32}$$

$$(v_7{}')^{(-1)} = L_2 \left(\frac{a^3\,v_1}{c^3b^3} + \frac{3\,a\,v_6}{b\,c\,v_2} + \frac{3\,b\,c\,v_3}{a} + \frac{b^3\,c^3}{a^3v_4}\right) \left(\frac{1}{a^3c^3b^3\,v_5} + \frac{3\,c\,v_6}{a\,b} + \frac{3\,a\,b\,c}{v_7} + \frac{a^3\,b^3v_8}{c^3}\right)^2 \tag{33}$$

$$v_8{}' = L_2 \left(\frac{1}{a^3c^3b^3\,v_5} + \frac{3\,c\,v_6}{a\,b} + \frac{3\,a\,b\,c}{v_7} + \frac{a^3\,b^3v_8}{c^3}\right)^3 \tag{34}$$

From the system of equations (28-34), we conclude that:

$$\left.\begin{aligned} v_2{}'^{(-1)} &= (v_1{}')^{\frac{2}{3}}\,(v_4{}')^{(\frac{-1}{3})} \\ v_3{}' &= (v_1{}')^{\frac{1}{3}}(v_4{}')^{(\frac{-2}{3})} \\ v_6{}' &= (v_5{}')^{(\frac{-2}{3})}(v_6{}')^{\frac{1}{3}} \\ (v_7{}')^{(-1)} &= (v_5{}')^{(\frac{-1}{3})}(v_8{}')^{\frac{2}{3}} \end{aligned}\right\} \quad (35)$$

Therefore, we can reduce the system of equations (28-34) to 4 equations with 4 independent variables: $v_1{}'$, $v_4{}'$, $v_5{}'$ and $v_8{}'$.

From the relations (35), we conclude the following remark.

Remark 3. 2.1. The compatibility condition (14) is satisfied if the vector valued boundary function $h = (h_1 ,\ h_2 , h_3 , h_4 , h_5 , h_6 , h_7 , h_8) \in R^8$ has the following form:

$$h = \left(m , \frac{n}{3} - \frac{2\,m}{3}, \frac{m}{3} - \frac{2\,n}{3}, n, o, \frac{s}{3} - \frac{2\,o}{3}, \frac{o}{3} - \frac{2\,s}{3}, s\right) \in R^8, n, o, m, s \in R$$

Proof. Assume $h_1 = m$ and $h_4 = n$ such that $m, n \in R$.Take the first equation of the system (35) , $\ {v_2}'^{(-1)} = ({v_1}')^{\frac{2}{3}}\, ({v_4}')^{(\frac{-1}{3})}$, then from (19, 20, 22),

$$e^{-h_2} = e^{\frac{2h_1}{3}} . e^{\frac{-1}{3}h_4} \quad \longrightarrow \quad -h_2 = \frac{2\,h_1}{3} + \frac{-1}{3} h_4 \ \longrightarrow \ h_2 = \frac{n}{3} - \frac{2\,m}{3}$$

In the same way, we can get the other component of the vector h.

If we assume $v_i = {u_i}^3,\ i = 1,2, \dots ,8$. Then, the system of equations (28-34) is reduced to the following four equations based on the relations (35) :

$${u_1}' = \sqrt[3]{L_2}\, \frac{(1 + a^2b^2c^2u_1u_4)^3}{a^3b^3c^3{u_4}^3} \qquad (36)$$

$$({u_4}')^{-1} = \sqrt[3]{L_2}\, \frac{(b^2 + a^2c^2u_5\, u_8)^3}{a^3b^3c^3{u_5}^3} \qquad (37)$$

$$({u_5}')^{-1} = \sqrt[3]{L_2}\, \frac{(b^2c^2 + a^2u_1\, u_4)^3}{a^3b^3c^3{u_4}^3} \qquad (38)$$

$${u_8}' = \sqrt[3]{L_2}\, \frac{(c^2 + a^2b^2u_5\, u_8)^3}{a^3b^3c^3{u_5}^3} \qquad (39)$$

Proof.

From (27), ${v_1}' = ({u_1}')^3 = L_2 \left(a^3b^3c^3{u_1}^3 + \frac{3\,a\,b\,c}{{u_2}^3} + \frac{3\,{u_3}^3}{a\,b\,c} + \frac{{u_4}^3}{a^3b^3c^3}\right)^3$.Then from (35) we have

$$(u_1')^3 = L_2\left(a^3b^3c^3u_1{}^3 + \frac{3\,a\,b\,c\,u_1{}^2}{u_4} + \frac{3\,u_1}{a\,b\,c\,u_4{}^2} + \frac{u_4{}^3}{a^3b^3c^3}\right)^3$$

$$\longrightarrow\ u_1{}' = \sqrt[3]{L_2}\ \left(a^3b^3c^3u_1{}^3 + \frac{3\,a\,b\,c\,u_1{}^2}{u_4} + \frac{3\,u_1}{a\,b\,c\,u_4{}^2} + \frac{u_4{}^3}{a^3b^3c^3}\right)$$

Which simplified to equation (36). In the same way we can prove the equations (37-39).

3.3 Translation-Invariant Gibbs Measures

In this section, we describe a subset from the set of translation-invariant Gibbs measures associated with our model. Consider the following definition:

Definition 3.3.1. [4] A function $h = \left\{h_{B_1(x),\sigma_i(1)} : i \in \{1,2,3,\dots\dots,16\}\right\}$ is considered as translation-invariant one if $h_{B_1(x),\sigma_i(1)} = h_{B_1(y),\sigma_i(1)}$ for all $y \in S(x)$ and $i \in \{1,2,\dots,16\}$, and a translation-invariant Gibbs measure is defined as a measure, μ_h, corresponding to a translation-invariant function h.

So the Gibbs measure for our model is translation invariant if $u_i{}' = u_i$ for all $i \in \{1,4,5,8\}$.

Now assume $u_i{}' = u_i, \forall\, i \in \{1,4,5,8\}$.

The analysis and solving the system of equations (36)-(39) is rather tricky. Below we consider a case in which this system of equations is solvable for the set:

$$A_1 = \{\,(u_1,u_4,u_5,u_8) \in R^{+4} :\ u_1\,u_4 = u_5\,u_8\}.$$

Next step, we find Gibbs measures for the previous considered case. To do this, we will introduce some notations.

Define the operator F by

$$F = (F_1, F_4, F_5, F_8) : R^{+4} \rightarrow R^{+4} \qquad (40)$$

with $u_1{}' = F_1(u_1, u_4, u_5, u_8)$, $u_4{}' = F_4(u_1, u_4, u_5, u_8)$, $u_5{}' = F_5(u_1, u_4, u_5, u_8)$, and $u'_8 = F_8(u_1, u_4, u_5, u_8)$.

Remark 3.3.1.[3, 4, 5] The fixed points of the equation $u = F(u)$ describe the translation-invariant Gibbs measures of the Ising model corresponding to the Hamiltonian (9), where $u = (u_1, u_4, u_5, u_8)$.

It's clear that the set A_1 is not invariant with respect to the operator F, i.e., $F(A_1)$ need not be a subset of A_1, see definition 2.2.15. As a result, we define the operator F on this set, which is used to describe the Gibbs distributions. The problem of finding an invariant set with respect to the operator F and with the parameters a, c and b not equal to one is left as interesting problem.

3.4 Phase Transitions

Remark 3.4.1. From definition 2.1.10 and definition 3.1.1, we can conclude that if there are more than one positive fixed point of the operator F.Then there is more than one Gibbs measure corresponding to these positive fixed points. In essence, a phase transition occurs for the Ising-Vannimenus model with Hamiltonian (9) if the system of equations ((36)–(39)) has more than one solution. The number of the solutions of these equations depends on the coupling constants $\{a, b, c\}$ and the inverse temperature $\beta = \frac{1}{T} > 0$.

Now we start to analyze the system of equations ((36)–(39)) in order to determine the number of positive fixed point of it.

We restrict the operator F (40) to the set A_1, i.e., assume $u_1\, u_4 = u_5\, u_8$.

Then, we reduce the nonlinear dynamical system of equations ((36)-(39)) to

one nonlinear dynamical equation as the following:

Divide equation (36) by (37), we get the following equation:

$$u_1 u_4 = \frac{(1 + a^2 b^2 c^2 u_1 u_4)^3 \, {u_5}^3}{(b^2 + a^2 c^2 u_5 \, u_8)^3 {u_4}^3} \tag{41}$$

Divide equation (38) by (39), we get the following equation:

$$\frac{1}{u_8 u_5} = \frac{(b^2 c^2 + a^2 u_1 \, u_4)^3 \, {u_5}^3}{(c^2 + a^2 b^2 u_5 \, u_8)^3 {u_4}^3} \tag{42}$$

Then from (41) and (42), we get

$$u_8 u_5 \; = \frac{(1 + a^2 b^2 c^2 \, u_1 u_4)^3 \, (c^2 + a^2 b^2 u_5 \, u_8)^3}{u_1 u_4 \, (b^2 + a^2 c^2 u_5 \, u_8)^3 (b^2 c^2 + a^2 \, u_1 u_4)^3} \tag{43}$$

Assume $u_1 \, u_4 = u_5 \, u_8 = x$.Then, the equation (43) is reduced to

$$x = \frac{(1+a^2 b^2 c^2 x)^3 \left(c^2+a^2 b^2 x\right)^3}{x \, (b^2+a^2 c^2 x)^3 (b^2 c^2+a^2 \, x)^3}$$

Then we define the following nonlinear dynamical function

$$f(x) = \frac{\left(1+a^2 b^2 c^2 x\right)^3 \left(c^2+a^2 b^2 x\right)^3}{x \, (b^2+a^2 c^2 x)^3 (b^2 c^2+a^2 \, x)^3}, x > 0 \tag{44}$$

In order to investigate the phase transition of the model, we analyze the positive fixed points of the rational function f with real coefficients as a dynamical system such that the number of phases for our model under the assumptions on the operator F equal to the number of positive roots for this function (44). We analyze this function in two cases :

Case1: Assume $c = b$, then the function f is reduced to

$$f(x) = \frac{\left(1+a^2b^4x\right)^3}{x\,(b^4+a^2x)^3} \tag{45}$$

By using the same method in [4, 5], we start to analyze the fixed points for this function. The first derivative for $f(x)$ is

$$f'(x) = -\frac{(1+a^2b^4x)^2(b^4a^4x^2+(4a^2-2a^2b^8)x+b^4)}{x^2(b^4+a^2x)^4}$$

$f'(x) > 0$ (increasing) iff $b^4a^4x^2 + (4a^2 - 2a^2b^8)x + b^4 < 0$

The roots for this equation are $\left\{\frac{-2+b^8}{a^2b^4} - \sqrt{\frac{4-5b^8+b^{16}}{a^4b^8}}\ ,\ \frac{-2+b^8}{a^2b^4} + \sqrt{\frac{4-5b^8+b^{16}}{a^4b^8}}\right\}$

The condition on this roots to be real is $4 - 5b^8 + b^{16} > 0$. The solution for the latest inequality is $0 < b < 1$ or $b > 1.189207$ by [57].

From the sign of the quadratic equation , the inequality $b^4a^4x^2 + (4a^2 - 2a^2b^4c^4)x + b^4 < 0$ is satisfied iff

$$x_1^* = \frac{-2+b^8}{a^2b^4} - \sqrt{\frac{4-5b^8+b^{16}}{a^4b^8}} < x < \frac{-2+b^8}{a^2b^4} + \sqrt{\frac{4-5b^8+b^{16}}{a^4b^8}} = x_2^*$$

Since $x > 0$. Then $-2 + \mathrm{b}^8$ must be greater than zero and this is satisfied if $b > 1.189207$.

As a result , we conclude that $f(x)$ is increasing ($f'(x) > 0$) iff $c > 0$, $a > 0$, $b > 1.189207$ and

$$0 < \frac{-2+b^8}{a^2b^4} - \sqrt{\frac{4-5b^8+b^{16}}{a^4b^8}} < x < \frac{-2+b^8}{a^2b^4} + \sqrt{\frac{4-5b^8+b^{16}}{a^4b^8}}$$

If one of these conditions is violated , then f is decreasing and there can only be one fixed point of $f(x)$, see Figure 3.4.1. Thus, we restrict ourselves to the case in which the previous conditions are satisfied.

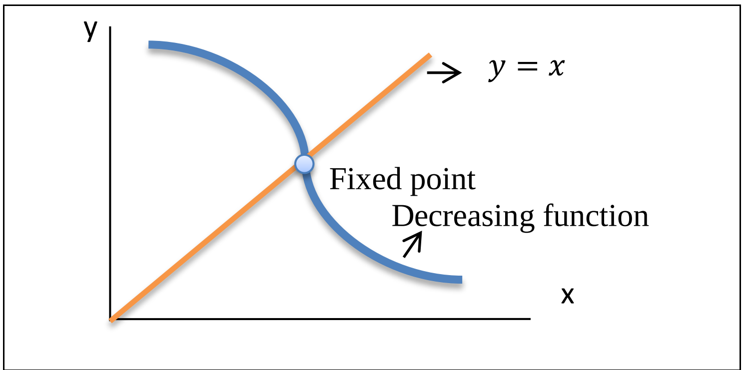

Figure 3.4.1 The uniqueness of fixed point for a decreasing function.

Remark 3.4.2. In this thesis we use Wolfarm Mathematica [57] to simplify the analysis process of equations and polynomials.

The second derivative for the function f is

$$f''(x) = \frac{2\left(10\,a^4x^2 + a^6b^{20}x^3 - 4a^8b^{16}x^4 + a^6b^{12}x^3(-9 + a^4x^2) + a^2b^4x(5 + 18a^4x^2) + b^8(1 + 9a^8x^4)\right)}{x^3(b^4 + a^2x)^5}$$

By Mathematica [57], we conclude that if $b > 1.164079$, then $f''(x)$ has two positive roots say R_1, $R_2 > 0$ such that $f''(x) > 0$ (concave up) $for\ x \in (0, R_1) \cup (R_2, \infty)$ and concave down otherwise.

Then, from the concavity information we conclude that There is at most 4 positive roots for $f(x)$, i.e., the number of phases for the model under the operator F (40) that defined on the set A_1 do not exceed 4. See Figure 3.4.2.

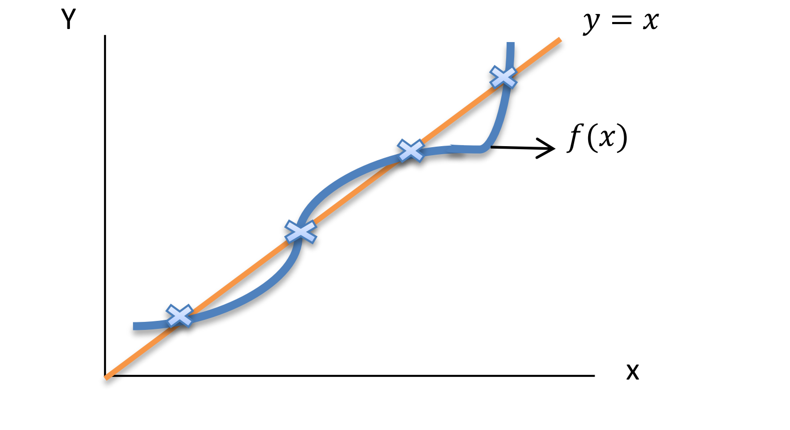

Figure 3.4.2. The maximum possible number of fixed points for an increaisng function $f(x)$ that changes it's concavity from up to down to up is 4.

To determine accurately the maximum possible number of positive roots for equation (45), we apply Descartes' Rule of signs, see definition 2.2.13. It's clear that the equation $f(x) = x$ (45) could be reduced to the following polynomial :

$$g(x) = a^6x^5 + 3a^4b^4x^4 + (3a^2b^8 - a^6b^{12})x^3 + (-3a^4b^8 + b^{12})x^2 - 3a^2b^4x - 1 = 0$$

Let $B = 3a^2b^8 - a^6b^{12}$ and $C = -3a^4b^8 + b^{12}$. Then, if $B < 0$ and $C > 0$, then there are 3 sign changes of $g(x)$ and according to Descartes' Rule it's possible to find 3 positive roots for $g(x)$. Now $B < 0$ and $C > 0$ iff

$$b > 0\,, b^2 > \sqrt{3} \; and \; \frac{\sqrt[4]{3}}{b} < a < \frac{b}{\sqrt[4]{3}} \qquad (46)$$

Then, we get the following remark:

Remark 3.4.3. The maximum possible number of positive fixed points for equation (45) are 3 according to Descartes's rule of signs**,** and this case is possible under the conditions in (46).

Note that the conditions in (46) can be reduced to

$$\beta(\,2\,J_p) > \ln(\sqrt{3}\,) \qquad \delta \quad \frac{\sqrt[4]{3}}{e^{\beta\,Jp}} < e^{\beta J} < \frac{e^{\beta\,Jp}}{\sqrt[4]{3}}$$

$$\rightarrow\ T < \frac{(2J_p)}{\ln(\sqrt{3})} \quad \delta \quad \frac{2J}{\frac{(2\ J_p)}{T} - \ln(\sqrt{3})} < T < \frac{2J}{\ln(\sqrt{3}) - \frac{(2J_p)}{T}}$$

So, if at least one from these conditions is violated. Then there is only one translation-invariant Gibbs measure corresponds to the unique positive root for the polynomial $g(x)$ according to Descartes rule, i.e., there is no phase transition.

Although the conditions in (46) guarantee the possibility of existence of phase transition, but to give an accurate result we use an analytical strategy as follows:

According to Preston [50, Proposition 10.7], there can be more than one solution for $f(x) = x$ if and only if there is more than one solution to $x\,f'(x) = f(x)$, which is reduced to the following equation:

$$\frac{(1 + a^2b^4x)^2(2\,b^4a^4x^2 + (5a^2 - a^2b^8)x + 2b^4)}{x(b^4 + a^2x)^4} = 0 \qquad (47)$$

To find the positive roots for this equation, consider

$$2\,b^4a^4x^2 + (5a^2 - a^2b^8)x + 2b^4 = 0,$$

Then, the roots for this quadratic equation are

$$x_1 = \frac{-5 + b^8}{4a^2b^4} - \frac{1}{4}\sqrt{\frac{25 - 26b^8 + b^{16}}{a^4b^8}}\ , x_2 = \frac{-5 + b^8}{4a^2b^4} + \frac{1}{4}\sqrt{\frac{25 - 26b^8 + b^{16}}{a^4b^8}}$$

x_1 and x_2 are positive roots for (47) iff $25 - 26b^8 + b^{16} > 0$ and $-5 + b^8 > 0$, these conditions satisfied iff $b > 1.49535$ and $a > 0$ by using [57]. Note that under the previous conditions, it's clear that $x_1^* < x_1$ and $x_2^* > x_2$, i.e., the positive roots x_1 and x_2 for equation (47) belongs to the interval (x_1^*, x_2^*) in which the function f in equation (45) increasing.

As a result, in the same way as Preston [50], we conclude the following proposition

Proposition 3.4.1. The equation

$$x = \frac{\left(1+a^2b^2c^2x\right)^3}{x\,(b^2c^2+a^2x)^3} \tag{48}$$

with $x > 0$, $b > 0$, $a > 0$, $c > 0$ and $c = b$ has one positive solution if $b < 1.189207$. If $b > 1.49535$, then there exists $\eta_1(a,b)$, $\eta_2(a,b) > 0$ such that equation (48) has at least two positive roots if $\eta_1(a,b) < 1 < \eta_2(a,b)$, where $\eta_1(a,b)$, $\eta_2(a,b)$ are the first derivative of the positive roots x_1 and x_2 of equation (47), i.e.,

$$\eta_1(a,b) =$$

$$-\frac{6a^4b^4(-1+b^8)(1+a^2b^4(\frac{1}{4}\left(\frac{-5+b^8}{a^2b^4}-\sqrt{\frac{25-26b^8+b^{16}}{a^4b^8}}\right)))^2}{(5-b^8+a^2b^4\sqrt{\frac{25-26b^8+b^{16}}{a^4b^8}})(b^4+a^2(\frac{1}{4}\left(\frac{-5+b^8}{a^2b^4}-\sqrt{\frac{25-26b^8+b^{16}}{a^4b^8}}\right)))^4}$$

$$\eta_2(a,b) =$$

$$\frac{6a^4b^4(-1+b^8)(1+a^2b^4(\frac{1}{4}\left(\frac{-5+b^8}{a^2b^4}+\sqrt{\frac{25-26b^8+b^{16}}{a^4b^8}}\right)))^2}{(-5+b^8+a^2b^4\sqrt{\frac{25-26b^8+b^{16}}{a^4b^8}})(b^4+a^2(\frac{1}{4}\left(\frac{-5+b^8}{a^2b^4}+\sqrt{\frac{25-26b^8+b^{16}}{a^4b^8}}\right)))^4}$$

Proof. If $b < 1.189207$, then the function f in equation (45) is a decreasing function and there is only one fixed point in this case [4], see Figure 3.4.1. Now for $b > 1.49535$, i.e., $\frac{J_P}{T} > \ln 1.49535$, then the function $f(x)$ is increasing and $f''(x)$ changes it's concavity from up to down to up as we have explained in the previous steps. Then, from Preston [Proposition 7.10, 50], there is more than one positive roots for equation (48) if

$$f'(x_1) < 1 \text{ and } f'(x_2) > 1 \quad (49)$$

where x_1 and x_2 are the positive roots for the equation $x\,f'(x) = f(x)$ which is reduced to the rational function (47). From the previous notation we conclude that (49) means

$$f(x_1) < x_1 \text{ and } f(x_2) > x_{\ 2} \quad (50)$$

It's clear that f is a continuous function $\forall\, x \in (0, \infty)$ and $\lim_{x\to 0} f(x) = \infty$. Let $M = 1$, from definition 2.2.14 $\exists\, \delta > 0$ such that if $|x| < \delta \longrightarrow \ f(x) \geq 1$. Let $0 < x_0 < 1$ such that $0 < x_0 < x_1 < x_{\ 2}$ and $x_0 \in (0, \delta)$, then it's clear that $f(x_0) > x_0$. As a result, the function f will intersect with the identity function while moving from x_0 to x_1, and the intersection will be recurred again from x_1 to x_2, see Figure 3.4.3.The proof is readily completed.

Theorem 3.4.1. Let $J_p > 0$ and $J_t > 0$ be the PNNN and TPNNN coupling constants, respectively. Then, if $J_p = J_t$ and $T < T_{cr}$ where $T_{cr} = \frac{J_p}{\ln(1.49535)}$, the Ising-Vannimenus model that corresponds to Hamiltonian (9) exhibits a phase transition.

Proof.

From proposition 3.4.1, there is more than one fixed point for the operator F that is defined on the set A_1 (The phase transition exists) if $\mathrm{b} > 1.49535$. Since $b = e^{\beta J_p} \ \rightarrow \ e^{\beta J_p} > 1.49535 \rightarrow \frac{J_p}{T} > \ln(1.49535)$.Then, the proof is readily completed.

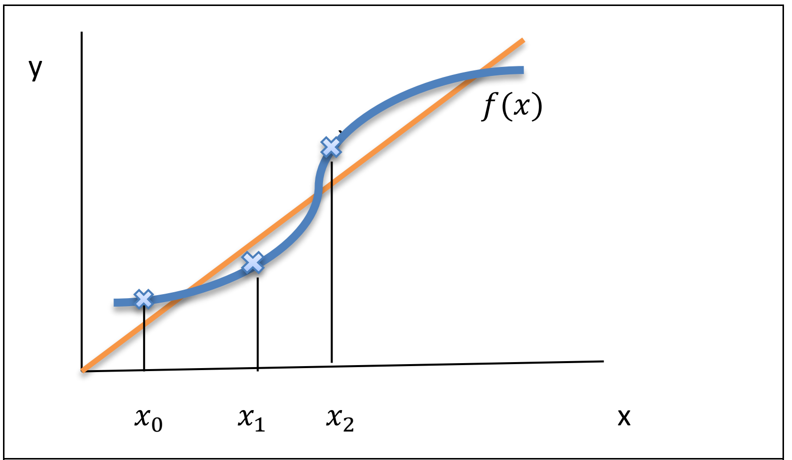

Figure 3.4.3. The existence of multiple fixed points for a function f if $f(x_0) > x_0$, $f'(x_1) < 1$ and $f'(x_2) > 1$.

Now, we give a numerical example for the existence of phase transition according to proposition 3.4.1 and Theorem 3.4.1.

Example 3.4.1

Let $b = 2$, then $b > 1.49535$, thus $f(x)$ is increasing and $f''(x)$ changes it's concavity from up to down to up. Now according to proposition 3.4.1, phase transition exists if

$$\eta_1(a,b) < 1 < \eta_2(a,b)$$

By substituting $b = 2$ in this equation , we get

$$-\frac{4194304a^4(-81+\sqrt{6545})}{-140378419+1722747\sqrt{6545}} < 1 < \frac{4194304a^4(81+\sqrt{6545})}{140378419+1722747\sqrt{6545}}$$

Then phase transition exists if

$$0.801153 < a < 1.248201$$

Let $a = 0.9$, then there are more than one positive solution for equation (48), the solutions are

$$x = 0.023104, x = 2.423943 \text{ and } x = 30.200382 \qquad (60)$$

As a result, we conclude that there are 3 phases for our model, i.e., Phase transition exists. See Figure 3.4.4.

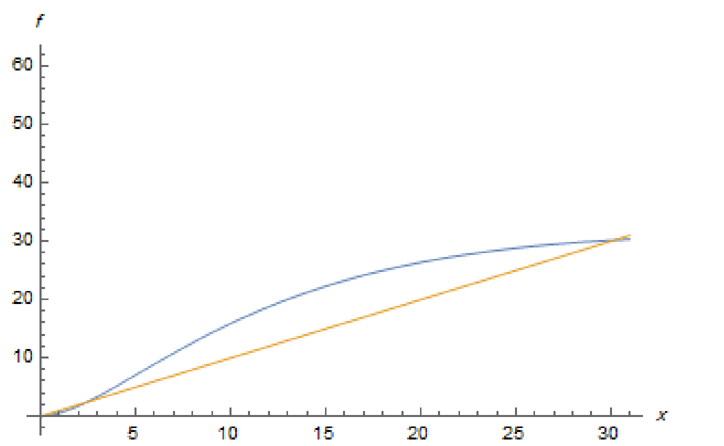

Figure 3.4.4. The existence of 3 positive fixed points of the function $f(x)$ for $a = 0.9, b = 2$ and $c = 2$.

Remark 3.4.4.[5, 40] The stable fixed points (roots) of equation (48) describe extreme Gibbs distributions.

From Remark 2.2.3 and Theorem 2.2.1, we conclude that a fixed point of $f(x)$ given in equation (45) is stable if the absolute value of its derivative less than one. From (60), we have $|f'(0.023104)| = 0.312213 < 1$ (stable), and $|f'(2.423943)| = 1.579549 > 1$ (unstable) and $|f'(30.200382)| = 0.178644 < 1$ (stable). Thus, we have two extreme translation-invariant Gibbs measures corresponding to the stable fixed points . As a result, there is a critical temperature $T_{cr} > 0$ such that for $T < T_{cr}$ the system of equations ((36)-(39)) has 3 positive solutions of boundary vectors : $h_1{}^*$, $h_2{}^*$ and $h_3{}^*$. We denote the translation-invariant Gibbs measures corresponding to the roots $h_1{}^*$, $h_2{}^*$ and $h_3{}^*$ respectively by $\mu_1{}^*$, $\mu_2{}^*$ and $\mu_3{}^*$ respectively.

Case2: Assume $c \neq b$, then the function f defined by equation (44) remains the same:

$$f(x) = \frac{\left(1+a^2b^2c^2x\right)^3 \left(c^2+a^2b^2x\right)^3}{x\,(b^2+a^2c^2x)^3(b^2c^2+a^2\,x)^3} \quad , \quad x > 0$$

In the same way as case 1, we find the positive fixed points for $f(x)$, i.e., we are interested in finding the positive solutions for the equation

$$x^2 = \frac{(1+a^2b^2c^2x)^3\,(c^2+a^2b^2x)^3}{(b^2+a^2c^2x)^3(b^2c^2+a^2\,x)^3},$$

Since $x > 0$, the previous equation can be written as

$$x = \sqrt{\frac{(1+a^2b^2c^2x)^3\,(c^2+a^2b^2x)^3}{(b^2+a^2c^2x)^3(b^2c^2+a^2\,x)^3}} \tag{61}$$

As a result, consider the new function g : $R^+ \to R^+$ such that

$$g(x) = \sqrt{\frac{(1+a^2b^2c^2x)^3\,(c^2+a^2b^2x)^3}{(b^2+a^2c^2x)^3(b^2c^2+a^2\,x)^3}} \tag{62}$$

Then,

$$g'(x) = \frac{3a^2(-1+b^4)c^2(c^2+a^2b^2x)^2(1+a^2b^2c^2x)^2(2a^2c^2x+2a^2b^4c^2x+b^2(1+c^4)(1+a^4x^2))}{2(b^2c^2+a^2x)^4(b^2+a^2c^2x)^4\sqrt{\frac{(c^2+a^2b^2x)^3(1+a^2b^2c^2x)^3}{(b^2c^2+a^2x)^3(b^2+a^2c^2x)^3}}}$$

It's clear that $g'(x) > 0$ (increasing) iff $b > 1$. So we restrict our self to the case when $b > 1$, that's to say $\frac{J_p}{T} > 0$ and hence $J_p > 0$.

Again from Preston [50, Proposition 10.7] , $x\,g'(x) = g(x)$ is reduced to the following equation :

$$\begin{aligned}&\left(a^2b^2x+a^2b^2c^4x+c^2(1+a^4b^4x^2)\right)^2\left(-8a^4c^4x^2+4a^4b^8c^4x^2\right.\\&\quad -5a^2b^2c^2(1+c^4)x(1+a^4x^2)+a^2b^6c^2(1+c^4)x(1+a^4x^2)\\&\quad \left.-2b^4(a^4x^2+a^4c^8x^2+c^4(1+a^4x^2)^2)\right)=0\end{aligned} \tag{63}$$

Then, from Mathematica Programming [57] there are two positive roots for the previous equation iff one of the following cases is satisfied:

1) For $0 < c \leq 0.4539076$ or $c > 2.203091$. Then

   a) If $\frac{(25-14c^4+25c^8)^{1/4}}{(1+34c^4+c^8)^{1/4}} \leq b \leq \frac{1}{2}\sqrt{\frac{1}{c^2}+c^2+\frac{\sqrt{1+34c^4+c^8}}{c^2}}$. Then, there are 4 positive roots for equation (63) :

$$
\begin{aligned}
x_1 = -\frac{1}{8a^2b^2c^2}\Big(&5+5c^4-b^4(1+c^4)+\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)} \\
&+\sqrt{2}a^2b^2c^2\sqrt{(}\frac{1}{a^4b^4c^4}(25+18c^4+25c^8+b^8(1+18c^4+c^8) \\
&+5\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)} \\
&+5c^4\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)}-b^4(18+18c^8 \\
&+\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)}+c^4(52 \\
&+\sqrt{(-1+b^4)(-25+14c^4-25c^8+b^4(1+34c^4+c^8))}))))\Big)
\end{aligned}
$$

$$
\begin{aligned}
x_2 = \frac{1}{8a^2b^2c^2}\Big(&-5+b^4-5c^4+b^4c^4-\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)} \\
&+\sqrt{2}a^2b^2c^2\sqrt{(}\frac{1}{a^4b^4c^4}(25+18c^4+25c^8+b^8(1+18c^4+c^8) \\
&+5\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)} \\
&+5c^4\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)}-b^4(18+18c^8 \\
&+\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)}+c^4(52 \\
&+\sqrt{(-1+b^4)(-25+14c^4-25c^8+b^4(1+34c^4+c^8))}))))\Big)
\end{aligned}
$$

$$
\begin{aligned}
x_3 = \frac{1}{8a^2b^2c^2}\Big(&-5+b^4-5c^4+b^4c^4+\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)} \\
&-\sqrt{2}a^2b^2c^2\sqrt{(}\frac{1}{a^4b^4c^4}(25+18c^4+25c^8+b^8(1+18c^4+c^8) \\
&-5\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)} \\
&-5c^4\sqrt{(-1+b^4)\big(-25+14c^4-25c^8+b^4(1+34c^4+c^8)\big)}+b^4(-18-18c^8 \\
&+\sqrt{(-1+b^4)(-25+14c^4-25c^8+b^4(1+34c^4+c^8))}+c^4(-52 \\
&+\sqrt{(-1+b^4)(-25+14c^4-25c^8+b^4(1+34c^4+c^8))})
\end{aligned}
$$

$$x_4 = \frac{1}{8a^2b^2c^2}(-5 + b^4 - 5c^4 + b^4c^4 + \sqrt{(-1 + b^4)(-25 + 14c^4 - 25c^8 + b^4(1 + 34c^4 + c^8))}$$
$$+ \sqrt{2}a^2b^2c^2\sqrt{(}\frac{1}{a^4b^4c^4}(25 + 18c^4 + 25c^8 + b^8(1 + 18c^4 + c^8)$$
$$- 5\sqrt{(-1 + b^4)(-25 + 14c^4 - 25c^8 + b^4(1 + 34c^4 + c^8))}$$
$$- 5c^4\sqrt{(-1 + b^4)(-25 + 14c^4 - 25c^8 + b^4(1 + 34c^4 + c^8))} + b^4(-18 - 18c^8$$
$$+ \sqrt{(-1 + b^4)(-25 + 14c^4 - 25c^8 + b^4(1 + 34c^4 + c^8))} + c^4(-52$$
$$+ \sqrt{(-1 + b^4)(-25 + 14c^4 - 25c^8 + b^4(1 + 34c^4 + c^8))})))))$$

b) If $\frac{1}{2}\sqrt{\frac{1}{c^2} + c^2 + \frac{\sqrt{1+34c^4+c^8}}{c^2}} < b$, then there are two positive roots for (63) , which are x_3 and x_4 above .

2) If $0.4539076 < c \leq 2.203091$, then there is two positive roots x_3 and x_4 for equation (63) if $b > \frac{1}{2}\sqrt{\frac{1}{c^2} + c^2 + \frac{\sqrt{1+34c^4+c^8}}{c^2}}$.

We conclude the following proposition:

Proposition 3.4.2. The equation

$$x = \sqrt{\frac{(1+a^2b^2c^2x)^3\,(c^2+a^2b^2x)^3}{(b^2+a^2c^2x)^3(b^2c^2+a^2\,x)^3}} \qquad (64)$$

with $x > 0$ $b > 0$, $a > 0$ and $c > 0$ has one positive solution if $b < 1$. If $b > 1$, then

a) If $c \in (0,0.4539079\,] \cup (2.203091, \infty)$, and if $b \geq \frac{(25-14c^4+25c^8)^{1/4}}{(1+34c^4+c^8)^{1/4}}$, then there exists $\eta_1(a, b, c)$, $\eta_2(a, b, c) > 0$ such that equation (64) has at least two positive roots if $\eta_1(a, b) < 1 < \eta_2(a, b)$, where $\eta_1(a, b, c)$, $\eta_2(a, b, c)$ are

$$\eta_i(a,b,c) = \frac{g(x_i{}^*)}{x_i{}^*}$$

Where $x_i{}^*,\ i = 1, 2$ are any positive solutions for the equation

$$(c^2 + a^2b^2x)^2(1 + a^2b^2c^2x)^2\big(-8a^4c^4x^2 + 4a^4b^8c^4x^2 - 5a^2b^2c^2(1 + c^4)x(1 + a^4x^2) + a^2b^6c^2(1 + c^4)x(1 + a^4x^2) - 2b^4(a^4x^2 + a^4c^8x^2 + c^4(1 + a^4x^2)^2)\big) = 0$$

b) If $c \in (0.4539076\,, 2.203091]$, and $b > \frac{1}{2}\sqrt{\frac{1}{c^2} + c^2 + \frac{\sqrt{1+34c^4+c^8}}{c^2}}$, then we get the same results that we have mentioned in (a).

Theorem 3.4.2. If proposition 3.4.2 is satisfied, then Ising-Vannimenus Model that corresponds to Hamiltonian (9) exhibits a phase transition, and at least two extreme Gibbs measures exists.

We give a numerical example for the existence of phase transition according to proposition 3.4.2 and Theorem 3.4.2.

Example 3.4.2

Let $c = 0.46 \in (0.4539076,\ 2.203091]$. Then, according to proposition 3.4.2 (b), we conclude that if $b > 1.763950$, the Phase transition exists if

$$\eta_1(a,b,c) < 1 < \eta_2(a,b,c)$$

Let $b = 3$, to find $\eta_1(a, 3{,}0.46)$ and $\eta_2(a, 3{,}0.46)$ we need to get x_1^* and x_2^* (the positive roots of polynomial (63)), which are (by Mathematica programming help [57])

$$x_1{}^* = \frac{0.0382549}{a^2}$$

$$x_2{}^* = \frac{26.1404561}{a^2}$$

As a result, phase transition exists if $\eta_1(a, 3, 0.46) = f'({x_1}^*) < 1 < f'({x_2}^*) = \eta_2(a, 3, 0.46)$

Now for $a > 0$, $f'\left(\frac{0.0382549}{a^2}\right) < 1$ iff $0.1644630a^2 < 1$ iff

$$0 < a < 2.4658457$$

In the same way, $f'\left(\frac{26.1404561}{a^2}\right) > 1$ iff $6.0803947\, a^2 > 1$ iff

$$a > 0.40554048$$

We can conclude that for $c = 0.45$ and $b = 3$, phase transition exists if

$$0.40554048 < a < 2.4658457 \qquad (65)$$

Let $a = 0.6$, then the positive solutions for equation (64) are

$$x = 0.001418, \ x = 9.447723 \ \text{ and } \ x = 534.710233$$

corresponding to 3 translations-invariant Gibbs measures, two of them are extreme one's. See Figure 3.4.5.

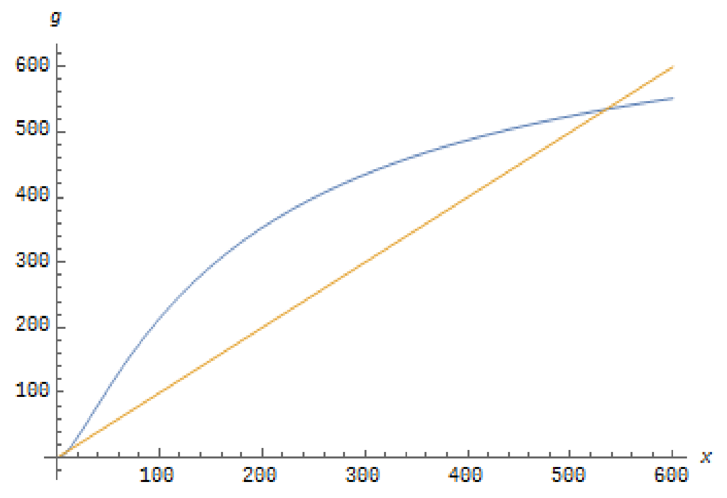

Figure 3.4.5. The existence of 3 positive fixed points of the function $g(x)$ (62) for $a = 0.6, b = 3$ and $c = 0.46$**.**

In the previous example, let take $a = 4 \notin (0.40554048\,, 2.4658457\;)$, i.e., outside the phase transition interval (65). Then the positive root for equation (64) is unique which is $x = \; 724.881067$. As a result, no phase transition exists for this case. See Figure 3.4.6.

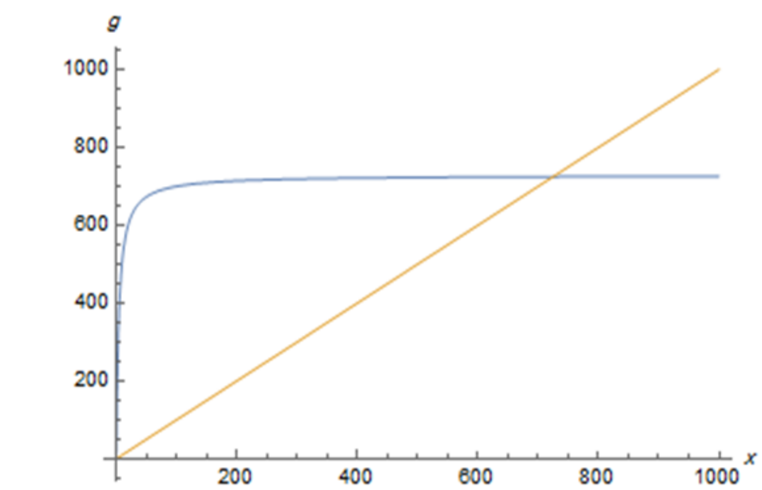

Figure 3.4.6. The Existence of unique fixed point for $g(x)$ (62) when $a = 4\,, b = 3$ and $c = 0.46$.

Chapter Four

The Second Model:

Gibbs Measures of an Ising-Vannimenus Model with One-Level Competing Interactions on Cayley tree of Order Four

In this chapter, we focus on the translation-invariant Gibbs measures with memory of length 2 associated to Ising-Vannimenus model on a Cayley tree of order four with one-level interactions. We use MRF method to achieve the following objectives: constructing the recurrence equations corresponding to a generalized NNN Ising-Vannimenus model; formulating the problem in terms of nonlinear recursion relations along the branches of a Cayley tree of order four; fulfilling the Kolmogorov consistency condition; describing the translation-invariant Gibbs measures for the model; showing that some measures are extreme Gibbs distributions and trying to give some numerical examples. Our model is a generalization for H. Akin work [3] for Cayley tree of order four by adding one level interaction. We study this model for the first time and attempt to determine when phase transition occurs.

4.1 Model Construction

Consider the Hamiltonian for Ising-Vannimenus Model (6) with one-level next nearest neighbor interaction (see definition 2.2.8). Therefore, the Hamiltonian is

$$H(\sigma) = -J_{oL} \sum_{\widehat{<x,y>}} \sigma(x)\sigma(y) - J_p \sum_{>x,y<} \sigma(x)\sigma(y) - J \sum_{<x,y>} \sigma(x)\sigma(y) \qquad (66)$$

This defines the Ising-Vannimenus model with competing NN , PNNN and One - Level NNN (OLNNN), where the sum in the first term ranges all OLNNN, and the sum in the second term ranges all the PNNN, and the third term ranges the

NN such that the spins $\sigma(x)$ and $\sigma(y)$ take values in the set Φ. Here, J_{oL}, J_P and $J \in R$ are coupling constants corresponding to OLNNN, PNNN and NN potentials, respectively. In brief, our Hamiltonian is the same as Vannimenus one (6) in addition to OL interaction on Cayley tree of order four.

In the same way as first model, let us consider a Cayley tree of order 4. Now let $x \in W_n$ for some $n \in N$ and $S(x) = \{y, z, w, t\}$ where $y, z, w, t \in W_{n+1}$ are the direct successors of x. Denote $B_1(x) = \{x, y, z, w, t\}$ a unite semi-ball with a center x, such that $S(x) = \{y, z, w, t\}$.

Remark 4.1.1. Denote the set of all configurations on unite semi-ball $B_1(x)$ on Cayley tree of order four by

$$\Phi^{B_1(x)} = \{\begin{pmatrix} & & i & & \\ j & l & & k & g \end{pmatrix} : \; l, i, j, g, k \in \Phi\}$$

It's clear that the set $\Phi^{B_1(x)}$ consists of 32 possible configurations.

In table 4.1.1 below, we denote the spin configurations belonging to $\Phi^{B_1(x)}$:

Table 4.1.1 The set of possible configurations ($\Phi^{B_1(x)}$) on cayley tree of order 4

$\boldsymbol{\sigma_1}^{(1)} = \begin{pmatrix} & & + & & \\ + & + & & + & + \end{pmatrix}$	$\boldsymbol{\sigma_2}^{(1)} = \begin{pmatrix} & & + & & \\ - & + & & + & + \end{pmatrix}$	$\boldsymbol{\sigma_3}^{(1)} = \begin{pmatrix} & & + & & \\ + & - & & + & + \end{pmatrix}$
$\boldsymbol{\sigma_4}^{(1)} = \begin{pmatrix} & & + & & \\ + & + & & - & + \end{pmatrix}$	$\boldsymbol{\sigma_5}^{(1)} = \begin{pmatrix} & & + & & \\ + & + & & + & - \end{pmatrix}$	$\boldsymbol{\sigma_6}^{(1)} = \begin{pmatrix} & & + & & \\ - & - & & + & + \end{pmatrix}$
$\boldsymbol{\sigma_7}^{(1)} = \begin{pmatrix} & & + & & \\ - & + & & - & + \end{pmatrix}$	$\boldsymbol{\sigma_8}^{(1)} = \begin{pmatrix} & & + & & \\ - & + & & + & - \end{pmatrix}$	$\boldsymbol{\sigma_9}^{(1)} = \begin{pmatrix} & & + & & \\ + & + & & - & - \end{pmatrix}$
$\boldsymbol{\sigma_{10}}^{(1)} = \begin{pmatrix} & & + & & \\ + & - & & + & - \end{pmatrix}$	$\boldsymbol{\sigma_{11}}^{(1)} = \begin{pmatrix} & & + & & \\ + & - & & - & + \end{pmatrix}$	$\boldsymbol{\sigma_{12}}^{(1)} = \begin{pmatrix} & & + & & \\ - & - & & - & + \end{pmatrix}$
$\boldsymbol{\sigma_{13}}^{(1)} = \begin{pmatrix} & & + & & \\ - & + & & - & - \end{pmatrix}$	$\boldsymbol{\sigma_{14}}^{(1)} = \begin{pmatrix} & & + & & \\ + & - & & - & - \end{pmatrix}$	$\boldsymbol{\sigma_{15}}^{(1)} = \begin{pmatrix} & & + & & \\ - & - & & + & - \end{pmatrix}$
$\boldsymbol{\sigma_{16}}^{(1)} = \begin{pmatrix} & & + & & \\ - & - & & - & - \end{pmatrix}$	$\boldsymbol{\sigma_{17}}^{(1)} = \begin{pmatrix} & & - & & \\ + & + & & + & + \end{pmatrix}$	$\boldsymbol{\sigma_{18}}^{(1)} = \begin{pmatrix} & & - & & \\ - & + & & + & + \end{pmatrix}$
$\boldsymbol{\sigma_{19}}^{(1)} = \begin{pmatrix} & & - & & \\ + & - & & + & + \end{pmatrix}$	$\boldsymbol{\sigma_{20}}^{(1)} = \begin{pmatrix} & & - & & \\ + & + & & - & + \end{pmatrix}$	$\boldsymbol{\sigma_{21}}^{(1)} = \begin{pmatrix} & & - & & \\ + & + & & + & - \end{pmatrix}$

$\boldsymbol{\sigma}_{22}^{(1)} = ({}_{-\;-}{}^{-}{}_{+\;+})$	$\boldsymbol{\sigma}_{23}^{(1)} = ({}_{-\;+}{}^{-}{}_{-\;+})$	$\boldsymbol{\sigma}_{24}^{(1)} = ({}_{-\;+}{}^{-}{}_{+\;-})$
$\boldsymbol{\sigma}_{25}^{(1)} = ({}_{+\;+}{}^{-}{}_{-\;-})$	$\boldsymbol{\sigma}_{26}^{(1)} = ({}_{+\;-}{}^{-}{}_{+\;-})$	$\boldsymbol{\sigma}_{27}^{(1)} = ({}_{+\;-}{}^{-}{}_{-\;+})$
$\boldsymbol{\sigma}_{28}^{(1)} = ({}_{-\;-}{}^{-}{}_{-\;+})$	$\boldsymbol{\sigma}_{29}^{(1)} = ({}_{+\;-}{}^{-}{}_{-\;-})$	$\boldsymbol{\sigma}_{30}^{(1)} = ({}_{-\;+}{}^{-}{}_{-\;-})$
$\boldsymbol{\sigma}_{31}^{(1)} = ({}_{-\;-}{}^{-}{}_{+\;-})$	$\boldsymbol{\sigma}_{32}^{(1)} = ({}_{-\;-}{}^{-}{}_{-\;-})$	

Now let us take a natural definition for the real valued boundary function quantities $h({}_{y\ z}{}^{x}{}_{w\ t})$ as $h_{B_1(x)}$. Consider the following definition for the vector valued function $h : V \to R^{32}$ defined by

$$h : < x, y, z, w, t > \to h_{B_1(x)} = \left(h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)\sigma(t)} : \sigma(x), \sigma(y), \sigma(z), \sigma(w), \sigma(t) \in \Phi\right) \qquad (67)$$

Where $h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)\sigma(t)} \in \mathrm{R}$, $x \in W_{n-1}$, $y, z, w, t \in S(x)$ and

$< x, y, z, w, t >$ denotes the vertices for the semi ball $B_1(x)$. Finally, we use the function $h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)\sigma(t)}$ to describe the Gibbs measure of any configuration $({}_{\sigma(y)\ \sigma(z)}{}^{\sigma(x)}{}_{\sigma(w)\ \sigma(t)})$ that belongs to $\Phi^{B_1(x)}$.

In this section, we present the general structure of Gibbs measures with a memory of length two on the Cayley tree of order four.

The finite-dimensional Gibbs probability measures (distributions) on the configuration space $\Omega^{V_n} = \{\sigma_n = \{\sigma(x) = \pm 1, x \in V_n\}\}$ at inverse temperature $\beta = \frac{1}{T}$ is defined by the formula

$$\mu_{\boldsymbol{h}}^{(\boldsymbol{n})}(\sigma) = \frac{1}{Z_n} \exp[-\beta\, H_n(\sigma) + \sum_{x \in W_{n-1}} \sum_{y,z,w,t\, \in S(x)} \sigma(x)\sigma(y)\sigma(w)\sigma(z)\sigma(t)\, h_{B_1(x),\sigma(x)\sigma(y)\sigma(w)\sigma(z)\sigma(t)}] \quad (68)$$

with the corresponding partition function defined by

$$Z_n = \sum_{\sigma_n \in \Omega^{V_n}} \exp[-\beta\, H_n(\sigma) + \sum_{x \in w_{n-1}} \sum_{y,z,w,t\, \in s(x)} \sigma(x)\sigma(y)\sigma(w)\sigma(z)\sigma(t)\, h_{B_1(x),\sigma(x)\sigma(y)\sigma(w)\sigma(z)\sigma(t)}] \quad (69)$$

Now we look for a new set of translation-invariant Gibbs measures and we consider a construction of an infinite volume distribution with given finite-dimensional distributions.

First step, we try to find the set of limiting Gibbs measures μ_h for a sequence of finite-dimensional compatible measures, i.e., we explain the splitting Gibbs measures μ_h with memory of length 2 corresponding to the Hamiltonian of our model (66) and the function h (67). See definition 3.1.1 and the condition (14).

We define the Hamiltonian (interaction energy) on V with the inner configuration $\sigma_{n-1} \in \Omega^{V_{n-1}}$ and the boundary condition $\eta \in \Omega^{W_n}$ as follows

$$\begin{aligned} H_n(\sigma_{n-1} \vee \eta) = & -J_{oL} \sum_{\widehat{<x,y>} \in V_{n-1}} \sigma(x)\sigma(y) - J_{oL} \sum_{\widehat{<x,y>} \in W_n} \eta(x)\eta(y) \\ & - J_p \sum_{>x,y< \in V_{n-1}} \sigma(x)\sigma(y) - J_p \sum_{x \in W_{n-2}} \sum_{y \in S^2(x)} \sigma(x)\eta(y) \\ & - J \sum_{<x,y> \in V_{n-1}} \sigma(x)\sigma(y) - J \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\eta(y) \end{aligned}$$

$$\begin{aligned} = H(\sigma_{n-1}) - J_{oL} \sum_{\widehat{<x,y>} \in W_n} \eta(x)\eta(y) - J_p \sum_{x \in W_{n-2}} \sum_{y \in S^2(x)} \sigma(x)\eta(y) \\ - J \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\eta(y) \end{aligned} \quad (70)$$

Thus , the compatibility condition (14) is satisfied if for the sequence of finite dimensional probability measures $\mu_h{}^{(n)},\ n \geq 1$ (68) for our model if

$$\sum_{\eta \in \Phi^{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \eta) = \mu_h^{(n-1)}(\sigma_{n-1})$$

Then, we get

$$\frac{1}{z_n}\sum_{\eta\in\Omega^{W_n}} exp[-\beta\, H_n(\sigma_{n-1}\vee\,\eta)$$

$$+\sum_{y,z,w,l\,\in w_{n-1}}\sum_{y_i\in S(y)}\sum_{z_i\in S(z)}\sum_{w_i\in S(w)}\sum_{t_i\in S(t)}[\sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)\,\eta(y_4)h_{\,B_1(y),\sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)\,\eta(y_4)}$$

$$+\sigma(\mathrm{z})\,\eta(\mathrm{z}_1)\,\eta(\mathrm{z}_2)\,\eta(\mathrm{z}_3)\,\eta(\mathrm{z}_4)\,\mathrm{h}_{\,\mathrm{B}_1(\mathrm{z}),\sigma(\mathrm{z})\,\eta(\mathrm{z}_1)\,\eta(\mathrm{z}_2)\,\eta(\mathrm{z}_3)\,\eta(\mathrm{z}_4)}$$

$$+\,\sigma(w)\,\eta(w_1)\eta(w_2)\,\eta(w_3)\,\eta(w_4)\,h_{\,B_1(w),\sigma(w)\,\eta(w_1)\,\eta(w_2)\,\eta(w_3)\,\eta(w_4)}$$

$$+\sigma(t)\,\eta(t_1)\,\eta(t_2)\ \eta(t_3)\ \eta(t_4)\,h_{\,B_1(t),\sigma(t)\,\eta(t_1)\,\eta(t_2)\,\eta(t_3)\,\eta(t_4)}\,]]$$

$$=\frac{1}{z_{n-1}}exp[-\beta\, H_n(\sigma_{n-1})$$

$$+\sum_{x\in w_{n-2}}\sum_{y,z,w,t\,\in S(x)}[\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(w)\,\sigma(t)h_{\,B_1(y),\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(w)\,\sigma(t)}]$$

Let $L_n=\frac{z_{n-1}}{z_n}$. From (70) This provides that

$$L_n\sum_{\eta\in\Omega^{W_n}} exp[-\beta\, H(\sigma_{n-1})+\ \beta\, J_{oL}\sum_{\widehat{<x,y>}\,\in\, w_n}\eta(x)\eta(y)+\beta\, J_p\sum_{x\in w_{n-2}}\sum_{y\,\in S^2(x)}\sigma(x)\eta(y)$$

$$+\,\beta\, J\sum_{x\in w_{n-1}}\sum_{y\,\in S(x)}\sigma(x)\eta(y)$$

$$+\sum_{y,z,w,t\,\in w_{n-1}}\sum_{y_i\in S(y)}\sum_{z_i\in S(z)}\sum_{w_i\in S(w)}\sum_{t_i\in S(t)}[\sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)\,\eta(y_4)h_{\,B_1(y),\sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)\,\eta(y_4)}$$

$$+\sigma(z)\,\eta(z_1)\,\eta(z_2)\,\eta(z_3)\,\eta(z_4)\,h_{\,B_1(z),\sigma(z)\,\eta(z_1)\,\eta(z_2)\,\eta(z_3)\,\eta(z_4)}$$

$$+\,\sigma(w)\,\eta(w_1)\eta(w_2)\,\eta(w_3)\,\eta(w_4)\,h_{\,B_1(w),\sigma(w)\,\eta(w_1)\,\eta(w_2)\,\eta(w_3)\,\eta(w_4)}$$

$$+\,\sigma(t)\,\eta(t_1)\,\eta(t_2)\ \eta(t_3)\ \eta(t_4)\,h_{\,B_1(t),\sigma(t)\,\eta(t_1)\,\eta(t_2)\,\eta(t_3)\,\eta(t_4)}\ ]$$

$$=\ exp[-\beta\, H_n(\sigma_{n-1})+\sum_{x\in w_{n-2}}\sum_{y,z,w,t\in S(x)}[\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(w)\,\sigma(t)h_{\,B_1(x),\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(w)\,\sigma(t)}]]$$

For all $i=1,2,3,4$. Now after simplifying we get

$$\prod_{x\in W_{n-2}}\prod_{y,z,w,t\,\in\,S(x)} e^{\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(w)\,\sigma(t)\,h_{\,B_1(x),\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(w)\,\sigma(t)}} =$$

$$L_n\prod_{x\,\in\,w_{n-2}}\prod_{y,z,w,t\,\in\,S(x)}\prod_{y_i\in S(y)}\prod_{z_i\in S(z)}\prod_{w_i\in S(w)}\prod_{t_i\in S(t)}\sum_{\eta(y_i),\eta(z_i),\eta(w_i),\eta(t_i)\in\{-1.+1\}} e^{\left[B\left(h,J,J_p,J_{oL}\right)\right]} \quad (71)$$

Where

$$B\left(h,J,J_p,J_{oL}\right) = \sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)\,\eta(y_4)h_{\,B_1(y),\sigma(y)\,\eta(y_1)\,\eta(y_2)\,\eta(y_3)\,\eta(y_4)}$$

$$+\sigma(z)\,\eta(z_1)\,\eta(z_2)\,\eta(z_3)\,\eta(z_4)\,h_{\,B_1(z),\sigma(z)\,\eta(z_1)\,\eta(z_2)\,\eta(z_3)\,\eta(z_4)}$$

$$+\sigma(w)\,\eta(w_1)\eta(w_2)\,\eta(w_3)\,\eta(w_4)\,h_{\,B_1(w),\sigma(w)\,\eta(w_1)\,\eta(w_2)\,\eta(w_3)\,\eta(w_4)}$$

$$+\sigma(t)\,\eta(t_1)\,\eta(t_2)\;\eta(t_3)\;\eta(t_4)\,h_{\,B_1(t),\sigma(t)\,\eta(t_1)\,\eta(t_2)\,\eta(t_3)\,\eta(t_4)}$$

$$+\beta\,J\left[\,\sigma(y)\big(\eta(y_1)+\eta(y_2)+\;\eta(y_3)+\;\eta(y_4)\big)+\;\;\sigma(z)\big(\;\eta(z_1)+\right.$$

$$+\eta(z_2)+\;\eta(z_3)+\;(z_4)\big)+\;\;\sigma(w)\big(\eta(w_1)+\eta(w_2)+\;\eta(w_3)$$

$$\left.+\,\eta(w_4)\big)+\;\;\sigma(t)(\eta(t_1)+\;\;\eta(t_2)+\;\eta(t_3)+\;\eta(t_4)\;)\right]$$

$$+\,\beta\,J_{OL}\left(\,\prod_{\boldsymbol{i=1}}^{\boldsymbol{4}}\eta(y_i)\;+\prod_{i=1}^{4}\eta(w_i)+\;\prod_{i=1}^{4}\eta(z_i)+\;\prod_{i=1}^{4}\eta(t_i)\right)$$

$$+\beta\,J_p[\,\sigma(x)\sum_{i=1}^{4}(\eta(y_i)+\;\eta(w_i)+\;\eta(z_i)+\eta(t_i))]$$

Next, let fix $< x,y >, < x,z >, < x,w >$ and $< x,t >$ by rewriting (71) for all values of $\sigma(x),\sigma(y),\sigma(z),\sigma(w),\sigma(t)\in\Phi$. In the same way as the first model, we assume $\sigma(x) = i,\ \sigma(y) = j,\sigma(z) = l,\sigma(w) = k,\sigma(t) = g,\eta(y_1) = u,\eta(y_2) = v,\eta(y_3) = f,\eta(y_4) = a,\eta(z_1) = r,\eta(z_2) = s,\ \eta(z_3) = q,\eta(z_4) = p,$ $\eta(w_1) = o,\ \eta(w_2) = n,\ \eta(w_3) = m,\eta(w_4) = h$ and $\eta(t_1) = b,\ \eta(t_2) = c,$ $\eta(\,t_3) = d,\eta(t_4) = e$.Where $i,j,l,k,g,u,v,f,a,r,s,q,p,o,n,m,h,b,c,d,e\in\Phi$. Then, (71) is reduced to

$$e^{ijlkg\;h_{B_1(X),i;j,k,l,g}} = L_2\sum_{m,n,o,,\ldots.} e^{\beta J_p i(u+v+f+a+r+s+q+p+o+n+m+h+b+c+d+e)}\;x$$

$$x\;e^{\beta J(l(r+s+q+p)+j(u+v+f+a)+k(o+n+m+h)+g(b+c+d+e)}$$

$$x\;e^{\beta J_{0L}\left((o(n+m+h)+n(m+h)+mh)+(r(s+q+p)+s(q+p)+qp)+(u(v+f+a)+v(f+a)+fa)+(b(c+d+e)+c(d+e)+de)\right)}$$

$$x\ e^{juvfa\ h_{B_1(y),j;u,v,f,a}}\ x\ e^{lrsqp\ h_{B_1(z),l;r,s,q,p}}\ x\ e^{konmh\ h_{B_1(w),k;o,n,m,h}}\ x\ e^{gbcde_{B_1(w),g;b,c,d,e}} \quad (72)$$

See Figure 4.1.1.

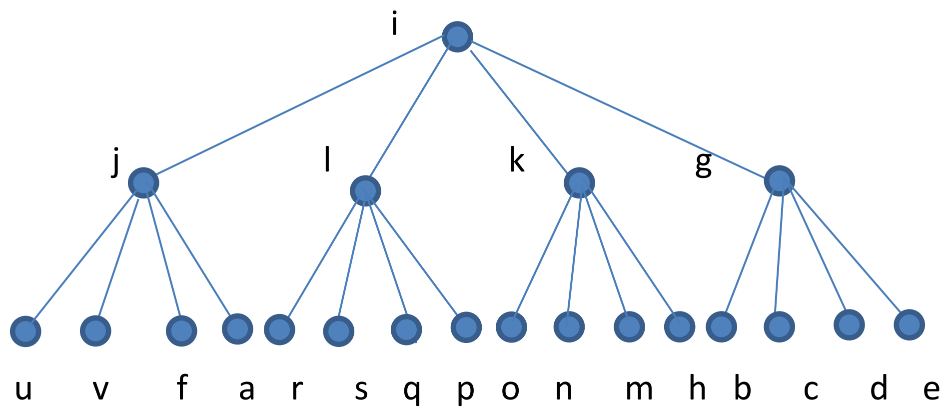

Figure 4.1.1. The set of Configurations on semi-infinite Cayley tree of order 4 with 2 levels**.**

Now let

$$h_1 = h_{B_1(x),\,\sigma_1{}^{(1)}} \quad (73)$$

$$h_2 = h_{B_1(x),\,\sigma_2{}^{(1)}} = h_{B_1(x),\,\sigma_3{}^{(1)}} = h_{B_1(x),\,\sigma_4{}^{(1)}} = h_{B_1(x),\,\sigma_5{}^{(1)}} \quad (74)$$

$$h_3 = h_{B_1(x),\,\sigma_6{}^{(1)}} = h_{B_1(x),\,\sigma_7{}^{(1)}} = h_{B_1(x),\,\sigma_8{}^{(1)}} = h_{B_1(x),\,\sigma_9{}^{(1)}} = h_{B_1(x),\,\sigma_{10}{}^{(1)}} = h_{B_1(x),\,\sigma_{11}{}^{(1)}} \quad (75)$$

$$h_4 = h_{B_1(x),\,\sigma_{12}{}^{(1)}} = h_{B_1(x),\,\sigma_{13}{}^{(1)}} = h_{B_1(x),\,\sigma_{14}{}^{(1)}} = h_{B_1(x),\,\sigma_{15}{}^{(1)}} \quad (76)$$

$$h_5 = h_{B_1(x),\,\sigma_{16}{}^{(1)}} \quad (77)$$

$$h_6 = h_{B_1(x),\,\sigma_{17}{}^{(1)}} \quad (78)$$

$$h_7 = h_{B_1(x),\,\sigma_{18}{}^{(1)}} = h_{B_1(x),\,\sigma_{19}{}^{(1)}} = h_{B_1(x),\,\sigma_{20}{}^{(1)}} = h_{B_1(x),\,\sigma_{21}{}^{(1)}} \quad (79)$$

$$h_8 = h_{B_1(x),\,\sigma_{22}{}^{(1)}} = h_{B_1(x),\,\sigma_{23}{}^{(1)}} = h_{B_1(x),\,\sigma_{24}{}^{(1)}} = h_{B_1(x),\,\sigma_{25}{}^{(1)}} = h_{B_1(x),\,\sigma_{26}{}^{(1)}} = h_{B_1(x),\,\sigma_{27}{}^{(1)}} \tag{80}$$

$$h_9 = h_{B_1(x),\,\sigma_{28}{}^{(1)}} = h_{B_1(x),\,\sigma_{29}{}^{(1)}} = h_{B_1(x),\,\sigma_{30}{}^{(1)}} = h_{B_1(x),\,\sigma_{31}{}^{(1)}} \tag{81}$$

$$h_{10} = h_{B_1(x),\,\sigma_{32}{}^{(1)}} \tag{82}$$

Therefore, the real vector valued function given in (67) can be redefined as

$$h = (h_1,\ h_2,\ h_3,\ h_4,\ h_5,\ h_6,\ h_7,\ h_8,\ h_9,\ h_{10}) \in R^{10}. \tag{83}$$

4.2 Basic Equations

Now we construct the recurrence equations that give explicit formula for Gibbs measures with memory of length two which satisfies consistency conditions (14) by means of equation (72). Assume that $a = e^{\beta J}$, $b = e^{\beta J_p}$ and $c = e^{\beta J_{OL}}$. As we have done before, by using the equations (73)-(83) we can assume that

$$v_i{}' = e^{h_{B_1(x),\sigma_j^{(1)}}} \text{ for } x \in W_{n-1} \text{ and } v_i = e^{h_{B_1(y),\sigma_j^{(1)}}} \text{ for } y \in S(x)$$

Where $i = 1, 2, \ldots, 10$ and $j \in \{1,2 \ldots\ldots,32\}$. From (72), through direct enumeration, we get the following 10 equations:

Let $v_1' = e^{h_{B_1(x),\,\sigma_1{}^{(1)}}} = e^{h_1}$ where h_1 is defined on the semi-ball $B_1(x), x \in W_0$.Then from (72) we get that :

$$v_1{}' =$$

$$L_2\,(a^{16}b^{16}c^{24}v_1^4 + \frac{256a^8b^8}{v_2^4} + \frac{256a^{10}b^{10}c^6v_1}{v_2^3} + \frac{96a^{12}b^{12}c^{12}v_1^2}{v_2^2} + \frac{16a^{14}b^{14}c^{18}v_1^3}{v_2} + 24a^{12}b^{12}c^{16}v_1^3v_3 +$$

$$\frac{1536a^6b^6v_3}{c^2v_2^3} + \frac{1152a^8b^8c^4v_1v_3}{v_2^2} + \frac{288a^{10}b^{10}c^{10}v_1^2v_3}{v_2} + 216a^8b^8c^8v_1^2v_3^2 + \frac{3456a^4b^4v_3^2}{c^4v_2^2} + \frac{1728a^6b^6c^2v_1v_3^2}{v_2}$$

$$+864a^4b^4v_1v_3^3+\frac{3456a^2b^2v_3^3}{c^6v_2}+\frac{1296v_3^4}{c^8}+\frac{256}{a^8b^8v_4^4}+\frac{256c^6v_1}{a^2b^2v_4^3}+\frac{1024}{a^4b^4v_2v_4^3}+\frac{1536v_3}{a^6b^6c^2v_4^3}+\frac{96a^4b^4c^{12}v_1^2}{v_4^2}+\frac{1536}{v_2^2v_4^2}$$

$$+\frac{768a^2b^2c^6v_1}{v_2v_4^2}+\frac{1152c^4v_1v_3}{v_4^2}+\frac{4608v_3}{a^2b^2c^2v_2v_4^2}+\frac{3456v_3^2}{a^4b^4c^4v_4^2}+\frac{16a^{10}b^{10}c^{18}v_1^3}{v_4}+\frac{1024a^4b^4}{v_2^3v_4}+\frac{768a^6b^6c^6v_1}{v_2^2v_4}+$$

$$\frac{192a^8b^8c^{12}v_1^2}{v_2v_4}+\frac{288a^6b^6c^{10}v_1^2v_3}{v_4}+\frac{4608a^2b^2v_3}{c^2v_2^2v_4}+\frac{2304a^4b^4c^4v_1v_3}{v_2v_4}+\frac{1728a^2b^2c^2v_1v_3^2}{v_4}+\frac{6912v_3^2}{c^4v_2v_4}+\frac{3456v_3^3}{a^2b^2c^6v_4}$$

$$+4a^8b^8c^{24}v_1^3v_5+\frac{256a^2b^2c^6v_5}{v_2^3}+\frac{192a^4b^4c^{12}v_1v_5}{v_2^2}+\frac{48a^6b^6c^{18}v_1^2v_5}{v_2}+72a^4b^4c^{16}v_1^2v_3v_5+\frac{1152c^4v_3v_5}{v_2^2}+$$

$$\frac{576a^2b^2c^{10}v_1v_3v_5}{v_2}+432c^8v_1v_3^2v_5+\frac{1728c^2v_3^2v_5}{a^2b^2v_2}+\frac{864v_3^3v_5}{a^4b^4}+\frac{256c^6v_5}{a^{10}b^{10}v_4^3}+\frac{192c^{12}v_1v_5}{a^4b^4v_4^2}+\frac{768c^6v_5}{a^6b^6v_2v_4^2}+$$

$$\frac{1152c^4v_3v_5}{a^8b^8v_4^2}+\frac{48a^2b^2c^{18}v_1^2v_5}{v_4}+\frac{768c^6v_5}{a^2b^2v_2^2v_4}+\frac{384c^{12}v_1v_5}{v_2v_4}+\frac{576c^{10}v_1v_3v_5}{a^2b^2v_4}+\frac{2304c^4v_3v_5}{a^4b^4v_2v_4}+\frac{1728c^2v_3^2v_5}{a^6b^6v_4}$$

$$+6c^{24}v_1^2v_5^2+\frac{96c^{12}v_5^2}{a^4b^4v_2^2}+\frac{48c^{18}v_1v_5^2}{a^2b^2v_2}+\frac{72c^{16}v_1v_3v_5^2}{a^4b^4}+\frac{288c^{10}v_3v_5^2}{a^6b^6v_2}+\frac{216c^8v_3^2v_5^2}{a^8b^8}+\frac{96c^{12}v_5^2}{a^{12}b^{12}v_4^2}+\frac{48c^{18}v_1v_5^2}{a^6b^6v_4}$$

$$+\frac{192c^{12}v_5^2}{a^8b^8v_2v_4}+\frac{288c^{10}v_3v_5^2}{a^{10}b^{10}v_4}+\frac{4c^{24}v_1v_5^3}{a^8b^8}+\frac{16c^{18}v_5^3}{a^{10}b^{10}v_2}+\frac{24c^{16}v_3v_5^3}{a^{12}b^{12}}+\frac{16c^{18}v_5^3}{a^{14}b^{14}v_4}+\frac{c^{24}v_5^4}{a^{16}b^{16}}\;)$$

Which is simplified (by [57]) to

$$v_1{}'=L_2\left(a^4b^4c^6v_1+\frac{4\,(a\,b)^2}{v_2}+\frac{6}{c^2}\,v_3+\frac{4}{(a\,b)^2\,v_4}+\frac{c^6v_5}{a^4b^4}\right)^4 \tag{84}$$

In the same way, we get the following recurrence equations

$$(v\,{}'_2)^{-1}=L_2\left(a^4b^4c^6v_1+\frac{4\,(a\,b)^2}{v_2}+\frac{6}{c^2}\,v_3+\frac{4}{(a\,b)^2\,v_4}+\frac{c^6v_5}{a^4b^4}\right)^3\left(\frac{b^4c^6}{a^4v_6}+\frac{4\,b^2}{a^2}v_7+\frac{6}{v_8\,c^2}+\frac{4\,a^2}{b^2}\,v_9+\frac{a^4c^6}{b^4v_{10}}\right) \tag{85}$$

$$v_3'=L_2\left(a^4b^4c^6v_1+\frac{4\,(a\,b)^2}{v_2}+\frac{6}{c^2}\,v_3+\frac{4}{(a\,b)^2\,v_4}+\frac{c^6v_5}{a^4b^4}\right)^2\left(\frac{b^4c^6}{a^4v_6}+\frac{4\,b^2}{a^2}v_7+\frac{6}{v_8\,c^2}+\frac{4\,a^2}{b^2}\,v_9+\frac{a^4c^6}{b^4v_{10}}\right)^2 \tag{86}$$

$$(v_4{}')^{-1}=L_2\left(\mathrm{a}^4\mathrm{b}^4\mathrm{c}^6\mathrm{v}_1+\frac{4\,(\mathrm{a\,b})^2}{\mathrm{v}_2}+\frac{6}{\mathrm{c}^2}\,\mathrm{v}_3+\frac{4}{(\mathrm{a\,b})^2\,\mathrm{v}_4}+\frac{\mathrm{c}^6\mathrm{v}_5}{\mathrm{a}^4\mathrm{b}^4}\right)^2\left(\frac{\mathrm{b}^4\mathrm{c}^6}{\mathrm{a}^4\mathrm{v}_6}+\frac{4\,\mathrm{b}^2}{\mathrm{a}^2}\mathrm{v}_7+\frac{6}{\mathrm{v}_8\,\mathrm{c}^2}+\frac{4\,\mathrm{a}^2\mathrm{v}_9}{\mathrm{b}^2}+\frac{\mathrm{a}^4\mathrm{c}^6}{\mathrm{b}^4\mathrm{v}_{10}}\right)^2 \tag{87}$$

$$v_5{}' = L_2\left(\frac{b^4c^6}{a^4v_6} + \frac{4\,b^2}{a^2}v_7 + \frac{6}{v_8\,c^2} + \frac{4\,a^2}{b^2}\,v_9 + \frac{a^4c^6}{b^4v_{10}}\right)^4 \tag{88}$$

$$(v_6{}')^{-1} = L_2\left(\frac{a^4c^6}{b^4}v_1 + \frac{4\,a^2}{b^2v_2} + \frac{6}{c^2}v_3 + \frac{4\,b^2}{a^2v_4} + \frac{b^4c^6}{a^4}v_5\right)^4 \tag{89}$$

$$(v_7{}') = L_2\left(\frac{a^4c^6}{b^4}v_1 + \frac{4\,a^2}{b^2v_2} + \frac{6}{c^2}v_3 + \frac{4\,b^2}{a^2v_4} + \frac{b^4c^6}{a^4}v_5\right)^3\left(\frac{c^6}{a^4b^4v_6} + \frac{4}{a^2b^2}v_7 + \frac{6}{c^2v_8} + 4\,a^2b^2v_9 + \frac{a^4b^4c^6}{v_{10}}\right) \tag{90}$$

$$(v_8{}')^{-1} = L_2\left(\frac{a^4c^6}{b^4}v_1 + \frac{4\,a^2}{b^2v_2} + \frac{6}{c^2}v_3 + \frac{4\,b^2}{a^2v_4} + \frac{b^4c^6}{a^4}v_5\right)^{\mathbf{2}}\left(\frac{c^6}{a^4b^4v_6} + \frac{4}{a^2b^2}v_7 + \frac{6}{c^2v_8} + 4\,a^2b^2v_9 + \frac{a^4b^4c^6}{v_{10}}\right)^2 \tag{91}$$

$$v_9{}' = L_2\left(\frac{a^4c^6}{b^4}v_1 + \frac{4\,a^2}{b^2v_2} + \frac{6v_3}{c^2} + \frac{4\,b^2}{a^2v_4} + \frac{b^4c^6v_5}{a^4}\right)\left(\frac{c^6}{a^4b^4v_6} + \frac{4\,v_7}{a^2b^2} + \frac{6}{c^2v_8} + 4\,a^2b^2v_9 + \frac{a^4b^4c^6}{v_{10}}\right)^3 \tag{92}$$

$$(v_{10}{}')^{-1} = L_2\left(\frac{c^6}{a^4b^4v_6} + \frac{4}{a^2b^2}v_7 + \frac{6}{c^2v_8} + 4\,a^2b^2v_9 + \frac{a^4b^4c^6}{v_{10}}\right)^4 \tag{93}$$

From the system of equations (84-93), we conclude that :

$$\left.\begin{aligned}
(v_2')^{-1} &= (v_1{}')^{\frac{3}{4}}\,(v_5{}')^{\frac{1}{4}}\\
v_3{}' &= (v_1{}')^{\frac{1}{2}}\,(v_5{}')^{\frac{1}{2}}\\
(v_4{}')^{-1} &= (v_1{}')^{\frac{1}{4}}\,(v_5{}')^{\frac{3}{4}}\\
v_7{}' &= (v_6{}')^{\frac{-3}{4}}\,(v_{10}{}')^{\frac{-1}{4}}\\
(v_8{}')^{-1} &= (v_6{}')^{\frac{-1}{2}}(v_{10}{}')^{\frac{-1}{2}}\\
v_9 &= (v_6{}')^{\frac{-1}{4}}\,(v_{10}{}')^{\frac{-3}{4}}
\end{aligned}\right\} \tag{94}$$

Therefore, we can only select the four independent real variables $v_1{}'$, $v_5{}'$, $v_6{}'$ and $v_{10}{}'$. Now the consistency (compatibility) condition satisfied if the following remark satisfied:

Remark 4.2.1. If the vector valued function h Given in (83) has the following form,

$$h = \left(p\,, -\left(\frac{3\,p+q}{4}\right),\ \frac{p+q}{2}\,, -\left(\frac{p+3q}{4}\right), \mathrm{q}\,, \mathrm{r}\,, -\left(\frac{3\ r+s}{4}\right),\ \frac{r+s}{2}\,, -\left(\frac{r+3\,s}{4}\right), s\right)$$

Then the consistency condition is satisfied , where $p, q, r, s \in R$.

Proof.

From (83), Let $h_1 = q,\ h_5 = q\,, h_6 = r$ and $h_{10} = s$ where $p, q, r, s \in R$.

Then from (94) , $(v_2')^{-1} = (v_1{}')^{\frac{3}{4}}\ (v_5{}')^{\frac{1}{4}} \quad \longrightarrow \quad e^{-h_2} = e^{\frac{3\,h_1+h_5}{4}}$

$$\longrightarrow \quad -h_2 = \frac{3\,h_1+h_5}{4} \quad \longrightarrow \quad h_2 = -\left(\frac{3\,p+q}{4}\right)$$

Now $v_3{}' = (v_1{}')^{\frac{1}{2}}\ (v_5{}')^{\frac{1}{2}}$. Then, $e^{h_3} = e^{\frac{h_1+h_5}{2}}$, i.e., $h_3 = \frac{p+q}{2}$

In the same way as the previous cases, we can get the other coordinates for h.

If we assume $u_i{}^4 = v_i$, then the system of equations (84-93) is reduced to

$$u_1{}' = \sqrt[4]{L_2}\left(\frac{4\,u_1 u_5{}^3}{a^2 b^2} + 4\,a^2 b^2 u_1{}^3\,u_5 + a^4 b^4 c^6 u_1{}^4 + \frac{6}{c^2}\,u_1{}^2 u_5{}^2 + \frac{c^6}{b^4 a^4}\,u_5{}^4\right) \tag{95}$$

$$u_5{}' = \sqrt[4]{L_2}\,\left(\frac{6}{c^2 u_6{}^2 u_{10}{}^2} + \frac{b^4 c^6}{a^4 u_6{}^4} + \frac{\mathrm{a}^4\mathrm{c}^6}{b^4 u_{10}{}^4} + \frac{4\,b^2}{a^2 u_6{}^3 u_{10}} + \frac{4\,a^2}{b^2 u_6 u_{10}{}^3}\right) \tag{96}$$

$$(u_6{}')^{(-1)} = \sqrt[4]{L_2}\left(\frac{4\,b^2 u_1 u_5{}^3}{a^2} + \frac{4\,a^2 u_1{}^3 u_5}{b^2} + \frac{\mathrm{a}^4\mathrm{c}^6}{b^4}\mathrm{u}_1{}^4 + \frac{6u_1{}^2 u_5{}^2}{c^2} + \frac{b^4 c^6}{a^4} u_5{}^4\right) \tag{97}$$

$$(u_{10}{}')^{(-1)} = \sqrt[4]{L_2}\left(\frac{6}{c^2{u_6}^2{u_{10}}^2}+\frac{c^6}{a^4b^4{u_6}^4}+\frac{\mathrm{a}^4\mathrm{b}^4\mathrm{c}^6}{{u_{10}}^4}+\frac{4}{a^2b^2{u_6}^3u_{10}}+\frac{4\ \mathrm{a}^2\mathrm{b}^2}{u_6{u_{10}}^3}\right) \qquad (98)$$

4.3 Translation-Invariant Gibbs Measures

In this section, we describe the set of translation-invariant Gibbs measures associated with our model that corresponds to Hamiltonian (66). Then, from definition 3.3.1, the vector field function

$$h = \{h_{B_1(x),\sigma_i{}^{(1)}} : i \in \{1,2,\dots,32\}\}$$

is considered as translation-invariant one if $h_{B_1(x),\sigma_i{}^{(1)}} = h_{B_1(y),\sigma_i{}^{(1)}}$ for all $y \in S(x)$ and $i \in \{1,2,\dots,32\}$, and the translation-invariant Gibbs measures are the measures that corresponds to a translation-invariant function h.

As a result, the Gibbs measure for our model is translation invariant if

$u_i{}' = u_i$ for all $i \in \{1,5,6,10\}$.

Assume $u_i{}' = u_i$, $\forall\, i \in \{1,5,6,10\}$.

Since the analysis and solving the system of equations (95)-(98) is rather tricky. Then, we define a set A_2 in which this system of equations is solvable , the set is:

$$A_2 = \left\{ (u_1,u_5,u_6,u_{10}) \in R^{+4} : u_5 = \frac{1}{u_6} \ , \ \frac{1}{u_{10}} = u_1 \right\} \qquad (99)$$

Then, we want to find Gibbs measures for the previous considered case. To do this, we introduce the operator

$$F(u) = (F_1(u), F_5(u), F_6(u), F_{10}(u)) : R^{+4} \to R^{+4}$$

Where $u = (u_1,u_5,u_6,u_{10}) \in R^{+4}$, $u_1' = F_1(u_1,u_5,u_6,u_{10})$

$u'_5 = F_5(u_1, u_5, u_6, u_{10})$, $u_6{}' = F_6(u_1, u_5, u_6, u_{10})$

and $u_{10}{}' = F_{10}(u_1, u_5, u_6, u_{10})$.

As we know, The fixed points of the equation $u = F(u)$ describe the translation-invariant Gibbs measures of the Ising-Vannimenus model corresponding to the Hamiltonian (66). It's clear that the set A_2 is invariant with respect to the operator F, i.e., $F(A_2) \subseteq A_2$. As a result, we consider this invariant subset for this operator, which will be used to describe the Gibbs distributions.

4.4 Phase Transitions

From Remark 3.4.1, we know that if there is more than one positive fixed point of the operator F. Then, there exist more than one Gibbs measure corresponding to these positive fixed points. In essence, a phase transition occurs for Ising-Vannimenus model with Hamiltonian (66) if the system of equations ((95)–(98)) has more than one solution. The number of the solutions of these equations depends on the coupling constants and the temperature T.

Note that if it is possible to find an exact value of temperature T_{cr} such that a phase transition occurs for all $T < T_{cr}$, then T_{cr} is called the critical temperature of the model .See definition 2.1.11.

Now we start to analyze the system of equations (95–98) in order to determine the number of positive fixed point for that system.

Assume $u_5 = \frac{1}{u_6}$ and $u_1 = \frac{1}{u_{10}}$, i.e., We restrict the operator F to the set A_2. Then, we reduce the nonlinear dynamical of equations (95–98) to the following equation :

$$\frac{u_1}{u_5} = \frac{(4c^2a^2b^2u_1u_5{}^3 + 4\,a^6b^6c^2u_1{}^3u_5 + a^8b^8c^8u_1{}^4 + 6\,a^4b^4u_1{}^2u_5{}^2 + c^8u_5{}^4)}{(6\,b^4a^4u_5{}^2u_1{}^2 + b^8c^8u_5{}^4 + a^8c^8u_1{}^4 + 4\,b^6a^2c^2u_5^3u_1 + 4\,a^6b^2c^2u_5u_1{}^3)}$$

Assume $\frac{u_1}{u_5} = x, x > 0$. Then, we define the following function

$$f(x) = \frac{(4c^2a^2b^2x + 4\,a^6b^6c^2x^3 + a^8b^8c^8x^4 + 6\,a^4b^4x^2 + c^8)}{(6\,b^4a^4x^2 + b^8c^8 + a^8c^8x^4 + 4\,b^6a^2c^2x + 4\,a^6b^2c^2x^3)} = x \quad (100)$$

In order to investigate the phase transition of the model, we analyze the positive fixed points of the rational function f with real coefficients as a dynamical system. We start the analysis with Mathematica Programming help [57].

For simplicity, assume $a^2 = a'$, $\mathrm{b}^2 = \mathrm{b}'$, $\mathrm{c}^2 = \mathrm{c}'$. Then, The function f is reduced to

$$f(x) = \frac{(4c'\,a'\,b'\,x + 4\,a'^3b'^3c'\,x^3 + a'^4b'^4c'^4x^4 + 6\,a'^2b'^2x^2 + c'^4)}{(6\,b'^2a'^2x^2 + b'^4c'^4 + a'^4c'^4x^4 + 4\,b'^3a'\,c'\,x + 4\,a'^3b'\,c'\,x^3)} \quad (101)$$

The fixed points of $f(x)$ are the roots of equation $f(x) = x$, the later could be reduced to the function

$$g(x) = a'^4c'^4x^5 + \left(4\,a'^3b'\,c' - a'^4b'^4c'^4\right)x^4 + \left(6\,b'^2a'^2 - 4\,a'^3b'^3c'\right)x^3 + \left(4\,b'^3a'\,c' - 6\,b'^2a'^2\right)x^2 + \left(b'^4c'^4 - 4\,a'\,b'\,c'\right)x - c'^4 = 0 \quad (102)$$

We use Descartes' Rule of Signs (see definition 2.2.13) to find the possible number of positive zeroes for the polynomial $g(x)$. Thus, we determine the number of positive and negative solutions to the equation (102) by the following steps :

To apply Descarts' Rule, as mentioned in definition 2.2.13, we find the negative case for the polynomial, i.e., $g(-x)$ which is reduced to

$$g(-x) = -a'^4c'^4x^5 + \left(4\,a'^3b'\,c' - a'^4b'^4c'^4\right)x^4 - \left(6\,b'^2a'^2 - 4\,a'^3b'^3c'\right)x^3 + \left(4\,b'^3a'\,c' - 6\,b'^2a'^2\right)x^2 - \left(b'^4c'^4 - 4\,a'\,b'\,c'\right)x - c'^4 = 0 \quad (103)$$

Let $B = \left(4\,a'^3b'\,c' - a'^4b'^4c'^4\right)$, $C = \left(6\,b'^2a'^2 - 4\,a'^3b'^3c'\right)$,

$D = \left(4\, b'^3 a'\, c' - 6\, b'^2 a'^2 \right)$ and $E = \left(b'^4 c'^4 - 4\, a'\, b'\, c' \right)$. Then :

$B = 0$ iff $a'^3 b'\, c' \left(4 - a'\, b'^3 c'^3\right) = 0$ iff $b'^3 c'^3 = \frac{4}{a'}$

$C = 0$ iff $6\, b'^2 a'^2 - 4\, a'^3 b'^3 c' = 0$ iff $b'\, c' = \frac{3}{2\, a'}$

$D = 0$ iff $4\, b'^3 a'\, c' - 6\, b'^2 a'^2 = 0$ iff $b'\, c' = \frac{3\, a'}{2}$

$E = 0$ iff $b'^4 c'^4 - 4\, a'\, b'\, c' = 0$ iff $b'^3 c'^3 = 4a'$

Consider the following table which counts the number of sign changes for the positive and negative case, respectively.

Table 4.4.1 Possible number of positive and negative roots for $g(x)$ depending on Descartes' Rule

B	C	D	E	Positive Roots	Negative Roots
+	+	+	+	1	0,2,4
+	+	+	-	1	0,2,4
+	+	-	+	Impossible	
+	+	-	-	1	0,2,4
+	-	-	+	Impossible	
+	-	+	-	1, 3	0,2
+	-	+	+	1,3	0,2
+	-	-	-	1	0,2,4
-	+	+	+	1, 3	0,2
-	+	+	-	Impossible	
-	+	-	+	Impossible	
-	+	-	-	Impossible	
-	-	-	+	1, 3	0,2
-	-	+	-	1 ,3	0,2
-	-	+	+	1 , 3	0,2
-	-	-	-	1	0,2,4

Since we are interested in the existence of more than one positive root for $g(x)$.Then, we consider these cases from Table 4.4.1 and search for the conditions on a', $b'$and c' such that these cases are satisfied.

To clarify, consider the following table

Table 4.4.2 The conditions under which there is possibly 3 positive roots for $g(x)$, $c' > 0$

Case number	B	C	D	E	Number of positive roots	conditions
1	+	-	+	-	1,3	- $0.958415 < a' \leq 1$ and $\frac{3}{2\,a'\,c'} < b' < \frac{\sqrt[3]{4\,a'}}{C'}$ - $1 < a' < 1.04339$ and $\frac{3\,a'}{2\,c'} < b' < \sqrt[3]{\frac{4}{a'}.\frac{1}{c'}}$
2	+	-	+	+	1,3	- $0.918559 < a' \leq 0.958415$ and $\frac{3}{2\,a'\,c'} < b' < \sqrt[3]{\frac{4}{a'}.\frac{1}{c'}}$ - $0.958415 < a' < 1$ and $\frac{\sqrt[3]{4\,a'}}{C'} < b' < \sqrt[3]{\frac{4}{a'}.\frac{1}{c'}}$
3	-	+	+	+	1,3	- $0 < a' < 0.918559$ and $\sqrt[3]{\frac{4}{a'}.\frac{1}{c'}} < b' < \frac{3}{2a'c'}$
4	-	-	+	+	1,3	- $0 < a' \leq 0.918559$ and $b' > \frac{3}{2a'c'}$ - $0.918559 < a' \leq 1$ and $b' > \sqrt[3]{\frac{4}{a'}.\frac{1}{c'}}$ - $1 < a' \leq 1.08866$ and $b' > \frac{\sqrt[3]{4\,a'}}{C'}$ - $a' > 1.08866$ and $b' > \frac{3a'}{2c'}$
5	-	-	-	+	1,3	- $a' > 1.08866$ and $\frac{\sqrt[3]{4\,a'}}{c'} < b' < \frac{3a'}{2c'}$
6	-	-	+	-	1,3	- $1 < a' \leq 1.04339$ and $\sqrt[3]{\frac{4}{a'}.\frac{1}{c'}} < b' < \frac{\sqrt[3]{4\,a'}}{C'}$ - $1.04339 < a' < 1.08866$ and $\frac{3a'}{2c} < b' < \frac{\sqrt[3]{4\,a'}}{C'}$

The analysis of these conditions is trivial, but to clarify this we analyze case1from Table 4.4.2 in details as follows :

Case1 is satisfied if $B > 0,\ C < 0\ ,\ D > 0$ and $E < 0$ which is reduced to

$$\frac{4}{a'} > b'^3c'^3 \ , \quad b'^3c'^3 > \frac{27}{8\,a'^3}\ , \quad b'^3c'^3 > \frac{27\,a'^3}{8} \quad \text{and} \quad b'^3c'^3 < 4\,a'$$

, i.e.,

$$\frac{27\,a'^3}{8} < b'^3c'^3 < \frac{4}{a'} \quad \text{and} \quad \frac{27}{8\,a'^3} < b'^3c'^3 < 4\,a' \qquad (*)$$

There are two possible cases considered in solving these inequalities :

1) If $a' > 1$, then the intersection of the inequalities in (*) is $\frac{27\,a'^3}{8} < b'^3c'^3 < \frac{4}{a'}$.

Also, we consider the condition $\frac{27\,a'^3}{8} < \frac{4}{a'}$ to be satisfied $\longrightarrow\ a'^4 < \frac{32}{27}$

$\longrightarrow a' < 1.04339$

So we conclude that the conditions for the this case are:

$$1 < a' < 1.04339 \quad \text{and} \quad \frac{3\,a'}{2\,c'} < b' < \sqrt[3]{\frac{4}{a'}}.\frac{1}{c'}\,,\quad c' > 0\ .$$

2) If $a' \le 1$, then the intersection of the inequalities in (*) is $\frac{27}{8\,a'^3} < b'^3c'^3 < 4\,a'$.

Again,we consider the condition $\frac{27}{8\,a'^3} < 4\,a'$ to be satisfied $\longrightarrow \frac{27}{32} < a'^4$

$\longrightarrow 0.958415 < a'$

So we conclude that the conditions for this case are :

$$0.958415 < a' \le 1 \quad \text{and} \quad \frac{3}{2\,a'\,c'} < b' < \frac{\sqrt[3]{4\,a'}}{c'},\ c' > 0$$

In the same way, we can prove the other cases.

Remark 4.4.1. The maximum number of positive roots for equation (102) that corresponds to a shifting-invariant phases for Ising-Vannimenus Model with Hamiltonian (66) and boundary function h (83) is three based on Descarte's Rule.

Next, we consider the following Examples on Descarte's Rule such that we confine the values of $a',\ b'$ and c' in which the phase transition possible to exists.

Example 4.4.1

Let $a' = 1.03$ and $c' = 0.5$. Then, if we try to find the conditions in which it is possible to find 3 roots for the fuction $g(x)$ (102), we conclude that the conditions are:

1) $3.143674 < b' < 3.20624$ (from case(6) in table 4.4.2) .
2) $b' > 3.20624$ (from case (4) in table 4.4.2) .
3) $3.09 < b' < 3.143674$ (from case(1) in table 4.4.2).

As a result, in order to detremine phase transition region we restrict the values of b' such that $3.09 < b'$, $b' \neq 3.143674, 3.20624$, i.e., when $a' = 1.03, c' = 0.5$ and $b' \leq 3.09$.Then, phase transition does not exists.

Example 4.4.2

Take $a' = 0.3 < 0.91856$ and $c' = 4$. Then, in the same way as Example 4.4.1 ,we find the conditions on b' in which it is possible to find 3 roots for the polynomial (102) and the results are :

1) $0.592816 < b' < 1.25$ (from case(3) in table 4.4.2).
2) $b' > 1.25$ (from case(4) in table 4.4.2).

Then, we can exclude the values of b' that are less than or equal 0.592816 .

Although Descartes' Rule is needed to determine the possible conditions for phase transition to exists, but in order to determine the exact conditions for phase transition we need further analysis as following :

Consider the first derivative for the function f (101) :

$$f'(x) = \frac{4a'b'\left(c' + 3a'b'x + 3a'^2b'^2c'x^2 + a'^3b'^3c'^4x^3\right)\left(b'^4c'^4 + 4a'b'^3c'x + 6a'^2b'^2x^2 + 4a'^3b'c'x^3 + a'^4c'^4x^4\right)}{(b'^4c'^4 + 4a'b'^3c'x + 6a'^2b'^2x^2 + 4a'^3b'c'x^3 + a'^4c'^4x^4)^2}$$
$$\frac{4a'\left(b'^3c' + 3a'b'^2x + 3a'^2b'c'x^2 + a'^3c'^4x^3\right)\left(6a'^2b'^2x^2 + 4c'\left(a'b'x + a'^3b'^3x^3\right) + c'^4\left(1 + a'^4b'^4x^4\right)\right)}{(b'^4c'^4 + 4a'b'^3c'x + 6a'^2b'^2x^2 + 4a'^3b'c'x^3 + a'^4c'^4x^4)^2}$$

Now $f'(x) > 0$ (f is increasing) if $c' > 0$, $b' > 1$, $a' > 0$ and $x > 0$.

If $b' < 1$, then f is decreasing and there is only one possible solution of $g(x)$ (102). Thus, we can restrict ourselves to the case in which $b' > 1$. As we have mentioned in chapter 3, there is more than one solution for $f(x) = x$ if and only if there is more than one solution to $x\, f'(x) = f(x)$, which reduced to the following equation:

$$\Big(5\, a'^4 c'^8 x^4 - 3a'^4 b'^8 c'^8 x^4 + 16a'^3 b' c'^5 x^3 (1 + a'^2 x^2) - 8a'^3 b'^7 c'^5 x^3 (1 + a'^2 x^2) + 8a' b'^3 c' x (c'^4 + 6a'^2 x^2 + 6a'^4 x^4 + a'^6 c'^4 x^6) + 6b'^2 (3a'^2 c'^4 x^2 + 8a'^4 c'^2 x^4 + 3a'^6 c'^4 x^6) - 2b'^6 (3a'^2 c'^4 x^2 + 8a'^4 c'^2 x^4 + 3a'^6 c'^4 x^6) + b'^4 \Big(36a'^4 x^4 + 16c'^2 (a'^2 x^2 + a'^6 x^6) + c'^8 (1 + a'^8 x^8)\Big)\Big) = 0 \qquad (104)$$

Then, by analyzing this equation using Mathematica Programming [57], there are two positive roots for this equation if the following conditions satisfied:

Let $c' > 0$ and $y(c') \equiv$ the positive root of the following equation for a specific value of $c' > 0$:

$$-5c'^4 - 12c'\, y - 6\, y^2 + 4c'\, y^3 + 3c'^4 y^4 = 0\,, y > 0 \qquad (105)$$

Then, there are two Positive roots for equation (104) iff $a' > 0$, $c' > 0$ and $b' > y(c')$. Assume, if these conditions are satisfied , the roots of polynomial (104) are x_1 and x_2 such that $0 < x_1 < x_2$. Then there are more than one fixed point for the function $f(x)$ if $f'(x_1) < 1 < f'(x_2)$. As a result, this construction provides the following proposition**:**

Proposition 4.4.1. The equation

$$x = \frac{\left(4c'a'\,b'\,x+4\,a'^3 b'^3 c'\,x^3+a'^4 b'^4 c'^4 x^4+6\,a'^2 b'^2 x^2+c'^4\right)}{\left(6\,b'^2 a'^2 x^2+b'^4 c'^4+a'^4 c'^4 x^4+4\,b'^3 a'\,c'\,x+4\,a'^3 b'\,c'\,x^3\right)} \qquad (106)$$

with $x > 0,\ a' > 0,\ c' > 0$ and $b' > 0$ has one positive solution if $\mathrm{b}' < 1$. If $b' > y(c')$ such that $\mathrm{y(c')} \equiv$ the positive root of the equation

$$-5c'^4 - 12c'\,y - 6\,y^2 + 4c'\,y^3 + 3c'^4 y^4 = 0,\ y > 0\ ,$$

Then, there exist $\eta_1(a', b', c')$, $\eta_2(a', b', c') > 0$ such that equation (106) has at least two positive roots if $\eta_1(a', b', c') < 1 < \eta_2(a', b', c')$, where

$$\eta_i(a', b', c') = \frac{f(x_i{}^*)}{x_i{}^*}$$

Where $x_i{}^*, i = 1, 2$ are the positive solutions for the equation

$$\Big(5\,a'^4 c'^8 x^4 - 3a'^4 b'^8 c'^8 x^4 + 16a'^3 b' c'^5 x^3\left(1 + a'^2 x^2\right) - 8a'^3 b'^7 c'^5 x^3\left(1 + a'^2 x^2\right)$$
$$+8a' b'^3 c' x\left(c'^4 + 6a'^2 x^2 + 6a'^4 x^4 + \mathrm{a}'^6 c'^4 x^6\right) + 6b'^2\left(3a'^2 c'^4 x^2 + 8a'^4 c'^2 x^4\right.$$
$$\left.+ 3a'^6 c'^4 x^6\right) - 2b'^6\left(3a'^2 c'^4 x^2 + 8a'^4 c'^2 x^4 + 3a'^6 c'^4 x^6\right) + b'^4\Big(36a'^4 x^4 +$$
$$16c'^2\left(a'^2 x^2 + a'^6 x^6\right) + c'^8\left(1 + a'^8 x^8\right)\Big)\Big) = 0,$$ and $f(x)$ is the function in (101).

Remark 4.4.2. If the function f in equation (101) changes it's concavity from up to down around $x = \alpha$, where $\alpha > 0$, and if $\eta_1(a, b, c) = 1$ or $\eta_2(a, b, c) = 1$.Then, there are exactly 2 positive roots for equation (34). Otherwise , if $\eta_1(a, b, c) < 1 < \eta_2(a, b, c)$, then there are 3 positive roots under the same concavity conditions.

As a result, we conclude that Ising-Vannimenus Model with Hamiltonian (66) exhibits a phase transition if there are more than one positive root for equation (106) due to proposition 4.4.1 Conditions.

4.5 Numerical Examples of Phase Transition

In this chapter, we present some numerical examples by using an elementary analysis , i.e., we obtain the fixed points of the function f that given in section 4.4 for a specific values of c', b' and a'. Thus, we use Wolfarm Mathematica [57] to solve this equation and try to find the conditions for phase transition to exists due to proposition 4.4.1. We only deal with positive fixed points, because of the positivity of exponential functions.

We try to check the existence of phase transition for different values of the coupling constants J, J_P, J_{OL} and the temperature T, i.e., for different values of a$'$, b$'$ and c', in the following Examples :

Example 4.5.1

Let $J_{OL} = -0.2$ and $T = 2$, then $c' = 0.45$. Then from proposition 4.4.1 we conclude that there are two positive roots x_1 and x_2 for equation (104) if b$' > 3.42051$ such that $y(0.45) = 3.42051$.

Let $b' = 3.43 > 3.42051$, then there is a phase transition if

$$f'(x_1) < 1 \ \ and \ \ f'(x_2) > 1 \ .$$

By using Mathematica Programming [57], this condition is satisfied iff

$$0.9994158351066276 < a' < 1.0005845067783965$$

Take $a' = 0.9995$, then the roots of polynomial (104) are

$$x_1 = 0.67619 \ \text{and} \ x_2 = 1.48035$$

As a result, there are 3 translation invariant Gibbs measures, i.e., there exists 3 positive fixed points for the function f in equation (101) and these fixed points (phases) are

$$x = 0.46181, \; x = 1.30138 \text{ and } x = 1.66284$$

See Figure 4.5.1.

Note that $f'(0.46181) = 0.99365 < 1$. Then, from Remark 2.2.3 we conclude that the positive fixed point $x = 0.46181$ is a stable one. As a result , the Gibbs measure at this point is extreme, and the same thing for $x = 1.66284$ such that $f'(0.46181) = 0.99849 < 1$. which means we have two extreme Gibbs measure. Also, $f'(1.30138) = 1.00123 > 1$, thus $x = 1.30138$ is not stable fixed point of $f(x)$ and the corresponding Gibbs measure for this point need not be extreme.

Example 4.5.2

Let $c' = 0.97$, then from Proposition 4.4.1, same as the previous Example, there is two positive roots x_1 and x_2 for equation (104) if $b' > 0.992233$ such that $y(0.97) = 0.992233$.

Take $b' = 2$, then the phase transition exists if $f'(x_1) < 1 < f'(x_2)$. This condition is satisfied iff

$$0.8146866021188426 < a' < 1.2274658714150852$$

Let $a' = 1.2$, then the positive roots of equation (104) are

$$x_1 = 0.269121 \text{ and } x_2 = 2.58042$$

As a result of Proposition 4.4.1, we have more than one positive fixed point for the function f, and these points are $x = 0.195253, x = 0.375999$ and $x = 9.74376$.

Thus, there are 3 translation invariant Gibbs measures corresponding to that points when $c' = 0.97$, $b' = 2$ and $a' = 1.2$. It's clear from Figure 4.5.2 that the first and third fixed points are stable, but the second one is unstable. We can make sure by using first derivative test as Example 4.5.1.

Example 4.5.3

Let $c' = 2$, then there are two positive roots x_1 and x_2 for equation (33) if $b' > 1.21224$ such that $y(2) = 1.21224$.

Consider $b' = 1.4$, then there is a phase transition if

$$0.7906262026419042 < a' < 1.264820210433788$$

Let take $a' = 0.8$, then the positive roots of equation (104) are

$$x_1 = 0.70685 \text{ and } x_2 = 2.21050$$

Then, we have more than one positive fixed point for the function f and these points are

$$x = 0.29259, x = 1.96282 \text{ and } x = 2.49463$$

It's clear from Figure 4.5.3 that the first and third fixed points are stable, but the second one is unstable. We can make sure by using first derivative test as mentioned in Remark 2.2.3.

Remark 4.5.1. If we take $a' = 0.7906262026419042$ in Example 4.5.3 such that $f'(x_2) = 1$ by [57] in this case. Then, there are exactly two fixed points for $f(x)$:

$$x = 0.291872 \text{ and } x = 2.236709$$

Thus, there are two translation invariant Gibbs measures corresponding for these points. That's what we noted in Remark 4.4.2 (see Figure 4.5.4).

4.6 Comparing the Results of Descartes' Rule with the Results of Proposition 4.4.1

As we have mentioned in section 4.4, Descartes' rule is needed to determine the intervals in which it's possible to find 1 or 3 fixed points for $f(x)$. In contrast, Proposition 4.4.1 gives exact results about the existence of phase transition, i.e., more than one fixed point for the function f are surely exists. As a result, it's clear that the values for a', b' and c' that satisfying proposition 4.4.1 conditions are surely contained in Descartes' rule intervals for the existence of phase transition.

To clarify, consider Example 4.4.2 such that $a' = 0.3$ and $c' = 4$. Then, from Proposition 4.4.1 and by analyzing equation (106) using Mathemaica Programming, the phase transition exists iff $b' \in [\,1.829331, \infty)$. Note that $[\,1.829331, \infty) \subseteq (1.25, \infty)$, where $(1.25, \infty)$ is the interval of b' that we have found from Descartes' rule conditions that is related to case(4) in table(4.4.2). So, Descartes' Rule helps to confine the intervals where the function f has one or three positive fixed points but on the other hand proposition 4.4.1 gives the exact conditions for the existence of more than one positive fixed point for the function f.

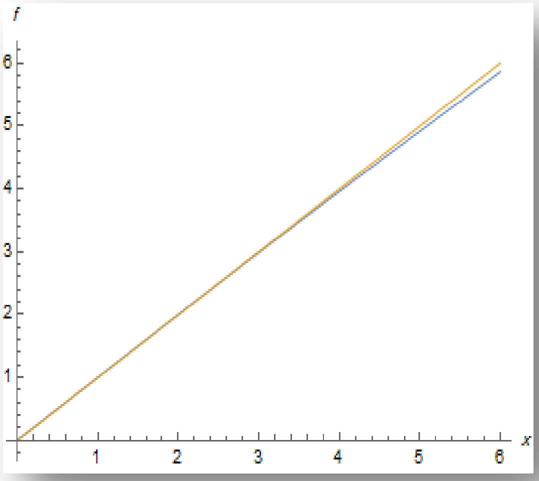

Figure 4.5.1. The existence of 3 positive fixed points of the function $f(x)$ for $a' = 0.4, b' = 3.43$ and $c' = 0.9995$

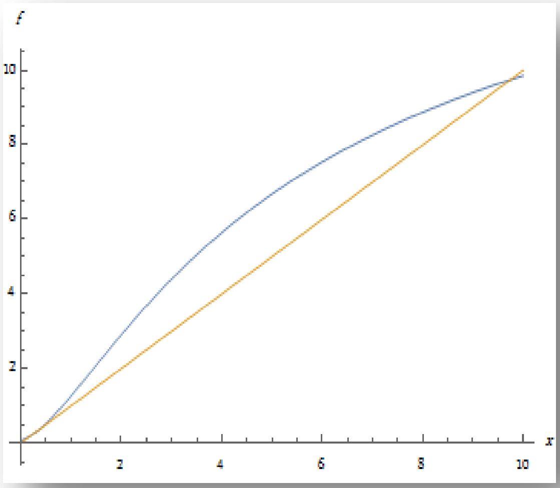

Figure 4.5.2. The existence of 3 positive fixed points of the function $f(x)$ for $a' = 1.2, b' = 2$ and $c' = 0.97$.

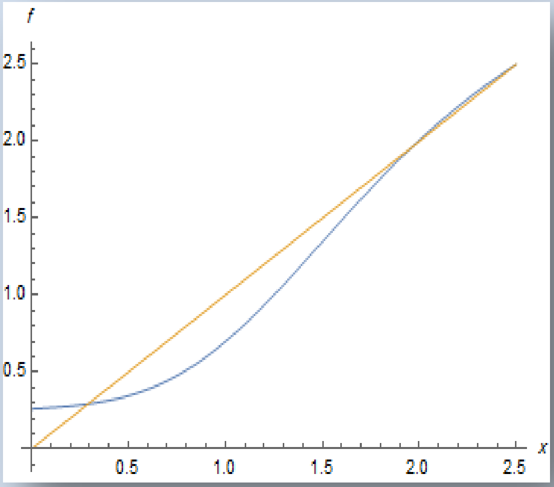

Figure 4.5.3. The existence of 3 positive fixed points of the function $f(x)$ for $a' = 0.8, b' = 1.4$ and $c' = 2$.

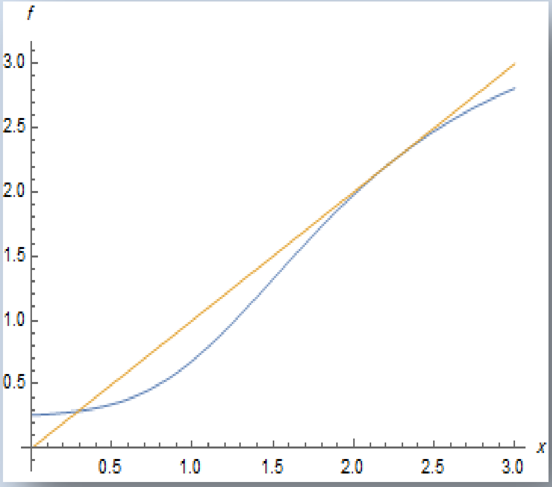

Figure 4.5.4. The existence of 2 fixed points of the function $f(x)$ for $a' = 0.7906262026419042, b' = 1.4$ and $c' = 2.$

Chapter Five

The Third Model:

5.1 A q-State Potts Model with Next-Nearest-Neighbor Ternary Interactions on a Third-Order Cayley Tree

It is natural to consider models with ternary interactions, i.e., interactions of triple neighbors as we have done in chapter 3. Monroe (1992) [45, 46] studied the Ising and the Potts models with ternary interactions only. Recently, Ganikhodjaev et al. (2011) $[11, 12]$ studied the Potts model with competing ternary interactions actions J_p and J_t and NN interactions J on the Cayley tree of second order with three states space $\{1,2,3\}$. In (2012) [24], the phase diagrams for the Potts model with restricted competing NN interactions J and ternary interactions J_t on a Cayley tree of arbitrary order k with 3 state space is constructed. In this section, we prove the existence of phase transition analytically for the same model with an arbitrary number of states (q), $q \in N$ on Cayley tree of order 3, i.e, we develop the model in [24] for q state space and determine its' effect on the phase transition conditions.

5.1.1 Model Construction

Consider a Cayley tree of order 3, let $W_1 = \{ x_1, x_2, x_3\}$ and let $l_1 =< x^{(0)}, x_1 >$, $l_2 =< x^{(0)}, x_2 >$ and $l_3 =< x^{(0)}, x_3 >$ be 3 edges emanating from $x^{(0)}$. It is evident that a semi-infinite Cayley tree Γ_+^{k} splits into 3 components , i.e., 3 single-trunk Cayley trees $\Gamma_+^K\ (l_i)$, $i = 1, 2, 3$. Let V^{l_i} be the set of vertices of a single-trunk Cayley tree $\Gamma_+^K\ (l_i)$, and $V_n^{l_i} = V^{l_i} \cap V_n$ be the set of vertices $x \in V^{l_i}$ with $d\big(x, x^{(0)}\big) \leq$ n.

Consider the Potts model (See Section 2.3.2) with spin values in

$$\Phi = \{1, 2, 3, .., q\}, q \in N, q \geq 2 \qquad (107)$$

, i.e., the spin variables $\sigma(x)$, $x \in V$ takes these values.

Then, relevant Hamiltonians with competing NN interactions J, and TPNNN interactions J_t have the forms

$$H(\sigma) = -J_t \sum_{>x,y,z<} \delta_{\sigma(x)\sigma(y)\sigma(z)} - J \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)} \tag{108}$$

Where $J_t,\ J \in R$ are coupling constant, δ in the second sum is the usual Kronecker symbol and δ in the first sum in (108) is the generalized Kronecker symbol defined as

$$\delta_{\sigma(x)\sigma(y)\sigma(z)} = \begin{cases} 1 & if\ \sigma(x) = \sigma(y) = \sigma(z) \\ 0 & otherwise \end{cases}$$

Conditional Gibbs measures with fixed boundary conditions

In this section, we will produce recurrence relations by fixing boundary configurations $\bar{\sigma}^n(V \setminus V_n)$ for the ball V_n such that $\bar{\sigma}^n(V \setminus V_n) = \{\sigma(x) \in \Phi : x \in V/V_n\}$, Φ is the state space in (107) and $n > 0$. Let $\bar{\sigma}^n(V \setminus V_n)$ be a fixed boundary configuration, i.e., $\bar{\sigma}^n(V \setminus V_n) = i,\ \ \forall\, n > 0,\ i \in \Phi$. From [25], the notion of total energy $H(\sigma(V_n)|\bar{\sigma}(V \setminus V_n))$ of configuration $\sigma(V_n)$ under boundary condition $\bar{\sigma}(V \setminus V_n)$ and partition function $Z_{V_n}(\bar{\sigma}(V \setminus V_n))$ in volume V_n under boundary condition $\bar{\sigma}(V \setminus V_n)$ is introduced as the following :

$$\begin{aligned} H\big(\sigma(V_n)\big|\bar{\sigma}(V \setminus V_n)\big) \; = \; & -J_t \sum_{\substack{>x,y,z< \\ x,y,z \in V_n}} \delta_{\sigma_n(x)\sigma_n(y)\sigma_n(z)} - J \sum_{\substack{<x,y> \\ x,y\, \in V_n}} \delta_{\sigma_n(x)\sigma_n(y)} \\ & - J_t \sum_{\substack{>x,y,z< \\ x,y \in V_n \\ z \in V \setminus V_n}} \delta_{\sigma_n(x)\sigma_n(y)\bar{\sigma}_n(z)} - J \sum_{\substack{<x,y> \\ x\, \in V_n \\ y \in V \setminus V_n}} \delta_{\sigma_n(x)\overline{\sigma_n}(y)} \end{aligned}$$

$$Z_{V_n}(\bar{\sigma}(V \setminus V_n)) = \sum_{\sigma(V_n) \in \Phi^{V_n}} exp(-\beta\, H(\sigma(V_n) | \bar{\sigma}(V \setminus V_n)))$$

Then, the conditional Gibbs measure μ_{V_n} of a configuration $\sigma(V_n)$ that defined on the ball V_n , $n > 0$ is defined as

$$\mu_{V_n}(\sigma(V_n) | \bar{\sigma}(V \setminus V_n)) = \frac{\exp(-\beta\, H(\sigma(V_n) | \bar{\sigma}(V \setminus V_n)))}{Z_{V_n}(\bar{\sigma}(V \setminus V_n))}$$

Here, the configuration $\sigma(V_n)$, the partition function $Z_{V_n}(\bar{\sigma}(V \setminus V_n))$ and the conditional Gibbs measure $\mu_{V_n}(\sigma(V_n) | \bar{\sigma}(V \setminus V_n))$ in volume (ball) V_n will be denoted as σ_n, $Z^{(n)}$ and μ_n, respectively. In this section, we fixed the boundary condition $\bar{\sigma}(V \setminus V_n) = 1, \forall\, n > 0$ and then build the recurrence equations for the conditional partition functions under this boundary condition.

In order to produce the recurrence equations, we use partition function method in the same way as [1, 24, 25]. This method depends on the relation of the partition function on the ball V_{n+1} to the partition function on its subsets V_n. Given the initial conditions on V_1, the recurrence equations will indicate how their influence propagates down the tree. We should take into account the partial partition functions for all possible configurations of the spins in two successive generations , i.e., all possible configurations on the ball V_1.

We consider the following partition functions:

- $Z^{(n)}(i,j)$ which represents a partition function on $V_n^{l_i}$ with the configuration $(i\ j)$ on an edge $< x^{(0)}, x >$, where $x \in W_1$ and $i,j = 1, 2, \cdots, q$.
- $Z^{(n)}\ (i_0, i_1, i_2, i_3)$ is a partition function on V_n where the spin in the root $x^{(0)}$ is i_0 and the 3 spins in the W_1 are $i_1,\ i_2\ and\ i_3$, respectively. Clearly i_0, i_1, i_2 and $i_3 \in \Phi$ (107).

From [21], it's clear that we can write the partition function in term of its component from single-trunk Cayley tree (see definition 2.2.3**)**

$$Z^{(n)}(i_0, i_1, i_2, i_3) = \prod_{j=1}^{3} Z^{(n)}(i_0, i_j) \tag{109}$$

There are q^4, $q \geq 2$, different partition function $Z^{(n)}(i_0, i_1, i_2, i_3)$ on the ball V_1.

The partition function $Z^{(n)}$ in volume V_n be written as

$$Z^{(n)} = \sum_{i_0, i_1, i_2, i_3 = 1}^{q} Z^{(n)}(i_0, i_1, i_2, i_3)$$

We consider the partition function in equation (108) , with a fixed spin i_0 in the root $x^{(0)}$, where $i_0 = 1, 2, 3, \dots q$ and let m_1 be the number of spin {1} and m_2 be the number of spin {2} and m_3 the number of spin {3} and …….. m_q the number of spin {q} on the same level W_1 such that $m_1 + m_2 + \cdots + m_q = 3$. It is evident that

$$\left.\begin{array}{l}
Z^{(n)}(1, i_1, i_2, i_3) = (Z^{(n)}(1,1))^{m_1} \, (Z^{(n)}(1,2))^{m_2} \dots \dots (Z^{(n)}(1,q))^{m_q} \\
Z^{(n)}(2, i_1, i_2, i_3) = (Z^{(n)}(2,1))^{m_1} \, (Z^{(n)}(2,2))^{m_2} \dots \dots (Z^{(n)}(2,q))^{m_q} \\
\qquad \qquad \dots\dots\dots \\
Z^{(n)}(q, i_1, i_2, i_3) = (Z^{(n)}(q,1))^{m_1} \, (Z^{(n)}(q,2))^{m_2} \dots \dots (Z^{(n)}(q,q))^{m_q}
\end{array}\right\} \tag{110}$$

In the same way as [24], we assume that $J_t \neq 0$, and we fix the boundary condition to be one , i.e., $\overline{\sigma_n} \equiv 1$ where $\overline{\sigma_n} = \{\sigma(x) \in \Phi : x \in V/V_n\}$.

Remark 5.1.1.1 Consider all partition functions in volume ${V_n}^{l_i}$ under the boundary condition $\overline{\sigma_n} \equiv 1$. Then we conclude that

$$\left.\begin{array}{l}
z^{(n)}(1,2) = z^{(n)}(1,3) = \cdots = z^{(n)}(1,q) = z^{(n)}(g,d) \text{, where } g \neq d \text{ , } 1 < g \leq q, \\
z^{(n)}(2,1) = z^{(n)}(3,1) = \cdots = z^{(n)}(q,1) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , 1 < d \leq q \\
z^{(n)}(2,2) = z^{(n)}(3,3) = \cdots .. = z^{(n)}(q,q)
\end{array}\right\} \tag{111}$$

Where $q \in N$, $q \geq 2, n \geq 1$ and $g, d \in \Phi$.

Proof.

Assume that $c = e^{\beta J}$, $b = e^{\beta J_t}$, and assume we have q states such that the states space is $\Phi = \{1, 2, 3, \dots \dots .., q\}$, $q \in N$, $q \geq 2$. Let $g, d \in \{2,3, \dots \dots .., q\}, g \neq d$

We prove this remark by induction $\forall n \geq 1$ as following :

- For $n = 1$, it's clear that

$$\left.\begin{aligned} &z^{(1)}(1,2) = z^{(1)}(1,3) =\dots\dots = z^{(1)}(1,q) = z^{(1)}(g,d) = 1 \\ &z^{(1)}(2,1) = z^{(1)}(3,1) = \cdots = z^{(1)}(q,1) = c^3 \\ &z^{(1)}(2,2) = z^{(1)}(3,3) = \cdots .. = z^{(1)}(q,q) = c \\ &z^{(1)}(1,1) = b^3 c^4 \end{aligned}\right\} \qquad (112)$$

Clearly it's satisfied for $n = 1$.

- Assume it's true for $n = k, k > 1$, i.e.,

$$z^{(k)}(1,2) = z^{(k)}(1,3) =\dots\dots = z^{(k)}(1,q) = z^{(k)}(l,m)$$

$$z^{(k)}(2,1) = z^{(k)}(3,1) = \cdots = z^{(k)}(q,1)$$

$$z^{(k)}(2,2) = z^{(k)}(3,3) = \cdots .. = z^{(k)}(q,q),$$

- We show it's true for $n = k + 1$

It's clear that $z^{(k+1)}(2,1) = z^{(k+1)}(3,1) = \cdots = z^{(k+1)}(q,1)$ under the boundary condition $\overline{\sigma_n} \equiv 1$.

To prove the other relations, take an arbitrary $g \in \{2,3, \dots \dots .., q\}, q \geq 2$. Then,

$$z^{(k+1)}(1,g) = \left(z^{(k)}(g,g)\right)^3 + \left(z^{(k)}(g,1)\right)^3 + \sum_{m \neq g, m \neq 1} \left(z^{(k)}(g,m)\right)^3$$

$$+\ 3\sum_{m\neq g}\left(z^{(k)}(g,m)\right)^2\left(z^{(k)}(g,g)\right)+3\sum_{m\neq g}\left(z^{(k)}(g,g)\right)^2\left(z^{(k)}(g,m)\right)$$

$$+\,3\sum_{\substack{m\\ m\neq g}}\sum_{\substack{l\\ l\neq m\\ l\neq g}}\left(z^{(k)}(g,m)\right)^2\left(z^{(k)}(g,l)\right)\ +\ 6\sum_{u}\sum_{\substack{l\\ l\neq g\\ l\neq u}}\sum_{\substack{m\\ m\neq g\\ m\neq u,m\neq l}}\left(z^{(k)}(g,u)\right)\left(z^{(k)}(g,l)\right)\left(z^{(k)}(g,m)\right)$$

Where $m,l,u\in\Phi$.

From the assumption when $n=k$. It's clear that $z^{(k+1)}(1,g)$ is the same value $\forall\ g\in\{2,3,4,\dots\dots\dots,q\},q\geq 2$. Now consider the following relations:

$$Z^{(k+1)}(g,d)\ =\ \left(z^{(k)}(d,d)\right)^3+\left(z^{(k)}(d,1)\right)^3+\sum_{\substack{m\neq d\\ m\neq 1}}\left(z^{(k)}(d,m)\right)^3+$$

$$3\sum_{d\neq m}\left(z^{(k)}(d,m)\right)^2\left(z^{(k)}(d,d)\right)\ +3\sum_{m\neq d}\left(z^{(k)}(d,d)\right)^2\left(z^{(k)}(d,m)\right)$$

$$+3\sum_{l\neq d,}\sum_{\substack{m\neq d\\ l\neq m}}\left(z^{(k)}(d,l)\right)^2\left(z^{(k)}(d,m)\right)\ +6\sum_{u}\sum_{\substack{l\\ l\neq d\\ l\neq u}}\sum_{\substack{m\\ m\neq d\\ m\neq l,m\neq u}}\ (z^{(k)}(d,u))\left(z^{(k)}(d,l)\right)\left(z^{(k)}(d,m)\right)$$

$$Z^{(k+1)}(g,g)=\ c[\,b^3\left(z^{(k)}(g,g)\right)^3+3\,b^2\sum_{m\neq g}\left(z^{(n)}(g,g)\right)^2\left(z^{(n)}(g,m)\right)$$

$$+3\,b\sum_{m\neq g}\left(z^{(n)}(g,m)\right)^2\left(z^{(n)}(g,g)\right)\ +\ 6\,\mathrm{b}\ \sum_{\substack{l\\ l\neq g}}\sum_{\substack{m\\ l\neq m\\ m\neq g}}(z^{(n)}(g,g))(z^{(n)}(g,l))(z^{(n)}(g,m))$$

$$+\sum_{l\neq q}\left(z^{(n)}(g,l)\right)^3\ +\ 3\sum_{l\neq m\neq q}\sum_{m\neq q}\left(z^{(n)}(g,l)\right)^2\left(z^{(n)}(g,m)\ \right)$$

$$+\,6\sum_{u}\sum_{\substack{l\\ l\neq g\\ l\neq u}}\sum_{\substack{m\\ m\neq g\\ m\neq l,m\neq u}}\ (z^{(k)}(g,u))\left(z^{(k)}(g,l)\right)\left(z^{(k)}(g,m)\right)]$$

Then, basing on our assumption , it's clear the equality satisfied for $n=k+1$.

As a result, the remark is proved by induction.

As a result of the relation (111), we can reduce the system (110) into the following system of 4 independent variables :

$$u_1{}^{(n)} = \sqrt[3]{z^{(n)}(1,1,1,1)} = \sqrt[3]{\left(z^{(n)}(1,1)\right)^3} = z^{(n)}(1,1)$$

$$u_2{}^{(n)} = \sqrt[3]{z^{(n)}(1,2,2,2)} = z^{(n)}(1,2)$$

$$u_3{}^{(n)} = \sqrt[3]{z^{(n)}(2,1,1,1)} = z^{(n)}(2,1)$$

$$u_4{}^{(n)} = \sqrt[3]{z^{(n)}(2,2,2,2)} = z^{(n)}(2,2)$$

Assume $u_k = u_k^{(n)}$ and $u_k' = u_k^{(n+1)}$, where $n \geq 1,\ k = 1,2,3,4$.

Then, we establish the following recurrence equations for the Potts model with Hamiltonian (108) with q state space , $q \geq 2$. These equations are:

$$u_{1'} = c\ (b^3 u_1{}^3 + 3\ b^2(q-1)u_2\, u_1{}^2 + 3\ b\ (q-1)^2 u_1\, u_2{}^2 \quad + u_2{}^3((q-1) + 6\binom{q-1}{3} + \ 6\binom{q-1}{2}))$$

$$= \ c\ (b^3 u_1{}^3 + 3\ b^2(q-1)u_2\, u_1{}^2 + 3\ b\ (q-1)^2 u_1\, u_2{}^2 + u_2{}^3(q-1)^3)$$

Then , $u_{1'} = \ c(b\, u_1 + (q-1)u_2)^3$ (113)

$$u_{2'} = \ u_3^3 + 3u_4 u_3^2 + 3u_4^2\, u_3 + 3(q-2)^2 u_4 u_2^2 + (q-2)^3 u_2^3 + u_4^3 + 3(q-2)u_2 u_4^2 + 3(q-2)u_2 u_3^2 + 3(q-2)^2 u_3 u_2^2 + 6(q-2)u_2 u_3 u_4\ .$$

Then, $u_2' = ((q-2)u_2 + u_3 + u_4)^3$ (114)

In the same way, we get

$u_{3'} = (\ u_1 + (q-1)u_2)^3$ (115)

$$u_{4'} = c(u_3^3 + b^3 u_4^3\ + 3b\ u_4\ u_3^2 + 3\ b^2 u_4^2 u_3 + 3(q-2)u_2 u_3^2 + 3(q-2)^2 u_2^2 u_3 + 3b^2\ (q-2)u_2\ u_4^2 + 3\ b\ (q-2)^2 u_2^2 u_4 + (q-2)^3 u_2^3 + 6\ b(q-2)u_2 u_4 u_3\ )$$

Then, $u_{4'} = c\,((q-2)u_2 + u_3 + b\,u_4)^3$ (116)

Now Define the following operator

$$F(u^{(n)}): u^{(n)} = \left(u_1^{(n)}, u_2^{(n)}, u_3^{(n)}, u_4^{(n)}\right) \in {R^+}^4 \rightarrow u^{(n+1)} = \left(u_1^{(n+1)}, u_2^{(n+1)}, u_3^{(n+1)}, u_4^{(n+1)}\right) \in {R^+}^4 \quad (117)$$

Thus, the recurrent equations (113-116) can be rewrite as $u^{(n+1)} = F(u^{(n)}), n \geq 1$.

Let define the operator F on the following shifting-invariant set

$$A_3 = \{ (u_1, u_2, u_3, u_4) \in {R^+}^4 : u_1 = u_4 \text{ and } u_3 = u_2 \},$$

Remark 5.1.1.2.[24, 26] It's known that the set of Gibbs measures that corresponds to solving the operator F (117) on the shifting invariant set A_3 is called Paramagnetic or high symmetry phases.

Thus, to investigate the paramagnetic phases, we reduce the system of equations (113-116) into only two recurrent equations as follows:

$$u_1^{(n+1)} = c(b\,u_1^{(n)} + (q-1)u_2^{(n)})^3 \quad (118)$$

$$u_2^{(n+1)} = (\,u_1^{(n)} + (q-1)u_2^{(n)})^3 \quad (119)$$

Let $x_{n+1} = \frac{u_1^{(n+1)}}{u_2^{(n+1)}}$. Then, the system of equations (118-119) is reduced to the following nonlinear dynamical recurrent equations

$$x_{n+1} = \frac{c\,(b\,x_n + (q-1))^3}{(\,x_n + (q-1))^3},\ n > 0 \quad (120)$$

It's easy to conclude that corresponding limit Gibbs measure is fully determined by fixed points of recurrent Equation (120), i.e., solutions of equation $F(u) = u$. So let define the following function $f : R^+ \rightarrow R^+$ such that

$$f(x) = \frac{(b\,x + (q-1))^3}{(\,x + (q-1))^3} \qquad (121)$$

Then, if there is more than one positive solutions for $f(x) = \frac{x}{c}$, a phase transition exists [4], i.e., there exists more than one paramagnetic phase for our model.

5.1.2 Phase transition

In this section we prove the existence of phase transition by analyzing the fixed points of the function f in equation (121).

It's is clear that $f(x)$ is continuous $\forall\, x > 0$, $f(0) = 1 > 0$ and f is a bounded function since $\lim_{x\to\infty} f(x) = b^3$. As a result , the curve $y = f(x)$ must intersect a line $y = m\,x$, $m = \frac{1}{c}$, $c > 0$. Therefore, this construction provides at least one element of the translation-invariant paramagnetic set of Gibbs measures corresponding to the model (108) for any $x \in R^+$. Now

$$f'(x) = \frac{3\ (b-1)(q-1)(q-1+bx)^2}{(-1+q+x)^4}$$

Note that $f'(x) > 0$ (f is increasing) iff $b > 1$, $q > 1$ and $c > 0$, so we restrict ourselves for the case when $b > 1$.

The second derivative:

$$f''(x) = \frac{6\ (b-1)(q-1)(2-2q+b\,(q-1-x))(-1+q+bx)}{(q-1+x)^5} = 0$$

Then , $\big(2 - 2q + b\,(q-1-x)\big) = 0 \ \longrightarrow\ x_{inf} = (q-1)(1-\frac{2}{b}) > 0$

It's clear that the inflection point $x_{inf} > 0$ iff $b > 2$, so we consider $b > 2$.

For $b > 2$, it's clear that $f''(x) > 0$ (f is concave up) iff $x < x_{inf}$

and $f''(x) < 0$ (f is concave down) iff $x > x_{inf}$

Then, there is at most 3 Positive fixed points for $f(x)$ (121) .

As we know from Preston [50], there is more than one solution for $f(x) = x$ iff there is more than one solution for $x\, f'(x) = f(x)$ which is reduced to the following equation:

$$-\frac{(q-1+bx)^2\left(bx^2-2\,(b-2)(q-1)x+(q-1)^2\right)}{(q-1+x)^4} = 0 \qquad (122)$$

Now we find the conditions for the existence of two positive roots for equation (122).

Consider the quadratic equation $b\,x^2 - 2\,(-2+b)(-1+q)x + (q-1)^2 = 0$. Then there is more than one real solution for this equation iff the discriminant $\Delta > 0$, it's clear that

$\Delta = 4\,(b-2)^2(q-1)^2 - 4\,b\,(q-1)^2 = 4\,(\,q-1)^2\,\,(b^2-3b+4) =$ $4(q-1)^2(b-4)(b+1) > 0$ iff $b > 4$ where $b > 0$.

As a result, under these conditions we have two positive roots for equation (122) which are

$$x_1 = \frac{2-b-2q+bq}{b} - \sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}}$$

$$x_2 = \frac{2-b-2q+bq}{b} + \sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}}$$

$x_1, x_2 > 0$. Thus, we can conclude the following proposition,

Proposition 5.1.2. The equation

$$\frac{x}{c} = \frac{(b\,x+(q-1))^3}{(\,x+(q-1))^3} \qquad (123)$$

with $x > 0$, $c > 0$, $b > 0$ and $q \geq 2$, $q \in N$ has one positive solution if $b < 1$. If $b > 4$ then there exists $\eta_1(b,q)$, $\eta_2(b,q) > 0$ such that equation (123) has three positive roots if $\eta_1(b,q) < \frac{1}{c} < \eta_2(b,q)$ and has two positive solutions if either $\eta_1(b,q) = \frac{1}{c}$ or $\eta_2(b,q) = \frac{1}{c}$, where

$$\eta_1(b,q) = \frac{3(q-1)\left(-1+q+b\left(\frac{2-b-2q+bq}{b}-\sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}}\right)\right)^2(-1+b))}{(-1+q+\frac{2-b-2q+bq}{b}-\sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}})^4}$$

$$\eta_2(b,q) = \frac{3(q-1\,)\left(-1+q+b\left(\frac{2-b-2q+bq}{b}+\sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}}\right)\right)^2(-1+b))}{(-1+q+\frac{2-b-2q+bq}{b}+\sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}})^4}$$

Proof.

Consider the function f (121). If $b < 1$, then f is decreasing and there is only one positive root for equation (123).

If $b > 4$, then f is increasing , changes it's concavity from up to down and there are two positive roots for $x\,f'(x) = f(x)$ in equation (122) such that the roots are x_1 and x_2 that we have founded in the previous page. Then, from Preston [50] we conclude that there are 3 positive roots for equation (123) if $f'(x_1) < \frac{1}{c} < f'(x_2)$ and there are two positive solutions if $f'(x_1) = \frac{1}{c}$ or $f'(x_2) = \frac{1}{c}$. By substituting x_1 and x_2, the proof is readily completed.

Note that the inequality $\eta_1(b,q) < \frac{1}{c} < \eta_2(b,q)$ could be reduced to the following inequality

$$\frac{(-1+q+\frac{2-b-2q+bq}{b}+\sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}})^4}{3(q-1)\left(-1+q+b\left(\frac{2-b-2q+bq}{b}+\sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}}\right)\right)^2(-1+b)} < c <$$
$$\frac{(-1+q+\frac{2-b-2q+bq}{b}-\sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}})^4}{3(q-1)\left(-1+q+b\left(\frac{2-b-2q+bq}{b}-\sqrt{\frac{4-5b+b^2-8q+10bq-2b^2q+4q^2-5bq^2+b^2q^2}{b^2}}\right)\right)^2(-1+b)} \quad (124)$$

From Proposition 5.1.2, we conclude that the phase transition exists if $b > 4$, i.e., $e^{\beta J_t} > 4 \longrightarrow \beta\, J_t > \ln(4)$ but $\beta = \frac{1}{T} \longrightarrow T < \frac{J_t}{\ln(4)}$, $J_t > 0$. Hence, we obtain the following Theorem

Theorem 5.1.2. Let $J_t > 0$ be the coupling constant that corresponds to PTNNN interactions and $T_{cr} = \frac{J_t}{\ln 4}$. If $T \geq T_{cr}$, then the model (108) has unique paramagnetic phase and if $T < T_{cr}$, then there are three translation-invariant paramagnetic Gibbs measures for the model , i.e., The phase transition exists.

Remark 5.1.2. If we take the limit of the function f in equation (121) as $q \to \infty$, i.e., $\lim_{q\to\infty} \frac{(b\,x+(q-1))^3}{(\,x+(q-1))^3} = 1$. Then there is only one solution for equation (123) and the solution is $x = c$. As a result, the paramagnetic Gibbs measure in this case is unique and the phase transition does not exist.

We consider some numerical examples for the existence of phase transition related to the model (108). In addition, we consider different values for the state space $q \geq 2$ and show the effect of increasing the number of states on phase transition.

Example 5.1.3

Consider $q = 6$ and $J_t = 2$. Then $T_{cr} = \frac{2}{\ln 4} = 1.4427,$ so phase transition exists if $T < 1.4427$. Let $T = 1.11622$, then $b = e^{\frac{2}{1.11622}} = 6$. Thus, from proposition 5.1.2 and inequality (124) we conclude that there are 3 phases (phase transition exists) if

$$0.115871 < c < 0.166479$$

Let $c = 0.12$ ($J = -2.36668$). Then, there are 3 solutions for the equation $f(x) = \frac{x}{c}$, and these solutions are

$$x = 0.2064714\,, x = 3.9056299 \text{ and } x = 8.8983138$$

As a result, there are 3 translation invariant paramagnetic Gibbs measures corresponding to these roots. See Figure 5.1.3.

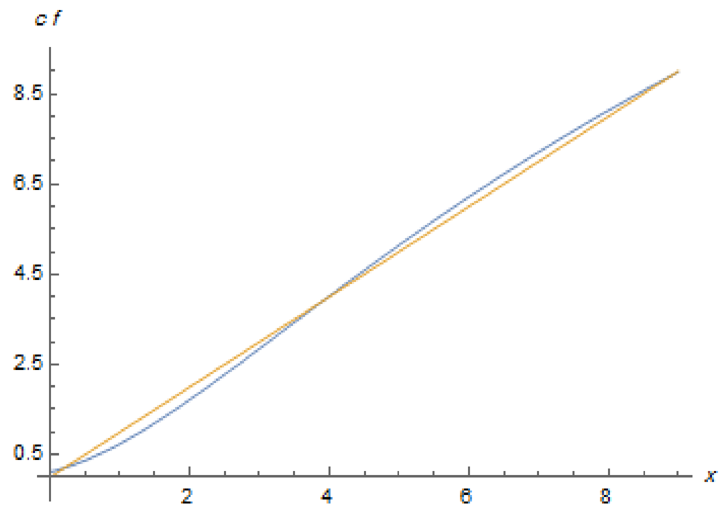

Figure 5.1.3. The existence of 3 translation-invariant paramagnetic phases when $q = 6, T = 1.11622$, $J_t = 2$ and $J = -2.36668$.

Now for the same value of b, i.e., for $b = 6$, we consider different values for the state space q, $q \geq 2$ and see the effect of increasing the number of state space on the values of c where the phase transition exists as the follows:

For $q = 3$ and according to proposition 5.1.2, there exists 3 positive roots for equation (123) if $0.0463484 < c < 0.0665916$

For $q = 4$, then $0.069522 < c < 0.0998875$

For $q = 5$, then $0.0847529 < c < 0.1331832$

For $q = 6$, then $0.1158710 < c < 0.1664791$

For $q = 7$, then $0.1390453 < c < 0.1997750$

For $q = 8$, then $0.1622196 < c < 0.2330708$

For $q = 100$, then $2.2942482 < c < 3.2962868$ ....

For $q = 1000$, then $23.15104991 < c < 33.2625303$

Now we take different value for b, suppose $b = 4.5$. Again, we try the following cases :

For $q = 3$, $0.0958231 < c < 0.1017976$

For $q = 4$, $0.1437354 < c < 0.1526965$

For $q = 5$, $0.1916469 < c < 0.2035955$ ....

Note that the length of the interval (124) is $L(b,q) = \frac{4(b-4)(8+b)\sqrt{\frac{(4-5b+b^2)(-1+q)^2}{b^2}}}{27\, b^3}$, and it's clear that for a fixed value of b, the length of the interval increases as q increases. In addition, the derivative for the left hand side for the interval (124) with respect to q is

$$d(p,q)=\frac{\left(2-2q+b\left(-1+\sqrt{\frac{(4-5b+b^2)(-1+q)^2}{b^2}}+q\right)\right)\left(2-2q+b\left(-2+\sqrt{\frac{(4-5b+b^2)(-1+q)^2}{b^2}}+2q\right)\right)^3}{b^4(-1+q)\left(1-q+b\left(-1+\sqrt{\frac{(4-5b+b^2)(-1+q)^2}{b^2}}+q\right)\right)^3}$$

Note that $d(p,q)>0, \forall b>4\ \delta\ q\geq 2$, i.e., the value of the interval starting point increases when q value increases. So we conclude the following remark :

Remark 5.1.3. For a fixed values of the coupling constants J_t and J for the model (108). Then, an increase in the value of q (state space) increases the phase transition range (paramagnetic region), but the value of the critical temperature decreases.

5.1.4 Periodic Gibbs Measure

It's known that investigating the existence of periodic points is one from the interesting problems in non-linear dynamic systems [27]. In statistical physics, these periodic points reveal the phase types corresponding to the given model [1]. In this section, we prove the existence of 2-periodic Gibbs measures for our model (108).

Definition5.1.4. [1] A point $u=(u_1,u_2,u_3,u_4)\in$ R^{+4} is called a periodic fixed point of the operator F if there exists $p\in N$ such that $F^p\,(u)=u$ where F^p is the p^{th} iterate of F.

It's known that the smallest positive integer p satisfying the above is called the prime period or least period of the point u. The set of periodic points with prime period p denoted by $Per_p(F)$.

Let describe the periodic points with $p=2$ for operator F (117). Then, in this case the equation $F(F(u))=u$ can be reduced to a description of 2-periodic points of the function f defined previously (121), i.e., to a solution of the equation $f(f(x)) = x$.

Remark5.1.4. [1] The 2-periodic positive fixed point of the operator F(117) defined on the set A_3 described a paramagnetic periodic Gibbs measure with p=2.

Consider $f(x) = \frac{c\,(b\,x+(q-1))^3}{(\,x+(q-1))^3}$. It's clear that the solutions for $f(x) = x$ are trivially a solutions for the equation $f(f(x)) = x$. Thus to exclude this case, i.e., to find $Per_2(F)$ we will consider the equation:

$$\frac{f(f(x)) - x}{f(x) - x} = 0 \qquad (125)$$

Hence, the solution for this equation are the fixed points with prime period =2, i.e., we will get the set $M = \{\,(u_1, u_2, u_3, u_4) \in \mathrm{R}^{+4} \;:\; f(f(x)) = x\,, f(x) \neq x\}$.

Now equation (125) is reduced to

$$\frac{1}{\left((-1+q)(-1+q+x)^3 + c(-1+q+bx)^3\right)^3}(-1+q+x)^3(15b^7c^3(-1+q)^2x^4 + b^9c^3x^6 + 6b^8x^5c^3(-1+q)$$

$$3b^2c(-1+q)^3(x(-1+q+x)^3(-1+q+3x) + c(-1+q)(-1+q^3+3q(-1+x)^2+3x-3x^2-4x^3+3q^2(-1+x)))$$

$$+3bc(-1+q)^4(1+q^4-6x+2(6+c)x^2-10x^3+3x^4+2q(-2+9x-(12+c)x^2+5x^3)+6q^2(1-3x$$

$$+2x^2)+q^3(-4+6x)) + (-1+q)^3(1+q^6+(-6-3c+c^2)x+5(3+2c)x^2-4(5+3c)x^3+3(5+2c)x^4-(6+c)x^5$$

$$+x^6+3q[-2+(10+4c-c^2)x-10(2+c)x^2+4(5+2c)x^3-2(5+c)x^4+2x^5]+3q^2[5+(-20-6c+c^2)x$$

$$-10(2+c)x^2+4(5+2c)x^3-2(5+c)x^4+2x^5)+3q^2(5+(-20-6c+c^2)x+10(3+c)x^2-4(5+c)\;x^3+5x^4)$$

$$+q^3(-20+(60+12c-c^2)x-10(6+c)x^2+20x^3]+3q^4(5-(10+c)x+5x^2)++q^3(-20+(60+12c-c^2)x$$

$$-10(6+c)x^2+20x^3)+3q^4(5-(10+c)x+5x^2)+6q^5(-1+x])+b^3c(-1+q)^2(c^2(-1+q)^4+x^2\quad(-1+q+x)^3$$

$$(-1+q+3x)+10c(-1+q)x(-1+q^3+3q(-1+x)^2+3x-3x^2-x^3+3q^2(-1+x)))+3b^4c^2x\quad(2c(-1+q)^5$$

$$-(-1+q)^2x\left(4-4q^3-12q(-1+x)^2-12x+12x^2+x^3-12q^2(-1+x)\right))+3b^5c^2x^2(5c(-1+q)^4+2(-1+q)^2x$$

$$(1+q^2-3x+3x^2+q(-2+3x)))+b^6c^2(-1+q)x^3(20c(-1+q)^2+x(1+q^2-3x+3x^2+q(-2\quad+3x+q(-2+3x))))$$

Since the previous equation is complicated, we use Mathematica [57] to investigate the conditions for the existence of more than one positive root , and the results that there are two positive roots (two periodic Gibbs measures with $p = 2$) if

$$0 < b < \frac{1}{4} \text{ and } \omega_1(b,q) < c < \omega_2(b,q),\ q \geq 2 \text{ where} \tag{126}$$

$$\omega_1(b,q) = -\frac{8 + 24b(-1+q) + 3b^2(-1+q) + 8b^3(-1+q) + 4b^4\sqrt{\frac{(-1+b)^3(2+b)^2(-1+4b)(-1+q)^2}{b^8}} - 8q}{27b^4}$$

$$\omega_2(b,q) = \frac{-8 - 24b(-1+q) - 3b^2(-1+q) - 8b^3(-1+q) + 4b^4\sqrt{\frac{(-1+b)^3(2+b)^2(-1+4b)(-1+q)^2}{b^8}} + 8q}{27b^4}$$

Thus we have proved the following

Proposition 5.1.4. The equation (125) has no positive solutions if $b \geq \frac{1}{4}$, and if $0 < b < \frac{1}{4}$ then there exist $\omega_1(b,q)$, $\omega_2(b,q)$ such that the equation has two positive solutions if $\omega_1(b,q) < c < \omega_2(b,q)$ and $q \geq 2$.

Theorem 5.1.4. Let $J_t < 0$, then there exists a critical temperature $T_{cr} = \frac{J_t}{\ln(0.25)} > 0$ such that if $T < T_{cr}$ and the conditions of the proposition5.4.1 are satisfied, then there are two periodic Gibbs measures with p=2 for the model (108), otherwise there is no periodic phases with p=2, i.e., no phase transition exists for period 2 paramagnetic phases.

Example 5.1.4

Let $q = 3$ and $b = \frac{1}{8}$. Then from (126) there are two periodic Gibbs measures with period 2 if

$$5.4783715 < c < 2990.6697767 \ \rightarrow \ 5.4783715 < e^{\frac{J}{T}} < 2990.6697767$$

$$\rightarrow \ 0 < \frac{J}{\ln(5.4783715)} < T < \frac{\mathrm{J}}{\ln(2990.6697767)}$$

Let $c = 6$, then the fixed points for equation (125) are $\{0.8173582, 2.4924612\}$,

and these points corresponds to periodic Gibbs measures with period 2. See Figure 5.1.4.

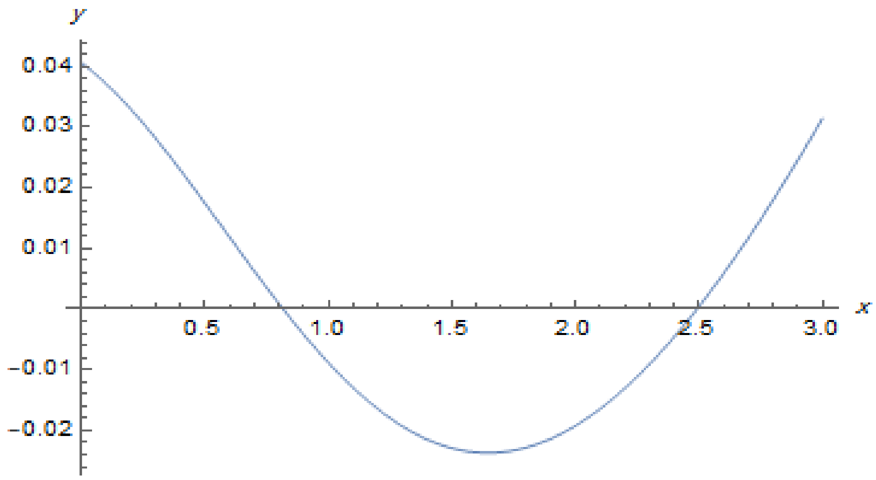

Figure 5.1.4. The existence of two periodic Gibbs measures (two positive roots for equation (125)) when $b = \frac{1}{8}$, $c = 6$ and $q = 3$.

5.2 A 3-State Potts Model with Next-Nearest-Neighbor Ternary and One-level Next-Nearest-Neighbor Interactions on a Second-Order Cayley Tree

In this section, we consider a Potts model with NN, TPNNN and OLNNN interactions on a Cayley tree of order 2. As we have done in section 5.1, we prove analytically for the first time the existence of phase transition for this model.

5.2.1 Model Construction

Consider a Cayley tree of order 2, let $W_1 = \{x_1, x_2\}$ and let $l_1 =< x^{(0)}, x_1 >$ and $l_2 =< x^{(0)}, x_2 >$ be 2 edges emanating from $x^{(0)}$. Then this semi-infinite Cayley tree Γ_+^2 could be splits into 2 single-trunk Cayley trees $\Gamma_+^2\,(l_i)$, $i = 1, 2$.

Let V^{l_i} be the set of vertices of a single-trunk Cayley tree $\Gamma_+^2\,(l_i),\ i = 1,\ 2,$ and $V_n^{l_i} = V^{l_i} \cap V_n$ be the set of vertices $x \in V^{l_i}$ with $d\big(x, x^{(0)}\big) \leq$ n. See Figure 5.2.1.

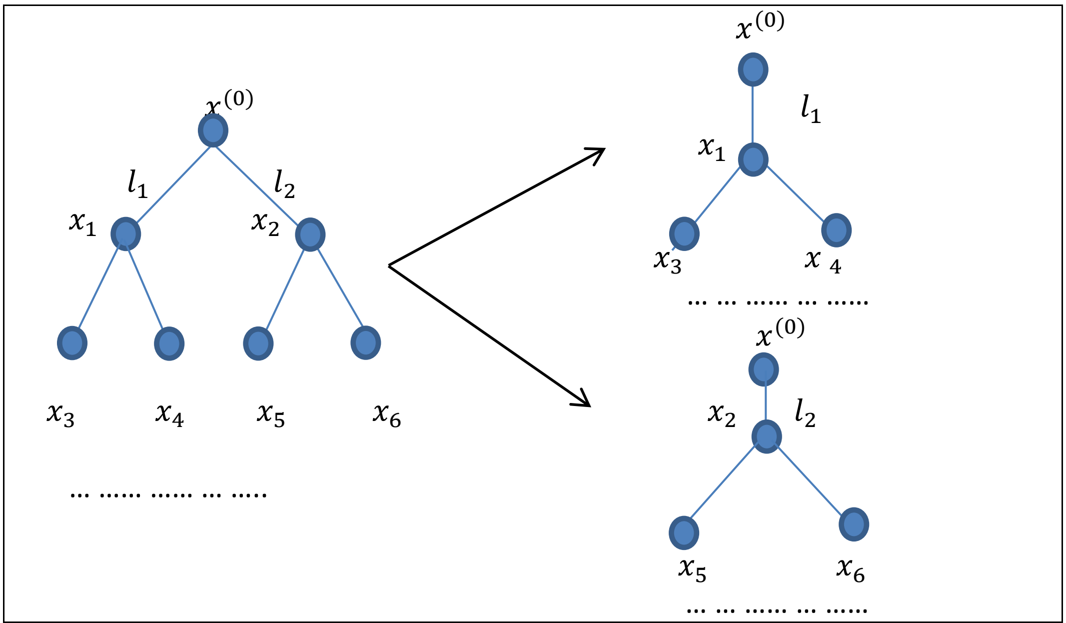

Figure 5.2.1. The semi-infinite Cayley tree Γ_+^2 decomposes into two single trunk Cayley trees : $\Gamma_+^2\,(l_1)$ and $\Gamma_+^2\,(l_2)$.

Consider the Potts model with spin values in $\Phi = \{1, 2, 3\}$, i.e., the spin variables $\sigma(x)$, $x \in V$, take these values.

Then, the relevant Hamiltonians with competing NN interactions J, TPNNN interactions J_t and OLNNN interactions J_{OL} have the forms

$$H(\sigma) = -J_t \sum\nolimits_{>x,y,z<} \delta_{\sigma(x)\sigma(y)\sigma(z)} - J \sum\nolimits_{<x,y>} \delta_{\sigma(x)\sigma(y)} - J_{oL} \sum_{\widetilde{<x,y>}} \delta_{\sigma(x)\sigma(y)} \quad (127)$$

Where $J_t, J, J_{oL} \in R$ are coupling constant, δ in the second and third sum is the usual Kronecker symbol and δ in the first sum is the generalized Kronecker symbol.

In the same way as the model in section 5.1, we fix the boundary condition $\bar{\sigma}(V \setminus V_n) = 1, \ \forall \, n > 0$ and then build the recurrence equations for the conditional partition functions under this boundary condition .

Now we use partition function method to construct a relation between the partition function on the ball V_{n+1} to the partition function on its subsets V_n. Given the initial conditions on V_1, the recurrence equations indicate how their influence propagates down the tree.

Then, it can be easily proved that under the interactions in (127) and with the boundary condition $\overline{\sigma_n} \equiv 1$, the partition functions in volume ${V_n}^{l_i}, \ i = 1, \ 2,$ satisfy the following relations :

$$z^{(n)}(1{,}2) = z^{(n)}(1{,}3) = z^{(n)}(2{,}3) = z^{(n)}(3{,}2)$$

$$z^{(n)}(2{,}1) = z^{(n)}(3{,}1)$$

$$z^{(n)}(2{,}2) = z^{(n)}(3{,}3)$$

The proof for these relations could be done in the same way as Remark 5.1.1.1 with fixing the value of q to be 3.

Assume $d = e^{\beta J_{oL}}, \ b = e^{\beta J_t}, \ c = e^{\beta J}$ and

$$u_1^{(n)} = \sqrt[3]{z^{(n)}(1{,}1{,}1)} = z^{(n)}(1{,}1)$$

$$u_2^{(n)} = \sqrt[3]{z^{(n)}(1{,}2{,}2)} = z^{(n)}(1{,}2)$$

$$u_3^{(n)} = \sqrt[3]{z^{(n)}(2{,}1{,}1)} = z^{(n)}(2{,}1)$$

$$u_4^{(n)} = \sqrt[3]{z^{(n)}(2{,}2{,}2)} = z^{(n)}(2{,}2)$$

After a direct calculation, we get the following recurrence equations:

$$u_1^{(n+1)} = c\left(d\, b^2 u_1^{(n)^2} + 4\, b\, u_1{}^{(n)}\, u_2{}^{(n)} + 2\,(d+1)\, u_2^{(n)^2}\right) \tag{128}$$

$$u_2^{(n+1)} = \left(d\, u_3^{(n)^2} + 2\, u_2{}^{(n)}\, u_3{}^{(n)} + 2\, u_4{}^{(n)}\, u_3{}^{(n)} + 2\, u_4{}^{(n)}\, u_2{}^{(n)} + \; d\, u_4^{(n)^2} + d\, u_2^{(n)^2}\right) \tag{129}$$

$$u_3^{(n+1)} = \left(d\, u_1^{(n)^2} + 4\, u_1{}^{(n)}\, u_2{}^{(n)} + 2(d+1) u_2^{(n)^2}\right) \tag{130}$$

$$u_4^{(n+1)} = c\left(d\, u_3^{(n)^2} + 2\, u_3^{(n)^2}\, u_2{}^{(n)} \; + 2\, b\, u_3{}^{(n)}\, u_4{}^{(n)} + 2\, b\, u_2{}^{(n)}\, u_4{}^{(n)} \; + d\, u_2^{(n)^2} \; + b^2\, d\, u_4^{(n)^2}\right) \tag{131}$$

We define an operator F (117) on the system of recurrent equations (128-131) such that this system can be rewritten as $u^{(n+1)} = F\left(u^{(n)}\right),\ n \geq 1$. We define this operator on the following shifting invariant set:

$$A = \{\, (u_1, u_2, u_3, u_4) \in R^{+^4} :\ u_1 = u_4 \ \text{ and } \ u_3 = u_2 \}\,,$$

From Remark 5.1.1.2, we conclude that the set of Gibbs measures that corresponds to solving the operator F (117) on the shifting invariant set A are called Paramagnetic Shifting-Invariant Phases.

Therefore, to investigate the paramagnetic phases, we will reduce the system of equations (128-131) into only two recurrence equations by defining the operator F on the set A. The equations are:

$$u_1{}^{(n+1)} = c\left(d\, b^2 u_1^{(n)^2} + 4\, b\, u_1{}^{(n)}\, u_2{}^{(n)} + 2\,(d+1) u_2^{(n)^2}\right) \tag{132}$$

$$u_2{}^{(n+1)} = \left(d\, u_1^{(n)^2} + 4\, u_1{}^{(n)}\, u_2{}^{(n)} + 2(d+1) u_2^{(n)^2}\right) \tag{133}$$

Let $x_{n+1} = \frac{u_1{}^{(n+1)}}{u_2^{(n+1)}}$. Then, the system of equations (132-133) is reduced to the following nonlinear dynamical system of recurrent equation

$$x_{n+1} = \frac{\mathrm{c}\left(\mathrm{d}\, b^2 x_n{}^2 + 4\, b\, x_n + 2(d+1)\right)}{(2(d+1) + d\, x_n{}^2 + 4 x_n)}\ ,\ n \geq 1 \tag{134}$$

In order to determine the corresponding limiting Gibbs measure , we find the fixed points of recurrence equation (134) , i.e., solutions of equation $F(u) = u$. So, define the following function $f : R^+ \to R^+$ such that

$$f(x) = \frac{c(d\, b^2 x^2 + 4\, b\, x + 2(d+1))}{(2(d+1) + d\, x^2 + 4x)} \qquad (135)$$

Then, if there is more than one positive fixed point for $f(x)$, a phase transition exists [1], i.e., there is more than one translation invariant paramagnetic phase for our model.

5.2.2 Phase Transition

In this section, we prove the existence of phase transition for Potts model with the Hamiltonian (127) by analyzing the fixed points of the function f (135). It's clear that $f(x)$ is continuous $\forall\, x > 0$, $f(0) = c > 0$ and f is a bounded function. As a result, the curve $y = f(x)$ must intersect the line $y = x$. Therefore, this construction provides at least one element of the translation-invariant paramagnetic set of Gibbs measures, corresponding to the model (127) for any $x \in R^+$. Let start with the first derivative, for $x > 0$:

$$f'(x) = \frac{4c\left(b^2 dx(1+d+x) - (1+d)(2+dx) + b\big(2 - d(-2+x^2)\big)\right)}{\big(2+4x+d(2+x^2)\big)^2}$$

$$= \frac{4\, c\, (d\, b(b-1)x^2 + d(d+1)(b^2-1)x + 2(1+d)(b-1)}{(2(d+1) + d\, x^2 + 4x)^2}$$

Notice that $f'(x) > 0$ (f is increasing) iff $b > 1, d > 0$ and $c > 0,$ so we restrict ourselves for the case when $b > 1$.

Now, the second derivative is:

$$f''(x) = \frac{4c(2b(-8-2d(4+3x)+d^2x(-6+x^2))+(1+d)(16+2d(-1+6x)+d^2(-2+3x^2))+b^2d(2+d^2(2-3x^2)+d(4-3x^2-2x^3)))}{(2+4x+d(2+x^2))^3} = 0$$

Then, there is an inflection point $x_{inf} > 0$ such that

$$f''(x) > 0 \text{ (} f \text{ is concave up)} \quad \text{if} \quad x < x_{inf}$$

$$\text{and } f''(x) < 0 \text{ (} f \text{ is concave down) if } x > x_{inf}$$

Iff $\left(0 < d \leq 1.5615528 \text{ and } b > \frac{8-d-d^2}{d+d^2}\right)$ or $(d > 1.5615528 \text{ and } b > 1)$

According to Preston [50], there is more than one solution for $f(x) = x$ iff there is more than one solution for $x\,f'(x) = f(x)$, which is reduced to the following equation:

$$-\frac{c(4+8d+4d^2+16(1+d)x+(16b+6d-2b^2d+6d^2-2b^2d^2)x^2+8bdx^3+b^2d^2x^4)}{(2+2d+4x+dx^2)^2} = 0 \qquad (136)$$

Then from [57], we conclude that there are two positive roots for the previous equation iff

$b > 2\sqrt{2}\left(\frac{4+3d+3d^2+\sqrt{2+3d+3d^2}}{d(1+d)}\right)$ and the roots are

$$x_1 = \frac{-4+\sqrt{2}\sqrt{(-1+b)(-8+(3+b)d+(3+b)d^2)}}{2bd} - \frac{1}{2}\sqrt{\frac{32-6d-6d^2+2b^2d(1+d)-4b(4+d+d^2)-8\sqrt{2}\sqrt{(-1+b)(-8+(3+b)d+(3+b)d^2)}}{b^2d^2}}$$

$$x_2 = \frac{-4+\sqrt{2}\sqrt{(-1+b)(-8+(3+b)d+(3+b)d^2)}}{2bd} - \frac{1}{2}\sqrt{\frac{32-6d-6d^2+2b^2d(1+d)-4b(4+d+d^2)-8\sqrt{2}\sqrt{(-1+b)(-8+(3+b)d+(3+b)d^2)}}{b^2d^2}}$$

It's clear that $2\sqrt{2}\left(\frac{4+3d+3d^2+\sqrt{2+3d+3d^2}}{d(1+d)}\right) > \frac{8-d-d^2}{d+d^2} > 1, \forall\, d > 0.$

Then, we can get the following proposition

Proposition 5.2.2. The equation

$$x = \frac{\mathrm{c}\left(\mathrm{d}\, b^2x^2+4\, b\, x+2(d+1)\right)}{(2(d+1)+d\, x^2+4x)} \tag{137}$$

with $x > 0,\ c > 0,\ b > 0$ and $d > 0$ has one positive solution if $\mathrm{b} < 1$. If $\mathrm{b} > 2\sqrt{2}\left(\frac{4+3d+3d^2+\sqrt{2+3d+3d^2}}{d(1+d)}\right)$, then there exists $\eta_1(b,d,c)$, $\eta_2(b,d,c) > 0$ such that equation (137) has three positive roots if $\eta_1(b,d,c) < 1 < \eta_2(b,d,c)$ and has two positive solutions if either $\eta_1(b,d,c) = 1$ or $\eta_2(b,d,c) = 1$, where $\eta_1(b,d,c) = f'(x_1^*)$, $\eta_2(b,d,c) = f'(x_2^*)$ and

$$x_1{}^* = \frac{-4+\sqrt{2}\sqrt{(-1+b)(-8+(3+b)d+(3+b)d^2)}}{2bd} - \frac{1}{2}\sqrt{\frac{32-6d-6d^2+2b^2d(1+d)-4b(4+d+d^2)-8\sqrt{2}\sqrt{(-1+b)(-8+(3+b)d+(3+b)d^2)}}{b^2d^2}}$$

$$x_2{}^* = \frac{-4+\sqrt{2}\sqrt{(-1+b)(-8+(3+b)d+(3+b)d^2)}}{2bd} + \frac{1}{2}\sqrt{\frac{32-6d-6d^2+2b^2d(1+d)-4b(4+d+d^2)-8\sqrt{2}\sqrt{(-1+b)(-8+(3+b)d+(3+b)d^2)}}{b^2d^2}}$$

Proof.

Consider the function f (135), if $b < 1$. Then f is decreasing and there is only one positive root for equation (137).

If $b > 2\sqrt{2}\left(\frac{4+3d+3d^2+\sqrt{2+3d+3d^2}}{d(1+d)}\right)$, then f is increasing , changes it's concavity from up to down around x_{inf} and there are two positive roots for $x\, f'(x) = f(x)$ in equation (136), the roots are $x_1{}^*$ and $x_2{}^*$, then from Preston [50] it's clear that there is 3 positive roots for equation (137) if $f'(x_1{}^*) < 1 < f'(x_2{}^*)$ and there is two positive solutions if $f'(x_1{}^*) = 1$ or $f'(x_2{}^*) = 1$. The proof is readily completed.

From Proposition 5.2.2, if $b > 2\sqrt{2}\left(\frac{4+3d+3d^2+\sqrt{2+3d+3d^2}}{d(1+d)}\right)$, then by substituting $b = e^{\beta J_t}$ and $d = e^{\beta J_{OL}}$, we get

$$e^{\beta J_t} > 2\sqrt{2}\left(\frac{4+3\,e^{\beta J_{OL}}+3e^{2\beta J_{OL}}+\sqrt{2+3e^{\beta J_{OL}}+3e^{2\beta J_{OL}}}}{e^{\beta J_{OL}}(1+e^{\beta J_{OL}})}\right), \quad \text{but } \beta = \frac{1}{T} > 0$$

$$\longrightarrow \qquad T < J_t \frac{1}{\ln(2\sqrt{2}\left(\frac{4+3\,e^{\frac{J_{OL}}{T}}+3e^{\frac{2J_{OL}}{T}}+\sqrt{2+3e^{\frac{J_{OL}}{T}}+3e^{\frac{2J_{OL}}{T}}}}{e^{\frac{J_{OL}}{T}}\left(1+e^{\frac{J_{OL}}{T}}\right)}\right))}$$

Then, we obtain the following theorem:

Theorem 5.2.2. Let $J_t > 0$, $J_{OL} \in R$ be the coupling constants that corresponds to PTNNN and OLNNN interactions respectively, and let $T > 0$ be the temperature. Then if

$$T < J_t \frac{1}{\ln(2\sqrt{2}\left(\frac{4+3\,e^{\frac{J_{OL}}{T}}+3e^{\frac{2J_{OL}}{T}}+\sqrt{2+3e^{\frac{J_{OL}}{T}}+3e^{\frac{2J_{OL}}{T}}}}{e^{\frac{J_{OL}}{T}}\left(1+e^{\frac{J_{OL}}{T}}\right)}\right))}$$

there are three translation-invariant paramagnetic Gibbs measures, i.e., there exists a phase transition. Otherwise, the phase transition does not exist.

We give a numerical example on phase transition for model (127) according to Theorem 5.2.2 and proposition 5.2.2.

Example 5.2.3

Let $d = 2$, then according to proposition 5.2.2, the phase transition exists if $b > 5.77485177$.

Assume $b = 5.8$, then, $x_1^* = 0.664086398$ and $x_2^* = 0.778876078$.

As a result , there are 3 positive roots for equation (137) if

$$f'(0.664086398\ ) < 1 < f'(0.778876078)$$

$$\longrightarrow \quad 8.06371795\, c < 1 < \quad 8.065427602\, c$$

$$\longrightarrow \quad 0.123985988 < c < 0.124012274$$

Let $c = 0.1240022$, then the three fixed points (phases) are

$$x = 0.63061104\,,\ x = \ 0.71025045\ \text{ and } x = 0.83057251$$

We conclude that there are 3 translation-invariant paramagnetic Gibbs measures corresponds to these positive roots. See Figure 5.2.3.

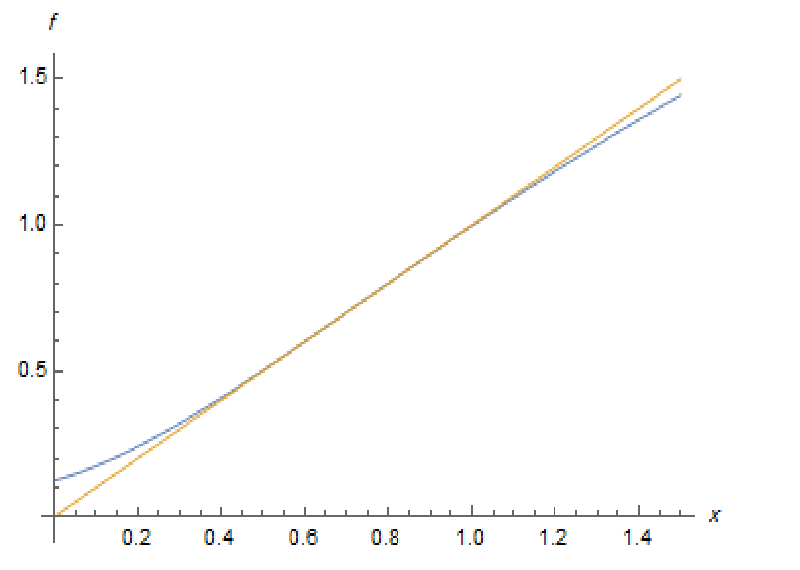

Figure 5.2.3. The existence of 3 translation invariant paramagnetic phases for model (127) when $d = 2, b = 5.8$ and $c = 0.1240022$.

5.3 A 3-State Potts Model with Next-Nearest-Neighbor Ternary and One-Level 3-Tuple Interactions on a Third-Order Cayley Tree

In this Section, we develop the model in section 5.2 such that we consider Potts model with NN interactions, TPNNN interactions and OL 3-tuple interactions on a Cayley tree of order 3, see definition 2.2.12. As we have done in section 5.1 and 5.2, we prove the existence of phase transition analytically for this model.

5.3.1 Model Construction

Consider a Cayley tree of order 3, Let $W_1 = \{x_1,\ x_2, x_3\}$ and let $l_1 =< x^{(0)}, x_1 >$, $l_2 =< x^{(0)},\ x_2 >$ and $l_3 =< x^{(0)},\ x_3 >$ be 3 edges emanating from $x^{(0)}$. Then this semi-infinite Cayley tree Γ_+^3 could be splits into 3 single-trunk Cayley trees $\Gamma_+^3\ (l_i),\ i\ = 1, 2, 3$. Then V^{l_i} are the set of vertices of a single-trunk Cayley tree $\Gamma_+^3\ (l_i), i = 1, 2, 3$.

Consider that model with spin values in $\Phi = \{1, 2, 3\}$. Then, the relevant Hamiltonians with competing NN interactions J, TPNNN interactions J_t and OL 3-tuple interactions J_O have the forms

$$H(\sigma) = -J_t \sum_{>x,y,z<} \delta_{\sigma(x)\sigma(y)\sigma(z)} - J \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)} - J_o \sum_{\substack{y_i \in S(x) \\ i=1,2,3}} \delta_{\sigma(y_1)\sigma(y_2)\sigma(y_3)} \qquad (138)$$

Where $J_t, J, J_o \in R$ are coupling constant.

In the same way as the models in section 5.1 and 5.2, we will fix the boundary condition $\bar{\sigma}(V \setminus V_n) = 1, \forall\, n > 0$ and then build the recurrence equations for the conditional partition functions under this boundary condition.

Remark 5.3.1. We can fixed the boundary condition for any value from state space Φ , i.e., $\bar{\sigma}(V \setminus V_n) = i$, $i = 1, 2, 3$ and the results will be the same for all $i \in \Phi$.

As we have done before, we use partition function method to construct a relation between the partition function on the ball V_{n+1} to the partition function on its subsets V_n. Now it can be easily proved that under the interactions in (138) and with the boundary condition $\overline{\sigma_n} \equiv 1,\ n \geq 1$. Then, the partition functions in volume ${V_n}^{l_i},\ i = 1\ ,2\ ,3$ satisfied the following relations :

$$\left.\begin{aligned} &z^{(n)}(1,2) = z^{(n)}(1,3) = z^{(n)}(2,3) = z^{(n)}(3,2) \\ &z^{(n)}(2,1) = z^{(n)}(3,1) \\ &z^{(n)}(2,2) = z^{(n)}(3,3) \end{aligned}\right\} \qquad (139)$$

The proof for these relations could be done in the same way as Remark 5.1.1.1.

Assume $d = e^{\beta J_o}, b = e^{\beta J_t},\ c = e^{\beta J}$ and

$$ {u_1}^{(n)} = \sqrt[3]{z^{(n)}(1,1,1,1)} = z^{(n)}(1,1)$$

$$ {u_2}^{(n)} = \sqrt[3]{z^{(n)}(1,2,2,2)} = z^{(n)}(1,2)$$

$$ {u_3}^{(n)} = \sqrt[3]{z^{(n)}(2,1,1,1)} = z^{(n)}(2,1)$$

$$ {u_4}^{(n)} = \sqrt[3]{z^{(n)}(2,2,2,2)} = z^{(n)}(2,2)$$

Then, we know that the construction of conditional partition function for Hamiltonian (137) under the condition $\bar{\sigma}(V \setminus V_n) = 1$ is defined by

$$Z_{V_n}(\bar{\sigma}(V \setminus V_n) = 1) = \sum_{\sigma(V_n) \in \Phi^{V_n}} exp(-\beta\ H(\sigma(V_n)|\bar{\sigma}(V \setminus V_n) = 1)) \qquad (140)$$

Where

$$H(\sigma(V_n)|\bar{\sigma}(V \setminus V_n) = 1) = -J_t \sum_{\substack{>x,y,z< \\ x,y,z \in V_n}} \delta_{\sigma_n(x)\sigma_n(y)\sigma_n(z)} - J \sum_{\substack{<x,y> \\ x,y \in V_n}} \delta_{\sigma_n(x)\sigma_n(y)}$$

$$- J_o \sum_{\substack{y_i \in S(x) \\ i=1,2,3 \\ x \in V_{n-1}}} \delta_{\sigma(y_1)\sigma(y_2)\sigma(y_3)} - J_t \sum_{\substack{>x,y,z< \\ x,y \in V_n \\ z \in V \setminus V_n}} \delta_{\sigma_n(x),\sigma_n(y),1} - J \sum_{\substack{<x,y> \\ x \in V_n \\ y \in V \setminus V_n}} \delta_{\sigma_n(x),1} \qquad (141)$$

Then, from (139-141) and after a direct calculations we get the following recurrence equations for the $\overline{\sigma_n} \equiv 1$ boundary conditions

$$u_1' = c(b^3\, d\, u_1^3 + 12\, b\, u_1\, u_2{}^2 + 6\, b^2 u_1^2\, u_2 + (\,2d\; + 6) u_2{}^3)$$

$$u_2' = (d\, u_3^3 + 3\, u_4\, u_3{}^2 + 3\, u_2\, u_3{}^2 + 3\, u_4{}^2\, u_3 + 3\, u_2{}^2\, u_3 + 6\, u_3\, u_4\, u_2 + 3\, u_4^2\, u_2 + 3\, u_4\, u_2^2 + d\, u_4^3 + d\, u_2^3)$$

$$u_3' = (d\, u_1^3 + 6\, u_1^2\, u_2 + 12\, u_2^2\, u_1 + (6\; + 2\, d)\, u_2^3)$$

$$u_4' = c(d\, b^3 u_4^3 + d\;\, u_2{}^3 + d\, u_3^3\; + 3\, b\, u_3{}^2\, u_4 + 3\, b^2\, u_4{}^2\, u_3 + 3\, b^2\, u_2\, u_4{}^2\; + 3\, u_2\, u_3{}^2 + 3\, u_2{}^2 u_3 + 6\, b\, u_4\, u_2\, u_3 + 3\mathrm{b} u_4 u_2^2)$$

Where $u_s = u_s^{(n)}$ and $u_s' = u_s^{(n+1)},\ s = 1, 2, 3, 4.$

In order to get the shifting-invariant paramagnetic phases for this model, we define the operator F (117) on the previous set of recurrence equations. In the same way as the previous sections (5.1 and 5.2), this operator will be restricted on the shifting-invariant set

$$A = \{\, (u_1, u_2, u_3, u_4) \in R^{+4} :\ u_1 = u_4 \ \text{ and } \ u_3 = u_2\}$$

As a result, to investigate the paramagnetic phases, we reduce the system of recurrence equations into only two recurrent equations by defining the operator F on the set A. which trivially reduced to the following nonlinear dynamical system of recurrence equation:

$$x_{n+1} = \frac{c\ (b^3\ d\ x_n{}^3 + 6\ b^2 x_n{}^2 + 12\ b\ x_n + (\ 2d\ + 6)\ )}{(\ d\ x_n{}^3 + 6\ x_n{}^2\ + 12\ x_n\ + (6\ + 2\ d\ )\ )}$$

Where $x_{n+1} = \frac{u_1{}^{(n+1)}}{u_2^{(n+1)}}$. Define the function $f : R^+ \to R^+$

$$f(x) = \frac{c\ (b^3\ d\ x^3 + 6\ b^2 x^2 + 12\ b\ x + (\ 2d\ + 6)\ )}{(\ d\ x^3 + 6\ x^2\ + 12\ x\ + (6\ + 2\ d\ )\ )} \qquad (142)$$

In order to determine the corresponding limiting Gibbs measure, we will find the fixed points of the function (142) , i.e., solutions of equation $F(u) = u$. Then, if there is more than one positive fixed point for $f(x)$, there is more than one translation-invariant paramagnetic phase for our model.

5.3.2 Phase Transition

In this section, we prove the existence of phase transition for Potts model with the Hamiltonian (138) by analyzing the fixed points of the function f (142).

It's clear that $f(x)$ is continuous $\forall\, x > 0, f(0) = c > 0$ and f is bounded function. As a result, the curve $y = f(x)$ must intersect a line $y = x$ and a Gibbs measure is exist, now let us search for the conditions of phase transition as we used to do before :

$$f'(x) = 6c\ (b-1)\Big(\frac{(1+b+b^2)d^2x^2 + 12(1+x)(1+bx) + d(4+4(1+b)x)}{(6(1+x)^2 + d(2+x^3))^2} + \frac{(3(1+b+b^2)x^2 + 4b(1+b)x^3 + b^2x^4)}{(6(1+x)^2 + d(2+x^3))^2}\Big)$$

It's clear that $f'(x) > 0$ iff $b > 1$. Note that this condition appears in all Potts models that we have studied, then we conclude if $\frac{J_t}{T} < 0$, the phase transition does not exists, i.e., for the phase transition to be exists, J_t and T must have the same signs. Now let consider the second derivative

$$f''(x) = \frac{1}{\left(6(1+x)^2 + d(2+x^3)\right)^3} 6(b-1)c(-6(4+4x+dx^2)((1+b+b^2)d^2x^2 + 12(1+x)(1+bx) + d(4+4(1+b)x + 3(1+b+b^2)x^2 + 4b(1+b)x^3 + b^2x^4)) + 2(6(1+x)^2 + d(2+x^3))((3+d)(2+dx) + b^2dx(3+d+6x + 2x^2) + b(6+12x+d^2x + d(2+3x+6x^2)))) = 0$$

From Mathematica [57], if $\left(0 < b < 3 \text{ and } d > \frac{9-3b}{1+b}\right)$ or $(b > 3 \text{ and } d > 0)$

. Then, there is an inflection point $x_{inf} > 0$ such that

$$f''(x) > 0 \ \ (f \text{ is concave up}) \qquad \text{if} \quad x < x_{inf}$$

$$\text{and} \ \ f''(x) < 0 \ (f \text{ is concave down}) \ \ \text{if} \ \ x > x_{inf}$$

According to Preston [50], there is more than one solution for $f(x) = x$ iff there is more than one solution for $x\,f'(x) = f(x)$, which reduced to the following equation

$$-\frac{c\left(36(1+x)(1+3x-(-4+b)bx^2+b^2x^3)+12d(2+4x-(-3+b^2)x^2-(-2+b^3)x^3-b(-3+b^2)x^4+b^2x^5)+d^2(4-4(-2+b^3)x^3+b^3x^6)\right)}{\left(6(1+x)^2+d(2+x^3)\right)^2} = 0 \quad (143)$$

Then from [57], we conclude that there are two positive roots for the previous equation iff one of the following cases is satisfied:

Let $c > 0$,

Case 1: If $1.587401 < b < 4.600482$, then there are two positive roots for equation (143) iff $d > d^*(b)$ where $d^*(b)$ is the second smallest real root for the polynomial (144) .

$$-419904b^2 + 1306368b^3 - 1399680b^4 + 559872b^5 - 46656b^6 + (1119744b - 3685824b^2$$

$$+5365440b^3 - 4758912b^4 + 2612736b^5 - 699840b^6 + 46656b^7)\mathrm{y} + (-186624 - 1150848b$$

$$+3748032b^2 - 3499200b^3 + 451008b^4 + 1613520b^5 - 1251936b^6 + 291600b^7 - 15552b^8)y^2$$

$$+ (324864 + 103680b - 393984b^2 - 1093824b^3 + 1710720b^4 - 436752b^5 - 533520b^6 +$$

$$358992b^7 - 55728b^8 + 1728b^9)y^3 + (-108864 + 48384b - 288576b^2 + 601344b^3 - 6480b^4$$

$$-340416b^5 + 34560b^6 + 166428b^7 - 60480b^8 + 4212b^9)y^4 + (-69696 + 8640b + 121536b^2$$

$$-8640b^3 - 189216b^4 + 23040b^5 + 52272b^6 + 26028b^7 - 31176b^8)y^5 + (44288 + 13248b +$$

$$11520b^2 - 59280b^3 - 10944b^4 + 37872b^5 - 588b^6 - 3276b^7 - 8568b^8 + 2485b^9)\,y^6 + (3904$$

$$-4032b - 10944b^2 + 7296b^3 + 7200b^4 + 7632b^5 - 6000b^6 - 1548b^7 - 1224b^8 + 848b^9)y^7$$

$$+(-6528 - 576b - 1152b^2 + 6048b^3 + 720b^4 + 576b^5 - 1800b^6 - 144b^7 - 72b^8 + 174b^9)y^8$$

$$+(-1280 + 960b^3 - 240b^6 + 20b^9)y^9 \quad +(-64 + 48b^3 - 12b^6 + b^9)y^{10} = 0\,, \qquad (144)$$

Case 2: If $4.600482 \le b < 9$, then there two positive roots for (143) iff $d > d^{**}(b) > 0$ where $d^{**}(b)$ the first real (positive) root for the polynomial (144).

Case 3: If $b \ge 9$, then the roots exists $\forall\, d > 0$.

Then, we conclude the following proposition:

Proposition 5.3.2. The equation

$$x = \frac{c\,(b^3\,d\,x^3 + 6\,b^2x^2 + 12\,b\,x + (2d + 6))}{(d\,x^3 + 6\,x^2 + 12\,x + (6 + 2\,d))} \qquad (145)$$

With $x > 0$, $c > 0$, $b > 0$ and $d > 0$ has one positive solution if b < 1. If one of the following conditions are satisfied:

a) $1.587401 < b < 4.600482$ and $d > d^{*}(b)$.

b) $4.600482 \le b < 9$ and $d > d^{**}(b)$.

c) $b \ge 9, \forall\, d > 0$.

Then, there exists $\eta_1(b,d,c)$, $\eta_2(b,d,c) > 0$ such that equation (145) has three positive roots if $\eta_1(b,d,c) < 1 < \eta_2(b,d,c)$ and has two positive solutions if either $\eta_1(b,d,c) = 1$ or $\eta_2(b,d,c) = 1$, where $\eta_1(b,d,c) = f'(x_1^*)$ and $\eta_2(b,d,c) = f'(x_2^*)$ such that x_1^* and x_2^* are the positive solutions of the equation

$$(36(1+x)(1+3x-(-4+b)bx^2+b^2x^3)+12d(2+4x-(-3+b^2)x^2-(-2+b^3)x^3 - b(-3+b^2)x^4+b^2x^5+d^2(4-4(-2+b^3)x^3+b^3x^6)) = 0$$

and $d^*(b)$, $d^{**}(b)$ are the second and the first smallest real roots for equation (144), respectively.

From the previous proposition, we conclude that if $b > 1.587401$, i .e., if

$T < \frac{J_t}{\ln(1.587401)}$, $J_t > 0$, then phase transition exists. So we get the following Theorem :

Theorem 5.3.2. Let $J_t > 0$ be the coupling constant that corresponds to TPNNN. Then, if $T < T_{cr}$ where $T_{cr} = \frac{J_t}{\ln(1.587401)}$, the model (138) has three translation-invariant paramagnetic Gibbs measures, i.e., there is a phase transition on the paramagnetic phases.

Example 5.3.3

Consider the first case in proposition 5.3.2, let $b = 1.7 \in (1.587401, 4.600482)$. Then by manipulating equation (144) in Mathematica [57] we get that the roots for this equation are

$$Y = \{-21.430312 - 7.225800\,i, -7.023260 - 0.012553\,i, -21.430312 + 7.225800\,i, -7.023260 + 0.012553\,i, -0.738280 - 0.010296\,i, -0.7382710 + 0.010296\,i, 0.905232 - 0.097915\,i, 0.905232 + 0.097915\,i, 1.096656, 35.476585\}$$

It's clear that $d^*(1.7) = 35.476585$. So we will take $d > 35.476585$.

Let $d = 36$. Then $x_1^* = 0.972753, x_2^* = 1.036391$, so according to proposition 5.3.2 there are 3 phases (phase transition exists) if

$f\,'(x_1^*) < 1 < f\,'(x_2^*) \rightarrow 0.451097 < c < 0.451128$. Let $c = 0.45111$, then the 3 positive roots (that corresponds for 3 paramagnetic Gibbs measures) for equation (144) are $\{0.948862, 1.007542, 1.058814\}$. See Figure 5.3.3.

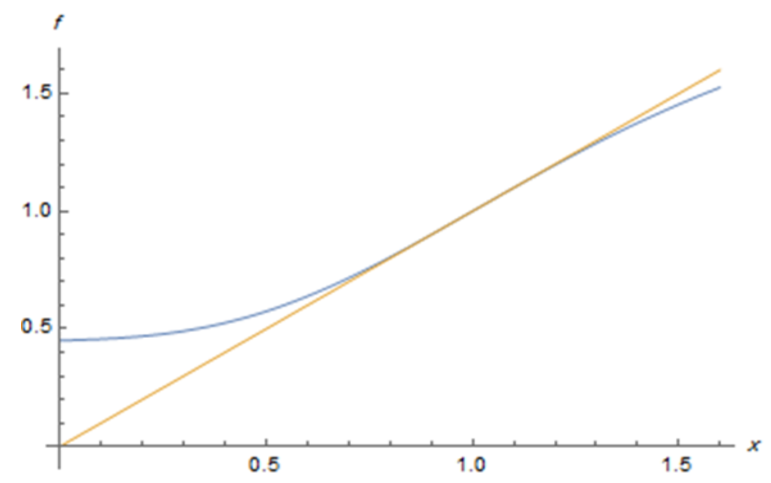

Figure 5.3.3. The existence of 3 translation-invariant paramagnetic phases for model (137) when $d = 36$, $b = 1.7$ and $c = 0.45111$.

Note that $f'(0.948862) = 0.996458 < 1$ and $f'(1.058814) = 0.997069 < 1$. Then, the fixed points $x = 0.948862, 1.058814$ are stable and the corresponding paramagnetic measures for this points are extreme.

Chapter Six

Conclusion and Comments

In this thesis, we have solved five different models of lattice spin systems. In the first model, we have obtained the set of Gibbs states for Ising-Vannimenus model on a Cayley tree of order three with TPNNN interaction. We have constructed the set of recurrence equations corresponding for the model and satisfying consistency condition (14). In addition, We have studied and analyzed the existence of translation-invariant measures by finding the fixed points for the operator F (40) defined on the set $A_1 = \left\{ (u_1, u_4, u_5, u_8) \in R^{+4} : u_1 u_4 = u_5 u_8 \right\}$. We have concluded that under accurate conditions on the coupling constants and the temperature, it's possible to find three translation-invariant Gibbs measures , two of them are extreme one's . As a result, the phase transition exists.

In the second model, the same strategy in the first model has been used to explain the existence of translation invariant Gibbs measures for the Vannimenus model on Cayley tree of order four with OLNNN interactions. We have concluded that If $\frac{J_p}{T} > 0$, and under specific conditions on J and J_{OL}, the phase transition exists with at most 3 translation invariant phases .

Finally, three Potts Models have been analyzed such that the existence of translation-invariant paramagnetic phases is proved. Firstly, we have analyzed the Potts model with NN and TPNNN Interactions on a Third-order Cayley Tree with q State Space. By using partition function method, we have concluded that for $T > T_{cr}$ where $T_{cr} = \frac{J_t}{\ln 4}$, there exist 3 paramagnetic phases , i.e., the phase transition exists. In addition, the existence of 2-periodic Gibbs measures for this model has been proved. The second model is the same as the first one with $q = 3$ and OLNNN interactions on Cayley tree of order two, the existence of phase transition has been proved. The third model is a development for the second one on a Cayley tree of order three with OL 3-tuple interactions. We conclude that if $T < \frac{J_t}{\ln(1.587401)}$ $, J_t > 0$, then there exist 3 paramagnetic phases (Phase transition).

Describing the set of all Gibbs measures corresponding to a given Hamiltonian is still a great problem. The complete description for this set has not been done yet, even for simple Hamiltonians. Although we have analyzed the shifting-invariant Gibbs measures corresponding for the Hamiltonians of our models, but there are still many unsolved problems related to these models which are suggested as a possible future work:

1) Defining the operator F on a different set and finding the corresponding Gibbs measures (this includes invariant and non-invariant sets).
2) Finding all periodic Gibbs measures corresponding to our models.
3) Constructing the phase diagrams for the models in this thesis.
4) Developing the models for arbitrary order Cayley tree.
5) Developing the first and second models for a memory of length n, $n > 2$.
6) The results obtained in our thesis can inspire to study Ising and Potts models over multi-dimensional lattices or the grid Z^d.
7) Trying to relate our results with physical or biological applications.

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Appendix

Analysis of the second derivative for the function in equation (45)

In[1]:= $D[D[\frac{(1+a^2b^4x)^3}{x(b^4+a^2x)^3},x],x]$

Out[1]= $\frac{6a^4b^8(1+a^2b^4x)}{x(b^4+a^2x)^3}-\frac{18a^4b^4(1+a^2b^4x)^2}{x(b^4+a^2x)^4}-\frac{6a^2b^4(1+a^2b^4x)^2}{x^2(b^4+a^2x)^3}+\frac{12a^4(1+a^2b^4x)^3}{x(b^4+a^2x)^5}+\frac{6a^2(1+a^2b^4x)^3}{x^2(b^4+a^2x)^4}+\frac{2(1+a^2b^4x)^3}{x^3(b^4+a^2x)^3}$

In[2]:= Simplify[%1]

Out[2]= $\frac{2(10a^4x^2+a^6b^{20}x^3-4a^8b^{16}x^4+a^6b^{12}x^3(-9+a^4x^2)+a^2b^4x(5+18a^4x^2)+b^8(1+9a^8x^4))}{x^3(b^4+a^2x)^5}$

In[3]:= Reduce[{$\frac{2(10a^4x^2+a^6b^{20}x^3-4a^8b^{16}x^4+a^6b^{12}x^3(-9+a^4x^2)+a^2b^4x(5+18a^4x^2)+b^8(1+9a^8x^4))}{x^3(b^4+a^2x)^5} == 0, b > 0, a > 0$}, x, PositiveReals]

Out[3]= $a>0$ &&(($b==1.16\cdots$&&$x==Root[b8+(5\,a2\,b4-a2\,b12)\,\#1+(10\,a4-5\,a4\,b8+a4\,b16)\,\#12+(8\,a6\,b4-4\,a6\,b12)\,\#13+a8\,b8\,\#14\&,1]$)||($b>1.16\cdots$&&($x==Root[b8+(5\,a2\,b4-a2\,b12)\#1+(10\,a4-5\,a4\,b8+a4\,b16)\#12+(8\,a6\,b4-4\,a6\,b12)\#13+a8\,b8\,\#14\&,1]$ ||$x==Root[b8+(5\,a2\,b4-a2\,b12)\,\#1+(10\,a4-5\,a4\,b8+a4\,b16)\,\#12+(8\,a6\,b4-4\,a6\,b12)\,\#13+a8\,b8\,\#14\&,2]$)))

In[4]:= Reduce[{$\frac{2(10a^4x^2+a^6b^{20}x^3-4a^8b^{16}x^4+a^6b^{12}x^3(-9+a^4x^2)+a^2b^4x(5+18a^4x^2)+b^8(1+9a^8x^4))}{x^3(b^4+a^2x)^5} > 0, b > 0, a > 0$}, x, PositiveReals]

Out[4]= $a>0$&&((0 < b < 1.16⋯&&$x>0$)||($b==1.16\cdots$&&($0<x<Root[b8+(5\,a2\,b4-a2\,b12)\,\#1+(10\,a4-5\,a4\,b8+a4\,b16)\,\#12+(8\,a6\,b4-4\,a6\,b12)\,\#13+a8\,b8\,\#14\&,1]$||$x>Root[b8+(5\,a2\,b4-a2\,b12)\,\#1+(10\,a4-5\,a4\,b8+a4\,b16)\,\#12+(8\,a6\,b4-4\,a6\,b12)\,\#13+a8\,b8\,\#14\&,1]$))||($b>1.16\cdots$&&($0<x<Root[b8+(5\,a2\,b4-a2\,b12)\,\#1+(10\,a4-5\,a4\,b8+a4\,b16)\,\#12+(8\,a6\,b4-4\,a6\,b12)\,\#13+a8\,b8\,\#14\&,1]$||$x>Root[b8+(5\,a2\,b4-a2\,b12)\,\#1+(10\,a4-5\,a4\,b8+a4\,b16)\,\#12+(8\,a6\,b4-4\,a6\,b12)\,\#13+a8\,b8\,\#14\&,2]$)))

Analysis of Example 3.4.1

In[1]:= $b=2$

Out[1]=2

In[2]:= $\eta_1=\frac{-6a^4(1+4(-\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}+\frac{251}{16a^2}})a^2)^2 16(255)}{(16+\frac{1}{4}(-\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}+\frac{251}{16a^2}})a^2)^4(-251+3\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2)}$

Out[2]= $-\frac{24480a^4(1+4(-\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}+\frac{251}{16a^2}})a^2)^2}{(16+\frac{1}{4}(-\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}+\frac{251}{16a^2}})a^2)^4(-251+3\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2)}$

In[3]:= $\text{Simplify}\left[-\dfrac{24480a^4\left(1+4\left(-\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}}+\frac{251}{16a^2}\right)a^2\right)^2}{\left(16+\frac{1}{4}\left(-\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}}+\frac{251}{16a^2}\right)a^2\right)^4\left(-251+3\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2\right)}\right]$

Out[3]= $\dfrac{4194304a^4(-81+\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2)}{-140378419+1722747\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2}$

In[4]:= $\eta_2 = \dfrac{6a^4 16(255)(1+a^2 16(\frac{1}{4}(\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}}+\frac{251}{16a^2})))^2}{(251+3\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2)(16+a^2(\frac{1}{4}(\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}}+\frac{251}{16a^2})))^4}$

Out[4]=

$$\frac{24480a^4(1+4(\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}}+\frac{251}{16a^2})a^2)^2}{(16+\frac{1}{4}(\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}}+\frac{251}{16a^2})a^2)^4(251+3\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2)}$$

In[5]:= $\text{Simplify}[\dfrac{24480a^4(1+4(\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}}+\frac{251}{16a^2})a^2)^2}{(16+\frac{1}{4}(\frac{3}{16}\sqrt{6545}\sqrt{\frac{1}{a^4}}+\frac{251}{16a^2})a^2)^4(251+3\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2)}]$

Out[5]= $\dfrac{4194304a^4(81+\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2)}{140378419+1722747\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2}$

In[6]:= $\text{Reduce}[\dfrac{4194304a^4(-81+\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2)}{-140378419+1722747\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2} < 1 < \dfrac{4194304a^4(81+\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2)}{140378419+1722747\sqrt{6545}\sqrt{\frac{1}{a^4}}a^2}, a, \text{PositiveReals}]$

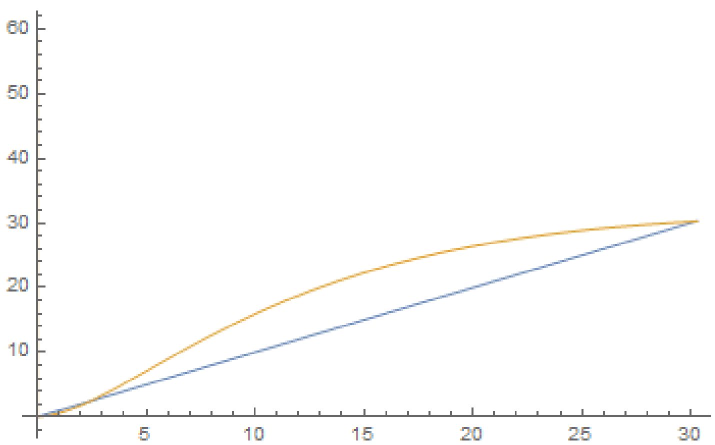
Out[6]= $\mathbf{0.801 \cdots < a < 1.25 \cdots}$

In[7]:= $\text{Reduce}[x == \frac{(1+a^2b^2c^2x)^3}{x(b^2c^2+a^2x)^3}, x, \text{PositiveReals}]$

Out[7]= $x == 0.023103967072251066 || x == 2.4239428766425655 || x == 30.200381861255842$

In[8]:= $\text{Plot}\left[\left\{x, \frac{(1+a^2b^2c^2x)^3}{x(b^2c^2+a^2x)^3}\right\}, \{x, 0, 30.3\}\right]$

Out[8]=



In[9]=FPrime[x_] $= -\frac{(1+a^2b^4x)^2(b^4a^4x^2+(4a^2-2a^2b^8)x+b^4)}{x^2(b^4+a^2x)^4}$

Out[9]:= $-\frac{(1+12.96x)^2(16-411.48x+10.497600000000002x^2)}{(16+0.81x)^4x^2}$

In[10]=FPrime[0.023104]

Out[10]:= -0.312213

In[11]= FPrime[2.423943]

Out[11]:= 1.57955

In[12]= FPrime[30.200382]

Out[12]:= 0.17864

Analysis of Example 3.4.2

In[1]:= $c = 0.46$

Out[1]= 0.46

In[2]:= $b > \frac{1}{2}\sqrt{\frac{1}{c^2} + c^2 + \frac{\sqrt{1+34c^4+c^8}}{c^2}}$

Out[2]= $b > 1.763957$

In[3]:= $b = 3$

Out[3]= 3

In[4]:= Solve[$(c^2 + a^2b^2x)^2(1 + a^2b^2c^2x)^2(-8a^4c^4x^2 + 4a^4b^8c^4x^2 - 5a^2b^2c^2(1 + c^4)x(1 + a^4x^2) + a^2b^6c^2(1 + c^4)x(1 + a^4x^2) - 2b^4(a^4x^2 + a^4c^8x^2 + c^4(1 + a^4x^2)^2)) == 0, x$]

Out[4]= $\{\{x \to -\frac{5.136824697217314}{a^2}\}, \{x \to -\frac{0.5250997689561017}{a^2}\}, \{x \to -\frac{0.5250997689561017}{a^2}\}, \{x \to -\frac{0.19467279086664457}{a^2}\},$
$\{x \to -\frac{0.023511111111111111}{a^2}\}, \{x \to -\frac{0.023511111111111111}{a^2}\}, \{x \to \frac{0.038254879641317234}{a^2}\}, \{x \to \frac{26.140456050996725}{a^2}\}\}$

In[5]:= FPrime[x_] $= \frac{25.392a^2(1+1.9044a^2x)^2(0.2116+9a^2x)^2(34.702400000000004a^2x+9.40297104(1+a^4x^2))}{(9+0.2116a^2x)^4(1.9044+a^2x)^4\sqrt{\frac{(1+1.9044a^2x)^3(0.2116+9a^2x)^3}{(9+0.2116a^2x)^3(1.9044+a^2x)^3}}}$

Out[5]= $\frac{25.392000000000003a^2(1+1.9044a^2x)^2(0.2116+9a^2x)^2(34.702400000000004a^2x+9.40297104(1+a^4x^2))}{(9+0.2116a^2x)^4(1.9044+a^2x)^4\sqrt{\frac{(1+1.9044a^2x)^3(0.2116+9a^2x)^3}{(9+0.2116a^2x)^3(1.9044+a^2x)^3}}}$

In[6]:= FPrime[$\frac{0.038255}{a^2}$]

Out[6]=$0.164463a^{2\cdots}$

In[7]:= FPrime[$\frac{26.1404560}{a^2}$]

Out[7]= $6.080395\, a^2$

In[8]:=Reduce$[6.080395\, a^2 < 1 < 6.080395\, a^2, a, \text{PositiveReals}]$

Out[8]= $0.40554 < a < 2.46585$

In[9]:= $a = 0.6$

Out[9]= 0.6

In[10]:= Reduce$\left[x == \sqrt{\frac{(1+\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}^2x)^3(\boldsymbol{c}^2+\boldsymbol{a}^2\boldsymbol{b}^2x)^3}{(\boldsymbol{b}^2+\boldsymbol{a}^2\boldsymbol{c}^2x)^3(\boldsymbol{b}^2\boldsymbol{c}^2+\boldsymbol{a}^2x)^3}}, x, \text{PositiveReals}\right]$

Out[10]= $x == 0.00141813 || x == 9.44772283 || x == 534.710233$

Analysis of equation (63)

In[1]:= Reduce$[\{(c^2 + a^2b^2x)^2(1 + a^2b^2c^2x)^2(-8a^4c^4x^2 + 4a^4b^8c^4x^2 - 2b^4(a^4x^2 + a^4c^8x^2 + c^4(1 + a^4x^2)^2) - 5a^2b^2c^2(1 + c^4)x(1 + a^4x^2) + a^2b^6c^2(1 + c^4)x(1 + a^4x^2)) == 0, a > 0, b > 0, c > 0\}, x, \text{PositiveReals}]$

Out[1]=

```
(0<c<0.454 …&&((b==Root[-25+14c^4-25 c^8+(1+34 c^4+c^8) #1^4&,2]&&a>0&&(x==Root[2
b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4
b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4
#1^4&,1]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4
c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6)
#1^3+2 a^8 b^4 c^4 #1^4&,3]))||(Root[-25+14 c^4-25 c^8+(1+34 c^4+c^8) #1^4&,2]<b<Root[-4
c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-
a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6
c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6
c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5
a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,2]||x==Root[2 b^4 c^4+(5 a^2
b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4
c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3]||x==Root[2
b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4
b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4
#1^4&,4]))||(b==Root[-4 c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]&&a>0&&(x==Root[2 b^4 c^4+(5 a^2
b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4
c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1]||x==Root[2
b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4
b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4
#1^4&,2]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4
c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6)
#1^3+2 a^8 b^4 c^4 #1^4&,4]))||(Root[-4 c^2+(-1-c^4) #1^2+2 c^2
#1^4&,2]<b<1/c&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6)
#1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6
b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2
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c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,2]))||(b>=1/c&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,4])))) ||

(0.454…<=c<0.669…&&((b==Root[-4 c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]&&a>0&&x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1])||(Root[-4 c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]<b<1/c&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,2]))||(b>=1/c&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,4]))))||

(c==0.669 …&&((b==1/c&&a>0&&x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3])||(b>1/c&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,4]))))||(0.669 …<c<=1.50 …&&((b==Root[-4 c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]&&a>0&&x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3])||(b>Root[-4 c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,4]))))||

(1.50 …<c<=2.20 …&&((b==Root[-4 c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]&&a>0&&x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1])||(Root[-4 c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]<b<c&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,2]))||(b>=c&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,4]))))
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||(c>2.20 …&&((b==Root[-25+14 c^4-25 c^8+(1+34 c^4+c^8) #1^4&,2]&&a>0&&(x==Root[2
b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4
b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4
#1^4&,1]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4
c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6)
#1^3+2 a^8 b^4 c^4 #1^4&,3]))||(Root[-25+14 c^4-25 c^8+(1+34 c^4+c^8) #1^4&,2]<b<Root[-4
c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-
a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6
c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6
c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5
a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,2]||x==Root[2 b^4 c^4+(5 a^2
b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4
c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3]||x==Root[2
b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4
b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4
#1^4&,4]))||(b==Root[-4 c^2+(-1-c^4) #1^2+2 c^2 #1^4&,2]&&a>0&&(x==Root[2 b^4 c^4+(5 a^2
b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4
c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1]||x==Root[2
b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4
b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4
#1^4&,2]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4
c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6)
#1^3+2 a^8 b^4 c^4 #1^4&,4]))||(Root[-4 c^2+(-1-c^4) #1^2+2 c^2
#1^4&,2]<b<c&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2
a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6
b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,1]||x==Root[2 b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6
c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5
a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,2]))||(b>=c&&a>0&&(x==Root[2 b^4 c^4+(5 a^2 b^2
c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4 b^8 c^4+2 a^4 b^4
c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4 #1^4&,3]||x==Root[2
b^4 c^4+(5 a^2 b^2 c^2-a^2 b^6 c^2+5 a^2 b^2 c^6-a^2 b^6 c^6) #1+(2 a^4 b^4+8 a^4 c^4+4 a^4 b^4 c^4-4 a^4
b^8 c^4+2 a^4 b^4 c^8) #1^2+(5 a^6 b^2 c^2-a^6 b^6 c^2+5 a^6 b^2 c^6-a^6 b^6 c^6) #1^3+2 a^8 b^4 c^4
#1^4&,4]))))
```

- **For determine the value of Root (like $\mathrm{Root}[-4c^2+(-1-c^4)\#1^2+2c^2\#1^4\&,2]$),we use the following code**

In[2]:= $\mathrm{Solve}[-4c^2+(-1-c^4)x^2+2c^2x^4==0,x]$

Out[2]= $\left\{x\to-\frac{1}{2}\sqrt{\frac{1}{c^2}+c^2-\frac{\sqrt{1+34c^4+c^8}}{c^2}}\right\},\left\{x\to\frac{1}{2}\sqrt{\frac{1}{c^2}+c^2-\frac{\sqrt{1+34c^4+c^8}}{c^2}}\right\},$

$$\{x\to-\frac{1}{2}\sqrt{\frac{1}{c^2}+c^2+\frac{\sqrt{1+34c^4+c^8}}{c^2}}\},\{x\to\frac{1}{2}\sqrt{\frac{1}{c^2}+c^2+\frac{\sqrt{1+34c^4+c^8}}{c^2}}\}\}$$

Since we deal with positive roots we choose

$$x\to\frac{1}{2}\sqrt{\frac{1}{c^2}+c^2+\frac{\sqrt{1+34c^4+c^8}}{c^2}}$$

Analysis of Example 4.5.1

In[1]:= $c = 0.45$

Out[1]=0.45

In[2]:= $\text{Reduce}[-5c^4 - 12cy - 6y^2 + 4cy^3 + 3c^4y^4 == 0, y, \text{PositiveReals}]$

Out[2]= $y = 3.420514006466678$

In[3]:= $b = 3.43$

Out[3]= 3.43

In[4]:= `Reduce[5 a`^{4} `c`^{8} `x`^{4}`-3a`^{4} `b`^{8} `c`^{8} `x`^{4}`+16a`^{3} `b c`^{5} `x`^{3}`(1+a`^{2} `x`^{2}`)-8a`^{3} `b`^{7} `c`^{5} `x`^{3}`(1+a`^{2} `x`^{2}`)+8a b`^{3} `c x(c`^{4}`+6a`^{2} `x`^{2}`+6a`^{4} `x`^{4}`+a`^{6} `c`^{4} `x`^{6}`)+6b`^{2}`(3a`^{2} `c`^{4} `x`^{2}`+8a`^{4} `c`^{2} `x`^{4}`+3a`^{6} `c`^{4} `x`^{6}`)-2b`^{6}`(3a`^{2} `c`^{4} `x`^{2}`+8a`^{4} `c`^{2}`x`^{4}`+3a`^{6} `c`^{4} `x`^{6}`)+b`^{4}`(36a`^{4} `x`^{4}`+16c`^{2}`(a`^{2} `x`^{2}`+a`^{6} `x`^{6}`)+c`^{8}`(1+a`^{8} `x`^{8}`))==0 ,x,PositiveReals]`

Out[4]= $a > 0\&\&(x == \text{Root}[8.21513811242005 \times 10^{34} + 2.102680052929581 \times 10^{36}a\#1 + 1.99395497 \times 10^{37}a^2\#1^2 + 1.698184095294822 \times 10^{37}a^3\#1^3 - 9.723533167137456 \times 10^{37}a^4\#1^4 + 1.698184095294822 \times 10^{37}a^5\#1^5 + 1.993954970053613 \times 10^{37}a^6\#1^6 + 2.102680052929581 \times 10^{36}a^7\#1^7 + 8.21513811242005 \times 10^{34}a^8\#1^8\&,3]||x == \text{Root}[8.21513811242005 \times 10^{34} + 2.102680052929581 \times 10^{36}a\#1 + 1.993954970053613 \times 10^{37}a^2\#1^2 + 1.698184095294822 \times 10^{37}a^3\#1^3 - 9.723533167137456 \times 10^{37}a^4\#1^4 + 1.698184095294822 \times 10^{37}a^5\#1^5 + 1.993954970053613 \times 10^{37}a^6\#1^6 + 2.102680052929581 \times 10^{36}a^7\#1^7 + 8.215138112 \times 10^{34}a^8\#1^8\&,4])$

In[5]:= $D[\frac{(4cabx+4a^3b^3cx^3+a^4b^4c^4x^4+6a^2b^2x^2+c^4)}{(6b^2a^2x^2+b^4c^4+a^4c^4x^4+4b^3acx+4a^3bcx^3)}, x]$

Out[5]= $\frac{a(19068.42443\ +473093.3546697245ax+3525157.780262701a^2x^2+6306664.4388a^3x^3+3525157.7802627a^4x^4+473093.35466972436a^5x^5+19068.424427733335a^6x^6)}{(138.41287201000003\ +1771.3517475994513ax+1721.4302697759488a^2x^2+150.56241426611794a^3x^3+1.a^4x^4)^2}$

In[6]:= $\text{FPrime}[x\_] =$

$$\frac{a(19068.42443\ +473093.3546697245ax+3525157.780262701a^2x^2+6306664.4388a^3x^3+3525157.7802627a^4x^4+473093.35466972436a^5x^5+19068.424427733335a^6x^6)}{(138.41287201000003\ +1771.3517475994513ax+1721.4302697759488a^2x^2+150.56241426611794a^3x^3+1.a^4x^4)^2}$$

Out[6]=

$$\frac{a(19068.42443\ +473093.3546697245ax+3525157.780262701a^2x^2+6306664.4388a^3x^3+3525157.7802627a^4x^4+473093.35466972436a^5x^5+19068.424427733335a^6x^6)}{(138.41287201000003\ +1771.3517475994513ax+1721.4302697759488a^2x^2+150.56241426611794a^3x^3+1.a^4x^4)^2}$$

In[7]:=$\text{Reduce}[\text{FPrime}\left[\text{Root}[8.21513811242005 \times 10^{34} + 2.102680052929581 \times 10^{36}a\#1 + 1.993954970053613 \times 10^{37}a^2\#1^2 + 1.698184095294822 \times 10^{37}a^3\#1^3 - 9.723533167137456 \times 10^{37}a^4\#1^4 + 1.698184095294822 \times 10^{37}a^5\#1^5 + 1.993954970053613 \times 10^{37}a^6\#1^6 + 2.102680052929581 \times 10^{36}a^7\#1^7 + 8.21513811242005 \times 10^{34}a^8\#1^8\&,3]\right] < 1, a, \text{PositiveReals}]$

Out[7]=$0 < a < 1.0005845067783963$

In[8]:=$\text{Reduce}\left[\text{FPrime}\left[\text{Root}[8.21513811242005 \times 10^{34} + 2.102680052929581 \times 10^{36}a\#1 + \right.\right.$

$1.993954970053613\times10^{37}a^2\#1^2+1.698184095294822\times10^{37}a^3\#1^3-9.723533167137456\times 10^{37}a^4\#1^4+1.698184095294822\times10^{37}a^5\#1^5+1.993954970053613\times10^{37}a^6\#1^6+ 2.102680052929581\times10^{36}a^7\#1^7+8.21513811242005\times10^{34}a^8\#1^8\&,4]\Big]>1,a,\mathrm{PositiveReals}\Big]$

Out[8]:= $a>0.9994158351066273$

Analysis of the model 5.3

- The positive roots for equation (143)

In[1]:=

$$\mathrm{Reduce}[\{\frac{c\left(36(1+x)(1+3x-(-4+b)bx^2+b^2x^3)+12d(2+4x-(-3+b^2)x^2-(-2+b^3)x^3-b(-3+b^2)x^4+b^2x^5)+d^2(4-4(-2+b^3)x^3+b^3x^6)\right)}{\left(6(1+x)^2+d(2+x^3)\right)^2}) == 0, c>0, b>0, d>0\}, x, \mathrm{PositiveReals}]$$

Out[1]=

```
(1.59 …<b<2.85 …&& ((d==Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576 b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176 b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276 b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9) #1^10&,2]&&c>0&&x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,1]) ||
(d>Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576 b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176 b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276 b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9) #1^10&,2]&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,1] ||
```

```
x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,2])))) ||
(b==2.85 …&&((d==Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576 b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176 b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276 b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9) #1^10&,4]&&c>0&&x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,1]) ||
(d > Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576 b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176 b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276 b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9) #1^10&,4]&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,1] ||
x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,2])))) ||
(2.85 …<b<4&&((d==Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576 b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176 b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276 b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9) #1^10&,2]&&c>0&&x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d
```

#1^{5}+b^{3} d^{2} #1^{6}&,1])||(d>Root[-419904 b^{2}+1306368 b^{3}-1399680 b^{4}+559872 b^{5}-46656 b^{6}+(1119744 b-3685824 b^{2}+5365440 b^{3}-4758912 b^{4}+2612736 b^{5}-699840 b^{6}+46656 b^{7}) #1+(-186624-1150848 b+3748032 b^{2}-3499200 b^{3}+451008 b^{4}+1613520 b^{5}-1251936 b^{6}+291600 b^{7}-15552 b^{8}) #1^{2}+(324864+103680 b-393984 b^{2}-1093824 b^{3}+1710720 b^{4}-436752 b^{5}-533520 b^{6}+358992 b^{7}-55728 b^{8}+1728 b^{9}) #1^{3}+(-108864+48384 b-288576 b^{2}+601344 b^{3}-6480 b^{4}-340416 b^{5}+34560 b^{6}+166428 b^{7}-60480 b^{8}+4212 b^{9}) #1^{4}+(-69696+8640 b+121536 b^{2}-8640 b^{3}-189216 b^{4}+23040 b^{5}+52272 b^{6}+26028 b^{7}-31176 b^{8}+4356 b^{9}) #1^{5}+(44288+13248 b+11520 b^{2}-59280 b^{3}-10944 b^{4}+37872 b^{5}-588 b^{6}-3276 b^{7}-8568 b^{8}+2485 b^{9}) #1^{6}+(3904-4032 b-10944 b^{2}+7296 b^{3}+7200 b^{4}+7632 b^{5}-6000 b^{6}-1548 b^{7}-1224 b^{8}+848 b^{9}) #1^{7}+(-6528-576 b-1152 b^{2}+6048 b^{3}+720 b^{4}+576 b^{5}-1800 b^{6}-144 b^{7}-72 b^{8}+174 b^{9}) #1^{8}+(-1280+960 b^{3}-240 b^{6}+20 b^{9}) #1^{9}+(-64+48 b^{3}-12 b^{6}+b^{9}) #1^{10}&,2]&&c>0&&(x==Root[36+24 d+4 d^{2}+(144+48 d) #1+(108+144 b-36 b^{2}+36 d-12 b^{2} d) #1^{2}+(144 b+24 d-12 b^{3} d+8 d^{2}-4 b^{3} d^{2}) #1^{3}+(36 b^{2}+36 b d-12 b^{3} d) #1^{4}+12 b^{2} d #1^{5}+b^{3} d^{2} #1^{6}&,1]||x==Root[36+24 d+4 d^{2}+(144+48 d) #1+(108+144 b-36 b^{2}+36 d-12 b^{2} d) #1^{2}+(144 b+24 d-12 b^{3} d+8 d^{2}-4 b^{3} d^{2}) #1^{3}+(36 b^{2}+36 b d-12 b^{3} d) #1^{4}+12 b^{2} d #1^{5}+b^{3} d^{2} #1^{6}&,2])))) ||

(4<=b<4.60 …&&((d==Root[-419904 b^{2}+1306368 b^{3}-1399680 b^{4}+559872 b^{5}-46656 b^{6}+(1119744 b-3685824 b^{2}+5365440 b^{3}-4758912 b^{4}+2612736 b^{5}-699840 b^{6}+46656 b^{7}) #1+(-186624-1150848 b+3748032 b^{2}-3499200 b^{3}+451008 b^{4}+1613520 b^{5}-1251936 b^{6}+291600 b^{7}-15552 b^{8}) #1^{2}+(324864+103680 b-393984 b^{2}-1093824 b^{3}+1710720 b^{4}-436752 b^{5}-533520 b^{6}+358992 b^{7}-55728 b^{8}+1728 b^{9}) #1^{3}+(-108864+48384 b-288576 b^{2}+601344 b^{3}-6480 b^{4}-340416 b^{5}+34560 b^{6}+166428 b^{7}-60480 b^{8}+4212 b^{9}) #1^{4}+(-69696+8640 b+121536 b^{2}-8640 b^{3}-189216 b^{4}+23040 b^{5}+52272 b^{6}+26028 b^{7}-31176 b^{8}+4356 b^{9}) #1^{5}+(44288+13248 b+11520 b^{2}-59280 b^{3}-10944 b^{4}+37872 b^{5}-588 b^{6}-3276 b^{7}-8568 b^{8}+2485 b^{9}) #1^{6}+(3904-4032 b-10944 b^{2}+7296 b^{3}+7200 b^{4}+7632 b^{5}-6000 b^{6}-1548 b^{7}-1224 b^{8}+848 b^{9}) #1^{7}+(-6528-576 b-1152 b^{2}+6048 b^{3}+720 b^{4}+576 b^{5}-1800 b^{6}-144 b^{7}-72 b^{8}+174 b^{9}) #1^{8}+(-1280+960 b^{3}-240 b^{6}+20 b^{9}) #1^{9}+(-64+48 b^{3}-12 b^{6}+b^{9}) #1^{10}&,2]&&c>0&&x==Root[36+24 d+4 d^{2}+(144+48 d) #1+(108+144 b-36 b^{2}+36 d-12 b^{2} d) #1^{2}+(144 b+24 d-12 b^{3} d+8 d^{2}-4 b^{3} d^{2}) #1^{3}+(36 b^{2}+36 b d-12 b^{3} d) #1^{4}+12 b^{2} d #1^{5}+b^{3} d^{2} #1^{6}&,5]) ||
(Root[-419904 b^{2}+1306368 b^{3}-1399680 b^{4}+559872 b^{5}-46656 b^{6}+(1119744 b-3685824 b^{2}+5365440 b^{3}-4758912 b^{4}+2612736 b^{5}-699840 b^{6}+46656 b^{7}) #1+(-186624-1150848 b+3748032 b^{2}-3499200 b^{3}+451008 b^{4}+1613520 b^{5}-1251936 b^{6}+291600 b^{7}-15552 b^{8}) #1^{2}+(324864+103680 b-393984 b^{2}-1093824 b^{3}+1710720 b^{4}-436752 b^{5}-533520 b^{6}+358992 b^{7}-55728 b^{8}+1728 b^{9}) #1^{3}+(-108864+48384 b-288576 b^{2}+601344 b^{3}-6480 b^{4}-340416 b^{5}+34560 b^{6}+166428 b^{7}-60480 b^{8}+4212 b^{9}) #1^{4}+(-69696+8640 b+121536 b^{2}-8640 b^{3}-189216 b^{4}+23040 b^{5}+52272 b^{6}+26028 b^{7}-31176 b^{8}+4356 b^{9}) #1^{5}+(44288+13248 b+11520 b^{2}-59280 b^{3}-10944 b^{4}+37872 b^{5}-588 b^{6}-3276 b^{7}-8568 b^{8}+2485 b^{9}) #1^{6}+(3904-4032 b-10944 b^{2}+7296 b^{3}+7200 b^{4}+7632 b^{5}-6000 b^{6}-1548 b^{7}-1224 b^{8}+848 b^{9}) #1^{7}+(-6528-576 b-1152 b^{2}+6048 b^{3}+720 b^{4}+576 b^{5}-1800 b^{6}-144 b^{7}-72 b^{8}+174 b^{9}) #1^{8}+(-1280+960 b^{3}-240 b^{6}+20 b^{9}) #1^{9}+(-64+48 b^{3}-12 b^{6}+b^{9}) #1^{10}&,2]<d<=1&&c>0&&(x==Root[36+24 d+4 d^{2}+(144+48 d) #1+(108+144 b-36 b^{2}+36 d-12 b^{2} d) #1^{2}+(144 b+24 d-12 b^{3} d+8 d^{2}-4 b^{3} d^{2}) #1^{3}+(36 b^{2}+36 b d-12 b^{3} d) #1^{4}+12 b^{2} d #1^{5}+b^{3} d^{2} #1^{6}&,5]||x==Root[36+24 d+4 d^{2}+(144+48 d) #1+(108+144 b-36 b^{2}+36 d-12 b^{2} d) #1^{2}+(144 b+24 d-12 b^{3} d+8 d^{2}-4 b^{3} d^{2}) #1^{3}+(36 b^{2}+36 b d-12 b^{3} d) #1^{4}+12 b^{2} d #1^{5}+b^{3} d^{2} #1^{6}&,6])) ||

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(d>1&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,1] ||
x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,2])))) ||

(b==4.60 …&&((d==Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576 b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176 b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276 b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9) #1^10&,1]&&c>0&&x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,5]) || (Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576 b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176 b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276 b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9) #1^10&,1]<d<=1&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,5] || x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,6])) ||(d>1&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,1]||x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,2])))) ||

(4.60 …<b<9& &((d==Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576
```

b^{2}+601344 b^{3}-6480 b^{4}-340416 b^{5}+34560 b^{6}+166428 b^{7}-60480 b^{8}+4212 b^{9}) #1^{4}+(-
69696+8640 b+121536 b^{2}-8640 b^{3}-189216 b^{4}+23040 b^{5}+52272 b^{6}+26028 b^{7}-31176
b^{8}+4356 b^{9}) #1^{5}+(44288+13248 b+11520 b^{2}-59280 b^{3}-10944 b^{4}+37872 b^{5}-588 b^{6}-3276
b^{7}-8568 b^{8}+2485 b^{9}) #1^{6}+(3904-4032 b-10944 b^{2}+7296 b^{3}+7200 b^{4}+7632 b^{5}-6000 b^{6}-
1548 b^{7}-1224 b^{8}+848 b^{9}) #1^{7}+(-6528-576 b-1152 b^{2}+6048 b^{3}+720 b^{4}+576 b^{5}-1800 b^{6}-
144 b^{7}-72 b^{8}+174 b^{9}) #1^{8}+(-1280+960 b^{3}-240 b^{6}+20 b^{9}) #1^{9}+(-64+48 b^{3}-12 b^{6}+b^{9})
#1^{10}&,1]&&c>0&&x==Root[36+24 d+4 d^{2}+(144+48 d) #1+(108+144 b-36 b^{2}+36 d-12 b^{2}
d) #1^{2}+(144 b+24 d-12 b^{3} d+8 d^{2}-4 b^{3} d^{2}) #1^{3}+(36 b^{2}+36 b d-12 b^{3} d) #1^{4}+12 b^{2} d
#1^{5}+b^{3} d^{2} #1^{6}&,3]) || (Root[-419904 b^{2}+1306368 b^{3}-1399680 b^{4}+559872 b^{5}-46656
b^{6}+(1119744 b-3685824 b^{2}+5365440 b^{3}-4758912 b^{4}+2612736 b^{5}-699840 b^{6}+46656 b^{7})
#1+(-186624-1150848 b+3748032 b^{2}-3499200 b^{3}+451008 b^{4}+1613520 b^{5}-1251936
b^{6}+291600 b^{7}-15552 b^{8}) #1^{2}+(324864+103680 b-393984 b^{2}-1093824 b^{3}+1710720 b^{4}-
436752 b^{5}-533520 b^{6}+358992 b^{7}-55728 b^{8}+1728 b^{9}) #1^{3}+(-108864+48384 b-288576
b^{2}+601344 b^{3}-6480 b^{4}-340416 b^{5}+34560 b^{6}+166428 b^{7}-60480 b^{8}+4212 b^{9}) #1^{4}+(-
69696+8640 b+121536 b^{2}-8640 b^{3}-189216 b^{4}+23040 b^{5}+52272 b^{6}+26028 b^{7}-31176
b^{8}+4356 b^{9}) #1^{5}+(44288+13248 b+11520 b^{2}-59280 b^{3}-10944 b^{4}+37872 b^{5}-588 b^{6}-3276
b^{7}-8568 b^{8}+2485 b^{9}) #1^{6}+(3904-4032 b-10944 b^{2}+7296 b^{3}+7200 b^{4}+7632 b^{5}-6000 b^{6}-
1548 b^{7}-1224 b^{8}+848 b^{9}) #1^{7}+(-6528-576 b-1152 b^{2}+6048 b^{3}+720 b^{4}+576 b^{5}-1800 b^{6}-
144 b^{7}-72 b^{8}+174 b^{9}) #1^{8}+(-1280+960 b^{3}-240 b^{6}+20 b^{9}) #1^{9}+(-64+48 b^{3}-12 b^{6}+b^{9})
#1^{10}&,1]<d<Root[-419904 b^{2}+1306368 b^{3}-1399680 b^{4}+559872 b^{5}-46656 b^{6}+(1119744 b-
3685824 b^{2}+5365440 b^{3}-4758912 b^{4}+2612736 b^{5}-699840 b^{6}+46656 b^{7}) #1+(-186624-
1150848 b+3748032 b^{2}-3499200 b^{3}+451008 b^{4}+1613520 b^{5}-1251936 b^{6}+291600 b^{7}-15552
b^{8}) #1^{2}+(324864+103680 b-393984 b^{2}-1093824 b^{3}+1710720 b^{4}-436752 b^{5}-533520
b^{6}+358992 b^{7}-55728 b^{8}+1728 b^{9}) #1^{3}+(-108864+48384 b-288576 b^{2}+601344 b^{3}-6480 b^{4}-
340416 b^{5}+34560 b^{6}+166428 b^{7}-60480 b^{8}+4212 b^{9}) #1^{4}+(-69696+8640 b+121536 b^{2}-
8640 b^{3}-189216 b^{4}+23040 b^{5}+52272 b^{6}+26028 b^{7}-31176 b^{8}+4356 b^{9}) #1^{5}+(44288+13248
b+11520 b^{2}-59280 b^{3}-10944 b^{4}+37872 b^{5}-588 b^{6}-3276 b^{7}-8568 b^{8}+2485 b^{9}) #1^{6}+(3904-
4032 b-10944 b^{2}+7296 b^{3}+7200 b^{4}+7632 b^{5}-6000 b^{6}-1548 b^{7}-1224 b^{8}+848 b^{9}) #1^{7}+(-
6528-576 b-1152 b^{2}+6048 b^{3}+720 b^{4}+576 b^{5}-1800 b^{6}-144 b^{7}-72 b^{8}+174 b^{9}) #1^{8}+(-
1280+960 b^{3}-240 b^{6}+20 b^{9}) #1^{9}+(-64+48 b^{3}-12 b^{6}+b^{9}) #1^{10}&,2]&&c>0&&(x==Root[36+24
d+4 d^{2}+(144+48 d) #1+(108+144 b-36 b^{2}+36 d-12 b^{2} d) #1^{2}+(144 b+24 d-12 b^{3} d+8
d^{2}-4 b^{3} d^{2}) #1^{3}+(36 b^{2}+36 b d-12 b^{3} d) #1^{4}+12 b^{2} d #1^{5}+b^{3} d^{2}
#1^{6}&,3]||x==Root[36+24 d+4 d^{2}+(144+48 d) #1+(108+144 b-36 b^{2}+36 d-12 b^{2} d)
#1^{2}+(144 b+24 d-12 b^{3} d+8 d^{2}-4 b^{3} d^{2}) #1^{3}+(36 b^{2}+36 b d-12 b^{3} d) #1^{4}+12 b^{2} d
#1^{5}+b^{3} d^{2} #1^{6}&,4]))||(Root[-419904 b^{2}+1306368 b^{3}-1399680 b^{4}+559872 b^{5}-46656
b^{6}+(1119744 b-3685824 b^{2}+5365440 b^{3}-4758912 b^{4}+2612736 b^{5}-699840 b^{6}+46656 b^{7})
#1+(-186624-1150848 b+3748032 b^{2}-3499200 b^{3}+451008 b^{4}+1613520 b^{5}-1251936
b^{6}+291600 b^{7}-15552 b^{8}) #1^{2}+(324864+103680 b-393984 b^{2}-1093824 b^{3}+1710720 b^{4}-
436752 b^{5}-533520 b^{6}+358992 b^{7}-55728 b^{8}+1728 b^{9}) #1^{3}+(-108864+48384 b-288576
b^{2}+601344 b^{3}-6480 b^{4}-340416 b^{5}+34560 b^{6}+166428 b^{7}-60480 b^{8}+4212 b^{9}) #1^{4}+(-
69696+8640 b+121536 b^{2}-8640 b^{3}-189216 b^{4}+23040 b^{5}+52272 b^{6}+26028 b^{7}-31176
b^{8}+4356 b^{9}) #1^{5}+(44288+13248 b+11520 b^{2}-59280 b^{3}-10944 b^{4}+37872 b^{5}-588 b^{6}-3276
b^{7}-8568 b^{8}+2485 b^{9}) #1^{6}+(3904-4032 b-10944 b^{2}+7296 b^{3}+7200 b^{4}+7632 b^{5}-6000 b^{6}-
1548 b^{7}-1224 b^{8}+848 b^{9}) #1^{7}+(-6528-576 b-1152 b^{2}+6048 b^{3}+720 b^{4}+576 b^{5}-1800 b^{6}-
144 b^{7}-72 b^{8}+174 b^{9}) #1^{8}+(-1280+960 b^{3}-240 b^{6}+20 b^{9}) #1^{9}+(-64+48 b^{3}-12 b^{6}+b^{9})
#1^{10}&,2]<=d<=1&&c>0&&(x==Root[36+24 d+4 d^{2}+(144+48 d) #1+(108+144 b-36 b^{2}+36

```
d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,5]||x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,6])) || (d>1&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,1]||x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,2])))) ||

(9<=b<√265. …&&((0<d<Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576 b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176 b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276 b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9) #1^10&,2]&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,3] ||  x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,4]))||(Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656 b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7) #1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936 b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576 b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176 b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276 b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9) #1^10&,2]<=d<=1&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,5] || x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,6])) ||
(d>1&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,1]||x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,2])))) ||
```

```
(b>=265. ⋯&&((0<d<Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656
b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7)
#1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936
b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-
436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576
b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-
69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176
b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276
b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-
1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-
144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9)
#1^10&,4]&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2
d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d
#1^5+b^3 d^2 #1^6&,3]||x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12
b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2
d #1^5+b^3 d^2 #1^6&,4]))||(Root[-419904 b^2+1306368 b^3-1399680 b^4+559872 b^5-46656
b^6+(1119744 b-3685824 b^2+5365440 b^3-4758912 b^4+2612736 b^5-699840 b^6+46656 b^7)
#1+(-186624-1150848 b+3748032 b^2-3499200 b^3+451008 b^4+1613520 b^5-1251936
b^6+291600 b^7-15552 b^8) #1^2+(324864+103680 b-393984 b^2-1093824 b^3+1710720 b^4-
436752 b^5-533520 b^6+358992 b^7-55728 b^8+1728 b^9) #1^3+(-108864+48384 b-288576
b^2+601344 b^3-6480 b^4-340416 b^5+34560 b^6+166428 b^7-60480 b^8+4212 b^9) #1^4+(-
69696+8640 b+121536 b^2-8640 b^3-189216 b^4+23040 b^5+52272 b^6+26028 b^7-31176
b^8+4356 b^9) #1^5+(44288+13248 b+11520 b^2-59280 b^3-10944 b^4+37872 b^5-588 b^6-3276
b^7-8568 b^8+2485 b^9) #1^6+(3904-4032 b-10944 b^2+7296 b^3+7200 b^4+7632 b^5-6000 b^6-
1548 b^7-1224 b^8+848 b^9) #1^7+(-6528-576 b-1152 b^2+6048 b^3+720 b^4+576 b^5-1800 b^6-
144 b^7-72 b^8+174 b^9) #1^8+(-1280+960 b^3-240 b^6+20 b^9) #1^9+(-64+48 b^3-12 b^6+b^9)
#1^10&,4]<=d<=1&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36
d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d)
#1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,5]||x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36
b^2+36 d-12 b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3
d) #1^4+12 b^2 d #1^5+b^3 d^2 #1^6&,6])) ||
(d>1&&c>0&&(x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12 b^2 d)
#1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2 d
#1^5+b^3 d^2 #1^6&,1]||x==Root[36+24 d+4 d^2+(144+48 d) #1+(108+144 b-36 b^2+36 d-12
b^2 d) #1^2+(144 b+24 d-12 b^3 d+8 d^2-4 b^3 d^2) #1^3+(36 b^2+36 b d-12 b^3 d) #1^4+12 b^2
d #1^5+b^3 d^2 #1^6&,2]))))
```

Analysis of the second derivative for the function in equation (142)

In[2]:= $\text{Reduce}[\{\frac{1}{(6(1+x)^2+d(2+x^3))^3}6(-1+b)c(-6(4+4x+dx^2)((1+b+b^2)d^2x^2+12(1+x)(1+bx)+d(4+4(1+b)x+3(1+b+b^2)x^2+4b(1+b)x^3+b^2x^4))+2(6(1+x)^2+d(2+x^3))((3+d)(2+dx)+b^2dx(3+d+6x+2x^2)+b(6+12x+d^2x+d(2+3x+6x^2))))>0, b>1, c>0, d>0\}, x, \text{PositiveReals}]$

```
Out[2]= (1<b<1.22 && ((Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,5]<d<(9-3 b)/(1+b)&&c>0&&Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,1]<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2]) ||

(d>=(9-3 b)/(1+b)&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2]))) ||

(b==1.22 ⋯&& ((Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,4]<d<(9-3 b)/(1+b)&&c>0&&Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,1]<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2])||

(d>=(9-3 b)/(1+b)&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2]))) ||

(1.22 ⋯<b<1.29 ⋯&&((Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-
```

```
124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,4]<d<=Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,5]&&c>0&&Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,3]<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,4])||

(Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,5]<d<(9-3 b)/(1+b)&&c>0&&Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,1]<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2])

||(d>=(9-3 b)/(1+b)&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2])))||

(b==1.29 …&&((Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416
```

b^{2}-762048 b^{3}+373248 b^{4}+104976 b^{5}) #1^{3}+(124416+787968 b+59616 b^{2}+360288 b^{3}+301968 b^{4}-27216 b^{5}-6561 b^{6}) #1^{4}+(-100224+98496 b-94176 b^{2}+107784 b^{3}-41040 b^{4}-54432 b^{5}-918 b^{6}) #1^{5}+(-28080-34848 b-63648 b^{2}-34128 b^{3}-41616 b^{4}-12816 b^{5}+2943 b^{6}) #1^{6}+(1152+864 b-864b^{2}-1656 b^{3}+432 b^{4}+2160 b^{5}+1800 b^{6}) #1^{7}+(400+1056 b+2112 b^{2}+2368 b^{3}+2112 b^{4}+1056 b^{5}+400 b^{6}) #1^{8}+(32+96 b+192 b^{2}+224 b^{3}+192 b^{4}+96 b^{5}+32 b^{6}) #1^{9}&,4]<d<(9-3 b)/(1+b)&&c>0&&Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,3]<x<Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,4]) ||

(d==(9-3 b)/(1+b)&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,4]) ||

(d>(9-3 b)/(1+b)&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,2]))) ||

(1.29 …<b<=1.55 …&&((Root[-1679616 b^{2}+(2239488 b+1119744 b^{2}+1679616 b^{3}) #1+(-373248-2612736 b+746496 b^{2}-1306368 b^{3}-629856 b^{4}) #1^{2}+(373248-466560 b-124416 b^{2}-762048 b^{3}+373248 b^{4}+104976 b^{5}) #1^{3}+(124416+787968 b+59616 b^{2}+360288 b^{3}+301968 b^{4}-27216 b^{5}-6561 b^{6}) #1^{4}+(-100224+98496 b-94176 b^{2}+107784 b^{3}-41040 b^{4}-54432 b^{5}-918 b^{6}) #1^{5}+(-28080-34848 b-63648 b^{2}-34128 b^{3}-41616 b^{4}-12816 b^{5}+2943 b^{6}) #1^{6}+(1152+864b-864 b^{2}-1656 b^{3}+432 b^{4}+2160 b^{5}+1800 b^{6}) #1^{7}+(400+1056 b+2112 b^{2}+2368 b^{3}+2112 b^{4}+1056 b^{5}+400 b^{6}) #1^{8}+(32+96 b+192 b^{2}+224 b^{3}+192 b^{4}+96 b^{5}+32 b^{6}) #1^{9}&,4]<d<(9-3 b)/(1+b)&&c>0&&Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,3]<x<Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,4])||

((9-3 b)/(1+b)<=d<=Root[-1679616 b^{2}+(2239488 b+1119744 b^{2}+1679616 b^{3}) #1+(-373248-2612736 b+746496 b^{2}-1306368 b^{3}-629856 b^{4}) #1^{2}+(373248-466560 b-124416 b^{2}-762048 b^{3}+373248 b^{4}+104976 b^{5}) #1^{3}+(124416+787968 b+59616 b^{2}+360288 b^{3}+301968 b^{4}-27216 b^{5}-6561 b^{6}) #1^{4}+(-100224+98496 b-94176 b^{2}+107784 b^{3}-41040 b^{4}-54432 b^{5}-918 b^{6}) #1^{5}+(-28080-34848 b-63648 b^{2}-34128 b^{3}-41616 b^{4}-12816 b^{5}+2943 b^{6}) #1^{6}+(1152+864 b-864 b^{2}-1656b^{3}+432 b^{4}+2160 b^{5}+1800 b^{6}) #1^{7}+(400+1056 b+2112 b^{2}+2368 b^{3}+2112 b^{4}+1056 b^{5}+400 b^{6}) #1^{8}+(32+96 b+192 b^{2}+224 b^{3}+192 b^{4}+96 b^{5}+32 b^{6}) #1^{9}&,5]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,4]) ||

(d>Root[-1679616 b^{2}+(2239488 b+1119744 b^{2}+1679616 b^{3}) #1+(-373248-2612736 b+746496 b^{2}-1306368 b^{3}-629856 b^{4}) #1^{2}+(373248-466560 b-124416 b^{2}-762048 b^{3}+373248 b^{4}+104976 b^{5}) #1^{3}+(124416+787968 b+59616 b^{2}+360288 b^{3}+301968 b^{4}-27216 b^{5}-6561 b^{6}) #1^{4}+(-100224+98496 b-94176 b^{2}+107784 b^{3}-41040 b^{4}-54432 b^{5}-918 b^{6}) #1^{5}+(-28080-34848 b-63648 b^{2}-34128 b^{3}-41616 b^{4}-12816 b^{5}+2943 b^{6}) #1^{6}+(1152+864 b-864 b^{2}-1656 b^{3}+432b^{4}+2160 b^{5}+1800 b^{6}) #1^{7}+(400+1056 b+2112 b^{2}+2368 b^{3}+2112 b^{4}+1056 b^{5}+400 b^{6}) #1^{8}+(32+96 b+192 b^{2}+224 b^{3}+192 b^{4}+96 b^{5}+32 b^{6}) #1^{9}&,5]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,2]))) ||

(1.55 ⋯<b<1.70 ⋯&&((Root[-1679616 b^{2}+(2239488 b+1119744 b^{2}+1679616 b^{3}) #1+(-373248-2612736 b+746496 b^{2}-1306368 b^{3}-629856 b^{4}) #1^{2}+(373248-466560 b-124416 b^{2}-762048 b^{3}+373248 b^{4}+104976 b^{5}) #1^{3}+(124416+787968 b+59616 b^{2}+360288 b^{3}+301968 b^{4}-27216 b^{5}-6561 b^{6}) #1^{4}+(-100224+98496 b-94176 b^{2}+107784 b^{3}-41040 b^{4}-54432 b^{5}-918 b^{6}) #1^{5}+(-28080-34848 b-63648 b^{2}-34128 b^{3}-41616 b^{4}-12816 b^{5}+2943 b^{6}) #1^{6}+(1152+864b-864 b^{2}-1656 b^{3}+432 b^{4}+2160 b^{5}+1800 b^{6}) #1^{7}+(400+1056 b+2112 b^{2}+2368 b^{3}+2112 b^{4}+1056 b^{5}+400 b^{6}) #1^{8}+(32+96 b+192 b^{2}+224 b^{3}+192 b^{4}+96 b^{5}+32 b^{6}) #1^{9}&,2]<d<(9-3 b)/(1+b)&&c>0&&Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,3]<x<Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-18 b d-18 b^{2} d-12 d^{2}-12 b d^{2}-12 b^{2} d^{2}-2 d^{3}-2 b d^{3}-2 b^{2} d^{3}) #1+(108+108 b+72 d-36 b^{2} d+12 d^{2}-12 b d^{2}-12 b^{2} d^{2}) #1^{2}+(72 b+48 d+24 b d-18 b^{2} d+16 d^{2}+16 b d^{2}+2 b^{2} d^{2}) #1^{3}+(36 b d+6 d^{2}+6 b d^{2}+6 b^{2} d^{2}+2 d^{3}+2 b d^{3}+2 b^{2} d^{3}) #1^{4}+(6 b d^{2}+6 b^{2} d^{2}) #1^{5}+b^{2} d^{2} #1^{6}&,4]) ||

((9-3 b)/(1+b)<=d<=Root[-1679616 b^{2}+(2239488 b+1119744 b^{2}+1679616 b^{3}) #1+(-373248-2612736 b+746496 b^{2}-1306368 b^{3}-629856 b^{4}) #1^{2}+(373248-466560 b-124416 b^{2}-762048 b^{3}+373248 b^{4}+104976 b^{5}) #1^{3}+(124416+787968 b+59616 b^{2}+360288 b^{3}+301968 b^{4}-27216 b^{5}-6561 b^{6}) #1^{4}+(-100224+98496 b-94176 b^{2}+107784 b^{3}-41040 b^{4}-54432 b^{5}-918 b^{6}) #1^{5}+(-28080-34848 b-63648 b^{2}-34128 b^{3}-41616 b^{4}-12816 b^{5}+2943 b^{6}) #1^{6}+(1152+864b-864 b^{2}-1656 b^{3}+432 b^{4}+2160 b^{5}+1800 b^{6}) #1^{7}+(400+1056 b+2112 b^{2}+2368 b^{3}+2112 b^{4}+1056 b^{5}+400 b^{6}) #1^{8}+(32+96 b+192 b^{2}+224 b^{3}+192 b^{4}+96 b^{5}+32 b^{6}) #1^{9}&,3]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^{2}-4 b d^{2}+(216+54 d-

18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,4])||

(d>Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,3]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2])))

||(1.70<=b<=1.80&&(((9-3b)/(1+b)<d<=Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560b-124416\b^2-762048b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,3]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,4])||(d>Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560b-124416b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,3]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2])))||

(1.80 <b<=2 && (((9-3 b)/(1+b)<d<=Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560b-124416b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-

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41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800
b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192
b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,1]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4
d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3)
#1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2
d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3)
#1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,4]) || (d>Root[-1679616 b^2+(2239488
b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856
b^4) #1^2+(373248-466560b-124416b^2-762048b^3+373248b^4+104976b^5)
#1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-
100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848
b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656
b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400
b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,1]&&c>0&&0<x<Root[108-
36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2
d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72
b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2
d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2]))) ||

(2<b<2.28 …&&(((9-3 b)/(1+b)<d<1&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4
b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3)
#1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2
d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3)
#1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,4])||(d==1&&c>0&&0<x<Root[128-64 b+(256-32
b-32 b^2) #1+(192+96 b-48 b^2) #1^2+(64+112 b-16 b^2) #1^3+(8+44 b+8 b^2) #1^4+(6 b+6
b^2) #1^5+b^2 #1^6&,6])||(1<d<=Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3)
#1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-
124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288
b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040
b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816
b^5+2943 b^6) #1^6+(1152+864b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056
b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96
b^5+32 b^6) #1^9&,1]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-
18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-
36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2
b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2)
#1^5+b^2 d^2 #1^6&,4])||

(d>Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736
b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048
b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-
27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-
918b^6) #1^5+(-28080-34848b-63648b^2-34128b^3-41616b^4-12816b^5+2943b^6) #1^6+(1152+864
b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112
b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6)
#1^9&,1]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18
b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12
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d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2])))||

(b==2.28 …&&(((9-3 b)/(1+b)<d<1&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,4])||

(d==1&&c>0&&0<x<Root[128-64 b+(256-32 b-32 b^2) #1+(192+96 b-48 b^2) #1^2+(64+112 b-16 b^2) #1^3+(8+44 b+8 b^2) #1^4+(6 b+6 b^2) #1^5+b^2 #1^6&,6])||

(1<d<=Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918b^6) #1^5+(-28080-34848b-63648b^2-34128b^3-41616b^4-12816b^5+2943b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056b^5+400b^6#1^8+(32+96b+192b^2+224b^3+192b^4+96b^5+32b^6) #1^9&,3]
&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,4]) ||

(d>Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496b^2-1306368b^3-629856b^4) #1^2+(373248-466560 b-124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400b^6) #1^8+(32+96b+192b^2+224b^3+192b^4+96b^5+32 b^6) #1^9&,3]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,2])))  ||

(2.28 …<b<=3&&(((9-3 b)/(1+b)<d<1&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,4])||

(d==1&&c>0&&0<x<Root[128-64 b+(256-32 b-32 b^2) #1+(192+96 b-48 b^2) #1^2+(64+112 b-16 b^2) #1^3+(8+44 b+8 b^2) #1^4+(6 b+6 b^2) #1^5+b^2 #1^6&,6])||

(1<d<=Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048 b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-
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27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-
918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6)
#1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112
b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32
b^6) #1^9&,1]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-
18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2
d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2)
#1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2
d^2 #1^6&,4])

||( d>Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736
b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048
b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-
27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-
918 b^6) #1^5+(-28080-34848 b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6)
#1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112
b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32
b^6) #1^9&,1]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-
18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2
d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2)
#1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2
d^2 #1^6&,2])))||

(b>3&&((0<d<1&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b
d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2
d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2)
#1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2
d^2 #1^6&,4])||

(d==1&&c>0 && 0<x<Root[128-64 b+(256-32 b-32 b^2) #1+(192+96 b-48 b^2)
#1^2+(64+112b-16b^2) #1^3+(8+44 b+8b^2) #1^4+(6 b+6b^2) #1^5+b^2 #1^6&,6])||(1<d<=Root[-
1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736 b+746496 b^2-
1306368 b^3-629856 b^4 #1^2+(373248-466560 b-124416 b^2-762048 b^3+373248 b^4+104976
b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-27216 b^5-6561 b^6) #1^4+(-
100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-918 b^6) #1^5+(-28080-34848
b-63648 b^2-34128 b^3-41616 b^4-12816 b^5+2943 b^6) #1^6+(1152+864 b-864 b^2-1656
b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112 b^2+2368 b^3+2112 b^4+1056 b^5+400
b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32 b^6) #1^9&,1]&&c>0&& 0<x<Root[108-
36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2
d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2 d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72
b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2) #1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2
d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2 d^2 #1^6&,4])||

(d>Root[-1679616 b^2+(2239488 b+1119744 b^2+1679616 b^3) #1+(-373248-2612736
b+746496 b^2-1306368 b^3-629856 b^4) #1^2+(373248-466560 b-124416 b^2-762048
b^3+373248 b^4+104976 b^5) #1^3+(124416+787968 b+59616 b^2+360288 b^3+301968 b^4-
27216 b^5-6561 b^6) #1^4+(-100224+98496 b-94176 b^2+107784 b^3-41040 b^4-54432 b^5-
918 b^6) #1^5+(-28080-34848 b-63648b^2-34128 b^3-41616b^4-12816 b^5+2943 b^6)
```

```
#1^6+(1152+864 b-864 b^2-1656 b^3+432 b^4+2160 b^5+1800 b^6) #1^7+(400+1056 b+2112
b^2+2368 b^3+2112 b^4+1056 b^5+400 b^6) #1^8+(32+96 b+192 b^2+224 b^3+192 b^4+96 b^5+32
b^6) #1^9&,1]&&c>0&&0<x<Root[108-36 b+24 d-24 b d-4 d^2-4 b d^2+(216+54 d-18 b d-
18 b^2 d-12 d^2-12 b d^2-12 b^2 d^2-2 d^3-2 b d^3-2 b^2 d^3) #1+(108+108 b+72 d-36 b^2
d+12 d^2-12 b d^2-12 b^2 d^2) #1^2+(72 b+48 d+24 b d-18 b^2 d+16 d^2+16 b d^2+2 b^2 d^2)
#1^3+(36 b d+6 d^2+6 b d^2+6 b^2 d^2+2 d^3+2 b d^3+2 b^2 d^3) #1^4+(6 b d^2+6 b^2 d^2) #1^5+b^2
d^2 #1^6&,2])))
```

Analysis of Example 5.3.3

In[1]:= $b = 1.7$

Out[1]=1.7

In[2]:=

$-419904b^2 + 1306368b^3 - 1399680b^4 + 559872b^5 - 46656b^6 + (1119744b - 3685824b^2 + 5365440b^3 - 4758912b^4 + 2612736b^5 - 699840b^6 + 46656b^7)y + (-186624 - 1150848b + 3748032b^2 - 3499200b^3 + 451008b^4 + 1613520b^5 - 1251936b^6 + 291600b^7 - 15552b^8)\mathrm{y}^2 + (324864 + 103680b - 393984b^2 - 1093824b^3 + 1710720b^4 - 436752b^5 - 533520b^6 + 358992b^7 - 55728b^8 + 1728b^9)\mathrm{y}^3 + (-108864 + 48384b - 288576b^2 + 601344b^3 - 6480b^4 - 340416b^5 + 34560b^6 + 166428b^7 - 60480b^8 + 4212b^9)\mathrm{y}^4 + (-69696 + 8640b + 121536b^2 - 8640b^3 - 189216b^4 + 23040b^5 + 52272b^6 + 26028b^7 - 31176b^8 + 4356b^9)\mathrm{y}^5 + (44288 + 13248b + 11520b^2 - 59280b^3 - 10944b^4 + 37872b^5 - 588b^6 - 3276b^7 - 8568b^8 + 2485b^9)\mathrm{y}^6 + (3904 - 4032b - 10944b^2 + 7296b^3 + 7200b^4 + 7632b^5 - 6000b^6 - 1548b^7 - 1224b^8 + 848b^9)\mathrm{y}^7 + (-6528 - 576b - 1152b^2 + 6048b^3 + 720b^4 + 576b^5 - 1800b^6 - 144b^7 - 72b^8 + 174b^9)\mathrm{y}^8 + (-1280 + 960b^3 - 240b^6 + 20b^9)\mathrm{y}^9 + (-64 + 48b^3 - 12b^6 + b^9)\mathrm{y}^{10}$

Out[2]= $337615. \; - 15813.943891201168y - 1164367.812928319y^2 + 245823.61610793322y^3 + 1150198.106012963y^4 - 327878.0257328283y^5 - 196429.24365663505y^6 - 23401.29814878397y^7 - 675.5236182419976y^8 + 15.220969940000487y^9 + 0.7610484969999902y^{10}$

In[3]:= $\mathrm{Reduce}[337615. \; - 15813.943891201168y - 1164367.812928319y^2 + 245823.61610793322y^3 + 1150198.106012963y^4 - 327878.0257328283y^5 - 196429.24365663505y^6 - 23401.29814878397y^7 - 675.5236182419976y^8 + 15.220969940000487y^9 + 0.7610484969999902y^{10} == 0, x]$

Out[3]= $y == -21.430312221711723 - 7.225800470752546i || y == -21.430312221711723 + 7.225800470752546i || y ==$

$-7.023260245462958 - 0.012552824637644264i||y ==$
$-7.023260245462958 + 0.012552824637644264i||y ==$
$-0.7382798495226569 - 0.010296162552550 04i||y ==$
$-0.7382798495226569 + 0.01029616255255004i||y ==$
$0.9052317152276892 \quad - 0.09791517436598843i||y ==$
$0.9052317152276892 \quad + 0.09791517436598843i||y == 1.096655982249612||y ==$
35.47658522068879

In[4]:= $d = 36$

Out[4]= 36

In[5]:= $\text{Reduce}[36(1 + x)(1 + 3x + 3.9099999999999997x^2 + 2.8899999999999997x^3) + 1236(2 + 4x + 0.11000000000000032x^2 - 2.9129999999999994x^3 + 2.8899999999999997x^5 - x^4 1.7(-0.11000000000000032)) + 1296(4 - 11.651999999999997x^3 + 4.912999999999999x^6) == 0, x, \mathbb{R}_{>\ 0}]$

Out[5]= $x == 0.9727532185202368||x == 1.0363909589027955$

In[6]:= FPrime[x_] =

$$\frac{4.199999999999999c\left(7244.639999999999x^2+12(1+x)(1+1.7x)+36\left(4+10.8x+16.77x^2+18.36x^3+2.8899999999999997x^4\right)\right)}{\left(6(1+x)^2+36(2+x^3)\right)^2}$$

Out[6]= $\frac{4.199999999999999c(7244.639999999999x^2+12(1+x)(1+1.7x)+36(4+10.8x+16.77x^2+18.36x^3+2.8899999999999997x^4))}{(6(1+x)^2+36(2+x^3))^2}$

In[7]:= FPrime[0.9727532185202368]

Out[7] $= 2.2166657c$

In[8]:=FPrime[1.0363909589027955]

Out[8]= $2.21682c$

In[9]:=$\text{Reduce}[2.2166657c < 1 < 2.21682c, c, \text{PositiveReals}]$

Out[9]= $0.451097 < c < 0.451128$

In[10]:= $c = 0.45111$

Out[10]= 0.45111

In[11]:=$\text{Reduce}[x == \frac{c(b^3dx^3+6b^2x^2+12bx+(2d+6))}{(dx^3+6x^2+12x+(6+2d))}, x, \text{PositiveReals}]$

Out[11]= $x == 0.9488619401440187||x == 1.0075420674671194||x == 1.0588138196681063$

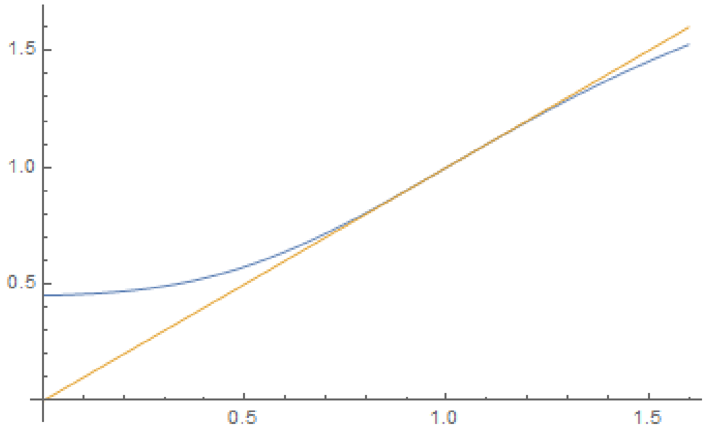
In[12]:=$\text{Reduce}[x == \frac{c(b^3dx^3+6b^2x^2+12bx+(2d+6))}{(dx^3+6x^2+12x+(6+2d))}, x, \text{PositiveReals}]$

Out[11]= $x == 0.9488619401440187||x == 1.0075420674671194||x == 1.0588138196681063$

In[12]:=

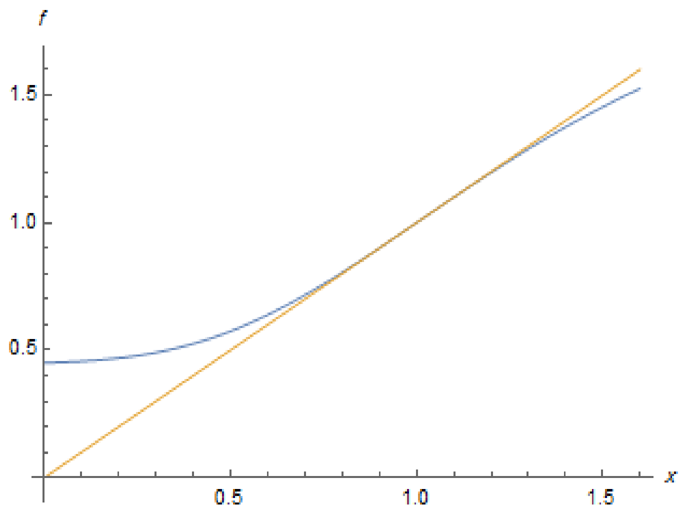
$$\text{Plot}[\{\frac{0.45111(78+20.4x+17.339999999999996x^2+176.86799999999997x^3)}{78+12x+6x^2+36x^3},x\},\{x,0,1.6\}]$$

Out[12]=



In[13]:= Show[%21, AxesLabel → {HoldForm[x], HoldForm[f]}, PlotLabel → None, LabelStyle → {GrayLevel[0]}]

Out[13] =



In[14]:=FPrime[0.948862]

Out[14]= 0.996458

In[15]:=FPrime[1.058814]

Out[15]= 0.997069

Simplifying the recurrence equations

Example : the simplifying process for equation (83)

In[1]:=Simplify$\Big[a^{16}b^{16}c^{24}L_2v_1^4+\frac{256a^8b^8L_2}{v_2^4}+\frac{256a^{10}b^{10}c^6L_2v_1}{v_2^3}+\frac{96a^{12}b^{12}c^{12}L_2v_1^2}{v_2^2}+\frac{16a^{14}b^{14}c^{18}L_2v_1^3}{v_2}+$

$24a^{12}b^{12}c^{16}L_2v_1^3v_3+\frac{1536a^6b^6L_2v_3}{c^2v_2^3}+\frac{1152a^8b^8c^4L_2v_1v_3}{v_2^2}+\frac{288a^{10}b^{10}c^{10}L_2v_1^2v_3}{v_2}+216a^8b^8c^8L_2v_1^2v_3^2+$

$\frac{3456a^4b^4L_2v_3^2}{c^4v_2^2}+\frac{1728a^6b^6c^2L_2v_1v_3^2}{v_2}+864a^4b^4L_2v_1v_3^3+\frac{3456a^2b^2L_2v_3^3}{c^6v_2}+\frac{1296L_2v_3^4}{c^8}+\frac{256L_2}{a^8b^8v_4^4}+\frac{256c^6L_2v_1}{a^2b^2v_4^3}+$

$\frac{1024L_2}{a^4b^4v_2v_4^3}+\frac{1536L_2v_3}{a^6b^6c^2v_4^3}+\frac{96a^4b^4c^{12}L_2v_1^2}{v_4^2}+\frac{1536L_2}{v_2^2v_4^2}+\frac{768a^2b^2c^6L_2v_1}{v_2v_4^2}+\frac{1152c^4L_2v_1v_3}{v_4^2}+\frac{4608L_2v_3}{a^2b^2c^2v_2v_4^2}+\frac{3456L_2v_3^2}{a^4b^4c^4v_4^2}+$

$\frac{16a^{10}b^{10}c^{18}L_2v_1^3}{v_4}+\frac{1024a^4b^4L_2}{v_2^3v_4}+\frac{768a^6b^6c^6L_2v_1}{v_2^2v_4}+\frac{192a^8b^8c^{12}L_2v_1^2}{v_2v_4}+\frac{288a^6b^6c^{10}L_2v_1^2v_3}{v_4}+\frac{4608a^2b^2L_2v_3}{c^2v_2^2v_4}+$

$\frac{2304a^4b^4c^4L_2v_1v_3}{v_2v_4}+\frac{1728a^2b^2c^2L_2v_1v_3^2}{v_4}+\frac{6912L_2v_3^2}{c^4v_2v_4}+\frac{3456L_2v_3^3}{a^2b^2c^6v_4}+4a^8b^8c^{24}L_2v_1^3v_5+\frac{256a^2b^2c^6L_2v_5}{v_2^3}+$

$\frac{192a^4b^4c^{12}L_2v_1v_5}{v_2^2}+\frac{48a^6b^6c^{18}L_2v_1^2v_5}{v_2}+72a^4b^4c^{16}L_2v_1^2v_3v_5+\frac{1152c^4L_2v_3v_5}{v_2^2}+\frac{576a^2b^2c^{10}L_2v_1v_3v_5}{v_2}+$

$432c^8L_2v_1v_3^2v_5+\frac{1728c^2L_2v_3^2v_5}{a^2b^2v_2}+\frac{864L_2v_3^3v_5}{a^4b^4}+\frac{256c^6L_2v_5}{a^{10}b^{10}v_4^3}+\frac{192c^{12}L_2v_1v_5}{a^4b^4v_4^2}+\frac{768c^6L_2v_5}{a^6b^6v_2v_4^2}+\frac{1152c^4L_2v_3v_5}{a^8b^8v_4^2}+$

$\frac{48a^2b^2c^{18}L_2v_1^2v_5}{v_4}+\frac{768c^6L_2v_5}{a^2b^2v_2^2v_4}+\frac{384c^{12}L_2v_1v_5}{v_2v_4}+\frac{576c^{10}L_2v_1v_3v_5}{a^2b^2v_4}+\frac{2304c^4L_2v_3v_5}{a^4b^4v_2v_4}+\frac{1728c^2L_2v_3^2v_5}{a^6b^6v_4}+6c^{24}L_2v_1^2v_5^2+$

$\frac{96c^{12}L_2v_5^2}{a^4b^4v_2^2}+\frac{48c^{18}L_2v_1v_5^2}{a^2b^2v_2}+\frac{72c^{16}L_2v_1v_3v_5^2}{a^4b^4}+\frac{288c^{10}L_2v_3v_5^2}{a^6b^6v_2}+\frac{216c^8L_2v_3^2v_5^2}{a^8b^8}+\frac{96c^{12}L_2v_5^2}{a^{12}b^{12}v_4^2}+\frac{48c^{18}L_2v_1v_5^2}{a^6b^6v_4}+\frac{192c^{12}L_2v_5^2}{a^8b^8v_2v_4}+$

$\frac{288c^{10}L_2v_3v_5^2}{a^{10}b^{10}v_4}+\frac{4c^{24}L_2v_1v_5^3}{a^8b^8}+\frac{16c^{18}L_2v_5^3}{a^{10}b^{10}v_2}+\frac{24c^{16}L_2v_3v_5^3}{a^{12}b^{12}}+\frac{16c^{18}L_2v_5^3}{a^{14}b^{14}v_4}+\frac{c^{24}L_2v_5^4}{a^{16}b^{16}}\Big]$

Out [1]= $\frac{L_2(4a^6b^6c^2v_4+v_2(4a^2b^2c^2+a^8b^8c^8v_1v_4+6a^4b^4v_3v_4+c^8v_4v_5))^4}{a^{16}b^{16}c^8v_2^4v_4^4}$