Modeling of Biological Population Using Fuzzy Differential Equations: Fuzzy Predator-Prey Models and Numerical Solutions

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نمذجة نمو مجتمعات حيوية باستخدام المعادلات التفاضلية الضبابية :

نماذج المفترس و الفريسة الضبابية والحلول العددية

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جامعة فلسطين التقنية– خضوري

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DEDICATION

I would like to dedicate my thesis:

To my father, who always urged me for more work.

To my mother, whose prays were with me all the way to success, who taught me to believe in myself.

To my husband, who has always been supporting and encouraging me.

To my friends, who stood next to me and were always a source of motivation.

To everyone, who always have inspired me.

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LIST OF CONTENTS

Subject **Page**

Committee Decision…………………………………………………….…………… I

Dedication ………………………………………...II

Acknowledgement ………………………………………….………………..…. …. III

List of Content …..………………………………….………………….………. IV

Abstract ………………...……………………………………………...…………VI

1.Introduction ……………………...……………….………………….……...…1

2. Basic Concepts ……………..……………………………………………….…....5

2.1: Preliminaries…………………………………………………………………...5

2.2: Fuzzy Differential Equations and Numerical Methods……..…………….….…10

2.3: Stability of the Equilibrium Point……………………..…………………….…11

2.4: Fuzzy Predator-Prey Models……………………………………………………12

3. An Application of Fuzzy Predator-Prey Model: The Simplest Model…….13

3.1: A Predator-Prey Model with Fuzzy Initial Conditions…...…………….….…13

3.2: A Predator-Prey Model with Fuzzy Parameters and Initial Conditions…...44

3.3: Summary………………………………………………………………………..114

4. Fuzzy Predator-Prey Model with a Functional Response of the Form Arctan(ax)…….……………...…………………..……………………………………115

4.1: Fuzzy Predator-Prey Model with a Functional Response of the Form Arctan(ax) and Fuzzy Initial Conditions…...…………………………………....115

4.2: Fuzzy Predator-Prey Model with a Functional Response of the Form Arctan(ax) and Fuzzy Parameters…………………………………………….…….123

4.3: Summary……………………………………………………………….…….132

5. Conclusions and Comments……………………..……………………...133

References ……………………………………………………….…………………...134
Appendix ...…………………………………………..……….……………………137

Modeling of Biological Population Using Fuzzy Differential Equations: Fuzzy Predator-Prey Models and Numerical Solutions

By: Doa'a Farekh

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Abstract

This thesis considers the application of fuzzy differential equations in modeling of predator and prey populations. When determining the initial populations of predator and prey, uncertainty can arise. We study a predator-prey model with different fuzzy initial populations using many cases of fuzzy numbers. The uncertainty can also arise when determining the birth and death rates of prey and predator, so we construct a fuzzy predator-prey model of fuzzy parameters. To the best of our knowledge, it is the first time to explore a fuzzy predator-prey model with functional response $arctan(ax)$ and we study it with fuzzy initial populations and then with fuzzy parameters. We use generalized Hukuhara derivative and solve all models numerically by Runge-Kutta method. Simulations are made and graphical representations are also provided to show the evolution of both populations over time.

At the end, we discuss the stability of the equilibrium points. From the simulations and graphs, we conclude that the fuzzy solution is not always better than the crisp solution biologically and sometimes they are unacceptable in fuzzy logic and some equilibrium points are unstable. We note that the solutions with triangular fuzzy numbers and shaped triangular fuzzy number are better than those with trapezoidal fuzzy numbers. As the initial populations of the prey and predator are closer to each other, the solution will be better since the lower and upper bounds are equal and positive. When we fuzzify the parameters of predator-prey model, we sometimes don't get a good fuzzy solution. However, as the endpoints of fuzzy numbers are closer, the solution is periodic and the equilibrium points are stable.

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إعداد: دعاء فرخ

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الملخص

تتناول هذه الأطروحة تطبيق المعادلات التفاضلية الضبابية في نمذجة نمو مجتمعات المفترس و الفريسة. من الممكن الحصول على بعض الغموض عند تحديد الشروط الابتدائية لهذه المجتمعات و بالتالي نتناول نموذجًا للمفترس و الفريسة وندرسه مع شروط ابتدائية ضبابية مختلفة لمجتمعاتهم باستخدام حالات مختلفة من الأرقام الغامضة. كما يمكن أن ينشأ بعض الغموض عند تحديد معدلات الولادة والوفاة للفريسة و المفترس ، لذلك نفرض ان هذه المعدلات اعداد ضبابية ونقوم بدراسة النماذج الناتجة. على حد علمنا ، هذه هي المرة الأولى لاستكشاف نموذج مفترس و فريسة جديد بدلالة اقتران معكوس الظل ونرسه بتحويل الشروط الابتدائية لمجتمعاتهم والمعاملات الى اعداد أولية ضبابية. في جميع الحالات نستخدم مشتقة (Generalized Hukuhara) لايجاد الحلول للنماذج، ونجد هذه الحلول عدديًا بطريقة (Runge–Kutta) ومن خلال المحاكاة نستعرض النتائج في جداول ورسومات بيانية لاظهار تطور نمو مجتمعات الفريسة والمفترس مع مرور الوقت. في النهاية ، نناقش استقرار نقاط الاتزان (equilibrium points) لجمبع النماذج.

بعد مناقشة النتائج ومقارنتها ببعضها نستنتج أن النموذج الضبابي ليس دائمًا أفضل من النموذج العادي بيولوجيًا وأحيانًا يكون غير مقبول بالمنطق الضبابي كما ان بعض نقاط الاتزان تكون غير مستقرة. وتبين ان الحلول مع استخدام الأرقام الضبابية المثلثية وشبه المثلثية أفضل من الأرقام الضبابية شبه المنحرفة ، وعندما تكون الاعداد الأولية لمجتمعات الفريسة والمفترس متقاربة ، تكون الحلول أفضل بحيث انه تكون الحدود الدنيا والعليا للحلول متساوية وموجبة مع مرور الزمن اي انها مقبولة بيولوجيا ومن قبل المنطق الضبابي. عندما نفرض معاملات نموذج الفريسة والمفترس أنها أعداد ضبابية، لم نحصل على حل ضبابي جيد في بعض الأوقات. ومع ذلك، كلما كانت أطراف الأعداد الاولية الضبابية أقرب الى المركز، يكون الحل دوريًا ونقاط الاتزان مستقرة.

Chapter 1

Introduction

Fuzzy set theory and its applications have become a subject of increasing interest for many authors. Many articles in different areas were published since introducing the concepts of Fuzzy sets and Probability Measure of Fuzzy Events by Zadeh in 1965 [33-34].

The basic arithmetic structure of fuzzy numbers was later developed by Zadeh [33], Kaufman and Gupta [19], Klir and Yuan [20] and Zimmerman [35]. Also the concepts of derivative of the fuzzy valued functions were introduced by Bede and Gal [5], Bede and Stefanini [7-9], Cano and Flores [11], Gomes and Barros [16], Pirzada and Vakaskar [29], Puri and Ralescu [30] and Stefanini [31].

Puri and Ralescu [30] defined the derivative for fuzzy functions based on the concept of Hukuhara derivative for set-valued functions. The first theorem of existence using this derivative was proposed by Kaleva [17]. In [29], Pirzada and Vakaskar discussed the existence of Hukuhara differentiability of fuzzy valued functions. But it soon appeared that the Hukuhara derivative has a shortcoming which fuzzifies the solution as time goes on. To overcome this situation and to solve this shortcoming, Bede and Gal [5] introduced and studied the generalized concepts of differentiability and as a result the concept of strongly generalized derivative was introduced.

Differential equations are commonly used for modeling real world phenomena. Unfortunately, every time uncertainty can appear with real world problems; the uncertainty can arise from deficient data, measurement errors or when determining initial conditions. Fuzzy set theory is a powerful tool to overcome these problems. The term fuzzy differential equation was used for the first time in 1980 by Kandel and Byatt [18]. Later on, many authors defined fuzzy differential equations with a derivative defined on Hukuhara derivative and its generalizations, see [5-8,11,15,17].

An initial value problem (IVP) is a system of ordinary differential equations together with an initial condition:

$$x'(t) = f\big(t, x(t)\big)\ , x(t_0) = x_0$$

where f is a function of t and $x\ and\ x_0$ is an initial value and $x'(t)$ is derivative of function x with respect to t. Assume that the initial value problem has an uncertain initial value modeled by a fuzzy interval, then we have the following initial value problem:

$$X'(t) = f\big(t, X(t)\big), \qquad X(t_0) = X_0$$

where $f: [0, T] \times R_F^n \to R_F^n$ is a fuzzy interval-valued function and $X_0 \in R_F^n$, R_F^n is the family of all fuzzy subsets of R^n.

Numerical methods have been developed to solve fuzzy differential equation, for example Euler's Method and Runge-Kutta Method, see [1,2,13,21,26].

Mathematical biology is one example employing mathematical tools to model biological phenomena, such as epidemiology problems, population dynamics, ecological systems and genetics, see [14]. As mentioned before, uncertainties are present in the process of modeling. To deal with uncertainties, we use fuzzy differential equations. The employment of fuzzy sets theory is present in many studies in biological problems, see for example [1-3], [15] and [27-28].

One of the mathematical biology models is the predator-prey model (predation), the predation is amongst the oldest in ecology. The Italian mathematician Volterra is said to have developed his ideas about predation from watching the rise and fall of Adriatic fishing fleets. When fishing was good, the number of fishermen increased, drawn by the success of others. After a time, the fish declined, perhaps due to over-harvest, and then the number of fishermen also declined. After some time, the cycle repeated [32].

An organism which feeds on another organism for their food is called predator while the organism that is fed upon is termed as the prey. This kind of interaction between the prey and predator is known as predation. Typically, a predator tends to be larger than that of the prey, and hence they consume many preys during their life cycle. During the act of predation often the death of prey will occur due to the absorption of the prey's tissue by the predator. Typical examples of predation are bats eating the insects, snakes eating mice, and the whales eating the krill [32].

Without the prey the predators will decrease, and without the predator the prey will increase. A mathematical model showing how an ecological balance can be maintained when both are present was proposed in 1925 by Lotka and Volterra. Let $X(t)$ and $Y(t)$ be the population of prey and predator, respectively, at time t. We have the following assumptions:

1. In the absence of the predator the prey grows without bound, thus $\frac{dX}{dt} = aX,\ a > 0$ for $Y = 0$.

2. In the absence of the prey the predator dies out, thus $\frac{dY}{dt} = -cY,\ c > 0$ for $X = 0$.
3. The increase in the number of predators is wholly dependent on the food supply (the prey) and the prey are consumed at a rate proportional to the number of encounters between predators and prey. Encounters decrease the number of prey and increase the number of predators. A fixed proportion of prey is killed in each encounter, and the rate of population growth of the predator is enhanced by a factor proportional to the amount of prey consumed.

As a consequence, we have the equations:

$$\frac{dX}{dt} = aX - bXY$$

$$\frac{dY}{dt} = -cY + dXY \tag{1}$$

The constants a, b, c and d are positive, a and c are the growth rate of the prey and the death rate of the predator, respectively, and b and d are measures of the effect of the interaction between the two species. System (1) is called the simplest model of predator –prey.

What happens for given initial values of $Y > 0$ and $X > 0$? Will the predators eat all of their prey and in turn die out? Will the predators die out because of a too low level of prey and then the prey grows without bound? Will an equilibrium state be reached, or will a cyclic fluctuation of prey and predator occur? [10].

Many articles were published about predator-prey models that answer the previous questions in different cases, for example see [4,14].

Many authors have studied a predator-prey model which takes into account the uncertainty in the initial populations of predator and prey. In their works, the authors gave numerical solutions to differential equations with fuzzy initial conditions and some of them discussed the stability of the solutions [1-3], [24] and [28].

Ahmed and Baets [1] studied a predator-prey population model with fuzzy initial populations of predator and prey. This model was solved numerically by means of a 4th-order Runge-Kutta method. Simulations were made and graphical representations were also provided to show the evolution of both populations over time. In addition to that, the stability of the equilibrium points was also described and they obtained fuzzy stable equilibrium points.

Ahmed and Hasan [2] solved the predator-prey model numerically by means of a fuzzy Euler method. The stability of the new fuzzy model was studied and was shown graphically in the fuzzy phase plane. At the beginning, they obtained unstable fuzzy equilibrium point. This problem arisen due to the cumulative errors generated in each step of the fuzzy Euler method. However, when they used a very small step size, the fuzzy equilibrium point became fuzzy stable.

Akin and Oruc [3] used the concept of generalized differentiability to solve the Lotka–Voltera model and obtained graphical solutions. The uniqueness of the solution of a fuzzy initial value problem was lost when they used the strongly generalized derivative concept, this situation was considered as a disadvantage. Actually, it is not a disadvantage because researchers can choose the best solution which reflects better the behavior of the system under consideration.

In this thesis, chapter 3, we study the fuzzy predator-prey model in [2] with different initial conditions, then give numerical and graphical solutions by using Runge-Kutta method in Matlab [25] and discuss the behavior and the stability of the solutions. Then we construct a predator and prey model with fuzzy birth and death rates. By using Matlab we make simulations and graphical representations and discuss the results.

In chapter 4, we follow the footsteps of [4] where the researchers dealt with the general predator prey model of the form

$$X'(t) = rX(1 - X) - Y\,tan^{-1}(aX)$$

$$Y'(t) = -dY + sY\,tan^{-1}(aX).$$

Where X and Y are the prey and the predator population sizes respectively, r, s, a and d are positive parameters. The researchers established the necessary and sufficient condition for the nonexistence of limit cycles of the model. For first time, we construct a numerical example for the model in [4], after number of attempts, to obtain a model satisfying the existence condition and has a periodic solution and then present the solution numerically and graphically. Then we convert the model to a fuzzy model with fuzzy initial conditions and discuss the results. Finally, we fuzzify the parameters of the model and find the numerical and graphical solutions.

In chapter 5, we give some conclusions and remarks.

Chapter 2

Basic Concepts

2.1 Preliminaries

Definition 1 A $fuzzy\, subset\ A$ of some set Ω is defined by its membership function written $A(x)$ which produces values in $[0,1]$ for all x in Ω. That is $A(x)$ is a function mapping Ω into$[0,1]$. If $A(x)$ is always equal to one or zero then the subset A is said to be crisp (classical) set. In the crisp case , $A(x)$ is called the characteristic function (or indicator function) and it is often denoted by χ_A . If $\chi_A(x)$= 0, then x does not belong to A, whereas if $\chi_A(x)$ = 1, then x belongs to A. The fuzzy subset is a generalization in which an element of Ω has partial membership to A characterized by a degree in the interval $[0,1]$, when $A(x) = 0.6$ we say the membership value of x in A is 0.6 .

Definition 2 Let A be a fuzzy subset of Ω. An $\alpha - level$ of A, written $[A]_\alpha$, is defined as $\{x\epsilon\, \Omega: A(x) \geq \alpha\}$ for $0 < \alpha \leq 1$. $[A]_0$, the $support$ of A is defined as the closure of the union of all the $[A]_\alpha$, for $0 < \alpha \leq 1$. The $core$ of A is the set of all elements in Ω with membership degree in A equal to 1.

Definition 3 A $fuzzy\, number\ N$ is a fuzzy subset of the real numbers satisfying:

1. $\exists x: N(x) = 1$.
2. $[N]_\alpha$is a closed and bounded interval for $0 \leq \alpha \leq 1$.

The family of all fuzzy numbers will be denoted by R_F.

A special type of fuzzy numbers M is called a triangular fuzzy number. M is defined by three numbers $a_1 < a_2 < a_3$ where:

1. $M(x) = 1$ at $x = a_2$.
2. The graph of $M(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_2, a_3]$ the graph is a straight line from $(a_2, 1)$ to $(a_3, 0)$ (3) $M(x) = 0$ for $x \leq a_1$ or $x \geq a_3$.

We write $M = (a_1, a_2, a_3)$ for triangular fuzzy number M. If at least one of the graphs described above is not a straight line (curve), then M is called triangular shaped fuzzy number and we write $M \approx (a_1, a_2, a_3)$.

Another special type of fuzzy numbers M is called a trapezoidal fuzzy number. Here M is defined by four numbers $a_1 < a_2 < a_3 < a_4$ where:

1. $M(x) = 1$ on $[a_2, a_3]$.
2. The graph of $M(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_3, a_4]$ the graph is a straight line from $(a_3, 1)$ to $(a_4, 0)$ (3) $M(x) = 0$ for $x \leq a_1$ or $x \geq a_4$.

We write $M = (a_1, a_2, a_3, a_4)$ for trapezoidal fuzzy number M. If at least one of the graphs described above is not a straight line (curve), then M is called trapezoidal shaped fuzzy number and we write $M \approx (a_1, a_2, a_3, a_4)$. If $M(x) = w < 1\ on\ [a_2, a_3]$, then it is called a generalized trapezoidal fuzzy number.

A fuzzy number is determined by its alpha cuts, $\alpha \in [0,1]$. These alpha cuts satisfy the relation if $\alpha_1 > \alpha_2$ then $[A]_{\alpha_1} \subset [A]_{\alpha_2}$, where $\alpha_1, \alpha_2 \in [0,1]$. More details, properties and operations can be found in [6,7], [10,11] and [20]. Other types of fuzzy numbers and their orders can be found in [12,13] and [22,23].

If u is a fuzzy number, then $[u]_\alpha = [u_{1\alpha}, u_{2\alpha}]$ where $u_{1\alpha} = \min\{s: s \in [u]_\alpha\}$ and $u_{2\alpha} = \max\{s: s \in [u]_\alpha\}$ for each $\alpha \in [0,1]$.

Theorem 1[6,7] Suppose that $u_1, u_2: [0,1] \to R$ satisfy the following conditions:

- u_1 is a bounded increasing function and u_2 is a bounded decreasing function with$u_{1\alpha} \leq u_{2\alpha}$ at $\alpha - level = 1$. .

- for each $k \in (0,1], u_1$ and u_2 are left-continuous functions at $\alpha = k$.

- u_1 and u_2 are right-continuous at $\alpha = 0$.

Then $u: R \to [0,1]$ defined by $u(s) = \sup\{\alpha: u_{1\alpha} \leq s \leq u_{2\alpha}\}$ is a fuzzy number with parameterization $[u_{1\alpha}, u_{2\alpha}]$.

Furthermore, if $u: R \to [0,1]$ is a fuzzy number with parameterization $[u_{1\alpha}, u_{2\alpha}]$, then the functions u_1 and u_2 satisfy the aforementioned conditions.

Definition 4 The complete metric structure on the set of all fuzzy numbers $R_F$is given by the Hausdorff distance mapping $D: R_F \times R_F \to [0, \infty)$ such that $D(u, v) = sup_{0 \leq \alpha \leq 1} \max\{|u_{1\alpha} - v_{1\alpha}|, |u_{2\alpha} - v_{2\alpha}|\}$ for arbitrary fuzzy numbers u and v.

Theorem 2 [6-7] If u and v are two fuzzy numbers, then for each $\alpha \in [0,1]$, we have:

- $[u + v]_\alpha = [u]_\alpha + [v]_\alpha = [u_{1\alpha} + v_{1\alpha}, u_{2\alpha} + v_{2\alpha}]$.

-$[\mu u]_\alpha = \mu[u]_\alpha = [\min\{\mu u_{1\alpha}, \mu\, u_{2\alpha}\}, \max\{\mu u_{1\alpha}, \mu\, u_{2\alpha}\}]$.

- $[uv]_\alpha =$

$[\min\{u_{1\alpha}v_{1\alpha}, u_{1\alpha}v_{2\alpha}, u_{2\alpha}v_{1\alpha}, u_{2\alpha}v_{2\alpha}\}, \max\{u_{1\alpha}v_{1\alpha}, u_{1\alpha}v_{2\alpha}, u_{2\alpha}v_{1\alpha}, u_{2\alpha}v_{2\alpha}\}]$.

Definition 5 Let $u, v \in R_F$. If there exists an element $w \in R_F$ such that $u = v + w$, then w is called the Hukuhara difference (H-difference) of u and v, denoted by $u \ominus v$.

Remark 1

1. This difference is not defined for pairs of fuzzy numbers such that the support of a fuzzy number has a bigger diameter than the one that is subtracted.
2. The H-difference has the property $u \ominus v = \{0\}$. So ,$u \ominus u = \{0\}$.
3. $(u + v) \ominus v = u$
4. The H-difference is unique and its α – level is $[u \ominus v]_\alpha = [u_{1\alpha} - v_{1\alpha}, u_{2\alpha -} v_{2\alpha}]$

Many authors proposed two new definitions for difference of fuzzy numbers, which generalize the H-difference.

Definition 6 Let $u, v \in R_F$. The generalized Hukuhara difference (gH-difference) $u \ominus_{gH} v = w$, where $w \in R_F$, if it exists, such that: (1) $u = v + w$ or (2) $v = u - w$.

Remark 2

1. The gH-difference is more general than H-difference. If the H-difference exists then the gH-difference will exist and $u \ominus_{gH} v = u \ominus v$.
2. $[u \ominus_{gH} v]_\alpha = [min\{u_{1\alpha} - v_{1\alpha}, u_{2\alpha -} v_{2\alpha}\}, \max\{u_{1\alpha} - v_{1\alpha}, u_{2\alpha -} v_{2\alpha}\}]$
3. The conditions for existence of $u \ominus_{gH} v = w$ are
   - Case(1): $c_{1\alpha} = u_{1\alpha} - v_{1\alpha}\ and\ c_{2\alpha} = u_{2\alpha} - v_{2\alpha}$ with $c_{1\alpha}\ increasing\ , c_{2\alpha}\ decreasing, c_{1\alpha} \le c_{2\alpha}\ , for\ all\ \alpha \in [0,1]$.
   - Case(2): $c_{1\alpha} = u_{2\alpha} - v_{2\alpha}\ and\ c_{2\alpha} = u_{1\alpha} - v_{1\alpha}$ with $c_{1\alpha}\ increasing\ , c_{2\alpha}\ decreasing, c_{1\alpha} \le c_{2\alpha}\ , for\ all\ \alpha \in [0,1]$.
4. $u \ominus_{gH} u = \{0\}$.
5. $(u + v) \ominus_{gH} v = u$.

Definition 7 Let $u, v \in R_F$. The generalized difference (g-difference) $u \ominus_g v = w$, where $w \in R_F$, if it exists, with $\alpha - level$ $\left[u \ominus_g v\right]_\alpha = cl\left(\cup_{\beta \geq \alpha} [u]_\beta \ominus_{gH} [v]_\beta\right), \forall \alpha \in [0,1]$.

Remark 3

1. The g-difference is more general than gH-difference. If the gH-difference exist, then the g-difference exists and it is the same.
2. $[u \ominus_g v]_\alpha = [\inf_{\beta \geq \alpha} min\{u_{1\alpha} - v_{1\alpha}, u_{2\alpha -} v_{2\alpha}\}, \sup_{\beta \geq \alpha} \max\{u_{1\alpha} - v_{1\alpha}, u_{2\alpha -} v_{2\alpha}\}]$.

Gomes and Barros in [16] showed that the g-difference is not defined for every pair of fuzzy numbers by a counter example. They also showed that a convexification is needed in order to assure that the result is a fuzzy number and they suggest a new definition for the g-difference using the convex hull (conv).

$$\left[u \ominus_g v\right]_\alpha = cl\left(conv \cup_{\beta \geq \alpha} [u]_\beta \ominus_{gH} [v]_\beta\right), \forall \alpha \in [0,1].$$

Definition 8 Let $f: [a, b] \to R_F$. f is Hukuhara differentiable (H- differentiable) at x_0 if the limits:

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} \text{ and } \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h}$$

exist and equal.

Remark 4 Let $f, g: [a, b] \to R_F$

1. $[f'_H(x_0)]_\alpha = [f'_{1\alpha}(x_0), f'_{2\alpha}(x_0)]$
2. Let f and g are H-differentiable, then
   - $(f + g)'_H = f'_H + g'_H$
   - $(\lambda f)'_H = \lambda f'_H$
3. The H-difference doesn't always exist, so the H-differentiable doesn't always exist.
4. Let $f(x) = c \odot g(x)$ where $f: [a, b] \to R_F$, $c \in R_F$, for all $x \in [a, b]$, and let $g: [a, b] \to R_+$ be differentiable at $x_0 \in [a, b] \subset R_+$.
   If $g'(x_0) > 0$ then f is $H - differentiable$ at x_0 $with$ $f'(x) = c \odot g'(x)$. But if $g'(x) < 0$ then f is not $H - differentiable$ [29].

Definition 9 Let $f: [a, b] \to R_F$. f is strongly generalized differentiable (GH-differentiable) at x_0 if the limits of some pair of the following exist and equal:

1. $\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h}$ and $\lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h}$.

2. $\lim_{h\to 0^+} \frac{f(x_0)\ominus f(x_0+h)}{-h}$ and $\lim_{h\to 0^+} \frac{f(x_0-h)\ominus f(x_0)}{-h}$.

3. $\lim_{h\to 0^+} \frac{f(x_0+h)\ominus f(x_0)}{h}$ and $\lim_{h\to 0^+} \frac{f(x_0-h)\ominus f(x_0)}{-h}$.

4. $\lim_{h\to 0^+} \frac{f(x_0)\ominus f(x_0+h)}{-h}$ and $\lim_{h\to 0^+} \frac{f(x_0)\ominus f(x_0-h)}{h}$.

More about Fuzzy calculus can be found in [15].

Definition 10 Let $f:[a,b]\to R_F$. f is (1)-differentiable on $[a\,,b]$ if f is differentiable in the sense (1) of definition 9. Similarly, f is (2)-differentiable on $[a\,,b]$ if f is differentiable in the sense (2) of definition 9.

Theorem 3 Let $f:[a,b]\to R_F$. Where $[f(x)]_\alpha = [f_{1\alpha}(x), f_{2\alpha}(x)]$ for each $\alpha \in [0,1]$

1. If f is (1)-differentiable , then $f_{1\alpha}$ and $f_{2\alpha}$ are differentiable functions and $[f'(x)]_\alpha = [f'_{1\alpha}(x), f'_{2\alpha}(x)]$.
2. If f is (2)-differentiable , then $f_{1\alpha}$ and $f_{2\alpha}$ are differentiable functions and $[f'(x)]_\alpha = [f'_{2\alpha}(x), f'_{1\alpha}(x)]$.

Definition 11 Let $f:[a,b]\to R_F$. $f\ is$ generalized Hukuhara differentiable (gH-differentiable) at x_0 if the limit $\lim_{h\to 0} \frac{f(x_0+h)\ominus_{gH} f(x_0)}{h}$ exist and belong to R_F and $f'_{gH}(x_0)$ is the generalized Hukuhara derivative (gH- derivative) of f at x_0.

Theorem 4 Let $f:[a,b]\to R_F$. Where $[f(x)]_\alpha = [f_{1\alpha}(x), f_{2\alpha}(x)]$ for each $\alpha \in [0,1]$, such that the functions $f_{1\alpha}(x)$ and $f_{2\alpha}(x)$ are real-valued functions, differentiable with respect to x, uniformly in $\alpha \in [0,1]$. Then the function $f(x)$ is gH-differentiable at a fixed $x \in [a,b]$ if and only if one of the following two cases holds:

a. $f'_{1\alpha}(x)$ is increasing, $,f'_{2\alpha}(x)$ is decreasing as functions of $\alpha,$ and $f'_{1\alpha}(x) \le f'_{2\alpha}(x)$ at $\alpha - level = 1$.

b. $f'_{1\alpha}(x)$ is decreasing, $,f'_{2\alpha}(x)$ is increasing as functions of $\alpha,$ and $f'_{2\alpha}(x) \le f'_{1\alpha}(x)$ at $\alpha - level = 1$.

Moreover, $\left[f'_{gH}(x)\right]_\alpha = \left[\min\left\{f'_{1\alpha}(x), f'_{2\alpha}(x)\right\}, \max\left\{f'_{1\alpha}(x), f'_{2\alpha}(x)\right\}\right]$.

Definition 12 Let $f:[a,b]\to R_F$ and $x_0 \in (a,b)$ with $f_{1\alpha}(x)$ and $f_{2\alpha}(x)$ both differentiable at x_0. We say that :

1. f is (1)-differentiable at x_0 if $\left[f'_{gH}(x_0)\right]_\alpha = \left[f'_{1\alpha}(x_0), f'_{2\alpha}(x_0)\right]$.
2. f is (2)-differentiable at x_0 if $\left[f'_{gH}(x_0)\right]_\alpha = \left[f'_{2\alpha}(x_0), f'_{1\alpha}(x_0)\right], \forall\alpha \in [0,1]$.

Remark 5 In [9], Bede and Stefanini showed that the gH-differentiability concept is more general than the GH-differentiability by giving a counter example.

Definition 13 Let $f: [a,b] \to R_F$. f *is* generalized differentiable (g- differentiable) at x_0 if the $\lim_{h\to 0} \frac{f(x_0+h)\ominus_g f(x_0)}{h}$ exist and belong to R_F and $f'_g(x_0)$ is the generalized derivative (g-derivative) of f at x_0. Moreover,

$$\left[f'_g(x)\right]_\alpha = \left[\inf_{\beta\geq\alpha} \min\left\{f'_{1\alpha}(x), f'_{2\alpha}(x)\right\}, \sup_{\beta\geq\alpha} \max\left\{f'_{1\alpha}(x), f'_{2\alpha}(x)\right\}\right].$$

For more details, they can be found in [5, 6,8,9,11,15,16,20,29,31].

2.2 Fuzzy Differential Equations and Numerical Methods

Consider that the classical initial value problem (IVP)

$$x'(t) = f\big(t, x(t)\big) \ , x(t_0) = x_0 \tag{2}$$

where f is a function of t and x *and* x_0 is an initial value and $x'(t)$ is derivative of function x with respect to t. Assume that the initial value x_0 is a fuzzy number, then we have the following fuzzy initial value problem (FIVP):

$$X'(t) = f\big(t, X(t)\big),\ X(t_0) = X_0 \tag{3}$$

where $f: [0,T] \times R_F^n \to R_F^n$ is a fuzzy interval-valued function and $X_0 \in R_F^n$.

The topics of numerical methods for solving fuzzy differential equations (FDE) have been rapidly growing in recent years. Some authors used numerical methods for FDE such as the fuzzy Euler method, Runge-Kutta method, as in [1,2,13,21,26]. In [13], they extended Runge-Kutta method for solving FDE numerically under generalized differentiability. They also compared the errors of generalized Runge-Kutta and Euler methods and observed that the error of generalized Runge-Kutta method was less than the generalized Euler method; that is, the generalized Runge-Kutta method was better than generalized Euler method.

In our thesis we will solve FDE's by converting a fuzzy system to a system of ODE's and use Matlab with solver ode45. ode45 can only solve a first order ODE. Therefore, to solve a higher order ODE, the ODE has to be first converted to a set of first order ODE's. It uses six stages, provides fourth and fifth order formulas of Runge-Kutta method. It compares methods of orders four and five to estimate error and determine step size. The fourth order Runge-Kutta method, the most widely used is the following:

Given the IVP: $x' = f\big(t, x(t)\big)$with $x(t_0) = x_0$ and h a step size, we compute:

$$k_1 = hf(t_i, x_i)$$
$$k_2 = hf(t_i + \frac{h}{2}, x_i + \frac{k_1}{2})$$
$$k_3 = hf(t_i + \frac{h}{2}, x_i + \frac{k_2}{2})$$
$$k_4 = hf(t_i + h, x_i + k_3)$$
$$x_{i+1} = x_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$\text{For } i = 0,1, \dots, n-1.$$

2.3 Stability of the Equilibrium Point

Definition 14 The system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

is called an **autonomous** system of differential equations. In such a system, the independent variable t is absent (i.e., t does not appear explicitly). The values of (x, y) for which $f(x, y) = 0$ and $g(x, y) = 0$ are called the **equilibrium points**, of the system. Hence, there is no change occurs in either the x or y. The **stability** discusses the behavior of the curves near an equilibrium point.

Proposition 1 [24] x is an equilibrium point of (2) if and only if $\chi_{\{x\}}$ is an equilibrium point of (3), where $\chi_{\{x\}}$ is the characteristic function of x.

In order to determine the stability of the equilibrium points of (3), start with fuzzy initial values near those equilibrium points. In this case, one of the following three possibilities can take place:

1. If the fuzzy initial values are sufficiently close to the fuzzy equilibrium points and stay close when t increases, then the fuzzy equilibrium points are said to be fuzzy stable.
2. If the fuzzy initial values are sufficiently close to the fuzzy equilibrium points and approach them when t approaches infinity, then the fuzzy equilibrium points are said to be asymptotically fuzzy stable.
3. If the fuzzy initial values are sufficiently close to the fuzzy equilibrium points and move away from them when t increases, then the fuzzy equilibrium points are said to be fuzzy unstable.

2.4 Fuzzy Predator-Prey Models

In data collection, both populations are nearly always affected by uncertainty. For the preliminary case, we assume that the initial populations of predator and prey are fuzzy and the parameters remain crisp numbers. Thus model (1) becomes:

$$\frac{dX}{dt} = aX - bXY$$

$$\frac{dY}{dt} = -cY + dXY$$

$$X(t_0) = X_0 \; and \; Y(t_0) = Y_0 \qquad (4)$$

where X_0 and Y_0 are fuzzy numbers and a, b, c and d are positive real (crisp) numbers.

In chapter 3 we study the fuzzy predator-prey model which was presented in [2] and solved by Euler method. We will study this model for different cases of fuzzy numbers for the initial conditions and analyze them.

Chapter 3

An Application of Fuzzy Predator-Prey Model: The Simplest Model.

3.1: A Predator-Prey Model with Fuzzy Initial Conditions

Consider the following predator- prey model:

$$x'(t) = x - 0.03xy$$

$$y'(t) = -0.4\,y + 0.01xy$$

With initial conditions:

$$x_o = 15\,,\ y_o = 15 \qquad (5)$$

Where $x(t)$ and $y(t)$ are numbers of prey and predator at time t, respectively. The equilibrium points of the model are the points at which the derivatives equal to zero. Solving the resulting system, model (5) has two equilibrium points $(0,0)$ and $(40,33.33)$. We solve the model numerically by Runge-Kutta method in Matlab. The solution for model (5) which we called it the crisp (classical) solution for the time interval $[0,100]$ is given in figure (3.1.1) and table (3.1.1).

Table (3.1.1): The crisp solution of (5)

Time	$x(t)$	$y(t)$
0.0000	15.0000	15.0000
5.0000	74.3290	65.9610
10.0000	8.6870	20.3800
15.0000	125.5500	30.9900
20.0000	6.2827	29.2910
25.0000	81.9850	14.2220
30.0000	6.7937	43.3010
35.0000	39.1410	11.4180
40.0000	13.1750	62.1540
45.0000	18.3480	13.1480
50.0000	48.1690	73.3960
55.0000	9.5843	17.9990
60.0000	126.3700	40.5810
65.0000	6.2694	26.4420
70.0000	95.1240	15.6750
75.0000	6.0738	40.0000
80.0000	44.4160	11.1530
85.0000	10.8280	59.3380
90.0000	19.6920	12.4410
95.0000	39.9810	74.8640
100.0000	9.5716	17.0560

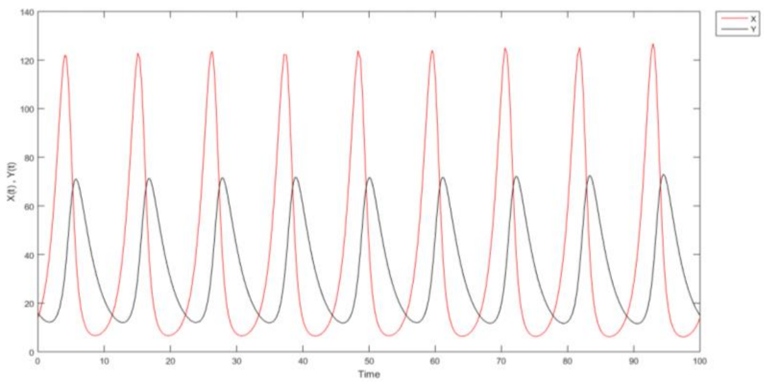


Figure (3.1.1): The crisp solution

The first equilibrium point is uninteresting because there are no populations to observe in the model. It means that the predator populations can only grow if there is not any predator to begin with, and the same holds for the prey populations. However, the second equilibrium point is of interest. From the previous table and figure, we can note that the solution is periodic about the equilibrium point $(40{,}33.33)$, so this point is stable.

Now, we want to convert model (5) to a fuzzy model by assuming that x_0 and y_0 are fuzzy numbers.

For case 1: we convert the initial conditions to triangular fuzzy numbers as follows: $[x_0]_\alpha = [14+\alpha, 16-\alpha]$, $[y_0]_\alpha = [14+\alpha, 16-\alpha]$. Let the α-level intervals of X(t) and Y(t) be $[X(t)]_\alpha = [u(t), v(t)]$ and $[Y(t)]_\alpha = [r(t), s(t)]$, respectively. First we find the generalized Hukuhara derivatives of X(t) and Y(t):

$$[X'_{gH}(t)]_\alpha = \left[\min_{x\epsilon[X(t)]_\alpha, y\epsilon[Y(t)]_\alpha}\{X-0.03XY\}, \max_{x\epsilon[X(t)]_\alpha, y\epsilon[Y(t)]_\alpha}\{X-0.03XY\}\right]$$

$$[Y'_{gH}(t)]_\alpha = \left[\min_{x\epsilon[X(t)]_\alpha, y\epsilon[Y(t)]_\alpha}\{-0.4Y+0.01XY\}, \max_{x\epsilon[X(t)]_\alpha, y\epsilon[Y(t)]_\alpha}\{-0.4Y-0.01XY\}\right]$$

Second, we assume that X(t) and Y(t) are (1)-differentiable, we called this form (1,1)-differentiable, then $[X'_{gH}(t)]_\alpha = [u'(t), v'(t)]$ and $[Y'_{gH}(t)]_\alpha = [r'(t), s'(t)]$. So, model (5) becomes a system of ordinary differential equations with four equations and four variables:

$$\begin{gathered} u' = u - 0.03vs \\ v' = v - 0.03ur \\ r' = -0.4s + 0.01ur \\ s' = -0.4r + 0.01vs \\ u_0 = 14+\alpha, v_0 = 16-\alpha, r_0 = 14+\alpha, s_0 = 16-\alpha \end{gathered} \tag{6}$$

Third we solve (6) by Runge-Kutta method in Matlab using the numerical solver ode45 at α-level= 0,0.5,1. The model has two fuzzy equilibrium points: $\chi_{(40,33.33)}$and $\chi_{(0,0)}$. At α-level = 0, the solution is table (3.1.2), where its graph is figure (3.1.2):

Table (3.1.2): The solution of (6) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	15.6670	18.8473	12.9332	15.3332
0.5000	17.4820	22.4482	11.9394	14.8656
0.7500	19.3670	27.0264	10.9946	14.6262
1.0000	21.1660	32.8839	10.0663	14.6636
1.2500	22.5880	40.4312	9.1092	15.0576
1.5000	23.0900	50.2313	8.0569	15.9429
1.7500	21.6840	63.0542	6.8114	17.5501
2.0000	16.5030	79.9389	5.2282	20.2979
2.2500	3.9817	102.2151	3.1076	24.9730
2.5000	-23.1180	131.3063	0.2211	33.1748
2.7500	-80.1550	167.8434	-3.5389	48.3322
3.0000	-200.9900	208.9989	-7.7090	78.0937
3.2500	-459.9900	243.2808	-10.6746	139.1784
3.5000	-996.8000	248.0578	-10.8598	261.0619
3.7500	-1958.0000	194.5493	-9.6853	462.4357
4.0000	-3141.1000	54.2850	-8.4039	644.4873
4.2500	-3811.0000	-154.5092	-6.3015	573.3541
4.5000	-3961.8000	-360.5598	-3.3223	299.5329
4.7500	-4387.0000	-535.8965	-1.0252	97.3433
5.0000	-5364.7000	-709.0471	-0.1807	20.6269

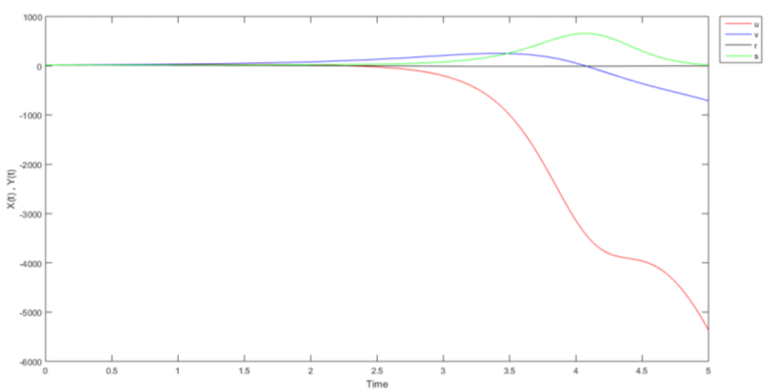


Figure (3.1.2): The solution of (6) at α=0

At α-level = 0.5, the solution is table (3.1.3) and figure (3.1.3):

Table (3.1.3): The solution of (5) at α=0.5

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.5000	15.5000	14.5000	15.5000
0.2500	16.4710	18.0610	13.5310	14.7310
0.5000	18.7500	21.2330	12.6640	14.1270
0.7500	21.3450	25.1750	11.8880	13.7040
1.0000	24.2330	30.0930	11.1860	13.4850
1.2500	27.3340	36.2610	10.5380	13.5130
1.5000	30.4630	44.0450	9.9108	13.8570
1.7500	33.2370	53.9510	9.2550	14.6320
2.0000	34.8990	66.6840	8.4874	16.0400
2.2500	33.9750	83.2450	7.4694	18.4430
2.5000	27.4970	105.0500	5.9723	22.5400
2.7500	9.3367	134.0200	3.6428	29.7540
3.0000	-34.1760	172.1200	0.0506	43.2790
3.2500	-135.3600	219.2200	-4.9244	70.7630
3.5000	-373.4000	265.9300	-9.8945	131.0700
3.7500	-924.6000	284.0100	-11.6260	264.8900
4.0000	-2032.8000	232.6300	-10.4330	516.5400
4.2500	-3501.1000	71.1710	-9.0457	775.6500
4.5000	-4214.1000	-178.8300	-6.7784	684.2800
4.7500	-4177.2000	-415.8800	-3.4036	322.2900
5.0000	-4553.8000	-608.5800	-0.9155	89.1400

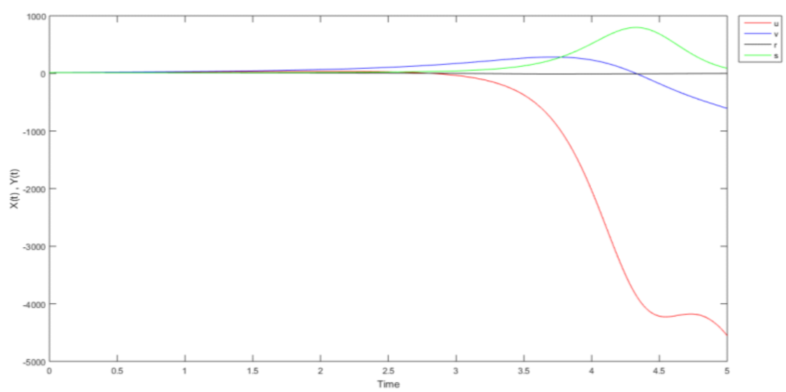


Figure (3.1.3): The solution of (6) at α=0.5

At α-level = 1, the solution is table (3.1.4) and figure (3.1.4):

Table (3.1.4): The solution of (6) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
5.0000	74.3290	74.3290	65.9610	65.9610
10.0000	8.6870	8.6870	20.3800	20.3800
15.0000	125.5500	125.5500	30.9900	30.9900
20.0000	6.2827	6.2827	29.2910	29.2910
25.0000	81.9850	81.9850	14.2220	14.2220

30.0000	6.7937	6.7937	43.3010	43.3010
35.0000	39.1410	39.1410	11.4180	11.4180
40.0000	13.1750	13.1750	62.1540	62.1540
45.0000	18.3480	18.3480	13.1480	13.1480
50.0000	48.1690	48.1690	73.3960	73.3960
55.0000	9.5843	9.5843	17.9990	17.9990
60.0000	126.3700	126.3700	40.5810	40.5810
65.0000	6.2694	6.2694	26.4420	26.4420
70.0000	95.1240	95.1240	15.6750	15.6750
75.0000	6.0738	6.0738	40.0000	40.0000
80.0000	44.4160	44.4160	11.1530	11.1530
85.0000	10.8280	10.8280	59.3380	59.3380
90.0000	19.6920	19.6920	12.4410	12.4410
95.0000	39.9810	39.9810	74.8640	74.8640
100.0000	9.5716	9.5716	17.0560	17.0560

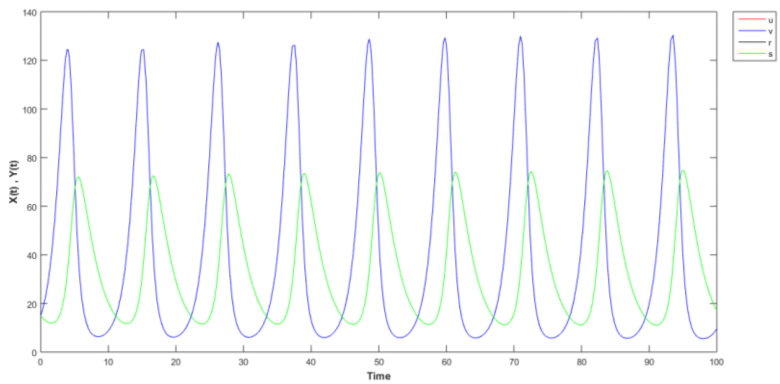


Figure (3.1.4): The solution of (6) at α=1

From previous tables and figures, we can note that when $\alpha < 1$, the solutions of $u(t), v(t) and\ r(t) \to -\infty\ as\ t \to \infty$. So, there are no acceptable solutions for $x(t)\ and\ y(t)$ since $x(t)\ and\ y(t)$ are numbers of populations which can't be negative. Also we can note that the interesting equilibrium point is fuzzy unstable. When $\alpha = 1$, we obtain solution equivalent to the crisp solution with stable equilibrium point $\chi_{(40,33.33)}$.

Now, If $x(t)$ is (1)-differentiable and $y(t)$ is (2)- differentiable, form (1,2)-differentiable, then $[X'(t)]_\alpha = [u'(t), v'(t)]$ and $[Y'(t)]_\alpha = [s'(t), r'(t)]$ then the model becomes as follow:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \tag{7}$$

We solve (7) by Runge-Kutta method in Matlab at different α-levels. At α-level = 0, the solution is figure (3.1.5)

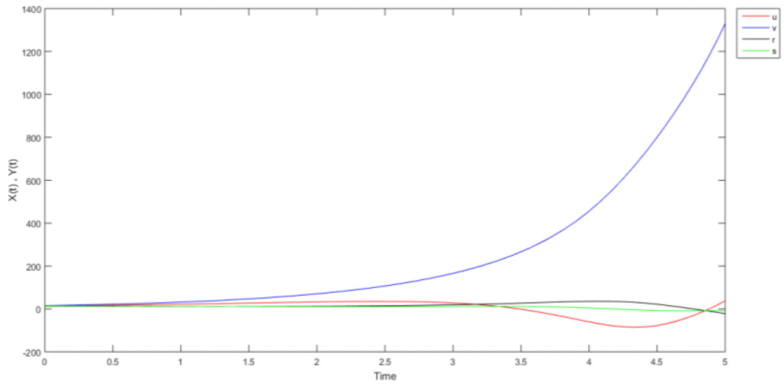


Figure (3.1.5): The solution of (7) at α=0

At α-level = 0.5, the solution is figure (3.1.6):

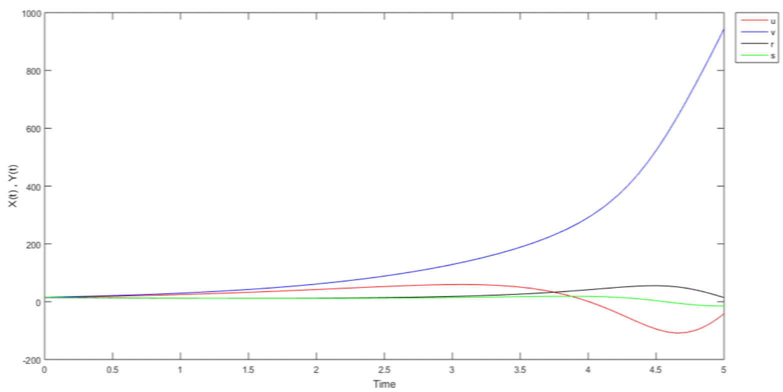


Figure (3.1.6): The solution of (7) at α=0.5

At α-level = 1, the solution is table (3.1.5) and figure (3.1.7):

Table (3.1.5): The solution of (6) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
5.0000	74.3290	74.3290	65.9610	65.9610
10.0000	8.6870	8.6870	20.3800	20.3800
15.0000	125.5500	125.5500	30.9900	30.9900
20.0000	6.2827	6.2827	29.2910	29.2910
25.0000	81.9850	81.9850	14.2220	14.2220
30.0000	6.7937	6.7937	43.3010	43.3010
35.0000	39.1410	39.1410	11.4180	11.4180
40.0000	13.1750	13.1750	62.1540	62.1540

45.0000	18.3480	18.3480	13.1480	13.1480
50.0000	48.1690	48.1690	73.3960	73.3960
55.0000	9.5843	9.5843	17.9990	17.9990
60.0000	126.3700	126.3700	40.5810	40.5810
65.0000	6.2694	6.2694	26.4420	26.4420
70.0000	95.1240	95.1240	15.6750	15.6750
75.0000	6.0738	6.0738	40.0000	40.0000
80.0000	44.4160	44.4160	11.1530	11.1530
85.0000	10.8280	10.8280	59.3380	59.3380
90.0000	19.6920	19.6920	12.4410	12.4410
95.0000	39.9810	39.9810	74.8640	74.8640
100.0000	9.5716	9.5716	17.0560	17.0560

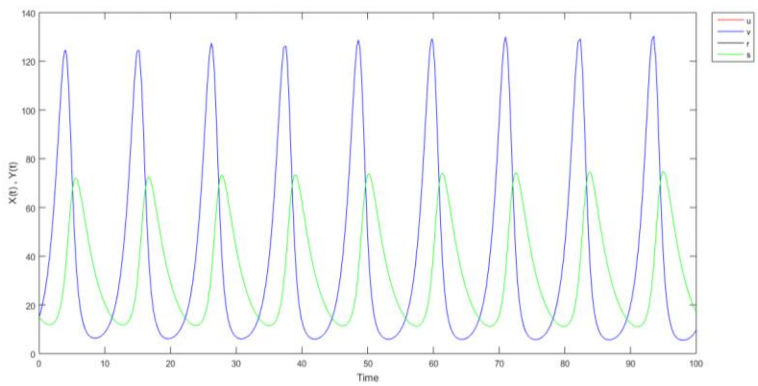


Figure (3.1.7): The solution of (7) at α=1

One can conclude from the previous figures that when $x(t)$ is (1)-differentiable and $y(t)$ is (2)- differentiable and for $\alpha < 1$ there is no fuzzy solution for $y(t)$ since $r(t) > s(t)$ for some time intervals but there is a fuzzy solution for $x(t)$. However, the solutions are unacceptable due to the presence of negative values and the equilibrium point $\chi_{(40,33.33)}$ is fuzzy unstable. When $\alpha = 1$, the solution is the crisp one and $\chi_{(40,33.33)}$ is fuzzy stable equilibrium point.

Now, If $x(t)$ is (2)-differentiable and $y(t)$ is (1) - differentiable, form (2,1)-differentiable, then $[X'(t)]_\alpha = [v'(t), u'(t)]$ and $[Y'(t)]_\alpha = [r'(t), s'(t)]$ then the model becomes:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (8)$$

We solve (8) by Runge-Kutta method in Matlab at α-levels= 0,0.5,1. At α-level = 0, the solution is figure (3.1.8).

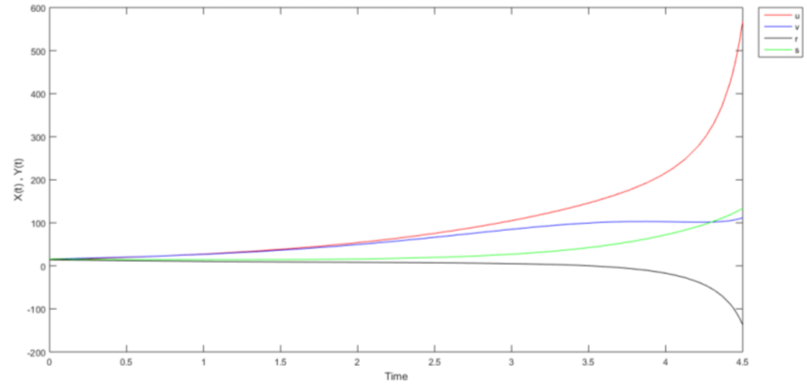


Figure (3.1.8): The solution of (8) at α=0

At α-level = 0.5, the solution is figure (3.1.9)

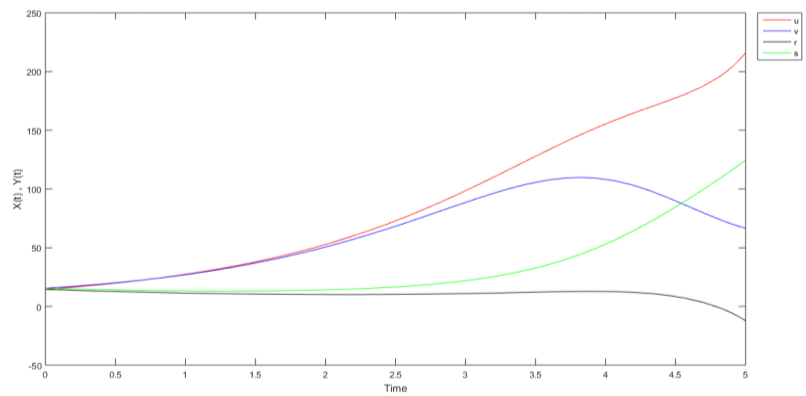


Figure (3.1.9): The solution of (8) at α=0.5

At α-level = 1, the solution is table (3.1.6) and figure (3.1.10)

Table (3.1.6): The solution of (6) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
5.0000	74.3290	74.3290	65.9610	65.9610
10.0000	8.6870	8.6870	20.3800	20.3800
15.0000	125.5500	125.5500	30.9900	30.9900
20.0000	6.2827	6.2827	29.2910	29.2910
25.0000	81.9850	81.9850	14.2220	14.2220
30.0000	6.7937	6.7937	43.3010	43.3010
35.0000	39.1410	39.1410	11.4180	11.4180
40.0000	13.1750	13.1750	62.1540	62.1540
45.0000	18.3480	18.3480	13.1480	13.1480
50.0000	48.1690	48.1690	73.3960	73.3960
55.0000	9.5843	9.5843	17.9990	17.9990
60.0000	126.3700	126.3700	40.5810	40.5810

65.0000	6.2694	6.2694	26.4420	26.4420
70.0000	95.1240	95.1240	15.6750	15.6750
75.0000	6.0738	6.0738	40.0000	40.0000
80.0000	44.4160	44.4160	11.1530	11.1530
85.0000	10.8280	10.8280	59.3380	59.3380
90.0000	19.6920	19.6920	12.4410	12.4410
95.0000	39.9810	39.9810	74.8640	74.8640
100.0000	9.5716	9.5716	17.0560	17.0560

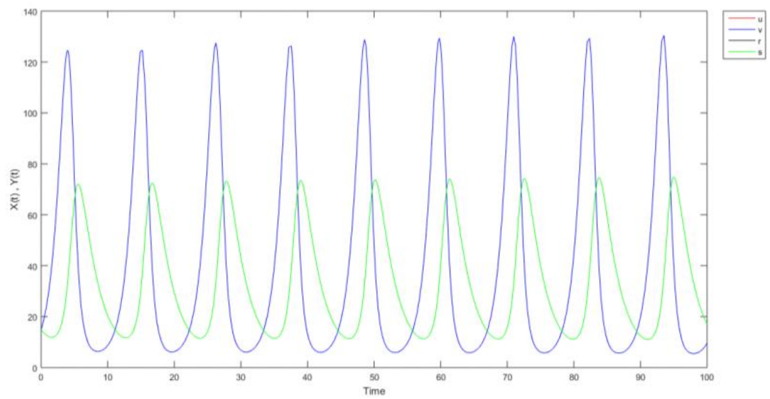


Figure (3.1.10): The solution of (8) at α=1

When $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable we note that for $\alpha < 1$ there is no fuzzy solution for $x(t)$ since $u(t) > v(t)$ for some time intervals but there is a fuzzy solution for $y(t)$ which is unacceptable since $r(t) \to -\infty \; as \; t \to \infty$. Here also $\chi_{(40,33.33)}$ is unstable fuzzy equilibrium point. When $\alpha = 1$ the solution is equivalent to the crisp solution and the equilibrium point is fuzzy stable.

Now, If $x(t)$ and $y(t)$ are (2)-differentiable, form (2,2)-differentiable, then $[X'(t)]_\alpha = [v'(t), u'(t)]$ and $[Y'(t)]_\alpha = [s'(t), r'(t)]$ and the model becomes:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (9)$$

We solve (9) by Runge-Kutta method in Matlab at α-levels= 0,0.5,1.At α-level = 0, the solution graphs are figure (3.1.11) and figure (3.1.12):

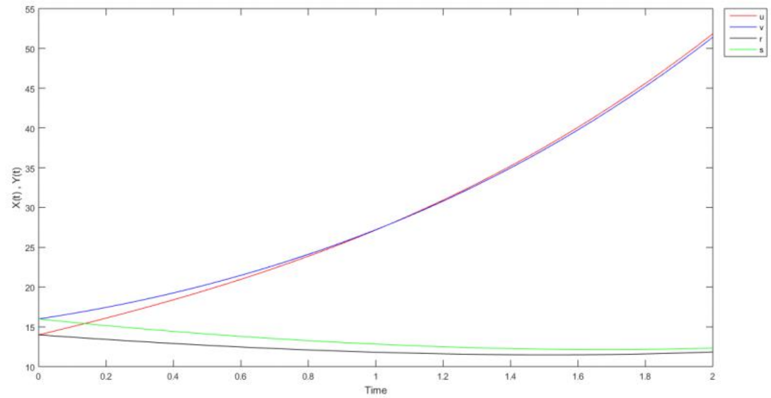


Figure (3.1.11): The solution of (9) at α=0 for short time period

The lower and upper bounds of $x(t)$ start with different points, similarly for $y(t)$. and $u(t) > v(t)\ for\ t < 15$. As time increases, the solution of lower and upper bounds of $x(t)\ and\ y(t)$ become identical as in the figure (3.1.12)

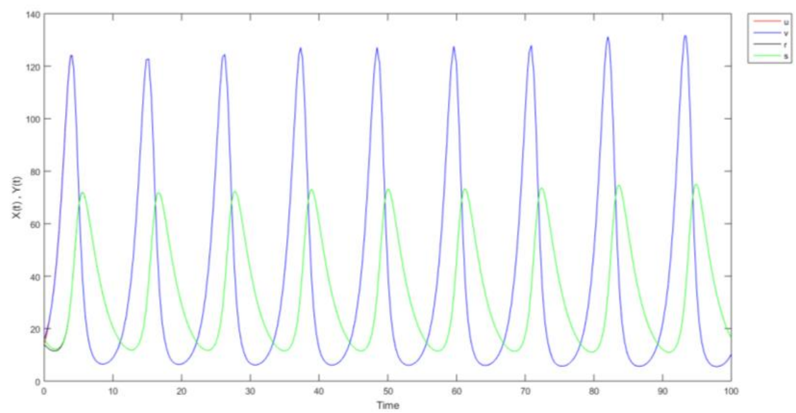


Figure (3.1.12): The solution of (9) at α=0 as time increases

At α-level = 0.5, the solution is figure (3.1.13) and figure (3.1.14):

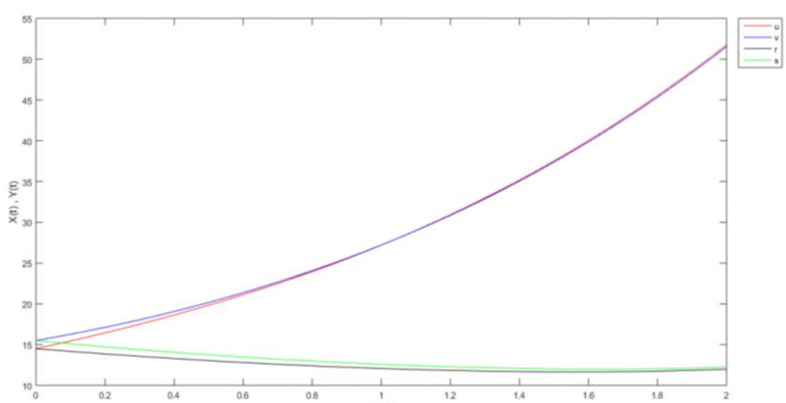


Figure (3.1.13): The solution of (9) at α=0.5 for short time period

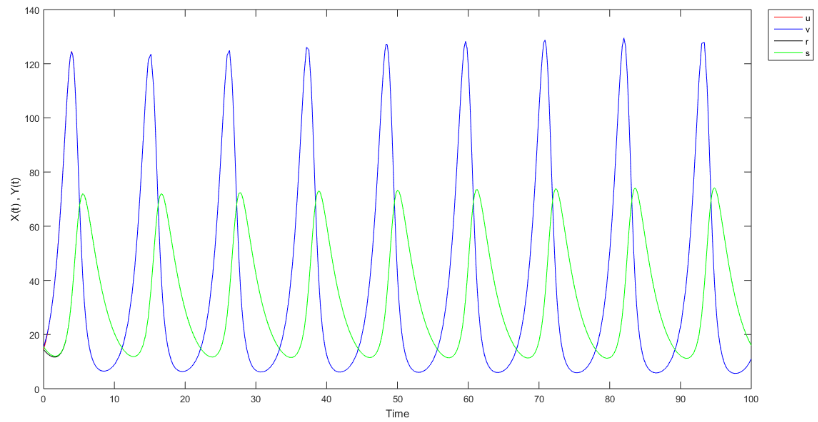


Figure (3.1.14): The solution of (9) at α=0.5 as time increases

For $t < 15$, $u(t) > v(t)$ and as time increases the lower and upper bound of $x(t)$ and $y(t)$ become identical.

At α-level = 1, the solution is table (3.1.7) and figure (3.1.15):

Table (3.1.7): The solution of (6) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
5.0000	74.3290	74.3290	65.9610	65.9610
10.0000	8.6870	8.6870	20.3800	20.3800
15.0000	125.5500	125.5500	30.9900	30.9900
20.0000	6.2827	6.2827	29.2910	29.2910
25.0000	81.9850	81.9850	14.2220	14.2220
30.0000	6.7937	6.7937	43.3010	43.3010
35.0000	39.1410	39.1410	11.4180	11.4180
40.0000	13.1750	13.1750	62.1540	62.1540
45.0000	18.3480	18.3480	13.1480	13.1480
50.0000	48.1690	48.1690	73.3960	73.3960
55.0000	9.5843	9.5843	17.9990	17.9990
60.0000	126.3700	126.3700	40.5810	40.5810
65.0000	6.2694	6.2694	26.4420	26.4420
70.0000	95.1240	95.1240	15.6750	15.6750
75.0000	6.0738	6.0738	40.0000	40.0000
80.0000	44.4160	44.4160	11.1530	11.1530
85.0000	10.8280	10.8280	59.3380	59.3380
90.0000	19.6920	19.6920	12.4410	12.4410
95.0000	39.9810	39.9810	74.8640	74.8640
100.0000	9.5716	9.5716	17.0560	17.0560

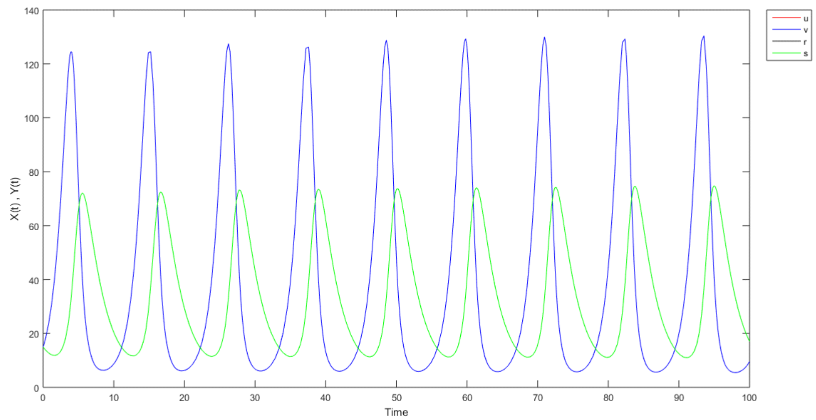


Figure (3.1.15): The solution of (9) at α=1

From previous graphs, we note that when we assume $x(t)$ and $y(t)$ are (2)-differentiable at any $\alpha < 1$, the lower and upper bounds of $x(t)$ and $y(t)$ become identical as time increases and oscillate about the equilibrium point $\chi_{(40,33.33)}$. So this point is fuzzy stable. However, when $\alpha = 1$ the solution is equivalent to the solution of the crisp case.

In figures (3.1.16) and (3.1.17), we plot the crisp solution with the solution of the fuzzy model (9) at α=0.

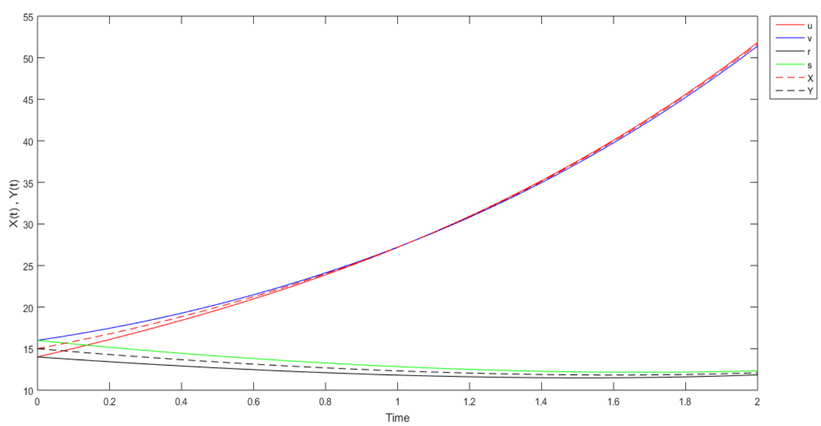


Figure (3.1.16): The solution of (9) at α=0 with the crisp case for short time period

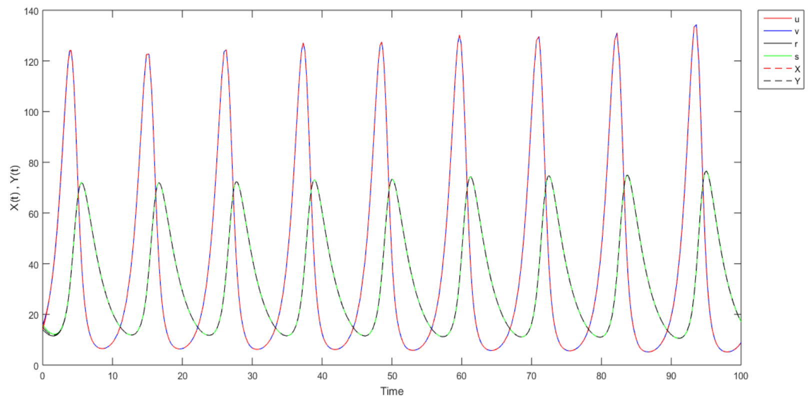


Figure (3.1.17): The solution of (9) at α=0 with the crisp case for long time period

From figures (3.1.16) and (3.1.17) we can note that for short time period, the solution of $x(t)$ lies between the solution of $u(t)$ and $v(t)$ then as time increases they become identical ($x(t) = u(t) = v(t)$). Also, the solution of $y(t)$ lies between the solution of $r(t)$ and $s(t)$ and as time increases they become identical ($y(t) = r(t) = s(t)$).

The model (5) was presented in [2] and they obtained fuzzy unstable equilibrium point by Euler method. However, we discuss this model and solve it using Runge-Kutta method. Thereafter, we conclude that when $x(t)$ and $y(t)$ are (1,1), (1,2) and (2,1)-differentiable there is some negative values for $\alpha < 1$, there is no meaning in this solution since it models population. At $\alpha = 1$, the core of the solution is the same as the solution of the crisp case, so it's stable. While, when $x\,(t)$ and $y(t)$ are (2)-differentiable, the curves of $x\,(t)$ and $y(t)$ become identical as $t \to \infty$ and the crisp solution lies between them. So, there is a fuzzy solution as $t \to \infty$,which is periodic about the equilibrium point. As prey population increases the predator population is minimum and as prey population decreases the predator population is maximum. So this solution is acceptable biologically and fuzzy stable. Therefore, the form (2)-differentiable for $x\,(t)$ and $y(t)$ gives solution better than the other forms.

Case 2: we try to change the initial conditions of (5) to be close to the equilibrium point $(40\ ,33.33)$. So, we let $x_o = 41$ and $y_o = 32$. Then we obtain the following model:

$$x'(t) = x - 0.03xy$$
$$y'(t) = -0.4\,y + 0.01xy$$
$$x_o = 41\ ,\ y_o = 32 \qquad (10)$$

We solve (10) by Matlab using Runge-Kutta method. The solution for model (10) is given in table (3.1.8) and figure (3.1.18).

Table (3.1.8): The solution of (10)

Time	$x(t)$	$y(t)$
0.0000	41.0000	32.0000
5.0000	38.8050	34.6630
10.0000	41.1660	32.0120
15.0000	38.6660	34.6370
20.0000	41.3210	32.0240
25.0000	38.5340	34.6150
30.0000	41.4580	32.0470
35.0000	38.4080	34.5940
40.0000	41.5870	32.0750
45.0000	38.2620	34.5680
50.0000	41.7460	32.1090
55.0000	38.0930	34.5230
60.0000	41.9130	32.1430
65.0000	37.9500	34.4690
70.0000	42.0640	32.1820
75.0000	37.8430	34.4200
80.0000	42.1900	32.2310
85.0000	37.7260	34.3760
90.0000	42.3130	32.2810
95.0000	37.5910	34.3240
100.0000	42.4600	32.3350

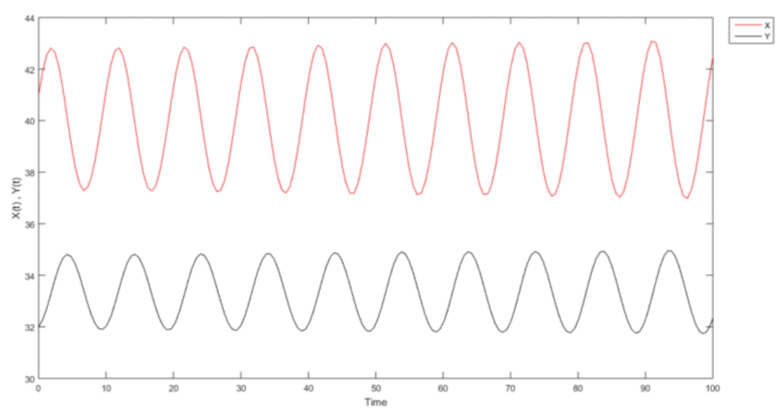


Figure (3.1.18): The solution of (10)

From table (3.1.8) and figure (3.1.18) we can notice that the crisp solution of $x(t)$ *and* $y(t)$ are periodic about the equilibrium point $\chi_{(40,33.33)}$. So, this interesting point is stable.

Here, we convert model (10) to a fuzzy model by assuming the initial conditions triangular fuzzy numbers. Let $[x_0]_\alpha = [40 + \alpha\,, 42 - \alpha]\ and\ [y_0]_\alpha = [31 + \alpha\,, 33 - \alpha]$. Then we solve the fuzzy model in the same manner as we did with the previous conditions. We assume that $x\ (t)\ and\ y(t)$ are (1)-differentiable, then we have the following model:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 40 + \alpha, v_0 = 42 - \alpha, r_0 = 31 + \alpha, s_0 = 33 - \alpha \qquad (11)$$

Thereafter, we solve (11) by Runge-Kutta method in Matlab at α-level= 0,0.5,1. At α-level = 0, the solution is table (3.1.9), where its graph is figure (3.1.19):

Table (3.1.9): The solution of (11) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	40.0000	42.0000	31.0000	33.0000
0.2500	39.2930	43.4760	30.7440	33.4560
0.5000	37.6510	45.7710	30.3120	34.1630
0.7500	34.3430	49.5090	29.5550	35.2840
1.0000	28.0270	55.7500	28.2260	37.1200
1.2500	16.1910	66.2400	25.9210	40.2420
1.5000	-6.1148	83.6090	22.0390	45.7790
1.7500	-48.9800	111.1900	15.8470	56.0550
2.0000	-134.5100	150.7900	6.9865	76.1760
2.2500	-311.5300	196.1300	-3.0815	117.3000
2.5000	-677.3000	223.1400	-9.7960	200.4900
2.7500	-1356.7000	198.9900	-10.5090	345.7300
3.0000	-2316.3000	104.4900	-9.1427	514.4800
3.2500	-3112.7000	-59.4850	-7.3671	551.350
3.5000	-3405.3000	-248.8000	-4.7514	375.1800
3.7500	-3640.0000	-419.9700	-2.0296	162.0200
4.0000	-4272.8000	-577.9300	-0.5146	46.6020
4.2500	-5354.2000	-751.9100	-0.0791	8.9054
4.5000	-6847.1000	-967.0900	-0.0073	1.0506
4.7500	-8788.4000	-1241.9000	-0.0004	0.0674
5.0000	-11284.0000	-1594.7000	0.0000	0.0020

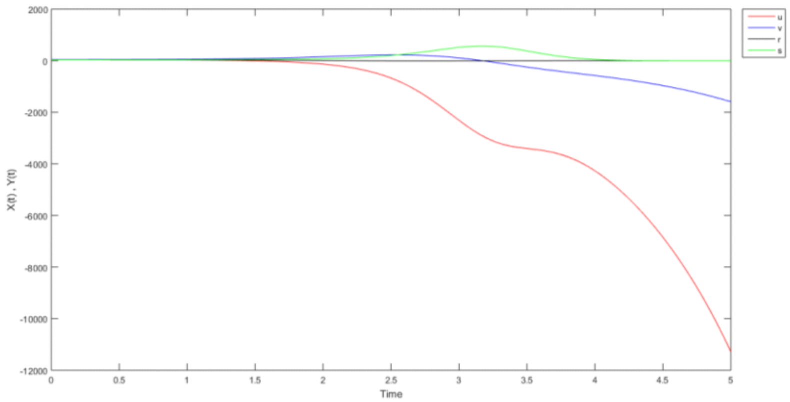


Figure (3.1.19): The solution of (11) at α=0

At α-level = 0.5, the solution graph is figure (3.1.20) in the appendix. At α-level = 1, the solution is table (3.1.10) and figure (3.1.21):

Table (3.1.10): The solution of (11) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	41.0000	41.0000	32.0000	32.0000
5.0000	38.8050	38.8050	34.6630	34.6630
10.0000	41.1660	41.1660	32.0120	32.0120
15.0000	38.6660	38.6660	34.6370	34.6370
20.0000	41.3210	41.3210	32.0240	32.0240
25.0000	38.5340	38.5340	34.6150	34.6150
30.0000	41.4580	41.4580	32.0470	32.0470
35.0000	38.4080	38.4080	34.5940	34.5940
40.0000	41.5870	41.5870	32.0750	32.0750
45.0000	38.2620	38.2620	34.5680	34.5680
50.0000	41.7460	41.7460	32.1090	32.1090
55.0000	38.0930	38.0930	34.5230	34.5230
60.0000	41.9130	41.9130	32.1430	32.1430
65.0000	37.9500	37.9500	34.4690	34.4690
70.0000	42.0640	42.0640	32.1820	32.1820
75.0000	37.8430	37.8430	34.4200	34.4200
80.0000	42.1900	42.1900	32.2310	32.2310
85.0000	37.7260	37.7260	34.3760	34.3760
90.0000	42.3130	42.3130	32.2810	32.2810
95.0000	37.5910	37.5910	34.3240	34.3240
100.0000	42.4600	42.4600	32.3350	32.3350

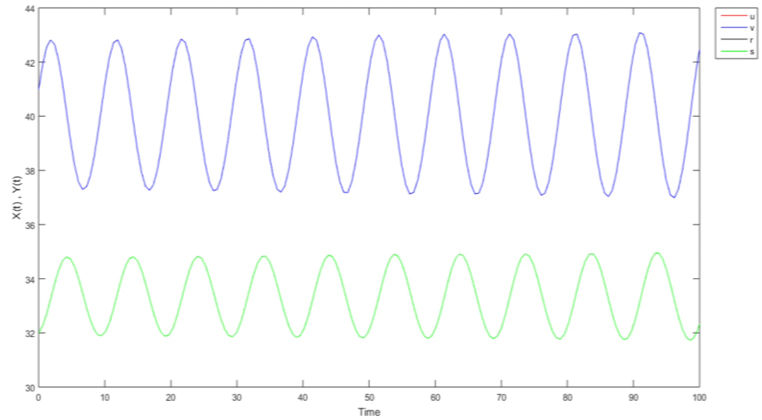


Figure (3.1.21): The solution of (11) at α=1

If $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, then the model becomes:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 40 + \alpha, v_0 = 42 - \alpha, r_0 = 31 + \alpha, s_0 = 33 - \alpha \qquad (12)$$

We solve (12) by Runge-Kutta method in Matlab at α-levels= 0,0.5,1. At α-level= 0, the solution is figure (3.1.22) as follow:

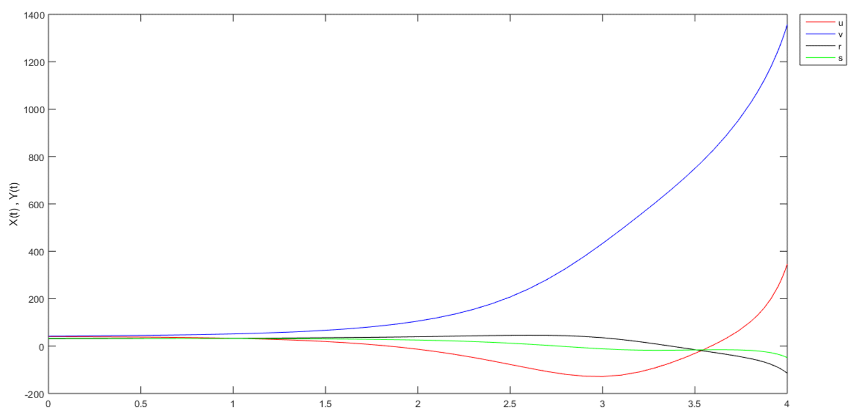


Figure (3.1.22): The solution of (12) at α=0

At α-level = 0.5, the solution figure (3.1.23) in the appendix. At α-level = 1, the solution is figure (3.1.24):

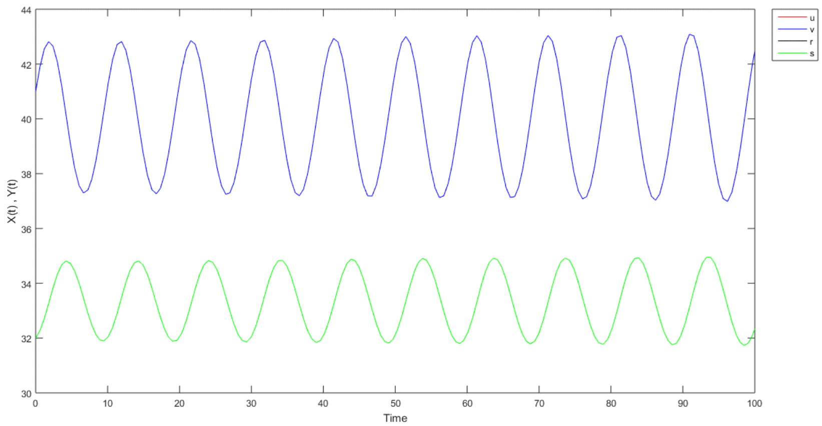


Figure (3.1.24): The solution of (12) at α=1

If $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable, then the model becomes:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 40 + \alpha, v_0 = 42 - \alpha, r_0 = 31 + \alpha, s_0 = 33 - \alpha \qquad (13)$$

We solve (13) by Runge-Kutta method in Matlab at α-levels= 0,0.5,1. At α-level = 0, the solution is figure (3.1.25):

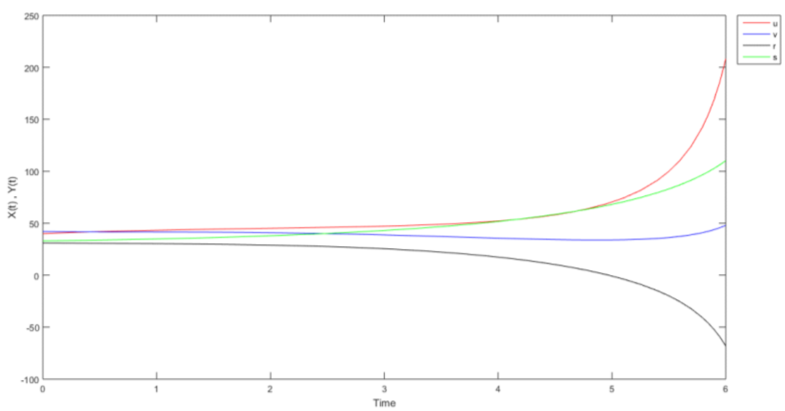


Figure (3.1.25): The solution of (13) at α=0

At α-level = 0.5, the solution is figure (3.1.26) in the appendix. At α-level = 1, the solution is figure (3.1.27):

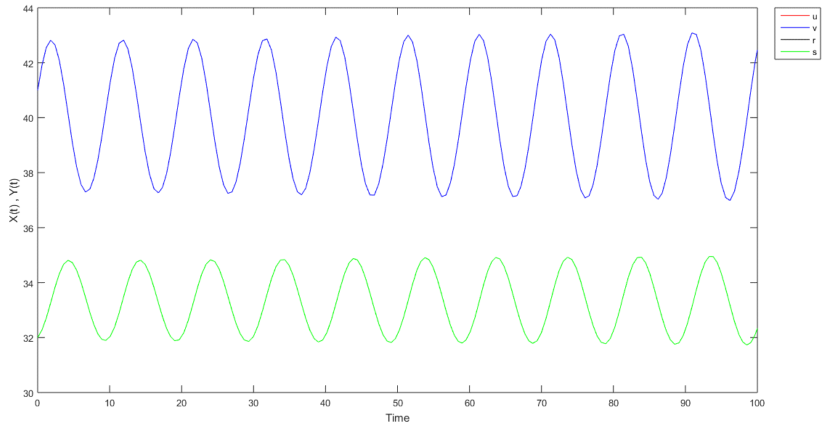


Figure (3.1.27): The solution of (13) at α=1

Now, If $x(t)$ *and* $y(t)$ are (2)-differentiable, then the model becomes:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 40 + \alpha, v_0 = 42 - \alpha, r_0 = 31 + \alpha, s_0 = 33 - \alpha \qquad (14)$$

We solve (14) by Runge-Kutta method in Matlab at α-levels= 0,0.5,1. At α-level = 0, the solution graphs are figure (3.1.28) and figure (3.1.29):

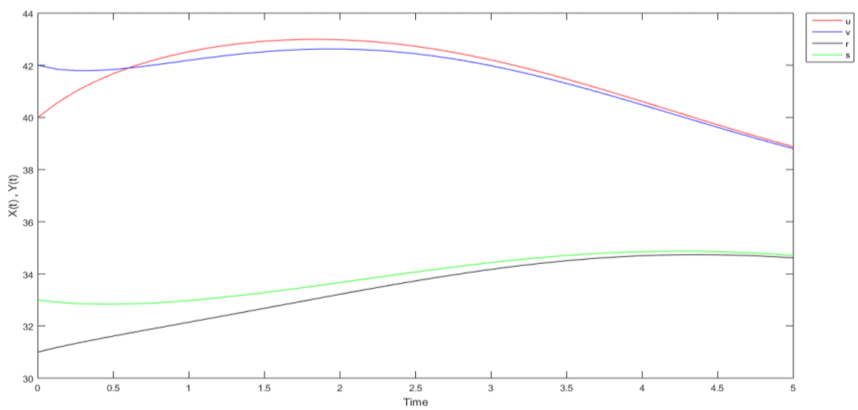


Figure (3.1.28): The solution of (14) at α=0 for short time period

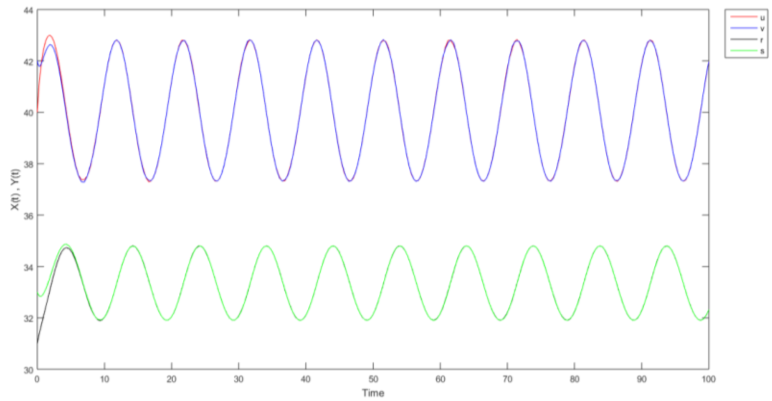


Figure (3.1.29): The solution of (14) at α=0 as time increases

At α= 0.5, the solution graphs are figure (3.1.30) and figure (3.1.31) in the appendix. At α = 1, the solution is figure (3.1.32):

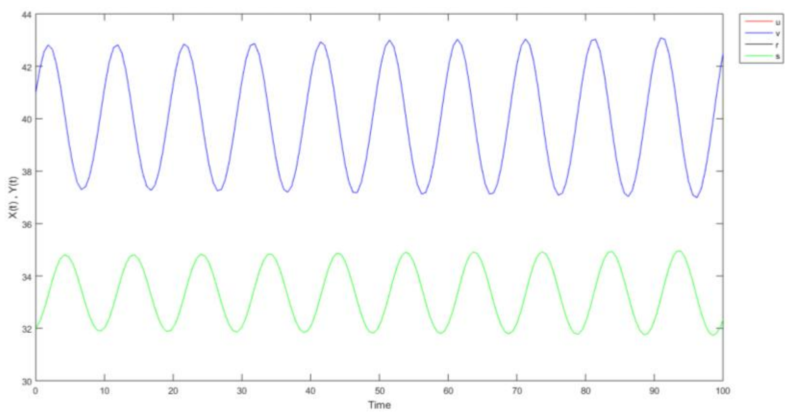


Figure (3.1.32): The solution of (14) at α=1

When we change the initial conditions to be close to the equilibrium point, we obtain the same results when $x(t)$ *and* $y(t)$ are (1,1), (1,2) and (2,1)-differentiable as in case 1. While, when $x\ (t)$ *and* $y(t)$ are (2)-differentiable , at any $\alpha < 1$ we note that $u(t) > v(t)$ and $r(t) > s(t)$ at some time intervals. So, there are no fuzzy solution for $x\ (t)$ *and* $y(t)$but the solution is periodic about the equilibrium point and stable. At α = 1, the solution is corresponding to the crisp solution and the interesting equilibrium point is stable. So, we can't say that the (2)-differentiable for $x\ (t)$ *and* $y(t)$with these initial conditions give good solution.

Case 3: we try to change the $\alpha - level$ of the initial conditions in model (5) using shaped triangular fuzzy number. Let $[X_0]_\alpha = [14 + \alpha^2,\ 16 - \alpha^2] = [Y_0]_\alpha$. Then we find the simulations and graphical solutions of the fuzzy predator prey model at different $\alpha - level$ by matlab using Runge-Kutta method. First, if $x(t)$ *and* $y(t)$ are (1)-differentiable then we obtain the following model:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha^2, v_0 = 16 - \alpha^2, r_0 = 14 + \alpha^2, s_0 = 16 - \alpha^2 \qquad (15)$$

At α-level = 0, the solution is table (3.1.11), where its graph is figure (3.1.33):

Table (3.1.11): The solution of (15) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	15.6670	18.8470	12.9330	15.3330
0.5000	17.4820	22.4480	11.9390	14.8660
0.7500	19.3670	27.0260	10.9950	14.6260
1.0000	21.1660	32.8840	10.0660	14.6640
1.2500	22.5880	40.4310	9.1092	15.0580
1.5000	23.0900	50.2310	8.0569	15.9430
1.7500	21.6840	63.0540	6.8114	17.5500
2.0000	16.5030	79.9390	5.2282	20.2980
2.2500	3.9817	102.2200	3.1076	24.9730
2.5000	-23.1180	131.3100	0.2211	33.1750
2.7500	-80.1550	167.8400	-3.5389	48.3320
3.0000	-200.9900	209.0000	-7.7090	78.0940
3.2500	-459.9900	243.2800	-10.6750	139.1800
3.5000	-996.8000	248.0600	-10.8600	261.0600
3.7500	-1958.0000	194.5500	-9.6853	462.4400
4.0000	-3141.1000	54.2850	-8.4039	644.4900
4.2500	-3811.0000	-154.5100	-6.3015	573.3500
4.5000	-3961.8000	-360.5600	-3.3223	299.5300
4.7500	-4387.0000	-535.9000	-1.0252	97.3430
5.0000	-5364.7000	-709.0500	-0.1807	20.6270

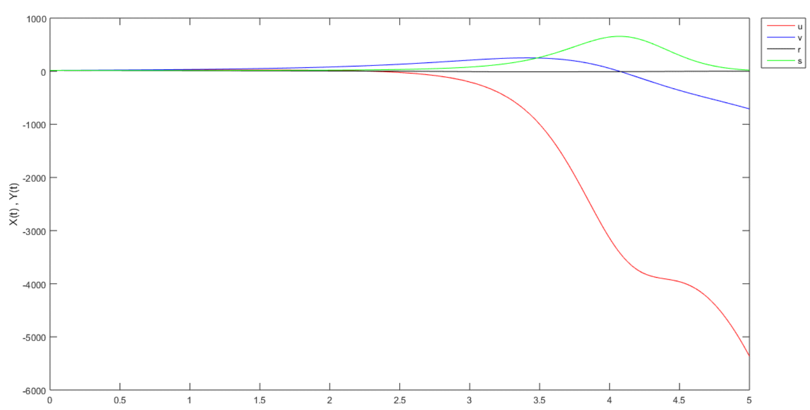


Figure (3.1.33): The solution of (15) at α=0

At α-level = 0.5, the solution is figure (3.1.34) in the appendix. At α-level = 1, the solution is table (3.1.12) and figure (3.1.35):

Table (3.1.12): The solution of (15) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
5.0000	74.3290	74.3290	65.9610	65.9610
10.0000	8.6870	8.6870	20.3800	20.3800
15.0000	125.5500	125.5500	30.9900	30.9900
20.0000	6.2827	6.2827	29.2910	29.2910
25.0000	81.9850	81.9850	14.2220	14.2220
30.0000	6.7937	6.7937	43.3010	43.3010
35.0000	39.1410	39.1410	11.4180	11.4180
40.0000	13.1750	13.1750	62.1540	62.1540
45.0000	18.3480	18.3480	13.1480	13.1480
50.0000	48.1690	48.1690	73.3960	73.3960
55.0000	9.5843	9.5843	17.9990	17.9990
60.0000	126.3700	126.3700	40.5810	40.5810
65.0000	6.2694	6.2694	26.4420	26.4420
70.0000	95.1240	95.1240	15.6750	15.6750
75.0000	6.0738	6.0738	40.0000	40.0000
80.0000	44.4160	44.4160	11.1530	11.1530
85.0000	10.8280	10.8280	59.3380	59.3380
90.0000	19.6920	19.6920	12.4410	12.4410
95.0000	39.9810	39.9810	74.8640	74.8640
100.0000	9.5716	9.5716	17.0560	17.0560

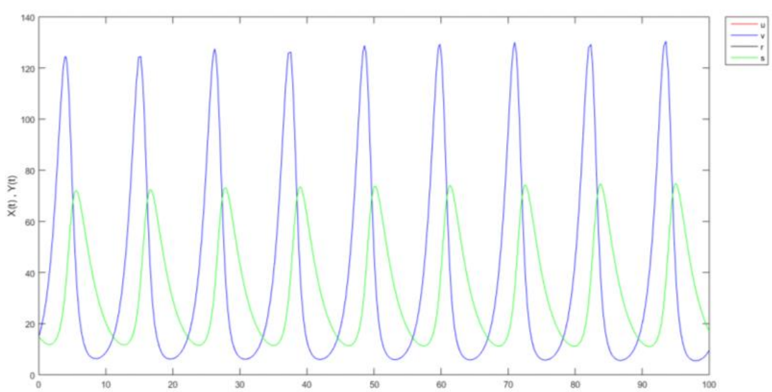


Figure (3.1.35): The solution of (15) at α=1

If $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, then the model becomes:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4r + 0.01vs$$

$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha^2, v_0 = 16 - \alpha^2, r_0 = 14 + \alpha^2, s_0 = 16 - \alpha^2 \qquad (16)$$
At α-level = 0, the solution is figure (3.1.36)

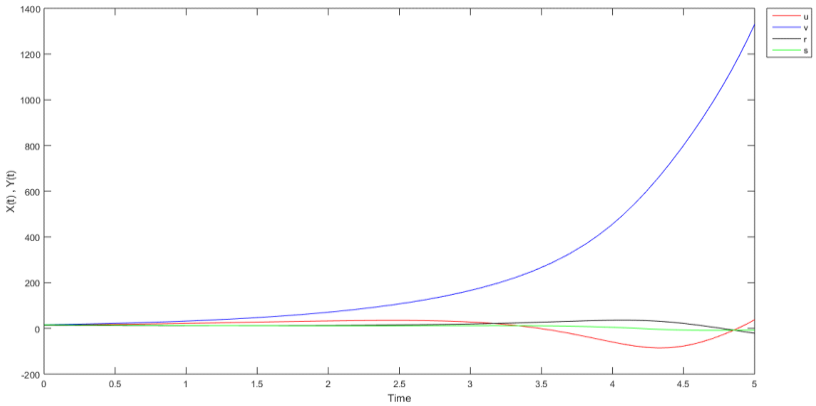


Figure (3.1.36): The solution of (16) at α=0

At α-level = 0.5, the solution is figure (3.1.37) in the appendix. At α-level = 1, the solution is figure (3.1.38):

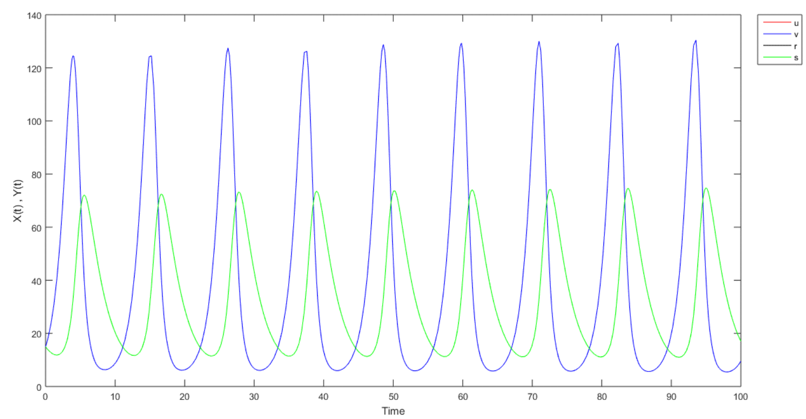


Figure (3.1.38): The solution of (16) at α=1

While $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable, then the model becomes:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha^2, v_0 = 16 - \alpha^2, r_0 = 14 + \alpha^2, s_0 = 16 - \alpha^2 \qquad (17)$$

At α-level = 0, the solution is figure (3.1.39)

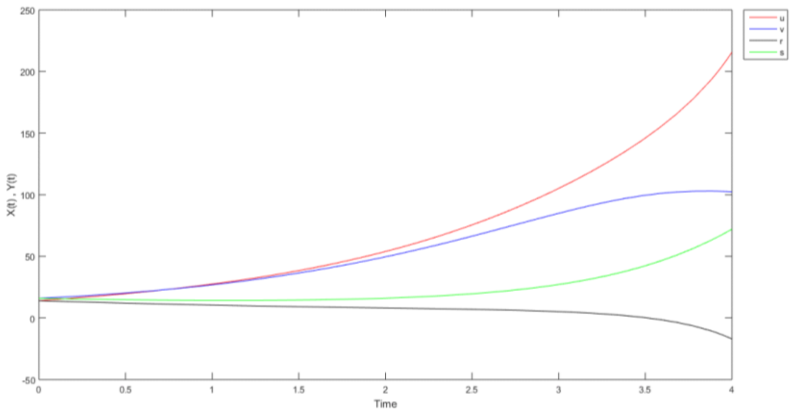


Figure (3.1.39): The solution of (17) at α=0

At α-level = 0.5, the solution is figure (3.1.40) in the appendix. At α-level = 1, the solution is figure (3.1.41):

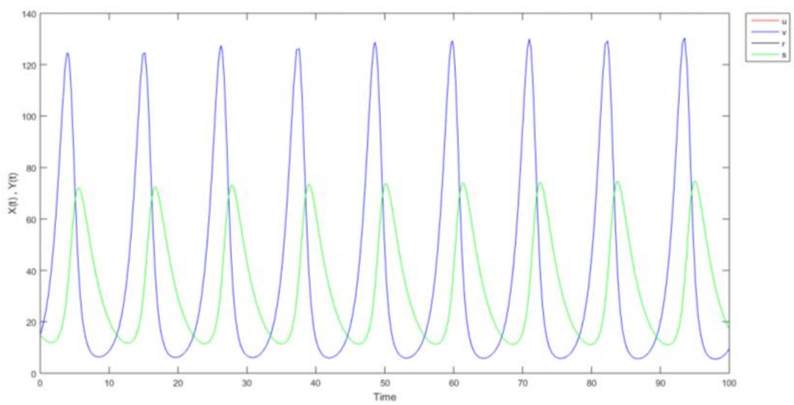


Figure (3.1.41): The solution of (17) at α=1

Now, If $x(t)\ and\ y(t)$ are (2)-differentiable, then the model becomes:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha^2, v_0 = 16 - \alpha^2, r_0 = 14 + \alpha^2, s_0 = 16 - \alpha^2 \qquad (18)$$

At α-level = 0, the solution is figure (3.1.42) and figure (3.1.43):

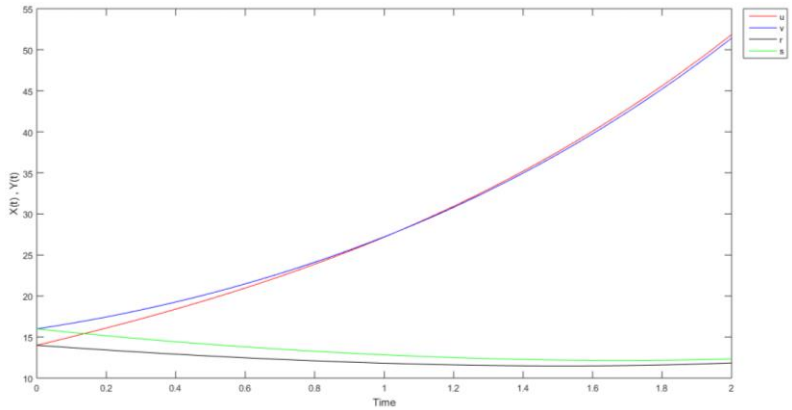


Figure (3.1.42): The solution of (18) at α=0 for short time period

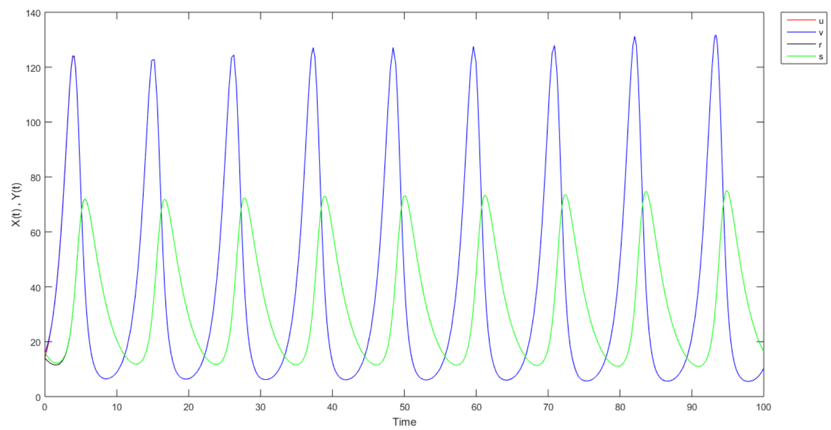


Figure (3.1.43): The solution of (18) at α=0 as time increases

At α-level = 0.5, the solution graphs are figure (3.1.44) and figure (3.1.45) in the appendix. At α-level = 1, the solution is figure (3.1.46):

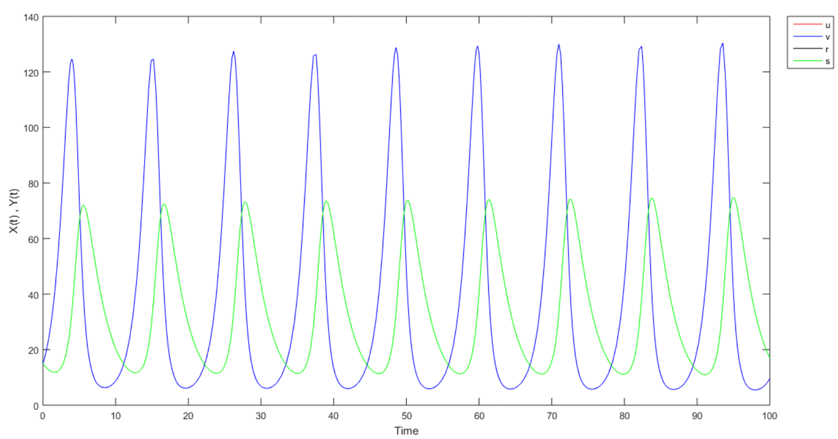


Figure (3.1.46): The solution of (18) at α=1

From previous tables and figures we can note that for case 3 we obtain the same results as in case 1. So, there is a fuzzy solution which is periodic about the equilibrium point only when $x(t)\ and\ y(t)$ are (2)-differentiable. Thus this equilibrium point is stable.

Case 4: We try to use trapezoidal fuzzy initial conditions. Therefore, we Let $x_0 = (14\ ,14.5\ ,15.5\ ,16) = y_0$ trapezoidal fuzzy numbers and there $\alpha - levels$ will be as follow: $[x_0]_\alpha = \left[14 + \frac{\alpha}{2},\ 16 - \frac{\alpha}{2}\right] = \ [y_0]_\alpha$.
Then if $x(t)\ and\ y(t)$ are (1)-differentiable the fuzzy model will be as follows:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \frac{\alpha}{2}, v_0 = 16 - \frac{\alpha}{2}, r_0 = 14 + \frac{\alpha}{2}, s_0 = 16 - \frac{\alpha}{2} \qquad (19)$$

And we solve (19) by Runge-Kutta method in Matlab at different α-level. At α-level = 0, the solution is table (3.1.13), where its graph is figure (3.1.47):

Table (3.1.13): The solution of (19) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	15.6670	18.8470	12.9330	15.3330
0.5000	17.4820	22.4480	11.9390	14.8660
0.7500	19.3670	27.0260	10.9950	14.6260
1.0000	21.1660	32.8840	10.0660	14.6640
1.2500	22.5880	40.4310	9.1092	15.0580
1.5000	23.0900	50.2310	8.0569	15.9430
1.7500	21.6840	63.0540	6.8114	17.5500
2.0000	16.5030	79.9390	5.2282	20.2980
2.2500	3.9817	102.2200	3.1076	24.9730
2.5000	-23.1180	131.3100	0.2211	33.1750
2.7500	-80.1550	167.8400	-3.5389	48.3320
3.0000	-200.9900	209.0000	-7.7090	78.0940
3.2500	-459.9900	243.2800	-10.6750	139.1800
3.5000	-996.8000	248.0600	-10.8600	261.0600
3.7500	-1958.0000	194.5500	-9.6853	462.4400
4.0000	-3141.1000	54.2850	-8.4039	644.4900
4.2500	-3811.0000	-154.5100	-6.3015	573.3500
4.5000	-3961.8000	-360.5600	-3.3223	299.5300
4.7500	-4387.0000	-535.9000	-1.0252	97.3430
5.0000	-5364.7000	-709.0500	-0.1807	20.6270

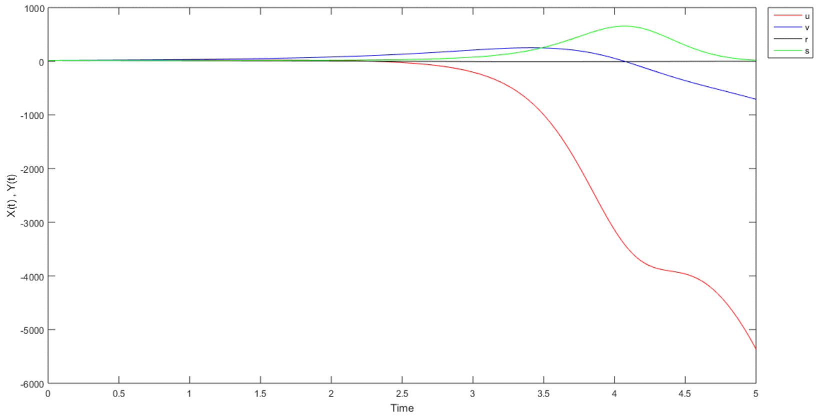


Figure (3.1.47): The solution of (19) at α=0

At α-level = 0.5, the solution is figure (3.1.48) in the appendix. At α-level = 1, the solution is table (3.1.14), where its graph is figure (3.1.49):

Table (3.1.14): The solution of (19) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.5000	15.5000	14.5000	15.5000
0.2500	16.4710	18.0610	13.5310	14.7310
0.5000	18.7500	21.2330	12.6640	14.1270
0.7500	21.3450	25.1750	11.8880	13.7040
1.0000	24.2330	30.0930	11.1860	13.4850
1.2500	27.3340	36.2610	10.5380	13.5130
1.5000	30.4630	44.0450	9.9108	13.8570
1.7500	33.2370	53.9510	9.2550	14.6320
2.0000	34.8990	66.6840	8.4874	16.0400
2.2500	33.9750	83.2450	7.4694	18.4430
2.5000	27.4970	105.0500	5.9723	22.5400
2.7500	9.3367	134.0200	3.6428	29.7540
3.0000	-34.1760	172.1200	0.0506	43.2790
3.2500	-135.3600	219.2200	-4.9244	70.7630
3.5000	-373.4000	265.9300	-9.8945	131.0700
3.7500	-924.6000	284.0100	-11.6260	264.8900
4.0000	-2032.8000	232.6300	-10.4330	516.5400
4.2500	-3501.1000	71.1710	-9.0457	775.6500
4.5000	-4214.1000	-178.8300	-6.7784	684.2800
4.7500	-4177.2000	-415.8800	-3.4036	322.2900
5.0000	-4553.8000	-608.5800	-0.9155	89.1400

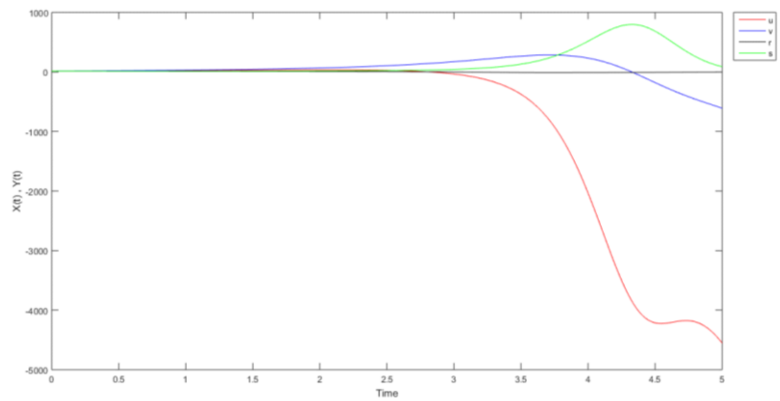


Figure (3.1.49): The solution of (19) at α=1

While $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, then the model becomes:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \frac{\alpha}{2}, v_0 = 16 - \frac{\alpha}{2}, r_0 = 14 + \frac{\alpha}{2}, s_0 = 16 - \frac{\alpha}{2} \qquad (20)$$

We solve (20) by Runge-Kutta method in Matlab at α-levels= 0,0.5,1. At α-level = 0, the solution is figure (3.1.50)

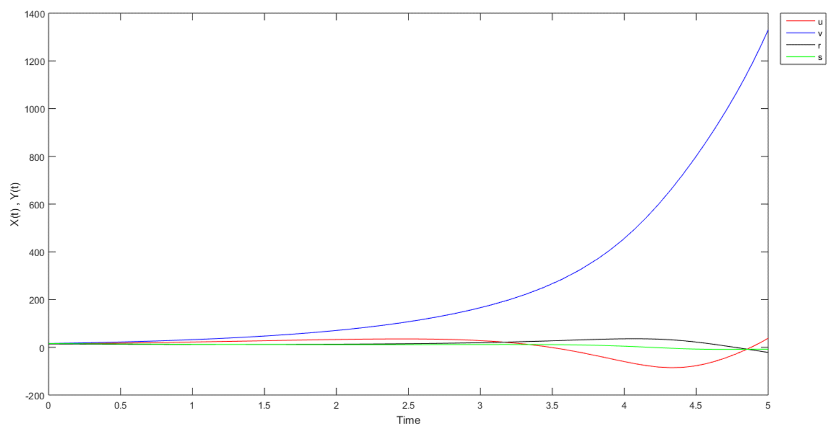


Figure (3.1.50): The solution of (20) at α=0

At α-level = 0.5, the solution is figure (3.1.51) in the appendix. At α-level = 1, the solution is figure (3.1.52):

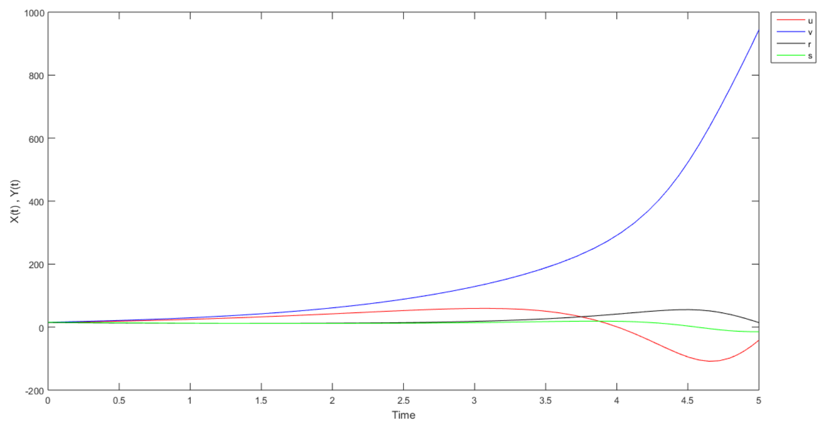


Figure (3.1.52): The solution of (20) at α=1

However, If $x(t)$ is (2)-differentiable and $y(t)$ is (1)- differentiable, then the model becomes:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \frac{\alpha}{2}, v_0 = 16 - \frac{\alpha}{2}, r_0 = 14 + \frac{\alpha}{2}, s_0 = 16 - \frac{\alpha}{2} \qquad (21)$$

We solve (21) by Runge-Kutta method in Matlab at α-levels= 0,0.5,1. At α-level = 0, the solution is figure (3.1.53)

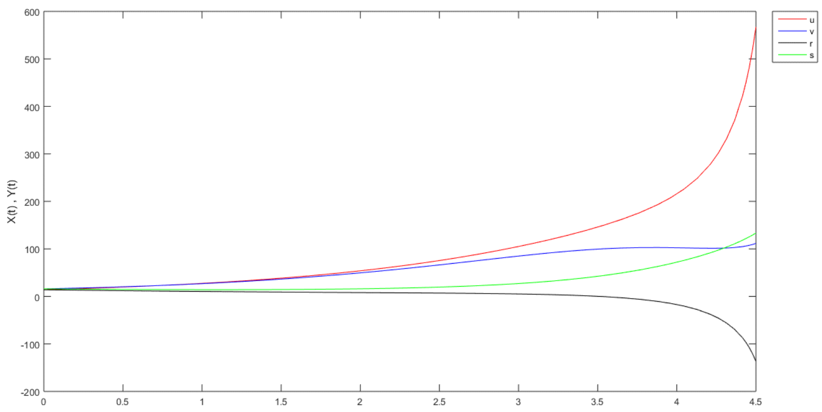


Figure (3.1.53): The solution of (21) at α=0

At α-level = 0.5, the solution is figure (3.1.54) in the appendix. At α-level = 1, the solution is figure (3.1.55):

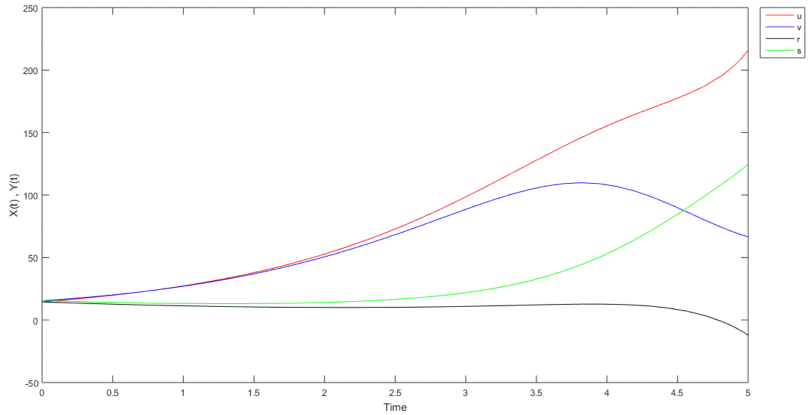


Figure (3.1.55): The solution of (21) at α=1

Now, If $x(t)$ *and* $y(t)$ are (2)-differentiable, then we obtain the following model:

$$u' = v - 0.03ur$$

$$v' = u - 0.03vs$$

$$r' = -0.4r + 0.01vs$$

$$s' = -0.4s + 0.01ur$$

$$u_0 = 14 + \frac{\alpha}{2}, v_0 = 16 - \frac{\alpha}{2}, r_0 = 14 + \frac{\alpha}{2}, s_0 = 16 - \frac{\alpha}{2} \qquad (22)$$

We solve (22) by Runge-Kutta method in Matlab at α-levels= 0,0.5,1. At α-level = 0, the solution graphs are figure (3.1.56) and figure (3.1.57):

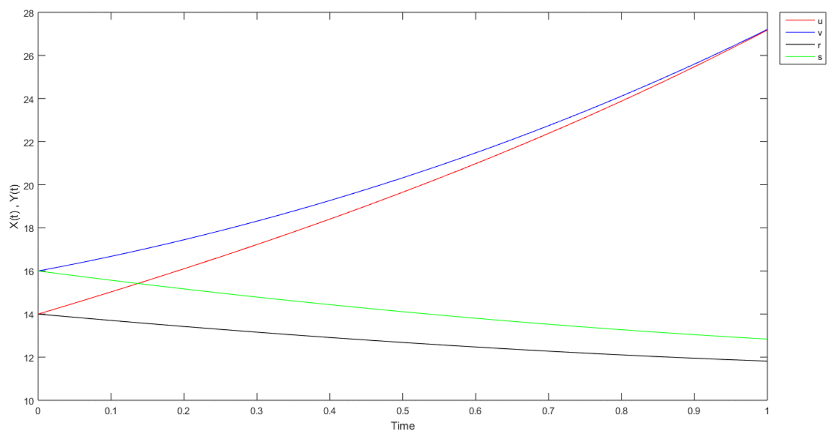


Figure (3.1.56): The solution of (22) at α=0 for short time period

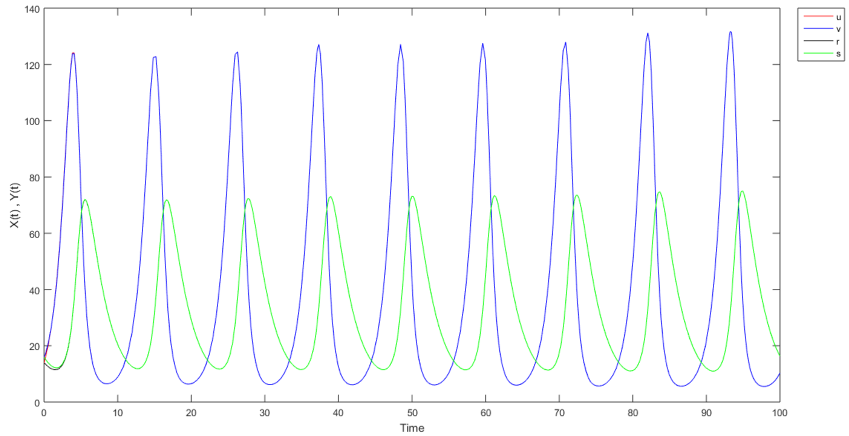


Figure (3.1.57): The solution of (22) at α=0 as time increases

At α-level = 0.5, the solution graphs are figure (3.1.58) and figure (3.1.59) in the appendix. At α-level = 1, the solution graphs are figure (3.1.60) and figure (3.1.61):

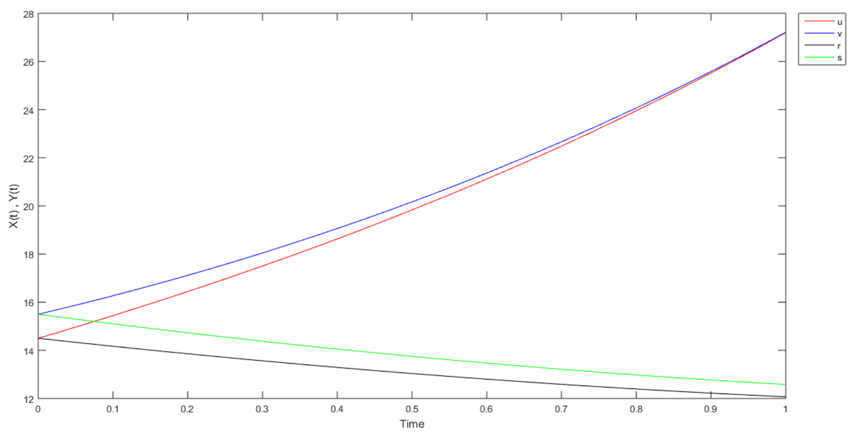


Figure (3.1.60): The solution of (22) at α=1 for short time period

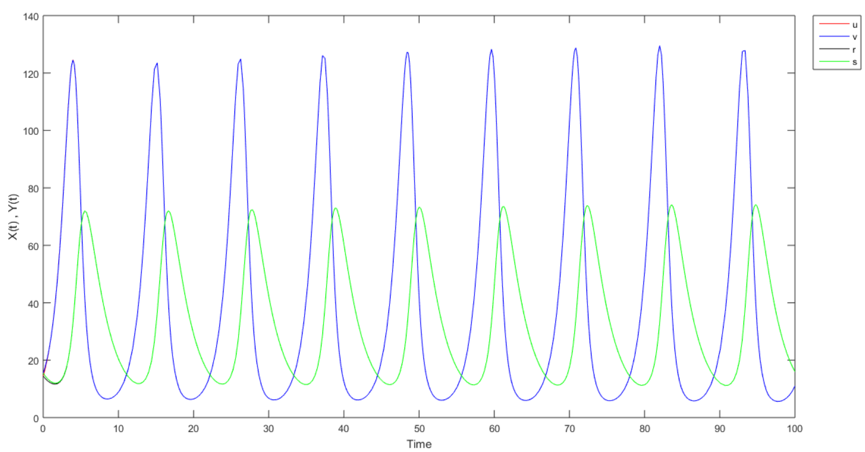


Figure (3.1.61): The solution of (22) at α=1 as time increases

For case 4, when $x(t)$ *and* $y(t)$ are (1,1), (1,2) and (2,1)-differentiable then the solution is incompatible with biological facts. At $\alpha = 1$, since $u_0 \neq v_0$ and $r_0 \neq s_0$, the solution isn't coincide with the crisp solution and the equilibrium points are fuzzy unstable. When $x(t)$ *and* $y(t)$ are (2)-differentiable then there is a fuzzy solution expect at small time interval at beginning, and as time increases the solution becomes periodic about the equilibrium point. So, the equilibrium point is fuzzy stable. While at $\alpha = 1$, Since $u_0 \neq v_0$ and $r_0 \neq s_0$, the solution isn't coincide with the crisp solution for short time period. Therefore, the triangular fuzzy initial condition is better than the trapezoidal one at least for $\alpha = 1$.

3.2: A Predator-Prey Model with Fuzzy Parameters and Initial Conditions.

In this section, we try to make the birth and death rates (parameters) of model (5) fuzzy numbers with fuzzy initial conditions. First, we want to fuzzify each parameter separately using a triangular fuzzy number and again using a trapezoidal fuzzy number.

Here we fuzzify $a = 1$. First, we let $a = (0.5\ ,1\ ,1.5)$ triangular fuzzy number. So, $[a]_\alpha = \left[0.5 + \frac{\alpha}{2},\ 1.5\ -\ \frac{\alpha}{2}\right]$ and we obtain the following model:

$$x'(t) = (0.5\ ,1\ ,1.5)x\ - 0.03xy$$
$$y'(t) = -0.4\ y + 0.01xy$$

With fuzzy initial conditions:

$$[x_0]_\alpha = [14 + \alpha, 16 - \alpha]\ , [y_0]_\alpha = [14 + \alpha, 16 - \alpha]$$

If $x(t)$ *and* $y(t)$are (1)-differentiable, then the model will be:

$$u' = \left(0.5 +\ \frac{\alpha}{2}\right)u - 0.03vs$$
$$v' = (1.5 - \frac{\alpha}{2})v - 0.03ur$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (23)$$

This model will change at any value of α, so the equilibrium points will also change as α changes. The first equilibrium point is (0,0,0,0) for any α –level. The second equilibrium point varies according to the α–level, as in the following table (3.2.1)

Table (3.2.1): The equilibrium points of (23)

α- level	u	v	r	s
0	57.69	27.7345	24.0375	34.6681
0.5	47.4252	33.7373	29.64	35.14
1	40	40	33.3333	33.3333

We solve (23) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is table (3.2.2), where its graph is figure (3.2.1):

Table (3.2.2): The solution of (23) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	13.5440	21.5820	12.8980	15.3830
0.5000	12.2620	29.9330	11.7780	15.1180
0.7500	9.5687	42.4120	10.5690	15.3580
1.0000	4.4171	61.0320	9.1713	16.3890
1.2500	-5.3118	88.7600	7.4323	18.8000
1.5000	-24.1360	129.8800	5.1219	23.8740
1.7500	-63.7690	190.2300	1.9065	34.9990
2.0000	-158.7600	276.3900	-2.5971	62.3590
2.2500	-433.9500	388.6900	-8.1709	143.3600
2.5000	-1413.1000	489.0700	-11.9510	436.7100
2.7500	-4788.7000	411.9600	-12.1050	1446.0000
3.0000	-8079.6000	-138.7100	-11.2450	2252.8000
3.2500	-4134.1000	-781.0800	-7.1838	677.9500
3.5000	-2646.1000	-1262.7000	-1.3098	52.6510
3.7500	-2828.8000	-1848.1000	-0.0408	1.1271
4.0000	-3201.8000	-2689.2000	-0.0002	0.0042
4.2500	-3628.1000	-3912.7000	0.0000	0.0000
4.5000	-4111.1000	-5692.9000	0.0000	0.0000
4.7500	-4658.5000	-8283.2000	0.0000	0.0000
5.0000	-5278.8000	-12052.0000	0.0000	0.0000

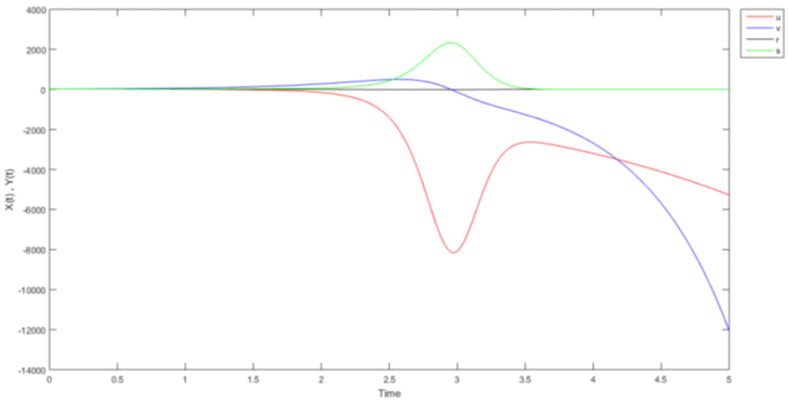


Figure (3.2.1): The solution of (23) at α=0

At α-level = 0.5, the solution is figure (3.2.2) in the appendix. At α-level = 1, the solution is table (3.2.3), where its graph is figure (3.2.3):

Table (3.2.3): The solution of (23) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
5.0000	74.3290	74.3290	65.9610	65.9610
10.0000	8.6870	8.6870	20.3800	20.3800
15.0000	125.5500	125.5500	30.9900	30.9900
20.0000	6.2827	6.2827	29.2910	29.2910
25.0000	81.9850	81.9850	14.2220	14.2220
30.0000	6.7937	6.7937	43.3010	43.3010
35.0000	39.1410	39.1410	11.4180	11.4180
40.0000	13.1750	13.1750	62.1540	62.1540
45.0000	18.3480	18.3480	13.1480	13.1480
50.0000	48.1690	48.1690	73.3960	73.3960
55.0000	9.5843	9.5843	17.9990	17.9990
60.0000	126.3700	126.3700	40.5810	40.5810
65.0000	6.2694	6.2694	26.4420	26.4420
70.0000	95.1240	95.1240	15.6750	15.6750
75.0000	6.0738	6.0738	40.0000	40.0000
80.0000	44.4160	44.4160	11.1530	11.1530
85.0000	10.8280	10.8280	59.3380	59.3380
90.0000	19.6920	19.6920	12.4410	12.4410
95.0000	39.9810	39.9810	74.8640	74.8640
100.0000	9.5716	9.5716	17.0560	17.0560

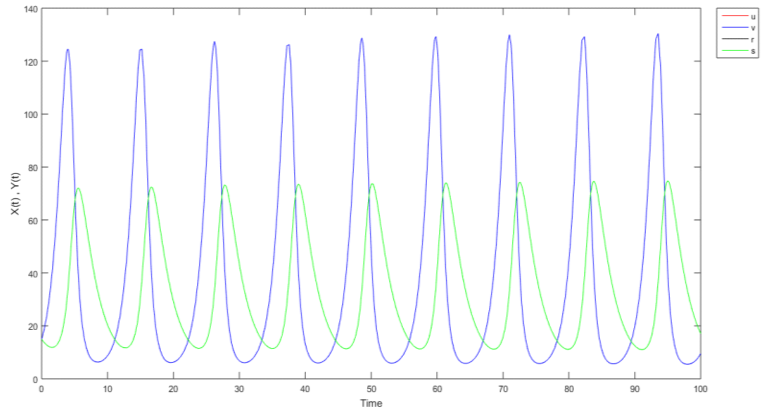


Figure (3.2.3): The solution of (23) at α=1

Whereas if $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, then the model will be:

$$u' = (0.5 + \frac{\alpha}{2})u - 0.03vs$$
$$v' = (1.5 - \frac{\alpha}{2})v - 0.03ur$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (24)$$

We solve (24) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.4):

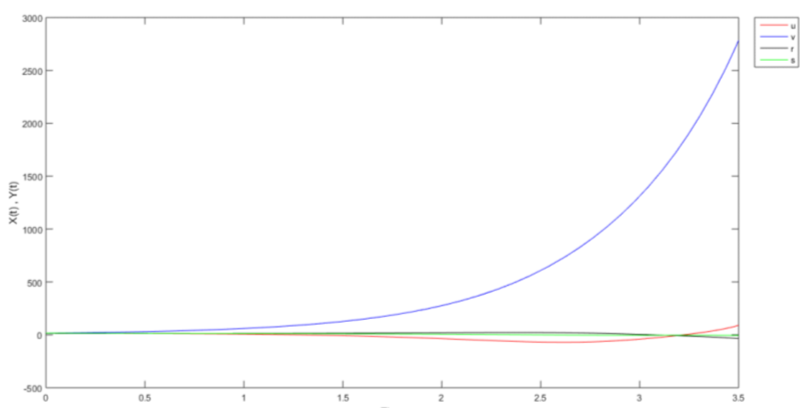


Figure (3.2.4): The solution of (24) at α=0

At α-level = 0.5, the solution is figure (3.2.5) in the appendix. At α-level = 1, the solution is figure (3.2.6):

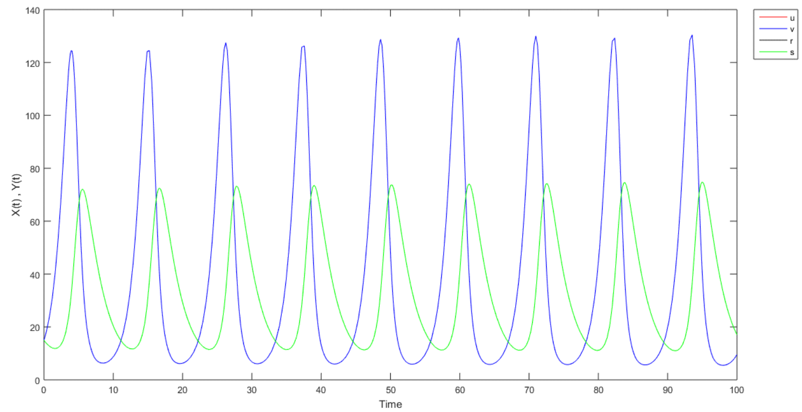


Figure (3.2.6): The solution of (24) at α=1

If $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable, then the model will be:

$$u' = (1.5 - \frac{\alpha}{2})v - 0.03ur$$
$$v' = (0.5 + \frac{\alpha}{2})u - 0.03vs$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$

With the initial conditions:

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (25)$$

We solve (25) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.7).

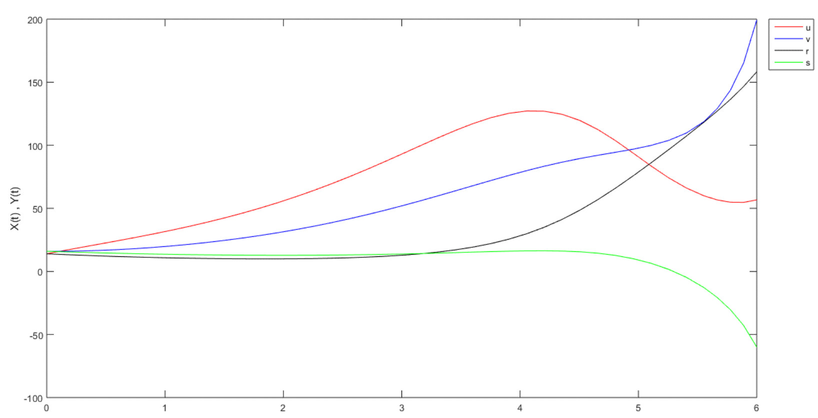


Figure (3.2.7): The solution of (25) at α=0

At α-level = 0.5, the solution is figure (3.2.8) in the appendix. At α-level = 1, the solution is figure (3.2.9):

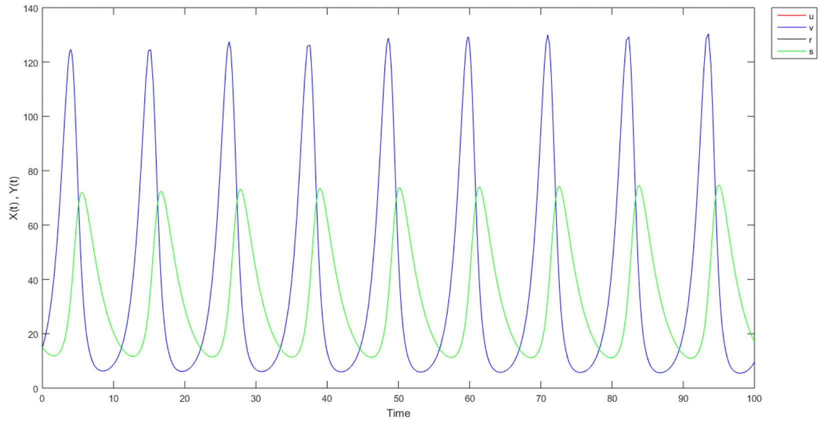


Figure (3.2.9): The solution of (25) at α=1

If $x(t)$ and $y(t)$ are (2)-differentiable, then the model will be:

$$u' = (1.5 - \frac{\alpha}{2})v - 0.03ur$$
$$v' = (0.5 + \frac{\alpha}{2})u - 0.03vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (26)$$

We solve (26) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution graphs are figure (3.2.10), figure (3.2.11) and figure (3.2.12):

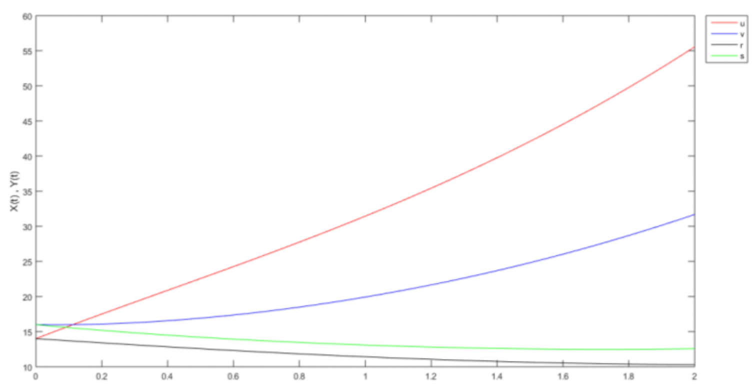
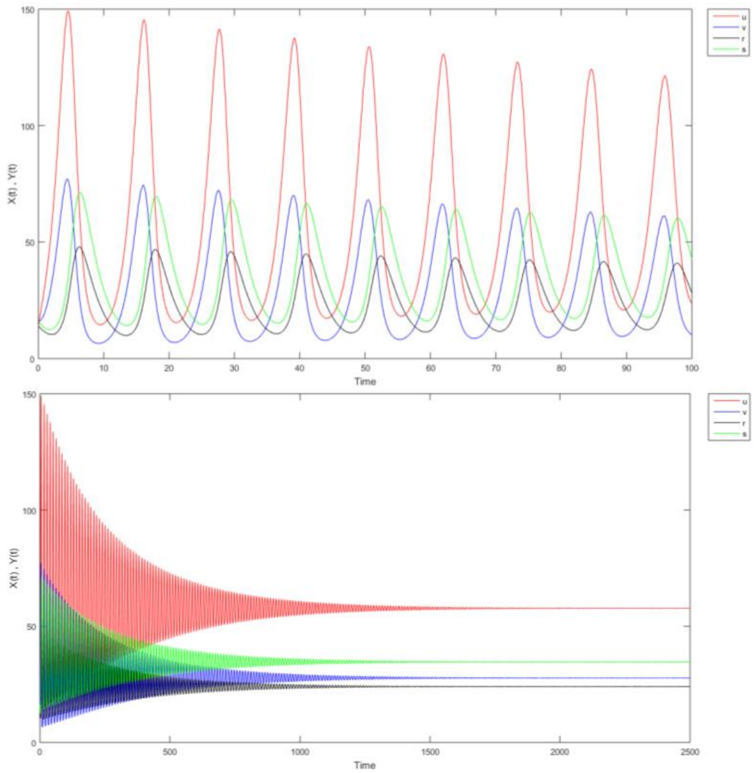


Figure (3.2.10): The solution of (26) at α=0 for short time period



Figures (3.2.11) and (3.2.12): The solution of (26) at α=0 as time increases

At α = 0, as $t \to \infty$, $u(t) \to 57.69$, $v(t) \to 27.73$, $r(t) \to 24.04$, $s(t) \to 34.67$. So, the solution is asymptotically stable for $y(t)$ but there is no fuzzy solution for $x(t)$ since $u(t) > v(t)$.

At α-level = 0.5, the solution graphs are figure (3.2.13), figure (3.2.14) and figure (3.2.15) in the appendix. At α-level = 1, the solution is figure (3.2.16):

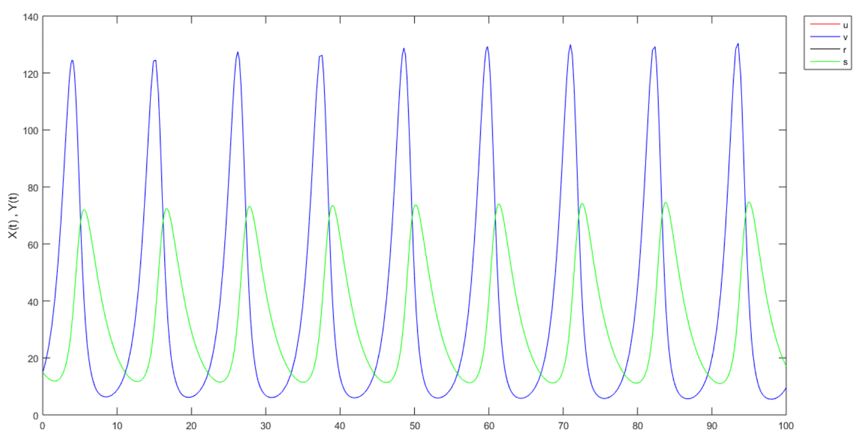


Figure (3.2.16): The solution of (26) at α=1

At α= 1, the solution is similar to the crisp solution and it is stable solution.

Second, we let $a = (0.25, 0.5, 1.5, 1.75)$ trapezoidal fuzzy number. So, $[a]_\alpha = \left[0.25 + \frac{\alpha}{4},\ 1.75 - \frac{\alpha}{4}\right]$ and we obtain the following model:

$$x'(t) = (0.25, 0.5, 1.5, 1.75)x - 0.03xy$$
$$y'(t) = -0.4\, y + 0.01xy$$

With fuzzy initial conditions:

$$[x_0]_\alpha = [14 + \alpha, 16 - \alpha], [y_0]_\alpha = [14 + \alpha, 16 - \alpha]$$

If $x(t)$ *and* $y(t)$ are (1)-differentiable, then the model will be:

$$u' = (0.25 + \frac{\alpha}{4})u - 0.03vs$$
$$v' = (1.75 - \frac{\alpha}{4})v - 0.03ur$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (27)$$

The model will change at any value of α, so the equilibrium points will also change as α changes. The first equilibrium point is (0,0,0,0) for any α –level. The second equilibrium point varies according to the α–level, as in the following table (3.2.4).

Table (3.2.4): The equilibrium points of (27)

α- level	*u*	*v*	*r*	*s*
0	76.5172	20.9103	15.9411	30.4942
0.5	65.213	24.535	20.3791	33.2245
1	57.69	27.7345	24.0375	34.6681

We solve (27) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is table (3.2.5) and its graph is figure (3.2.17):

Table (3.2.5): The solution of (27) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	12.5720	23.0810	12.8820	15.4100
0.5000	10.0430	34.4060	11.7050	15.2610
0.7500	5.5735	52.4330	10.3810	15.8120
1.0000	-2.4648	80.9900	8.7757	17.5710
1.2500	-18.0330	126.0700	6.6624	21.7880
1.5000	-51.3560	196.7600	3.6563	31.6960
1.7500	-138.0100	305.8100	-0.8783	58.6080
2.0000	-434.1300	463.5800	-7.4426	152.4400
2.2500	-1849.9000	624.7200	-12.6950	605.5700

2.5000	-8392.3000	451.4500	-12.8500	2694.3000
2.7500	-8817.7000	-550.5700	-11.9730	2616.9000
3.0000	-1938.6000	-1335.8000	-6.4648	229.6600
3.2500	-1351.0000	-2116.9000	-0.7328	3.2733
3.5000	-1427.7000	-3281.8000	-0.0267	0.0049
3.7500	-1519.8000	-5083.1000	-0.0007	0.0000
4.0000	-1617.8000	-7872.9000	0.0000	0.0000
4.2500	-1722.1000	-12194.0000	0.0000	0.0000
4.5000	-1833.2000	-18886.0000	0.0000	0.0000
4.7500	-1951.4000	-29251.0000	0.0000	0.0000
5.0000	-2077.3000	-45305.0000	0.0000	0.0000

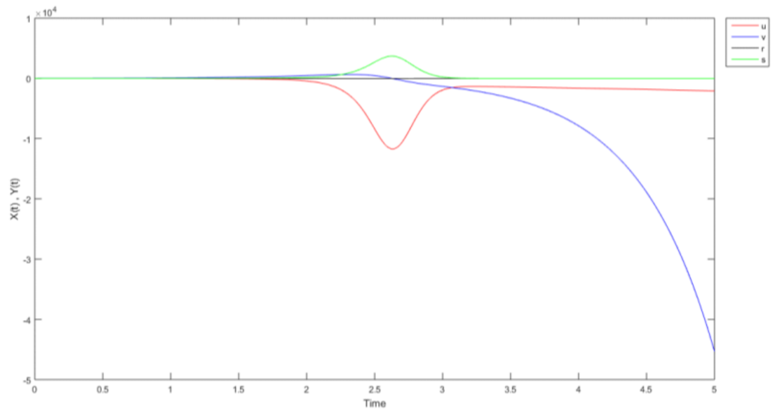


Figure (3.2.17): The solution of (27) at α=0

At α-level = 0.5, the solution is figure (3.2.18) in the appendix. At α-level = 1, the solution is table (3.2.6) and graph is figure (3.2.19):

Table (3.2.6): The solution of (27) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
0.2500	14.9960	19.8320	14.0900	14.1740
0.5000	14.4340	27.0140	13.2070	13.6120
0.7500	12.9220	37.6960	12.2990	13.4140
1.0000	9.7830	53.5930	11.2870	13.7600
1.2500	3.7136	77.2630	10.0490	15.0080
1.5000	-7.9541	112.5000	8.3919	17.9100
1.7500	-31.7310	164.8200	5.9996	24.3420
2.0000	-85.7130	241.8100	2.3669	39.6230
2.2500	-232.6900	351.2100	-3.1089	82.5540
2.5000	-743.8900	484.9300	-9.6621	235.6000
2.7500	-2900.6000	545.8000	-12.2590	890.2000
3.0000	-8648.5000	191.6700	-11.9300	2572.4000
3.2500	-6566.2000	-607.9800	-9.5522	1519.8000
3.5000	-3021.6000	-1168.6000	-2.8989	158.6000

3.7500	-2917.0000	-1729.9000	-0.1336	4.3821
4.0000	-3291.0000	-2517.9000	-0.0010	0.0232
4.2500	-3729.1000	-3663.5000	0.0000	0.0000
4.5000	-4225.6000	-5330.3000	0.0000	0.0000
4.7500	-4788.3000	-7755.6000	0.0000	0.0000
5.0000	-5425.8000	-11284.0000	0.0000	0.0000

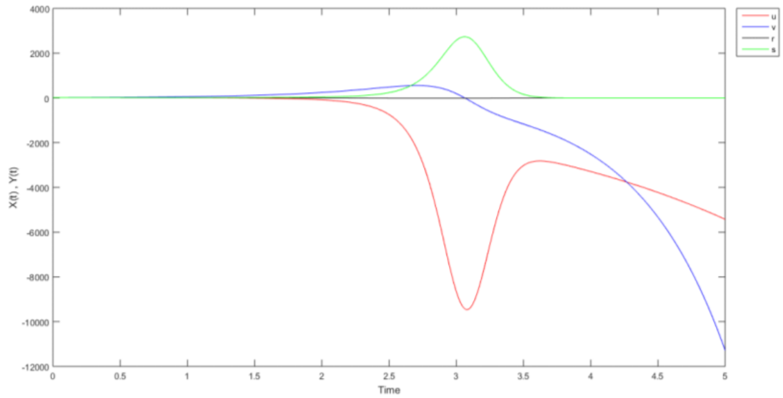


Figure (3.2.19): The solution of (27) at α=1

If $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, then the model will be:

$$u' = (0.25 + \frac{\alpha}{4})u - 0.03vs$$

$$v' = (1.75\ -\ \frac{\alpha}{4})v - 0.03ur$$

$$r' = -0.4r + 0.01vs$$

$$s' = -0.4s + 0.01ur$$

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \tag{28}$$

We solve (28) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.20):

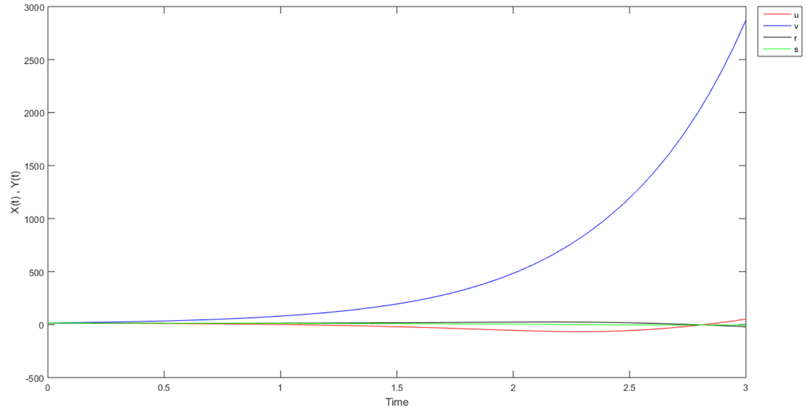


Figure (3.2.20): The solution of (28) at α=0

At α-level = 0.5, the solution is figure (3.2.21) in the appendix. At α-level = 1, the solution is figure (3.2.22):

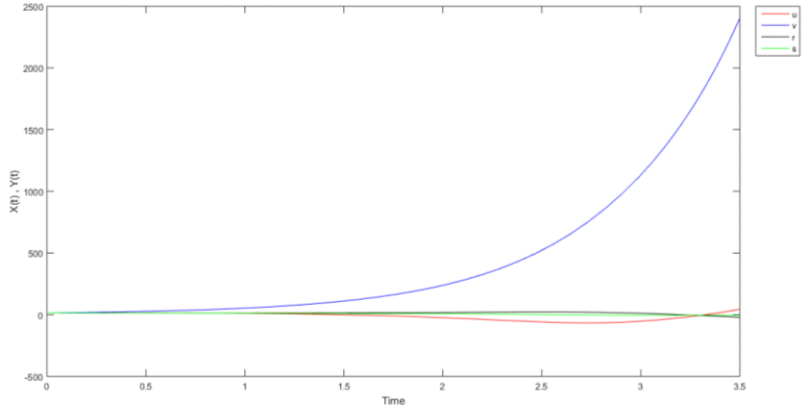


Figure (3.2.22): The solution of (28) at α=1

Now, If $x(t)$ is (2)-differentiable and $y(t)$is (1)-differentiable, then the model will be:

$$u' = (1.75 - \frac{\alpha}{4})v - 0.03ur$$
$$v' = (0.25 + \frac{\alpha}{4})u - 0.03vs$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (29)$$

We solve (29) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.23):

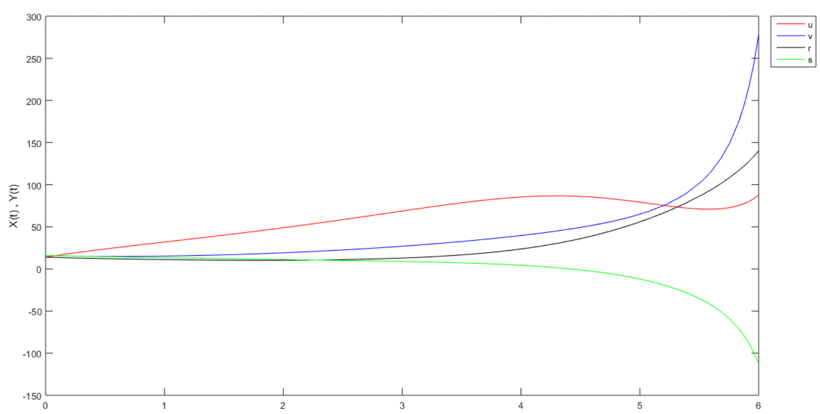


Figure (3.2.23): The solution of (29) at α=0

At α-level = 0.5, the solution is figure (3.2.24) in the appendix. At α-level = 1, the solution is figure (3.2.25):

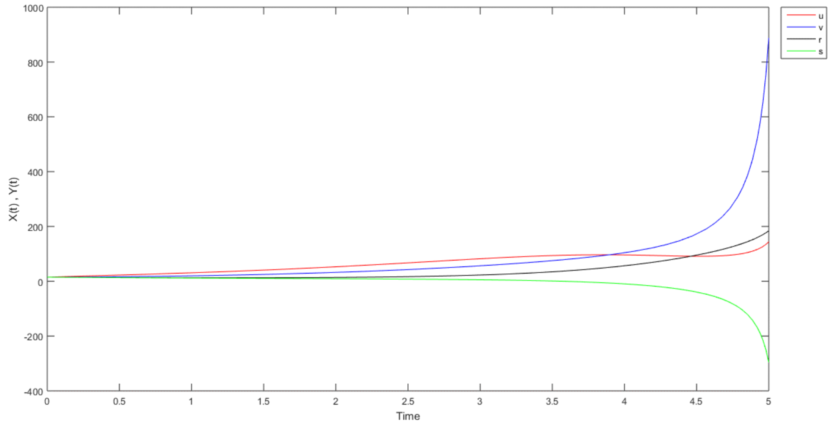


Figure (3.2.25): The solution of (29) at α=1

If $x(t)$ and $y(t)$ are (2)-differentiable, then the model will be:

$$u' = (1.75 - \frac{\alpha}{4})v - 0.03ur$$
$$v' = (0.25 + \frac{\alpha}{4})u - 0.03vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (30)$$

We solve (30) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution graphs are figure (3.2.26), figure (3.2.27) and figure (3.2.28):

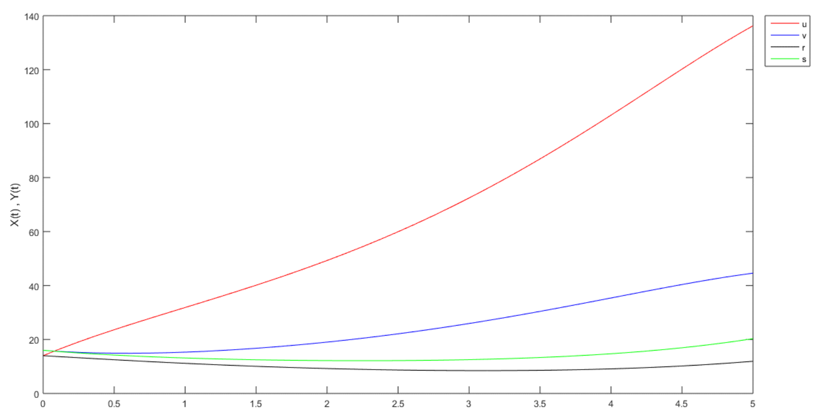
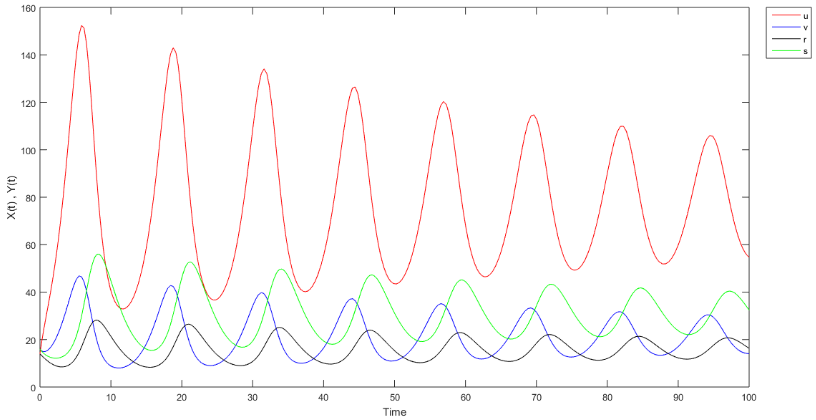
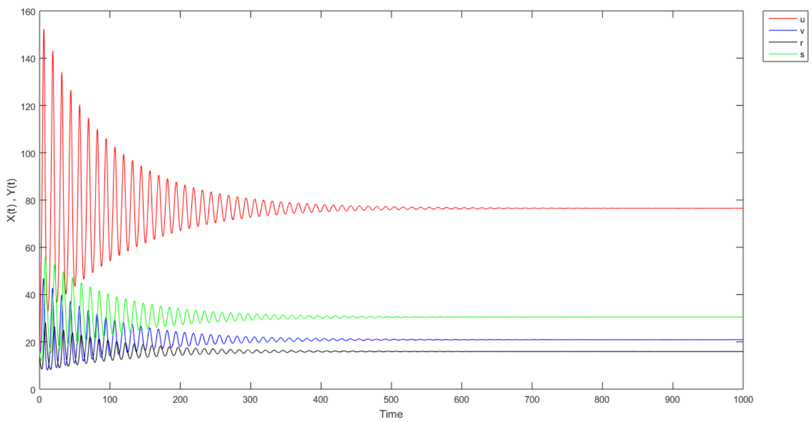


Figure (3.2.26): The solution of (30) at α=0 for short time period





Figures (3.2.27) and (3.2.28): The solution of (30) at α=0 as time increases

At $\alpha = 0$, as $t \rightarrow \infty$, $u(t) \rightarrow 76.5$, $v(t) \rightarrow 20.92$, $r(t) \rightarrow 15.94$, $s(t) \rightarrow 30.49.$ So the solution is asymptotically stable.

At α-level = 0.5, the solution is figure (3.2.29) in the appendix. At α-level = 1, the solution graphs are figure (3.2.30), figure (3.2.31) and figure (3.2.32):

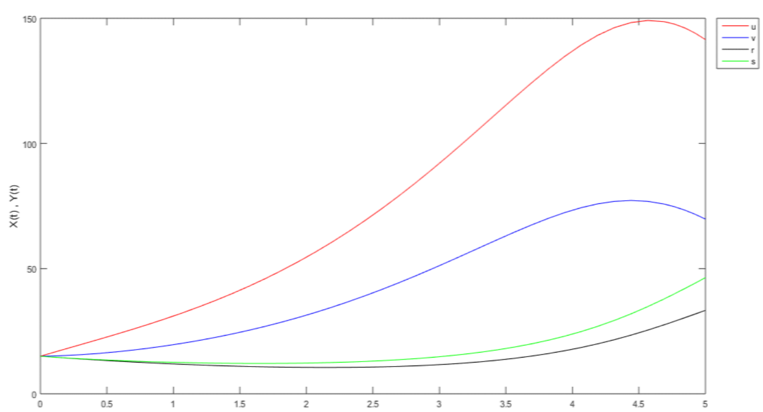
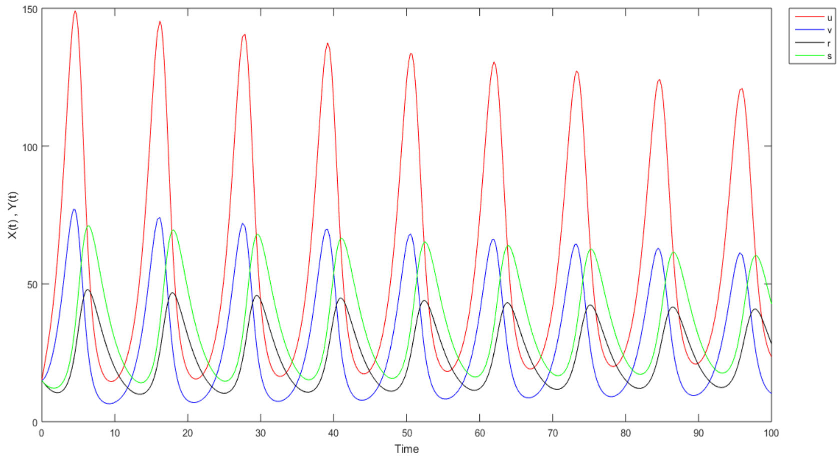
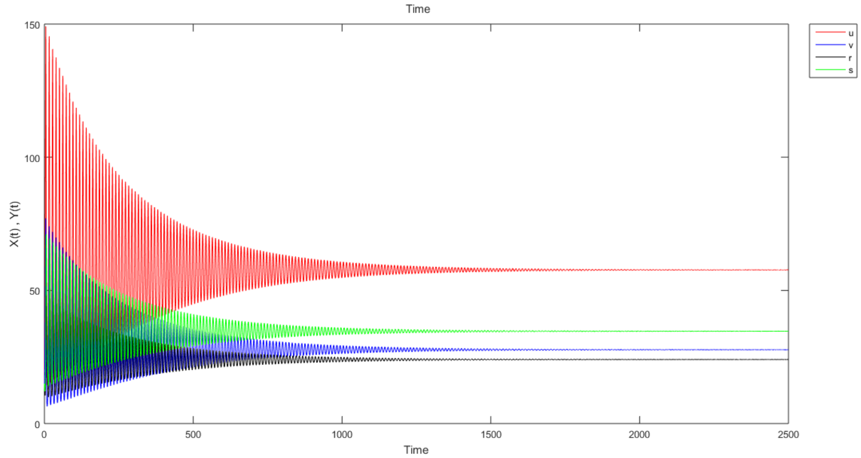


Figure (3.2.30): The solution of (30) at α=1 for short time period





Figures (3.2.31) and (3.2.32): The solution of (30) at α=1 as time increases

At α-level = 1, the solution is asymptotically stable since $as\, t \to \infty$, $u(t) \to 57.69$, $v(t) \to 27.74$, $r(t) \to 24.04$, $s(t) \to 34.67$.

Firstly, we assume (a) a triangular fuzzy number then we note that when $x(t)$ *and* $y(t)$ are (1,1), (1,2) and (2,1)-differentiable, we obtain unacceptable and unstable solution, but at $\alpha = 1$ the solution is the same as the solution of the crisp case and it is stable. While, when $x(t)$ and $y(t)$ are (2)-differentiable, the solution is asymptotically stable. However, we note that $u(t) > v(t)$ for $t \to \infty$, so there is no fuzzy solution for $x(t)$ but this solution is acceptable biologically. At $\alpha = 1$ the solution is the same as the solution of the crisp case. Secondly, we assume (a) a trapezoidal fuzzy number then we obtain the same results for all cases of derivatives else at $\alpha = 1$, we have a solution not similar to the crisp case. So, we deduce that the triangular fuzzy number is better than the trapezoidal fuzzy number. Therefore, we want to discover the solution for the model (5) with fuzzy initial conditions when a is a triangular fuzzy number with small support and then with large support. Thereafter, we compare between them. We choose form (2,2)-

differentiable since we haven't got a fuzzy solution for the rest forms of the derivative.

We fuzzify a by a triangular fuzzy number with small support. So, we let $a = (0.9999, 1, 1.0001)\ with\ [a]_\alpha = \left[0.9999 + \frac{\alpha}{10000},\ 1.0001 - \frac{\alpha}{10000}\right]$. Then we have the following model:

$$u' = (1.0001 - \frac{\alpha}{10000})v - 0.03ur$$
$$v' = (0.9999 + \frac{\alpha}{10000})u - 0.03vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s = 16 - \alpha \qquad (31)$$

We solve (31) by Runge-Kutta method in Matlab at α-level=0. The solution is table (3.2.7), where its graph is figure (3.2.33):

Table (3.2.7): The solution of (31) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
5.0000	74.4350	74.3420	65.8540	65.9210
10.0000	8.8663	8.8636	20.4870	20.4970
15.0000	124.3000	124.2800	31.5100	31.5120
20.0000	6.5222	6.5213	29.1380	29.1400
25.0000	83.8760	83.8660	14.7660	14.7670
30.0000	6.9470	6.9459	42.2360	42.2390
35.0000	42.0810	42.0770	11.6830	11.6840
40.0000	12.2560	12.2550	60.0920	60.0970
45.0000	20.0960	20.0930	12.9540	12.9550
50.0000	40.8910	40.8840	73.2820	73.2870
55.0000	10.4940	10.4920	17.2980	17.2990
60.0000	118.7900	118.7700	46.5670	46.5700
65.0000	6.7008	6.6999	25.0360	25.0380
70.0000	104.0900	104.0800	17.8040	17.8050
75.0000	6.0236	6.0227	37.2960	37.2990
80.0000	51.6560	51.6500	11.4840	11.4840
85.0000	9.3957	9.3941	55.2940	55.2980
90.0000	23.3490	23.3460	12.0560	12.0570
95.0000	29.7430	29.7380	73.3640	73.3700
100.0000	11.3660	11.3640	15.9880	15.9900

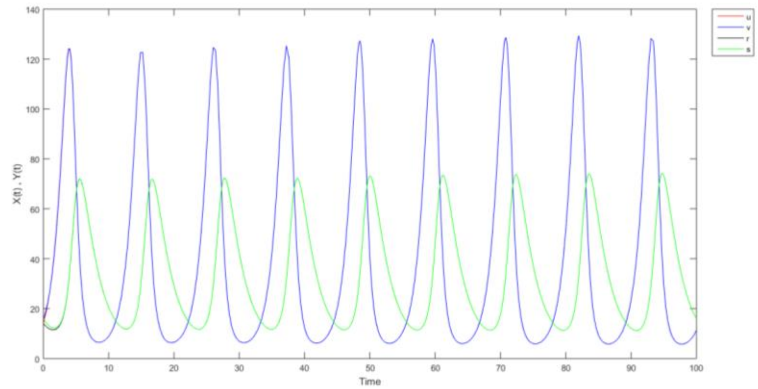


Figure (3.2.33): The solution of (31) at α=0

We try to understand the behavior of the solution for long time intervals. So, we note that $u(t) > v(t)\ as\ t \to \infty$ but the difference $u(t) - v(t)$ ≈0.0005.

Then, we fuzzify a by a triangular fuzzy number with large support. So, we let $a = (0.02, 1, 1.98)\ with\ [a]_\alpha = [0.02 + 0.98\alpha,\ 1.98\ - 0.98\alpha]$. Then we have the following model:

$$u' = (1.98\ - 0.98\alpha)v - 0.03ur$$
$$v' = (0.02 + 0.98\alpha)u - 0.03vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (32)$$

We solve (32) by Runge-Kutta method in Matlab at α-level=0. The solution is table (3.2.8), where its graph is figure (3.2.34):

Table (3.2.8): The solution of (32) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
50.0000	173.8800	6.5124	3.5855	18.3930
100.0000	181.0000	8.3602	3.0222	14.1360
150.0000	184.8100	8.6684	3.0637	14.1650
200.0000	185.1100	8.6557	3.0834	14.2590
250.0000	185.0600	8.6470	3.0845	14.2680
300.0000	185.0500	8.6449	3.0849	14.2640
350.0000	185.0600	8.6433	3.0857	14.2610
400.0000	185.0400	8.6462	3.0843	14.2660
450.0000	185.0400	8.6466	3.0841	14.2670
500.0000	185.0600	8.6427	3.0860	14.2600
550.0000	185.0400	8.6477	3.0835	14.2690
600.0000	185.0300	8.6485	3.0831	14.2700
650.0000	185.0400	8.6473	3.0837	14.2680

700.0000	185.0400	8.6481	3.0834	14.2690
750.0000	185.0100	8.6527	3.0810	14.2780
800.0000	185.0400	8.6475	3.0836	14.2690
850.0000	185.0100	8.6536	3.0806	14.2790
900.0000	185.0600	8.6440	3.0854	14.2620
950.0000	185.0600	8.6426	3.0861	14.2600
1000.0000	185.0400	8.6473	3.0837	14.2680

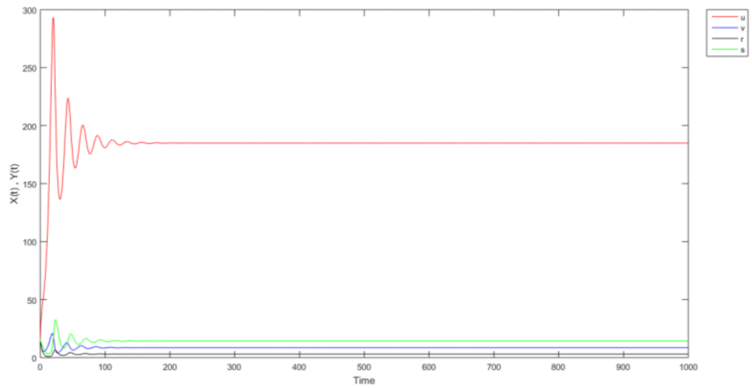


Figure (3.2.34): The solution of (32) at α=0

From previous table and graph we note that $u(t) > v(t)\ as\ t \to \infty$ with large difference.

We compare between a triangular fuzzy number of small support with other of large support, and we note that when the support is large the difference between $u(t)$ and $v(t)$ is clear but when the support is small the difference between $u(t)$ and $v(t)$ isn't clear and close to the solution of the model with crisp a. Therefore, as the support of the triangular fuzzy number is small and close to the crisp number, the solution will be more periodic and closer to the crisp solution.

Finally, we try to discover the behavior of the solution of model (5) with fuzzy initial conditions by assuming (a) a triangular fuzzy number with support such that the distance between its endpoints and the core is unequal. Figure (3.2.35) and figure (3.2.36) show the solution of model (5) with initial conditions $[x_0]_\alpha = [14+\alpha, 16-\alpha] = [y_0]_\alpha$ when $x(t)$ and $y(t)$ are (2)-differentiable at $\alpha - level = 0$, for $a = (0.2,1,1.2)$ and $a = (0.95,1,1.8)$, respectively.

When $a = (0.2,1,1.2)$, we get the following model:

$$\begin{gathered} x'(t) = (0.2,1,1.2)x - 0.03xy \\ y'(t) = -0.4\,y + 0.01xy \\ [x_0]_\alpha = [14+\alpha, 16-\alpha] = [y_0]_\alpha \end{gathered} \tag{33}$$

When $a = (0.95,1,1.8)$, we get the following model:

$$\begin{gathered} x'(t) = (0.95,1,1.8)x \; - 0.03xy \\ y'(t) = -0.4\, y + 0.01xy \\ [x_0]_\alpha = [14 + \alpha, 16 - \alpha] = [y_0]_\alpha \end{gathered} \tag{34}$$

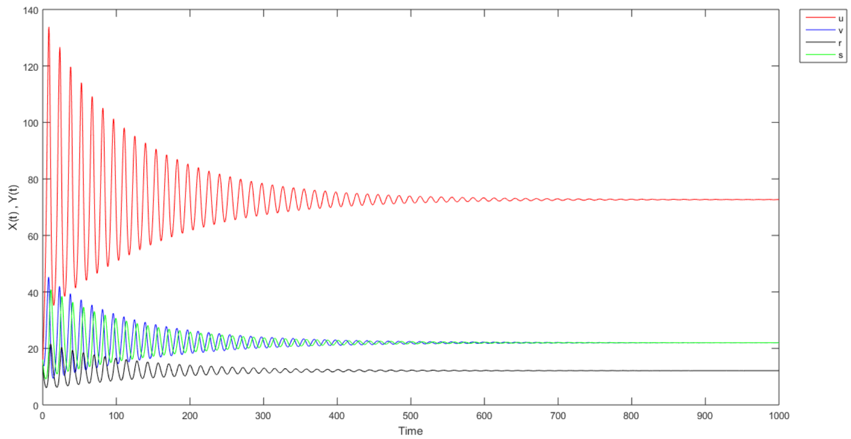


Figure (3.2.35): The solution of (33) at α=0 for $x(t)$*and* $y(t)$ are (2)-differentiable

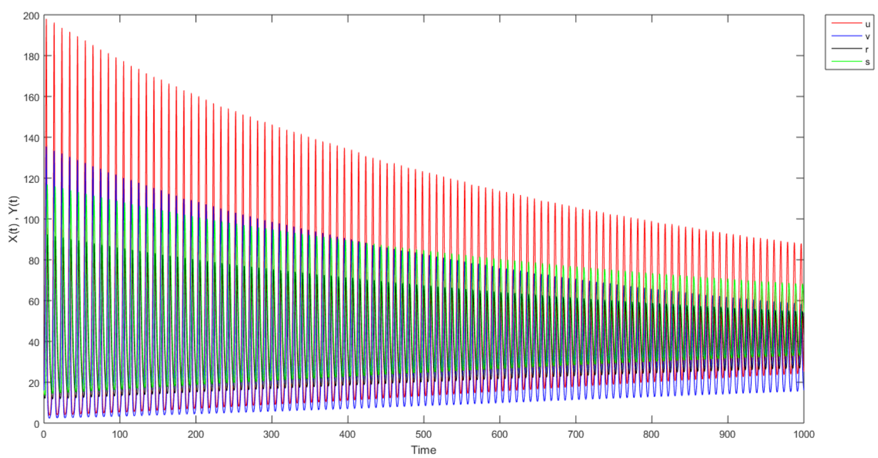


Figure (3.2.36): The solution of (34) at α=0 for $x(t)$*and* $y(t)$ are (2)-differentiable

From figures (3.2.35) and (3.2.36), we note that the solutions are asymptotically stable. So, we conclude that for a fuzzy number $a = (a_1, a_2, a_3)$ whenever at least one of the differences $(a_2 - a_1), (a_2 - a_3)$ increased then the solution will be asymptotically stable. And vice versa, when a_1 *and* a_3 are closer to the core a_2, the solution is closer to the crisp case.

Now, we try to fuzzify $b = 0.03$. Initially using triangular fuzzy number, we let $b = (0.01\ ,0.03\ , 0.05)$. So, $[b]_\alpha = \left[0.01 + \frac{\alpha}{50},\ 0.05 - \frac{\alpha}{50}\right]$ and we have the following model:

$$x'(t) = x - (0.01\ ,0.03\ , 0.05)xy$$
$$y'(t) = -0.4\,y + 0.01xy$$

With fuzzy initial conditions:

$$[x_0]_\alpha = [14 + \alpha, 16 - \alpha] = [y_0]_\alpha$$

If we consider $x(t)$ and $y(t)$ are (1)-differentiable, then the model will be:

$$u' = u - (0.05 - \frac{\alpha}{50})vs$$
$$v' = v - (0.01 + \frac{\alpha}{50})ur$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (35)$$

The first equilibrium point of (35) is (0,0,0,0) for any α –level. The second one varies according to the α–level, as in the following table (3.2.9)

Table (3.2.9): The equilibrium points of (35)

$\alpha - level$	u	v	r	s
0	68.399	23.3921	34.1995	58.4804
0.5	50.3968	31.748	31.498	39.685
1	40	40	33.3333	33.3333

we solve (35) by Runge-Kutta method in Matlab at α-level=0,0.5,1. At α-level = 0, the solution is table (3.2.10) and its graph is figure (3.2.37):

Table (3.2.1): The solution of (35) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	14.0100	20.0060	12.9070	15.3560
0.5000	13.1760	25.2070	11.8160	14.9700
0.7500	10.9280	31.9730	10.6690	14.9050
1.0000	6.3389	40.7970	9.3889	15.2620
1.2500	-2.1152	52.3140	7.8782	16.2140
1.5000	-17.0440	67.3380	6.0241	18.0560
1.7500	-43.0430	86.8490	3.7276	21.3220

2.0000	-88.6620	111.9200	0.9821	27.0220
2.2500	-170.6300	143.5000	-1.9743	37.1870
2.5000	-323.6500	182.0000	-4.4848	56.1750
2.7500	-625.4900	226.9400	-5.7279	94.2230
3.0000	-1265.1000	276.7900	-5.6989	177.5400
3.2500	-2741.2000	325.8100	-5.5265	378.4800
3.5000	-6377.7000	351.9200	-5.5885	891.9700
3.7500	-14713.0000	294.3400	-5.5922	2056.9000
4.0000	-26366.0000	64.2780	-5.1018	3353.6000
4.2500	-29651.0000	-280.6100	-3.4867	2562.6000
4.5000	-28443.0000	-556.3500	-1.2642	881.1500
4.7500	-32360.0000	-768.2100	-0.2127	167.6300
5.0000	-40674.0000	-995.0200	-0.0189	18.6630

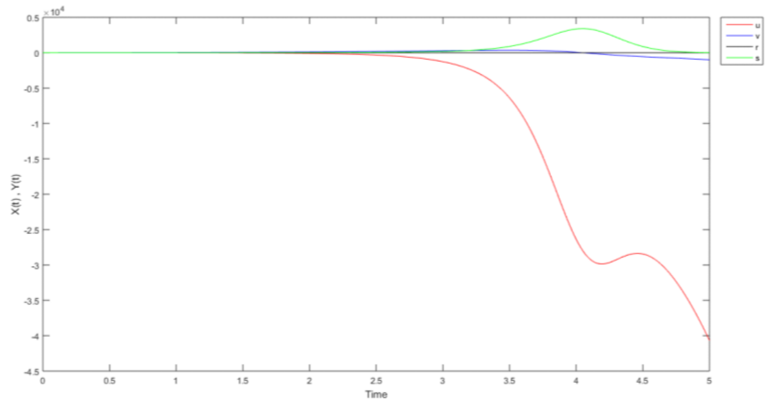


Figure (3.2.37): The solution of (35) at α=0

At α-level = 0.5, the solution is figure (3.2.38) in the appendix. At α-level = 1, the solution is figure (3.2.39):

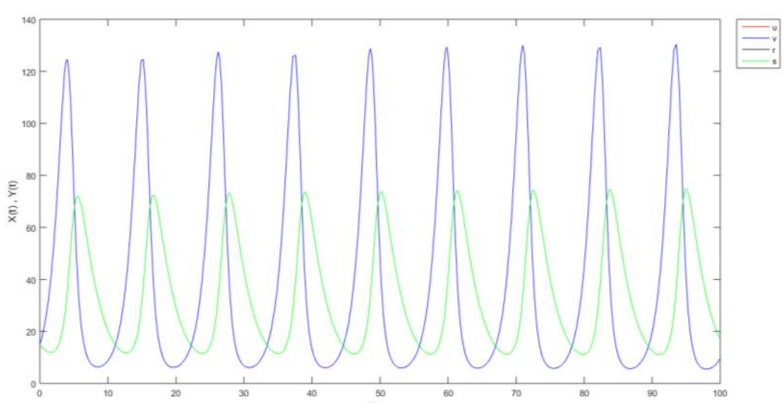


Figure (3.2.39): The solution of (35) at α=1

While $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, then the model will be:

$$u' = u - (0.05 - \frac{\alpha}{50})vs$$
$$v' = v - (0.01 + \frac{\alpha}{50})ur$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (36)$$

We solve (36) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is table (3.2.11), where its graph is figure (3.2.40):

Table (3.2.11): The solution of (36) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.5000	13.4700	25.1650	12.8300	13.9430
1.0000	8.8983	40.5230	12.4000	12.0750
1.5000	-3.8096	66.4900	12.7820	10.0690
2.0000	-30.4190	110.9000	13.8750	7.2584
2.5000	-73.7510	187.4200	14.5780	2.5670
3.0000	-116.8800	317.2500	10.8370	-3.7206
3.5000	-121.0900	527.5900	-1.6578	-5.8170
4.0000	-121.6100	864.2700	-11.5930	-0.5437
4.5000	-287.0500	1418.7000	4.2701	3.2886
5.0000	-415.1800	2345.3000	-7.2862	-0.1676
5.5000	-713.0700	3865.3000	-5.6272	0.7536
6.0000	-1216.4000	6375.8000	-1.4206	-1.9897
6.5000	-2024.1000	10510.0000	1.6793	-1.5380
7.0000	-3336.6000	17325.0000	2.8722	0.5984
7.5000	-5478.6000	28564.0000	0.2206	1.1360
8.0000	-9026.5000	47095.0000	-0.9066	0.8421
8.5000	-14898.0000	77648.0000	1.7127	0.1173
9.0000	-24555.0000	128020.0000	1.2120	-0.3156
9.5000	-40480.0000	211070.0000	1.1247	0.0875
10.0000	-66728.0000	348000.0000	-0.7788	-0.2136

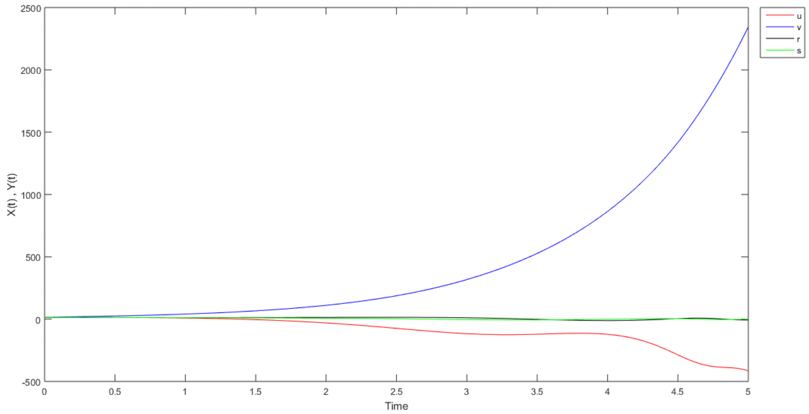


Figure (3.2.40): The solution of (36) at α=0

At α-level = 0.5, the solution is figure (3.2.41) in the appendix. At α-level = 1, the solution is figure (3.2.42):

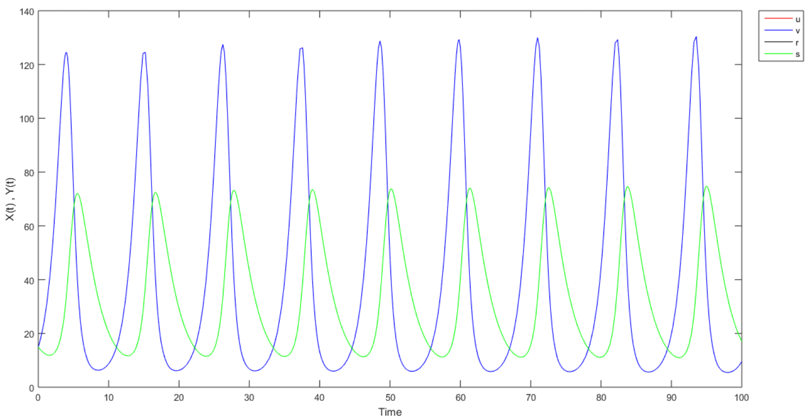


Figure (3.2.42): The solution of (36) at α=1

Now, if $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable, then the model will be:

$$u' = v - (0.01 + \frac{\alpha}{50})ur$$
$$v' = u - (0.05 - \frac{\alpha}{50})vs$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (37)$$

We solve (37) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.43):

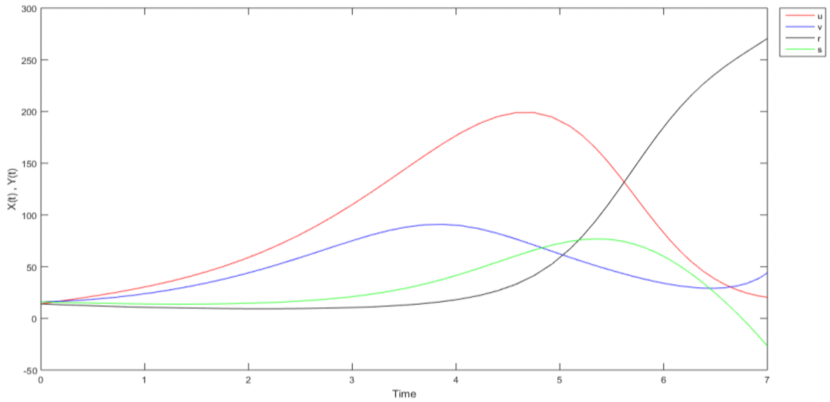


Figure (3.2.43): The solution of (37) at α=0

At α-level = 0.5, the solution is figure (3.2.44) in the appendix. At α-level = 1, the solution is figure (3.2.45):

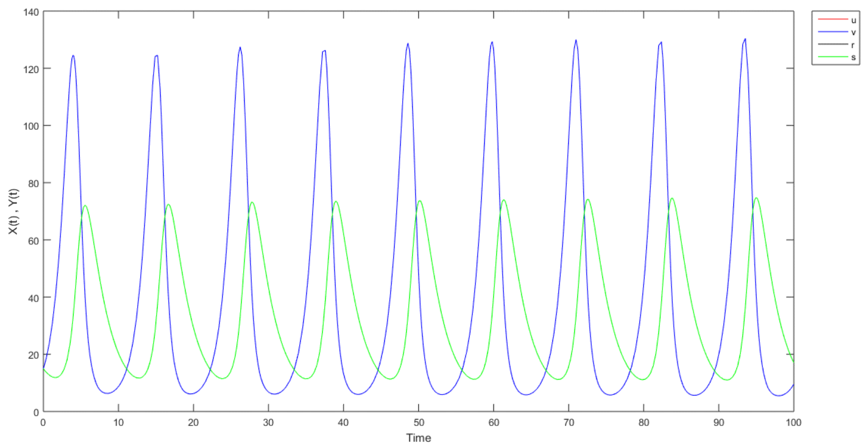


Figure (3.2.45): The solution of (37) at α=1

If $x(t)$ and $y(t)$ are (2)-differentiable, then the model will be:

$$u' = v - (0.01 + \frac{\alpha}{50})ur$$
$$v' = u - (0.05 - \frac{\alpha}{50})vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (38)$$

We solve (38) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution graphs are figure (3.2.46) and figure (3.2.47):

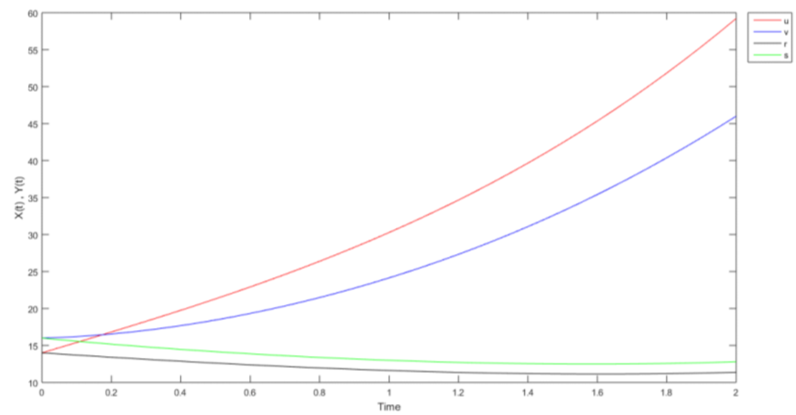


Figure (3.2.46): The solution of (38) at α=0 for short time period

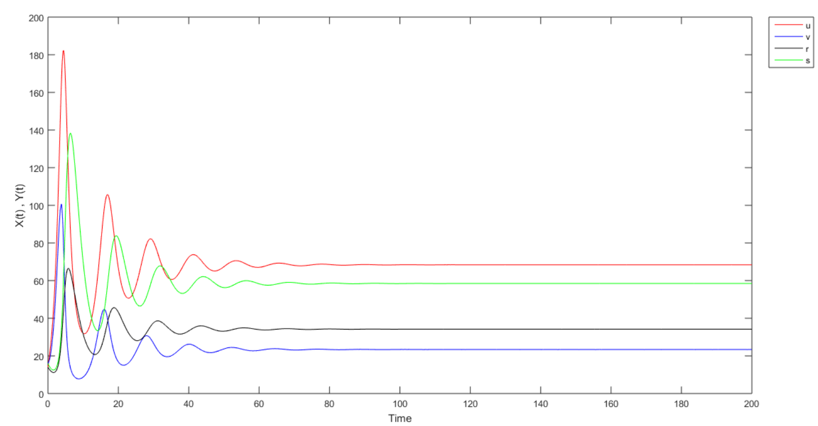


Figure (3.2.47): The solution of (38) at α=0 as time increases

At $\alpha = 0$, as $t \to \infty$, $u(t) \to 68.40$, $v(t) \to 23.39$, $r(t) \to 34.20$ and $s(t) \to 58.48$. So the solution is asymptotically stable for $y(t)$ but there is no fuzzy solution for $x(t)$.

At α-level = 0.5, the solution graphs are figure (3.2.48) and figure (3.2.49) and figure (3.2.50) in the appendix. At α-level = 1, the solution is figure (3.2.51):

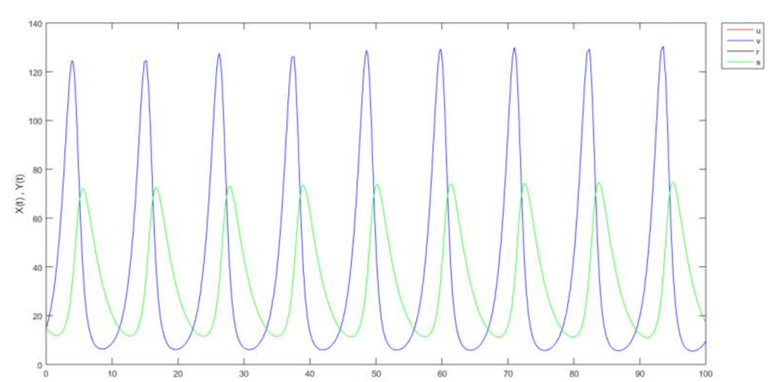


Figure (3.2.51): The solution of (38) at α=1

Now, we let $b = (0.01, 0.025, 0.035, 0.04)$ a trapezoidal fuzzy number. So, $[b]_\alpha = \left[0.015 + \frac{\alpha}{100}\ ,\ 0.045 - \frac{\alpha}{100}\right]$ and we obtain the following model:

$$x'(t) = x - (0.01, 0.025, 0.035, 0.04)xy$$
$$y'(t) = -0.4\, y + 0.01xy$$

With fuzzy initial conditions:

$$[x_0]_\alpha = [14 + \alpha, 16 - \alpha]\ , [y_0]_\alpha = [14 + \alpha, 16 - \alpha]$$

If $x(t)$ and $y(t)$ are (1)-differentiable, then the model will be:

$$u' = u - (0.045 - \frac{\alpha}{100})vs$$
$$v' = v - (0.015 + \frac{\alpha}{100})ur$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (39)$$

The first equilibrium point is (0,0,0,0) for any α –level but the second equilibrium point varies according to the α–level, as in the following table (3.2.12)

Table (3.2.12): The equilibrium points of (39)

α- level	*u*	*v*	*r*	*s*
0	57.69	27.7345	32.05	46.2241
0.5	50.3968	31.748	31.498	39.685
1	44.7476	35.7561	31.9625	35.7561

We solve (39) by Runge-Kutta method in Matlab at α-level= 0,0.5,1. At α-level = 0, the solution is table (3.2.13), where its graph is figure (3.2.52):

Table (3.2.13): The solution of (39) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	14.4330	19.7270	12.9140	15.3500
0.5000	14.2900	24.5680	11.8480	14.9450
0.7500	13.1340	30.8770	10.7520	14.8400
1.0000	10.2460	39.1270	9.5600	15.1240
1.2500	4.4370	49.9420	8.1833	15.9510
1.5000	-6.3388	64.1270	6.5108	17.5840
1.7500	-25.6770	82.6750	4.4207	20.4940
2.0000	-60.2670	106.6800	1.8319	25.5690
2.2500	-123.1100	137.0700	-1.1711	34.5800
2.5000	-240.9100	173.9500	-4.1255	51.2660
2.7500	-471.8200	215.7000	-6.1079	84.0460
3.0000	-949.3200	258.5200	-6.5279	152.9700

3.2500	-1990.8000	294.5400	-6.2192	307.2000
.5000	-4299.9000	301.0600	-6.0920	654.4300
3.7500	-8773.2000	226.0200	-5.9265	1296.5000
4.0000	-13942.0000	17.0820	-5.2176	1807.2000
4.2500	-15552.0000	-263.9200	-3.4695	1327.2000
4.5000	-15737.0000	-497.7300	-1.3279	505.1700
4.7500	-18162.0000	-688.9200	-0.2630	114.3500
5.0000	-22800.0000	-894.2000	-0.0292	15.9290

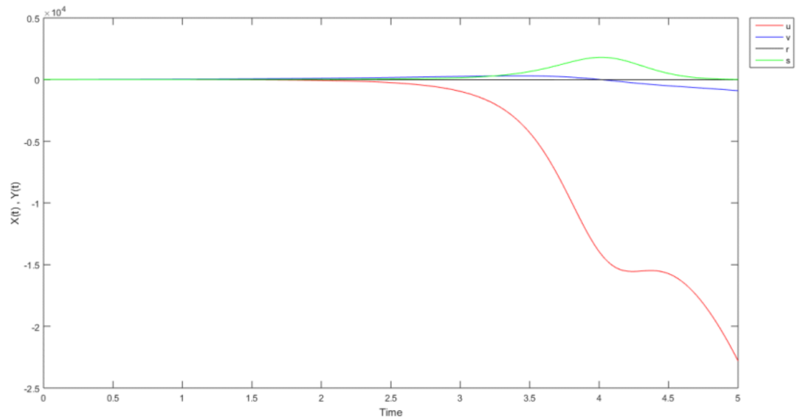


Figure (3.2.52): The solution of (39) at α=0

At α-level = 0.5, the solution is figure (3.2.53) in the appendix. At α-level = 1, the solution is table (3.2.14), where its graph is figure (3.2.54):

Table (3.2.14): The solution of (39) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
0.2500	16.9140	17.6170	14.1240	14.1360
0.5000	19.1010	20.8700	13.3640	13.4220
0.7500	21.5580	24.9230	12.7120	12.8680
1.0000	24.2490	29.9860	12.1570	12.4910
1.2500	27.0840	36.3360	11.6820	12.3230
1.5000	29.8650	44.3390	11.2640	12.4160
1.7500	32.2090	54.4920	10.8640	12.8610
2.0000	33.3740	67.4800	10.4110	13.8160
2.2500	31.9440	84.2720	9.7847	15.5680
2.5000	25.1020	106.2600	8.7682	18.6720
2.7500	7.0723	135.4100	7.0017	24.2700
3.0000	-35.0690	174.2200	3.9696	34.9550
3.2500	-132.6000	224.4200	-0.7368	57.1450
3.5000	-367.6600	281.8500	-6.2956	108.0400
3.7500	-958.8600	325.8500	-9.3968	233.6200
4.0000	-2392.3000	316.0100	-9.0121	531.3700
4.2500	-5047.8000	186.6400	-8.2443	1030.9000

4.5000	-7146.5000	-96.7960	-6.7433	1182.3000
4.7500	-6992.5000	-401.0100	-3.7767	626.4700
5.0000	-7133.9000	-629.7400	-1.0548	170.6200

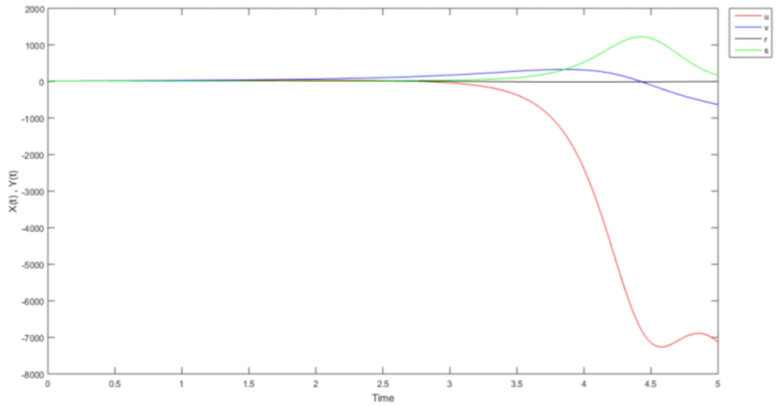


Figure (3.2.54): The solution of (39) at α=1

While $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, then the model will be as follow:

$$u' = u - (0.045 - \frac{\alpha}{100})vs$$
$$v' = v - (0.015 + \frac{\alpha}{100})ur$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (40)$$

We solve (40) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.55):

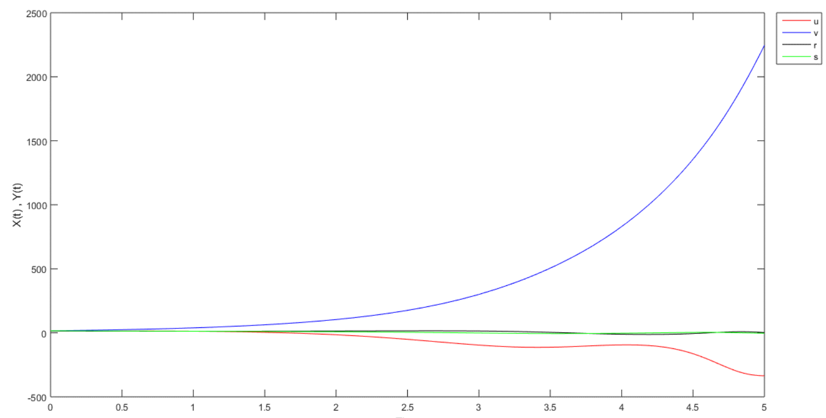


Figure (3.2.55): The solution of (40) at α=0

At α-level = 0.5, the solution is figure (3.2.56) in the appendix. At α-level = 1, the solution is figure (3.2.57):

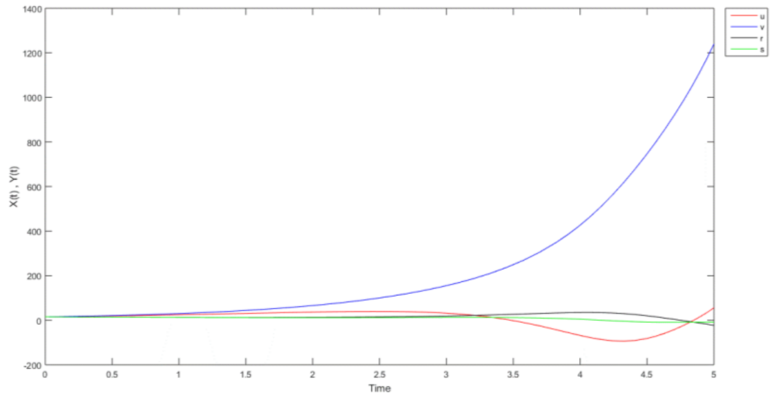


Figure (3.2.57): The solution of (40) at α=1

If $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable, then we have the following model:

$$u' = v - (0.015 + \frac{\alpha}{100})ur$$
$$v' = u - (0.045 - \frac{\alpha}{100})vs$$
$$r' = -0.4s + 0.01ur$$
$$s' = -0.4r + 0.01vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (41)$$

We solve (41) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.58):

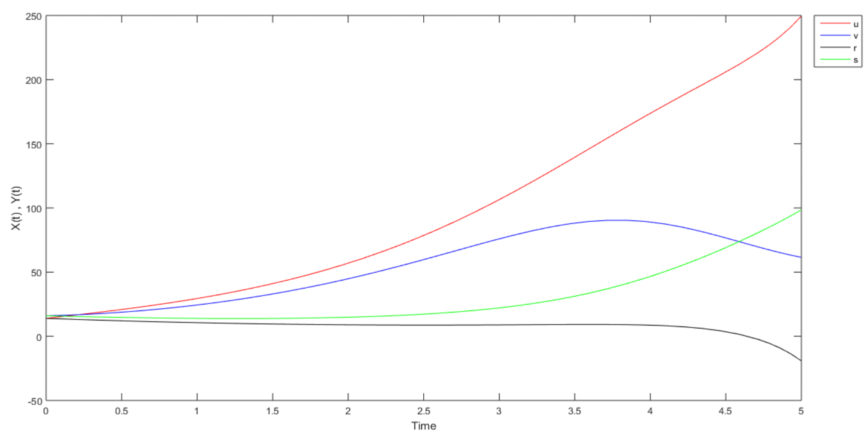


Figure (3.2.58): The solution of (41) at α=0

At α-level = 0.5, the solution is figure (3.2.59) in the appendix. At α-level = 1, the solution is figure (3.2.60):

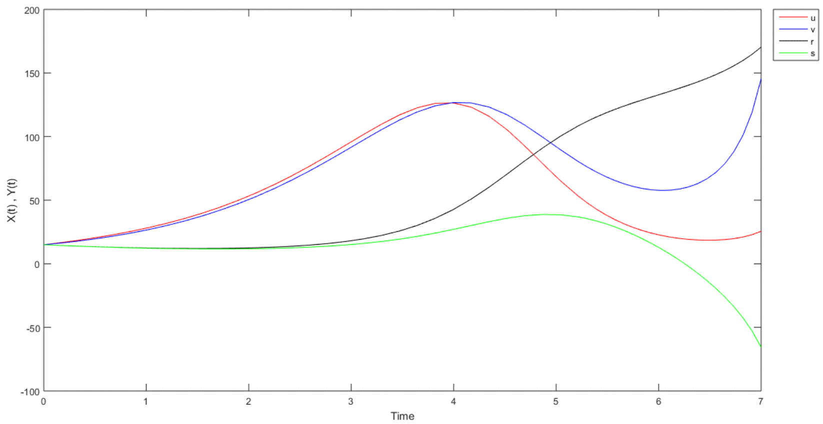


Figure (3.2.60): The solution of (41) at α=1

If $x(t)$ and $y(t)$ is (2)-differentiable, then we have the following model:

$$u' = v - (0.015 + \frac{\alpha}{100})ur$$
$$v' = u - (0.045 - \frac{\alpha}{100})vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (42)$$

We solve (42) by Runge-Kutta method in Matlab at α-level= $0, 0.5, 1$. At α-level = 0, the solution graphs are figure (3.2.61) and figure (3.2.62):

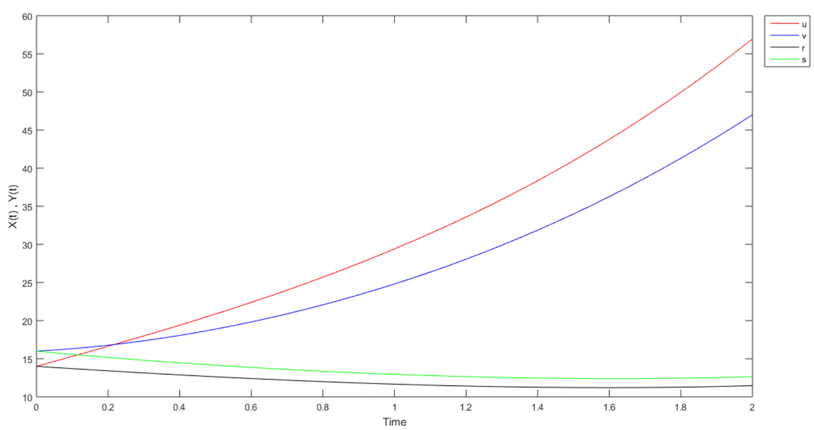


Figure (3.2.61): The solution of (42) at α=0 for short time period

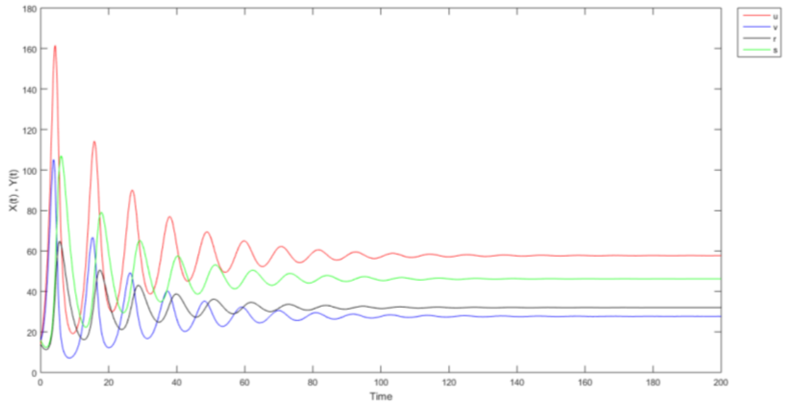


Figure (3.2.62): The solution of (42) at α=0 as time increases

We can note that the solution of $y(t)$ is asymptotically stable while there is no fuzzy solution for $x(t)$ since as $t \to \infty$, $u(t) \to 57.7$, $v(t) \to 27.74$, $r(t) \to 32.50$, $s(t) \to 46.22$ and $u(t) > v(t)$.

At α-level = 0.5, the solution graphs are figure (3.2.63), figure (3.2.64) and figure (3.2.65) in the appendix. At α-level = 1, the solution graphs are figure (3.2.66), figure (3.2.67) and figure (3.268):

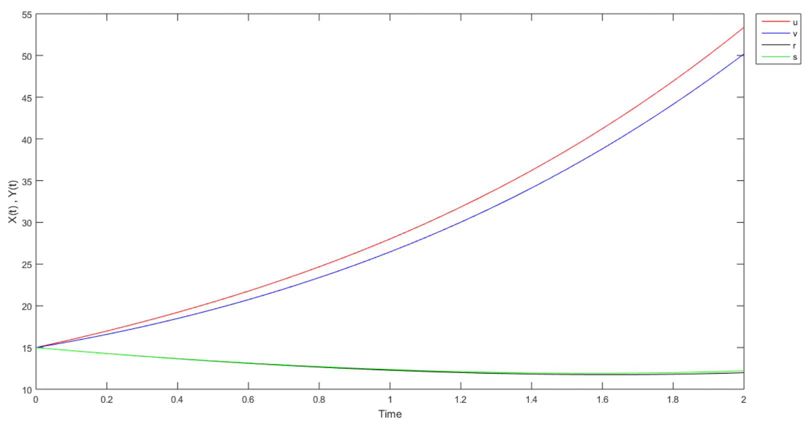
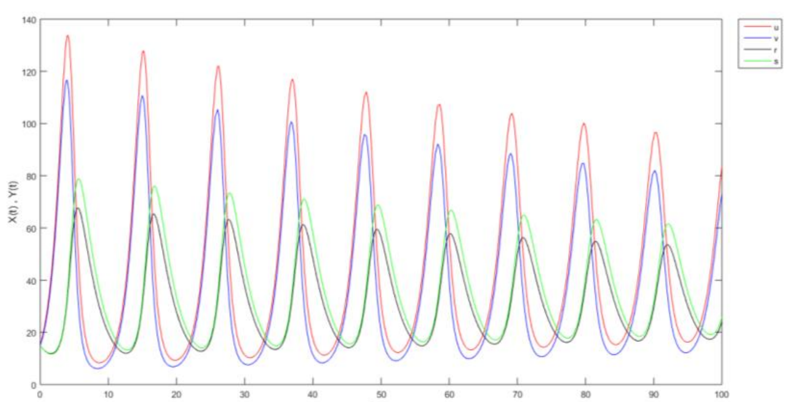
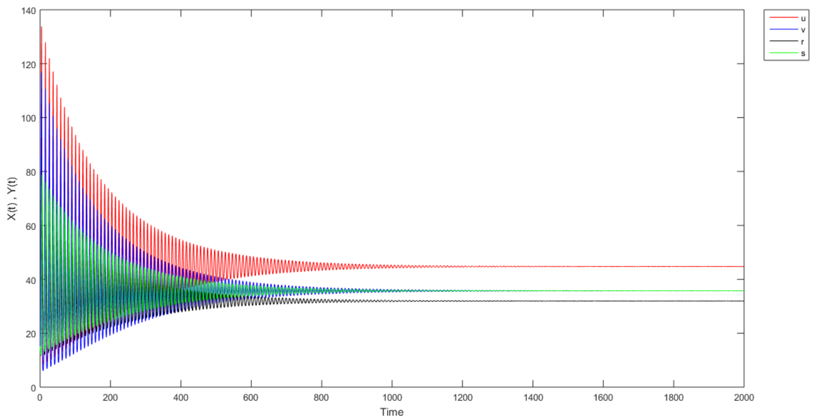


Figure (3.2.66): The solution of (42) at α=1 for short time period





Figures (3.2.67) and (3.2.68): The solution of (42) at α=1 as time increases

At α-level=1, $as\ t \to \infty$, $u(t) \to 44.74$, $v(t) \to 35.76$, $r(t) \to 31.96$ and $s(t) \to 35.76$. So the solution is asymptotically stable but there is no fuzzy solution for $x(t)$ and the solution isn't equal to the crisp one.

For $b = 0.03$. Firstly, we assume it a triangular fuzzy number, then we obtain fuzzy unacceptable and unstable solution when $x(t)$ and $y(t)$ are (1,1), (1,2) and (2,1)-differentiable, but at $\alpha = 1$ the solution is equivalent to the crisp case. While, when $x(t)$ and $y(t)$ are (2)-differentiable, the solution is asymptotically stable, but we note that $u(t) > v(t)$ $as\ t \to \infty$, so there is no fuzzy solution for $x(t)$ but this solution is acceptable biologically. At $\alpha = 1$ the solution is the same as the solution of the crisp case. Secondly, we assume b a trapezoidal fuzzy number then we obtain the same results for all cases of derivatives else at $\alpha = 1$ the solution not similar to the crisp case. So the triangular fuzzy number is better than the trapezoidal fuzzy number. So we compare between a triangular fuzzy number of small support with other of large support. We choose the case when $x(t)$ and $y(t)$ are (2)-differentiable since we haven't get a fuzzy solution for the rest forms of the derivative.

Therefore, we let $b = (0.029, 0.03, 0.031)$ a triangular fuzzy number with small support. So, $[b]_\alpha = \left[0.029 + \frac{\alpha}{1000},\ 0.031 - \frac{\alpha}{1000}\right]$. Then we have the following model for $x(t)$ and $y(t)$ are (2)-differentiable:

$$u' = v - (0.029 + \frac{\alpha}{1000})ur$$
$$v' = u - (0.031 - \frac{\alpha}{1000})vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (43)$$

We solve (43) by Runge-Kutta method in Matlab at α-level=0. The solution is figure (3.2.69) and figure (3.2.70):

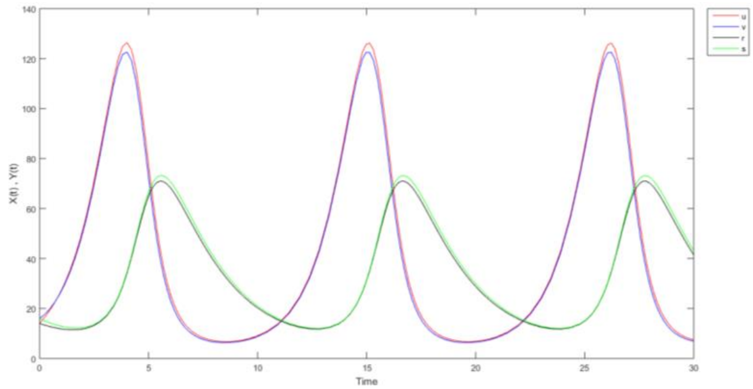


Figure (3.2.69): The solution of (43) at α=0 for short time period

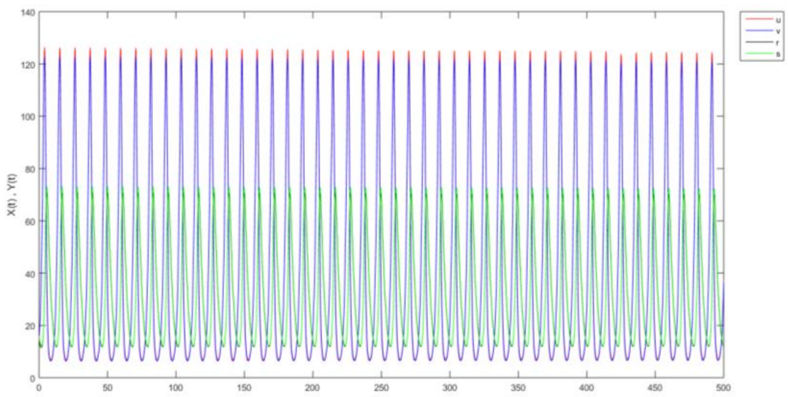


Figure (3.2.70): The solution of (43) at α=0 as time increases

From previous figures we can note that the solution is periodic and stable but it's clear that $u(t) > v(t)$ as $t \to \infty$. So this solution is fuzzy unacceptable.

While when $b = (0.005, 0.03, 0.055)$ a triangular fuzzy number of large support with $\alpha - level$ $[b]_\alpha = \left[0.005 + \frac{\alpha}{40}, 0.055 - \frac{\alpha}{40}\right]$ we have the following model for $x(t)$ and $y(t)$ are (2)-differentiable:

$$u' = v - (0.005 + \frac{\alpha}{40})ur$$
$$v' = u - (0.055 - \frac{\alpha}{40})vs$$
$$r' = -0.4r + 0.01vs$$
$$s' = -0.4s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (44)$$

We solve (44) in Matlab at α-level= 0. The solution is figure (3.2.71):

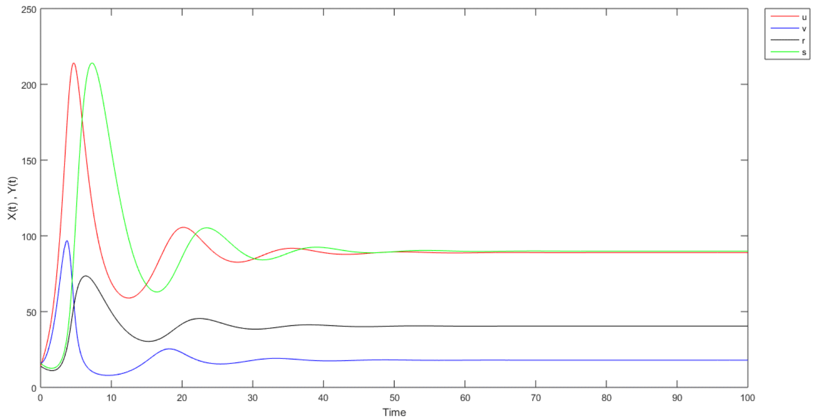


Figure (3.2.71): The solution of (44) at α=0

Here we can see that the solution is asymptotically stable but $u(t) > v(t)\ as\ t \to \infty$ with large difference. So this solution is fuzzy unacceptable.

In addition, we assume b a fuzzy triangular fuzzy number with support such that the distance between it endpoints and the core unequal. For example $b = (0.01,0.03,0.035)$ and $b = (0.025,0.03,0.05)$. Figure (3.2.72) and figure (3.2.73) show the solution with fuzzy initial conditions $[x_0]_\alpha = [14 + \alpha, 16 - \alpha] = [y_0]_\alpha$ and fuzzy number b when $x(t)$ and $y(t)$ are (2)-differentiable at $\alpha = 0$.

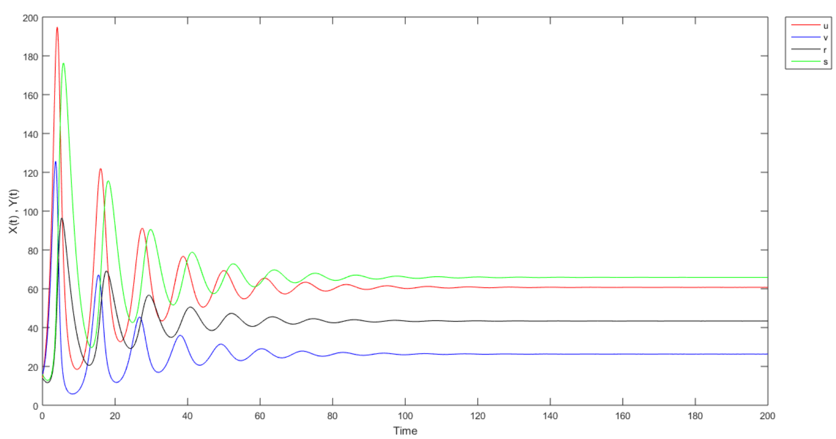


Figure (3.2.72) : The solution when $b = (0.01,0.03,0.035)$

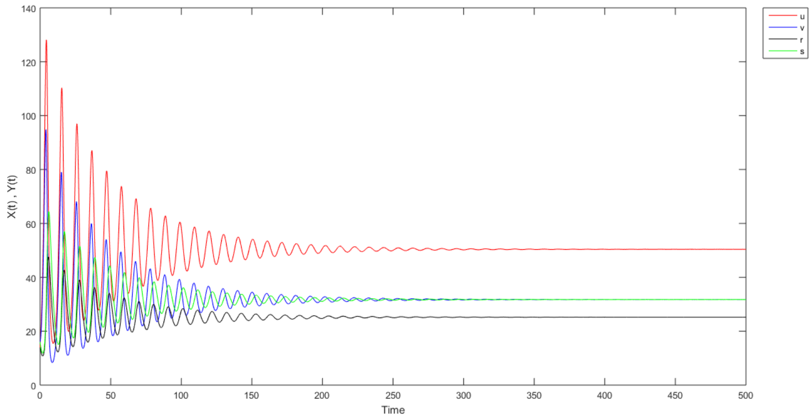


Figure (3.2.73) : The solution when $b = (0.025, 0.03, 0.05)$

From figures (3.2.72) and (3.2.73), we note that the solutions are asymptotically stable. So, we conclude that for any fuzzy number $b = (b_1, b_2, b_3)$ whenever at least one of the differences $(b_2 - b_1), (b_2 - b_3)$ increased then the solution will be asymptotically stable. And when b_1 *and* b_3 are closer to the core b_2, the solution will be periodic with small difference between $u(t)$ and $v(t)$.

Now, we want to make c a fuzzy number using triangular fuzzy number and then using trapezoidal fuzzy number. First, We assume that $c = (0.3, 0.4, 0.5)$ a triangular fuzzy number. Therefore , $[c]_\alpha = \left[0.3 + \frac{\alpha}{10}, \ 0.5 - \frac{\alpha}{10}\right]$.

If $x(t)$ and $y(t)$ are (1)-differentiable, then we have the following model:

$$\begin{aligned} u' &= u - 0.03vs \\ v' &= v - 0.03ur \\ r' &= -(0.5 - \frac{\alpha}{10})s + 0.01ur \\ s' &= -(0.3 + \frac{\alpha}{10})r + 0.01vs \end{aligned}$$

With the initial conditions:

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (45)$$

The first equilibrium point is (0,0,0,0) for any α –level. The second equilibrium point varies according to the α–level, as in the following table (3.2.15)

Table (3.2.15): The equilibrium points of (45)

$\alpha - level$	u	v	r	s
0	35.5689	42.1716	39.521	28.1144
0.5	38.0583	41.3839	36.246	30.6547
1	40	40	33.3333	33.3333

We solve (45) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is table (3.2.16), where its graph is figure (3.2.74):

Table (3.2.16): The solution of (45) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	15.6400	18.8740	12.5120	15.6940
0.5000	17.3420	22.5800	11.0420	15.6170
0.7500	18.9540	27.4000	9.5478	15.8210
1.0000	20.1910	33.7140	7.9689	16.3860
1.2500	20.5370	42.0430	6.2245	17.4430
1.5000	19.0460	53.0800	4.2053	19.2130
1.7500	14.0070	67.7020	1.7694	22.0770
2.0000	2.2023	86.8740	-1.2425	26.7250
2.2500	-22.4530	111.2900	-4.9405	34.4490
2.5000	-71.8880	140.1800	-9.1796	47.7560
2.7500	-168.9300	168.9800	-13.1690	71.3570
3.0000	-353.9000	186.2200	-15.2740	113.1000
3.2500	-678.4800	175.5100	-14.4500	180.5400
3.5000	-1159.2000	123.5700	-12.2340	266.1300
3.7500	-1687.6000	24.0060	-10.1170	324.1100
4.0000	-2063.1000	-112.8500	-7.6342	292.1300
4.2500	-2280.7000	-258.9700	-4.5698	183.7400
4.5000	-2574.3000	-397.6000	-1.8955	80.8910
4.7500	-3112.2000	-536.8000	-0.5059	25.2500
5.0000	-3926.8000	-696.5800	-0.0870	5.4459

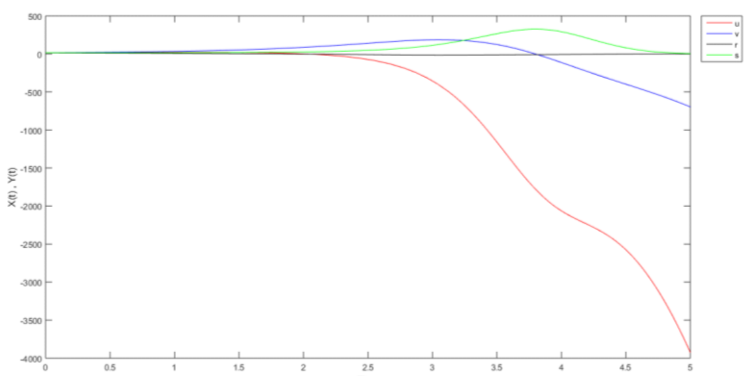


Figure (3.2.74): The solution of (45) at α=0

At α-level = 0.5, the solution is figure (3.2.75) in the appendix. At α-level = 1, the solution is figure (3.2.76):

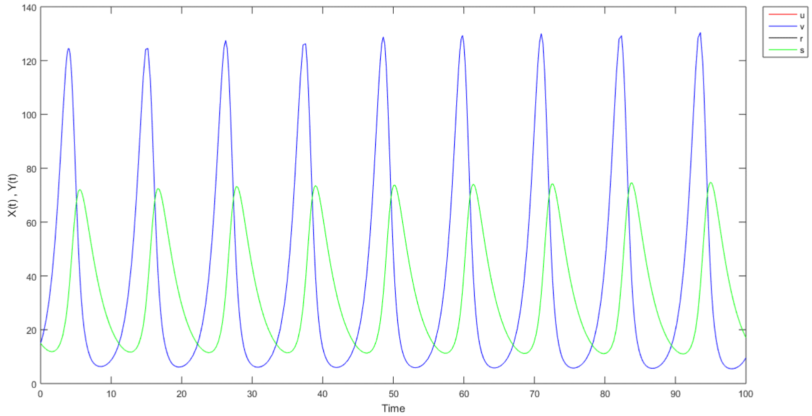


Figure (3.2.76): The solution of (45) at α=1

If $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, then we have the following model:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -(0.3 + \frac{\alpha}{10})r + 0.01vs$$
$$s' = -(0.5 - \frac{\alpha}{10})s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (46)$$

We solve (46) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is table (3.2.17), where its graph is figure (3.2.77):

Table (3.2.17): The solution of (46) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	15.7230	18.8030	13.6270	14.6020
0.5000	17.7560	22.2290	13.3310	13.4150
0.7500	20.1410	26.4170	13.1210	12.4270
1.0000	22.9140	31.5350	13.0090	11.6270
1.2500	26.1050	37.7890	13.0100	11.0100
1.5000	29.7240	45.4270	13.1470	10.5730
1.7500	33.7450	54.7510	13.4520	10.3230
2.0000	38.0730	66.1270	13.9730	10.2670
2.2500	42.5050	79.9960	14.7780	10.4220
2.5000	46.6490	96.9050	15.9650	10.8090
2.7500	49.8270	117.5400	17.6770	11.4500
3.0000	50.9060	142.8300	20.1220	12.3470
3.2500	48.1020	174.1600	23.5860	13.4500
3.5000	38.8070	213.9400	28.4200	14.5430
3.7500	19.6630	266.5900	34.9540	15.0400
4.0000	-11.9930	340.6500	42.9980	13.6420

4.2500	-53.2220	450.1400	50.5310	8.3645
4.5000	-84.9120	609.2100	51.2630	-1.3572
4.7500	-71.7490	814.8800	36.5320	-10.0670
5.0000	-2.6443	1056.3000	8.3466	-11.4290

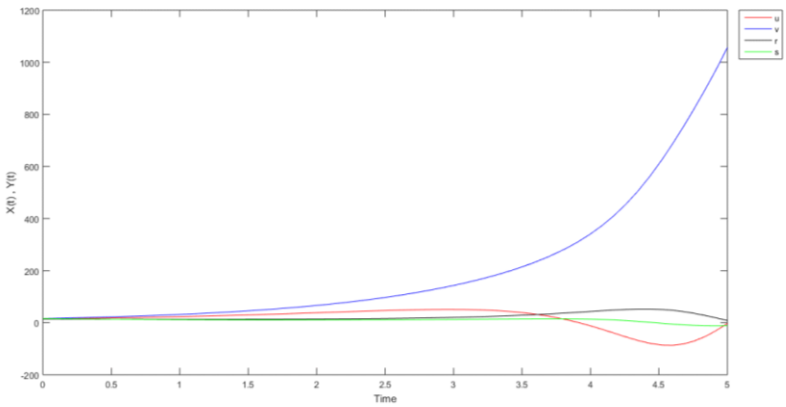


Figure (3.2.77): The solution of (46) at α=0

At α-level = 0.5, the solution is figure (3.2.78) in the appendix. At α-level = 1, the solution is figure (3.2.79):

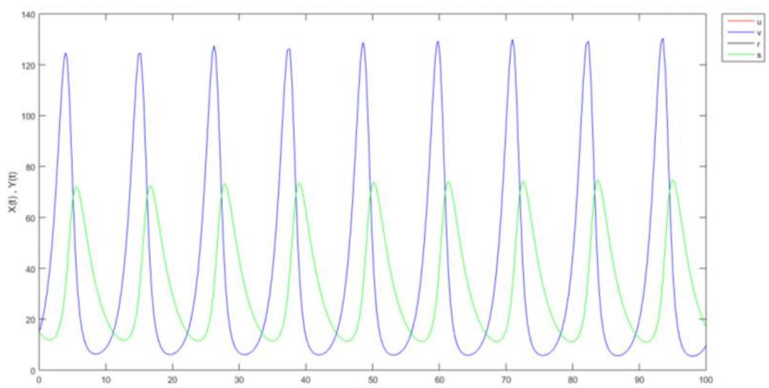


Figure (3.2.79): The solution of (46) at α=1

If $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable, then we have the following model:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -(0.5 - \frac{\alpha}{10})s + 0.01ur$$
$$s' = -(0.3 + \frac{\alpha}{10})r + 0.01vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (47)$$

At α-level = 0, the solution is figure (3.2.80):

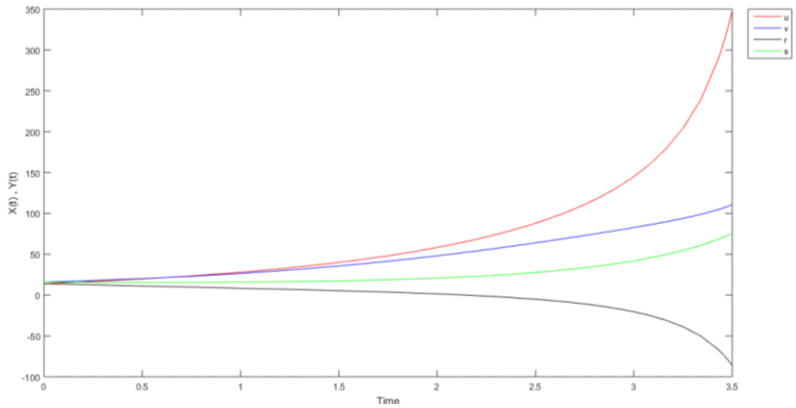


Figure (3.2.80): The solution of (47) at α=0

At α-level = 0.5, the solution is figure (3.2.81) in the appendix. At α-level = 1, the solution is figure (3.2.82):

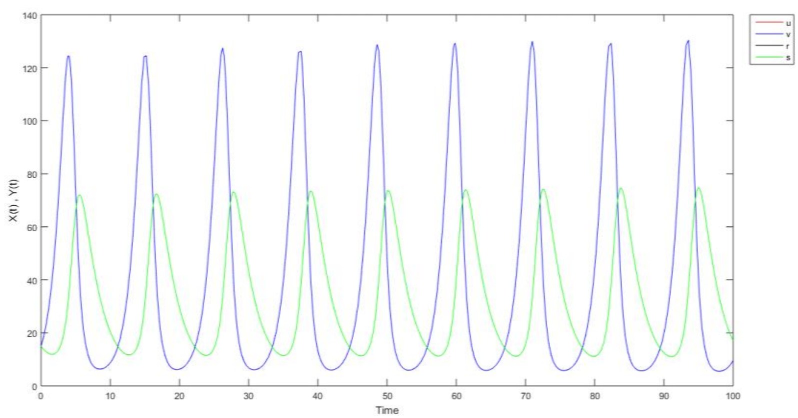


Figure (3.2.82): The solution of (46) at α=1

If $x(t)$ and $y(t)$ are (2)-differentiable, then we have the following model:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -(0.3 + \frac{\alpha}{10})r + 0.01vs$$
$$s' = -(0.5 - \frac{\alpha}{10})s + 0.01ur$$

With the initial conditions:

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (48)$$

We solve this model by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution graphs are figure (3.2.83), figure (3.2.84) and figure (3.2.85):

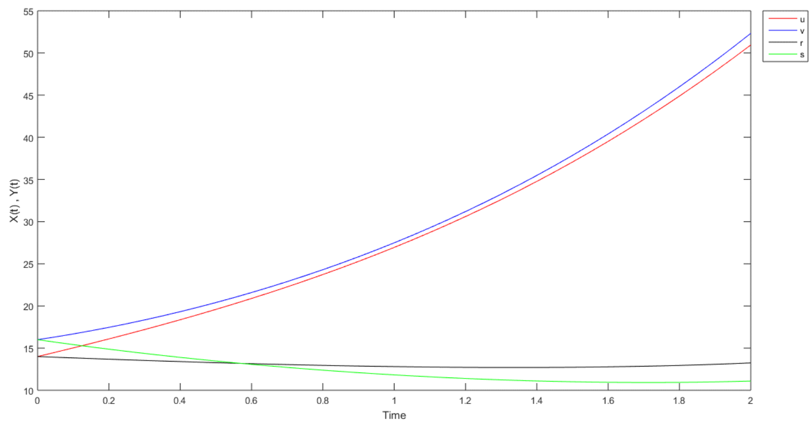
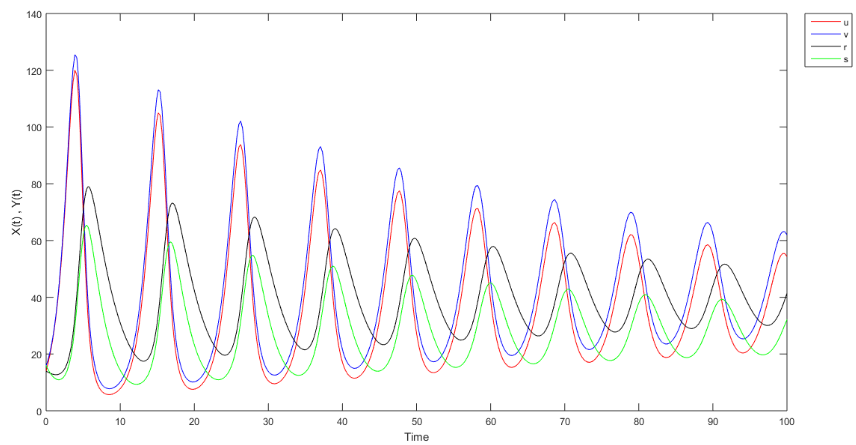
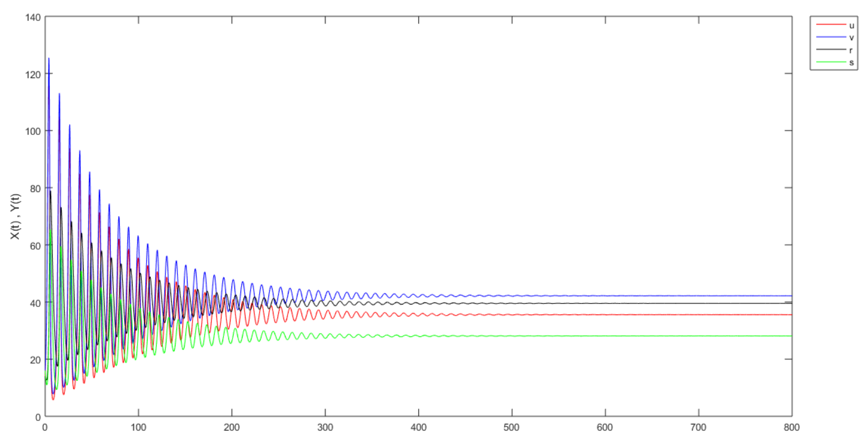


Figure (3.2.83): The solution of (48) at α=0 for short time period





Figures (3.2.84) and (3.2.85): The solution of (48) at α=0 as time increases

At $\alpha - level = 0$, the solution is asymptotically stable and there is no fuzzy solution for $y(t)$ since $as\ t \to \infty$, $u(t) \to 35.57$, $v(t) \to 42.17$, $r(t) \to 39.52$ and $s(t) \to 28.2$, and $r(t) > s(t)\ as\ t \to \infty$.

At α-level = 0.5, the solution graphs are figure (3.2.86), figure (3.2.87) and figure (3.2.88) in the appendix. At α-level = 1, the solution is figure (3.2.89):

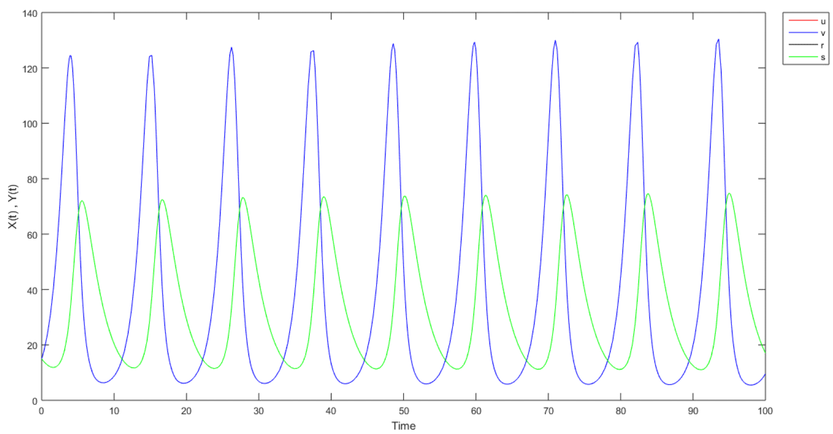


Figure (3.2.89): The solution of (48) at α=1

Then, we assume $c = (0.25, 0.35, 0.45, 0.55)$ a trapezoidal fuzzy number with α-level $[c]_\alpha = \left[0.25 + \frac{\alpha}{10},\ 0.55 - \frac{\alpha}{10}\right]$. If $x(t)$ and $y(t)$ are (1)-differentiable, then we have the following model:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -(0.55 - \frac{\alpha}{10})s + 0.01ur$$
$$s' = -(0.25 + \frac{\alpha}{10})r + 0.01vs$$

With the initial conditions:

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (49)$$

The equilibrium points of (49) is (0,0,0,0) for any α –level. However, the model has another equilibrium point which varies according to the α–level, as in the following table (3.2.18).

Table (3.2.18): The equilibrium points of (49)

$\alpha\ - level$	u	v	r	s
0	32.5148	42.2885	42.353	25.6294
0.5	35.5689	42.1716	39.521	28.1144
1	38.0583	41.3839	36.246	30.6547

We solve model (49) by Runge-Kutta method in Matlab at α-level= $0, 0.5, 1$. At α-level = 0, the solution is table (3.2.19) and figure (3.2.90):

Table (3.2.19): The solution of (49) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	15.6270	18.8870	12.2980	15.8700
0.5000	17.2740	22.6470	10.5810	15.9770
0.7500	18.7540	27.5880	8.7972	16.3810
1.0000	19.7200	34.1290	6.8738	17.1760
1.2500	19.5470	42.8340	4.7148	18.5170
1.5000	17.0990	54.4330	2.1959	20.6630
1.7500	10.3210	69.7840	-0.8310	24.0600
2.0000	-4.6197	89.6510	-4.4973	29.5160
2.2500	-34.8550	114.0700	-8.7980	38.5050
2.5000	-93.8280	140.6800	-13.3050	53.7550
2.7500	-205.1200	162.1200	-16.7830	79.8230
3.0000	-402.6200	164.8200	-17.5550	122.1400
3.2500	-710.9100	134.9300	-15.5090	180.3200
3.5000	-1097.0000	66.1940	-12.7320	234.7800
3.7500	-1446.2000	-38.8330	-10.0990	245.2400
4.0000	-1674.0000	-165.1200	-7.0575	190.9200
4.2500	-1864.8000	-293.8800	-3.8209	107.7300
4.5000	-2175.1000	-420.5200	-1.4306	44.2160
4.7500	-2684.7000	-556.3400	-0.3533	13.1130
5.0000	-3410.7000	-718.6800	-0.0570	2.6875

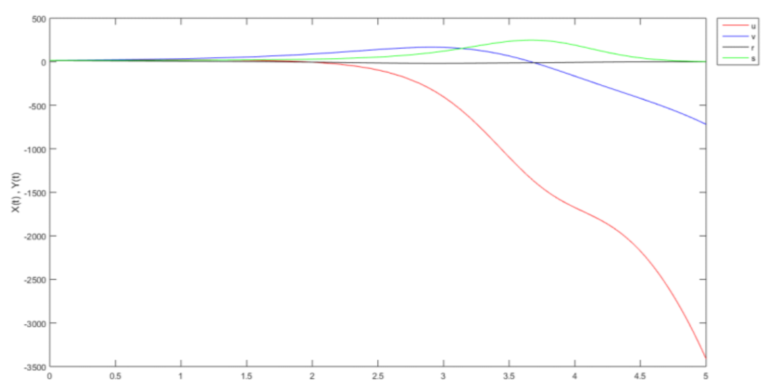


Figure (3.2.90): The solution of (49) at α=0

At α-level = 0.5, the solution is figure (3.2.91) in the appendix. At α-level = 1, the solution is table (3.2.20) and figure (3.2.92):

Table (3.2.20): The solution of (49) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
0.2500	17.2550	17.2820	13.9330	14.3240
0.5000	19.9320	20.0700	12.9760	13.8020
0.7500	23.0830	23.4820	12.1170	13.4440
1.0000	26.7500	27.6740	11.3450	13.2680
1.2500	30.9490	32.8430	10.6430	13.3070
1.5000	35.6420	39.2560	9.9883	13.6120
1.7500	40.6780	47.2700	9.3452	14.2660
2.0000	45.6910	57.3850	8.6537	15.4080
2.2500	49.8990	70.3170	7.8113	17.2760
2.5000	51.6660	87.1200	6.6382	20.3080
2.7500	47.6250	109.3500	4.8266	25.3440
3.0000	30.4630	139.1400	1.8913	34.1450
3.2500	-16.5190	178.5200	-2.7400	50.7070
3.5000	-133.2000	225.6600	-9.0076	84.5930
3.7500	-414.4700	263.7300	-14.2540	158.2300
4.0000	-1033.1000	253.9100	-14.3050	308.4300
4.2500	-2074.1000	159.5900	-11.8900	529.3500
4.5000	-3030.2000	-32.9040	-9.7244	632.4100
4.7500	-3232.1000	-267.1000	-6.5245	434.6900
5.0000	-3245.0000	-471.1000	-2.7735	171.5300

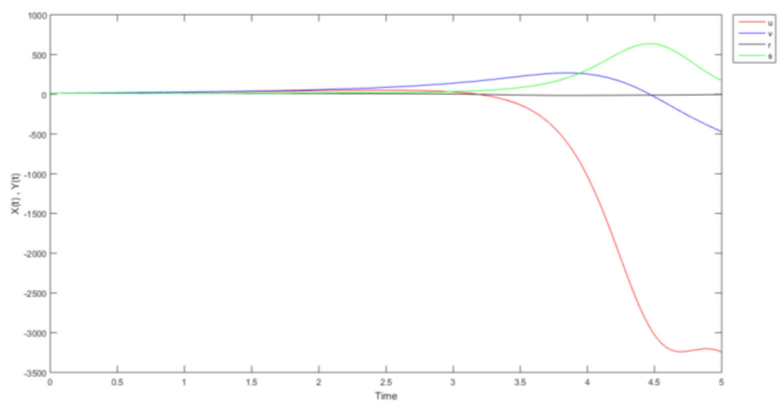


Figure (3.2.92): The solution of (49) at α=1

While when $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, we have the following model:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -(0.25 + \frac{\alpha}{10})r + 0.01vs$$
$$s' = -(0.55 - \frac{\alpha}{10})s + 0.01ur$$

With the initial conditions:

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (50)$$

We solve (50) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.93):

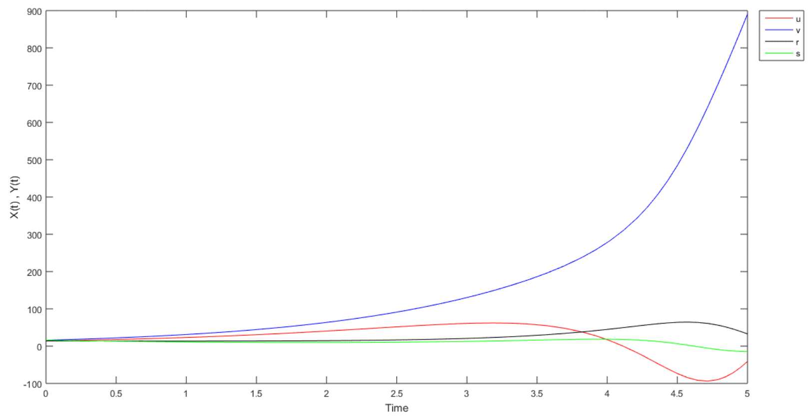


Figure (3.2.93): The solution of (50) at α=0

At α-level = 0.5, the solution is figure (3.2.94) in the appendix. At α-level = 1, the solution is figure (3.2.95):

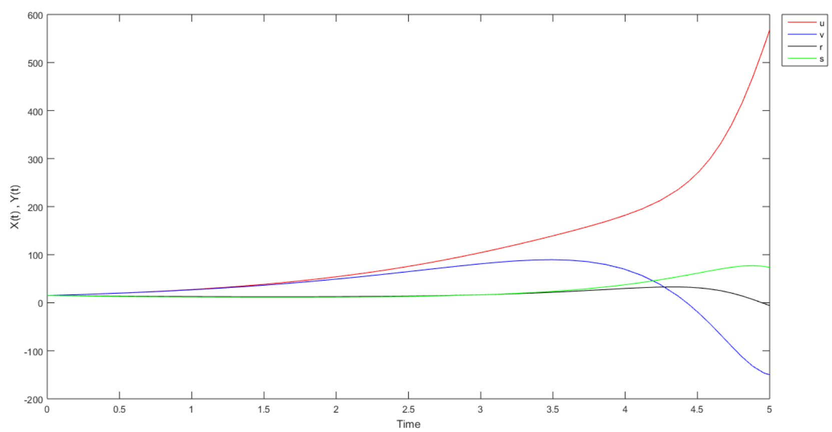


Figure (3.2.95): The solution of (50) at α=1

If $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable, then we have the following model:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -(0.55 - \frac{\alpha}{10})s + 0.01ur$$
$$s' = -(0.25 + \frac{\alpha}{10})r + 0.01vs$$

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (51)$$

At α-level = 0, the solution is figure (3.2.96):

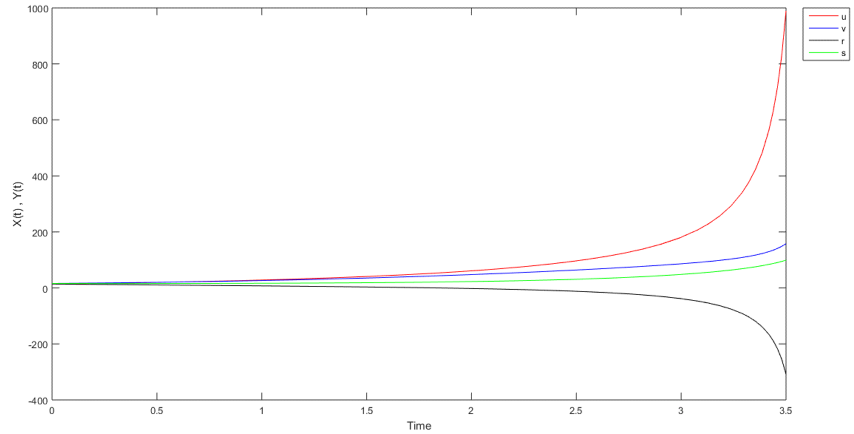


Figure (3.2.96): The solution of (51) at α=0

At α-level = 0.5, the solution is figure (3.2.97) in the appendix. At α-level = 1, the solution is figure (3.2.98):

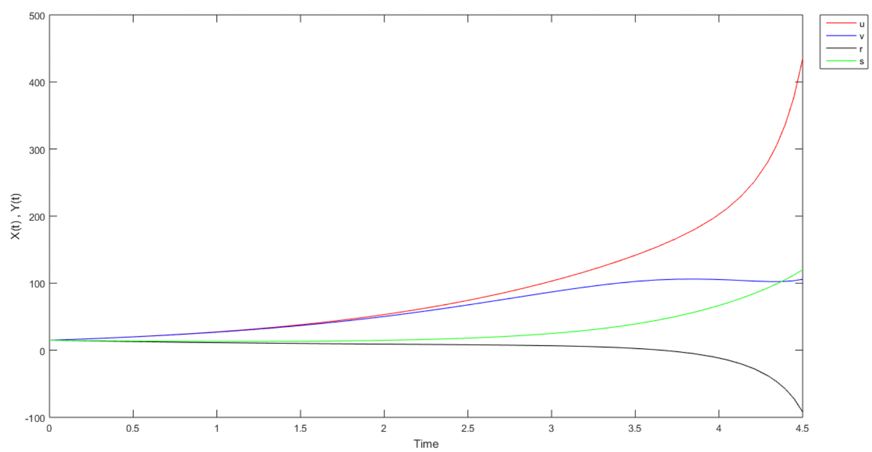


Figure (3.2.98): The solution of (51) at α=1

If $x(t)$ and $y(t)$ are (2)-differentiable, then we have the following model:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -(0.25 + \frac{\alpha}{10})r + 0.01vs$$
$$s' = -(0.55 - \frac{\alpha}{10})s + 0.01ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (52)$$

We solve this model by Runge-Kutta method in Matlab at α-level= $0, 0.5, 1$. At α-level = 0, the solution graphs are figure (3.2.99), figure (3.2.100) and figure (3.2.101):

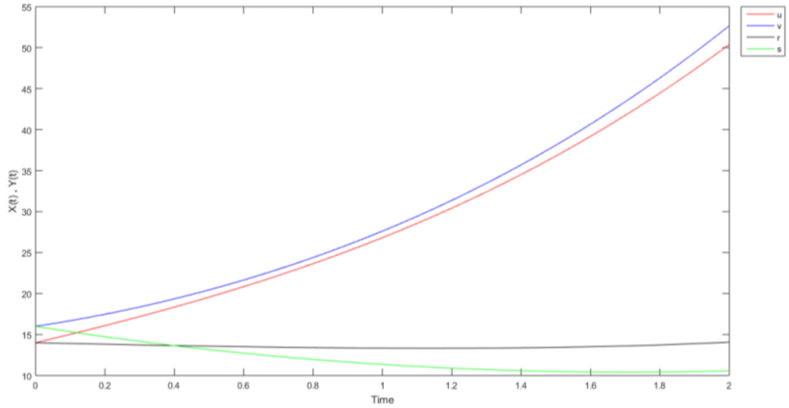
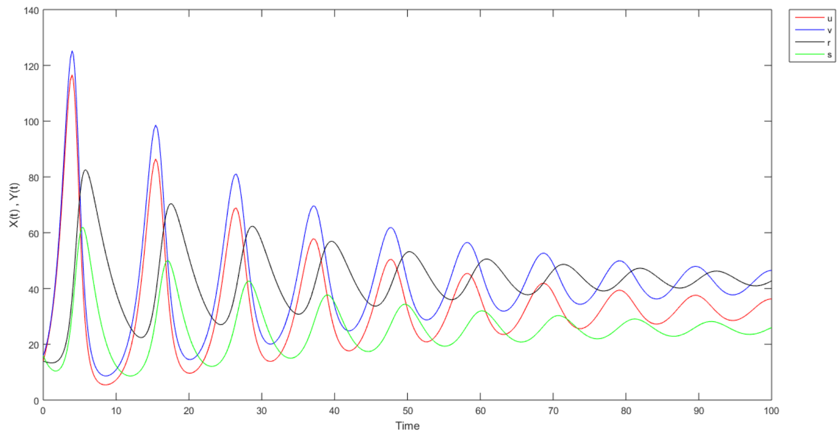
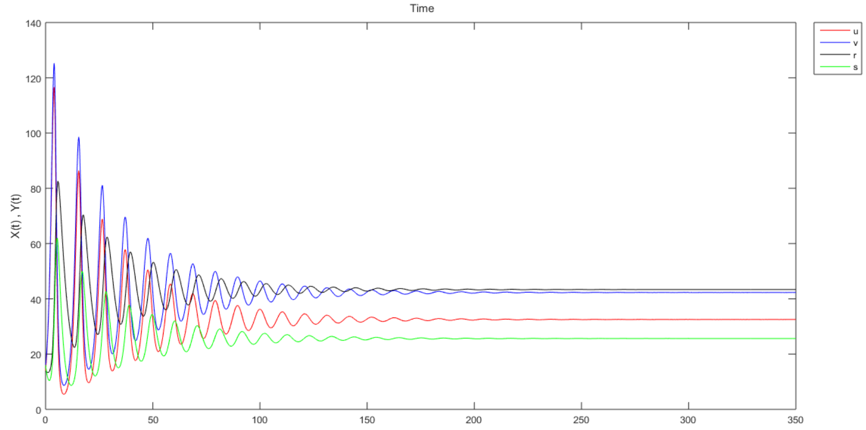


Figure (3.2.99): The solution of (52) at α=0 for short time period





Figures (3.2.100) and (3.2.101): The solution of (52) at α=0 as time increases

At α-level = 0, $as\ t \to \infty$, $u(t) \to 32.51$, $v(t) \to 42.29, r(t) \to 43.35$, $s(t) \to 25.63$. So the solution is asymptotically stable but there is no fuzzy solution for $y(t)$ since $r(t) > s(t)$.

At α-level = 0.5, the solution graphs are figure (3.2.102), figure (3.2.103) and figure (3.2.104) in the appendix. At α-level = 1, the solution graphs are figure (3.2.105), figure (3.2.106) and figure (3.2.107):

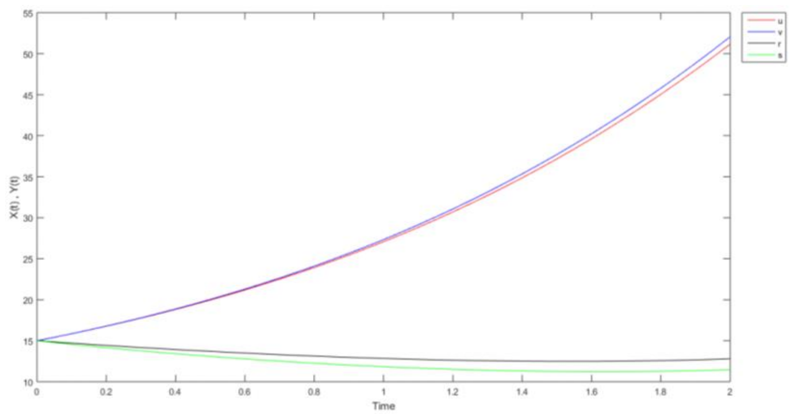
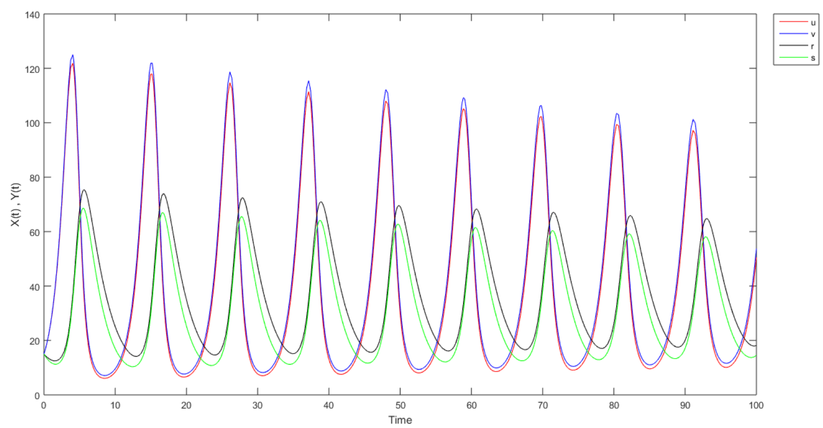
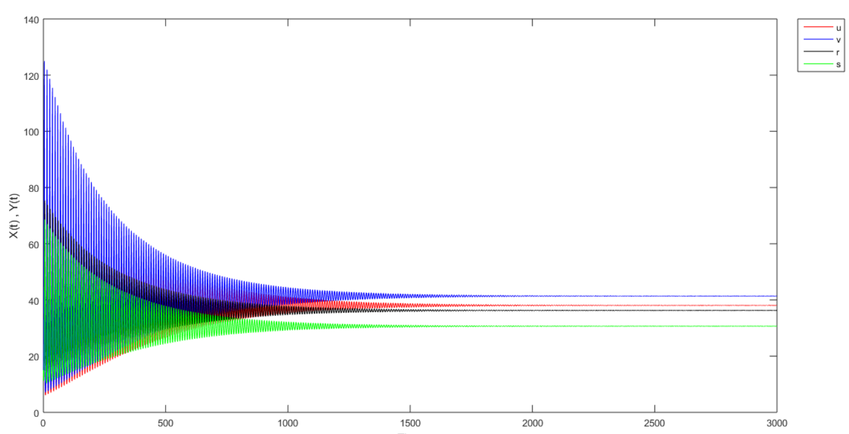


Figure (3.2.105): The solution of (52) at α=1 for short time period





Figures (3.2.106) and (3.2.107): The solution of (52) at α=1 as time increases

At α-level = 1, $as\ t \to \infty$, $u(t) \to 38.06$, $v(t) \to 41.39, r(t) \to 36.24,\ s(t) \to 30.66$. Therefore, the solution is asymptotically stable and there is no fuzzy solution for $y(t)$.

From previous work, at $\alpha < 1$ we conclude that we obtain a biologically acceptable solution only when $x(t)\ and\ y(t)$ are (2)-differentiable which is asymptotically stable to the equilibrium point whether for trapezoidal or triangular fuzzy number but we note that $r(t) > s(t)\ for\ t \to \infty$, so there is no fuzzy solution for $y(t)$. At $\alpha = 1$ the solution is the same as the solution of the crisp case when c a triangular fuzzy number but it isn't when c a trapezoidal fuzzy number. So, one more time the triangular fuzzy number is better than the trapezoidal fuzzy number.

If we fuzzify c by a triangular fuzzy number with small support , for example $c = (0.3999, 0.4, 0.4001)$ then the solution will be periodic and stable, but with large support, for example $c = (0.1, 0.4, 0.7)$ the solution will be asymptotically stable. As in figures (3.2.108) and (3.2.109) we plot the solution of $x(t)$ and $y(t)$ when they are (2)-differentiable at $\alpha = 0$.

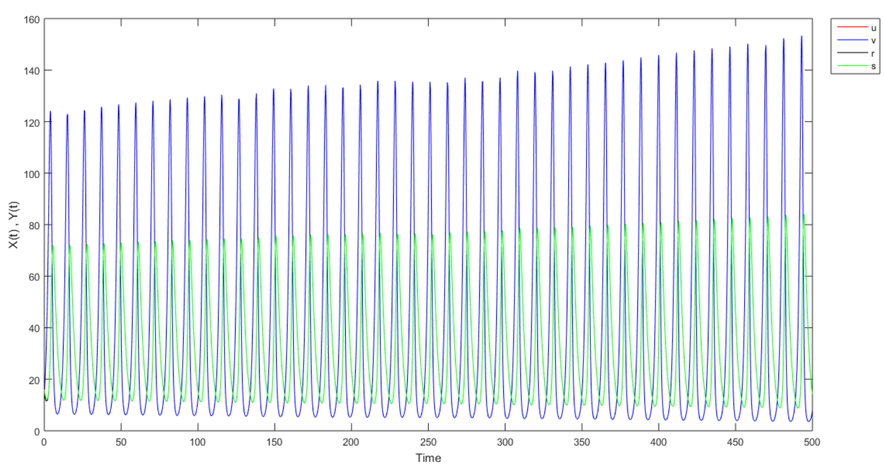


Figure (3.2.108): The solution of X(t) and Y(t) when $c = (0.3999, 0.4, 0.4001)$

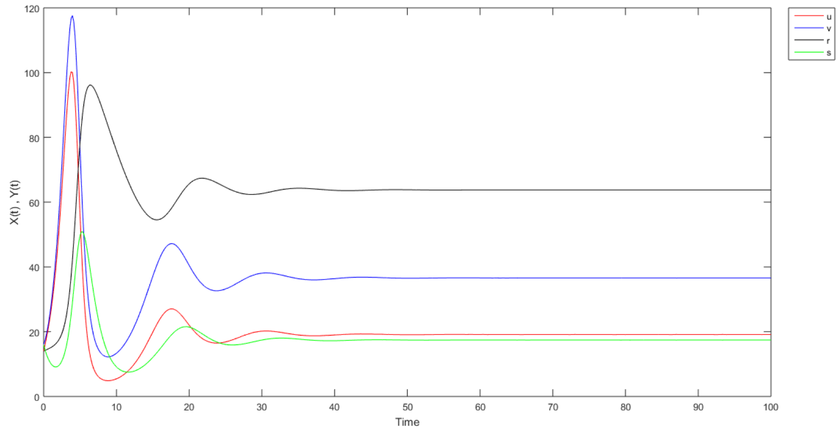


Figure (3.2.109): The solution of X(t) and Y(t) when $c = (0.1, 0.4, 0.7)$

In figure (3.2.108) the solution is oscillated about the equilibrium point so the solution is stable. We try to solve this in Matlab for too long time period and then we note that $r(t) > s(t)$ with unclear difference but the difference is clear when c is a triangular fuzzy number with large support.

Finally, we assume c a triangular fuzzy number with support such that the distance between its endpoints and the core unequal. Figure (3.2.110) and figure (3.2.111) show the solution of $x(t)$ and $y(t)$ when they are (2)-differentiable at α-level= 0 when $c = (0.1, 0.4, 0.405)$ and $c = (0.395, 0.4, 0.6)$, respectively.

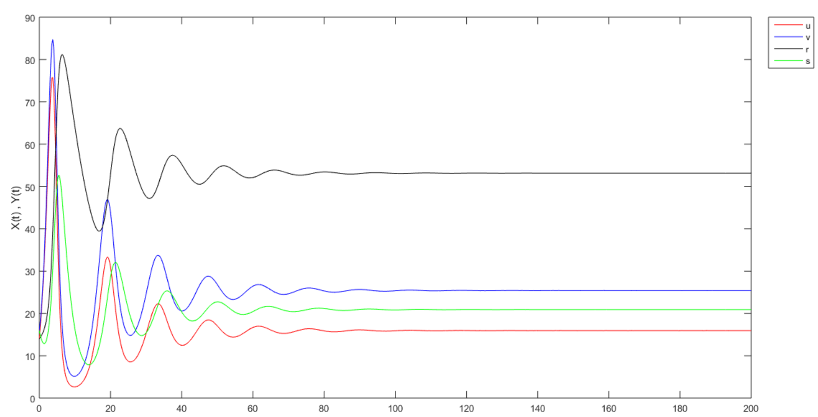


Figure (3.2.110): The solution of X(t) and Y(t) when $c = (0.1, 0.4, 0.405)$

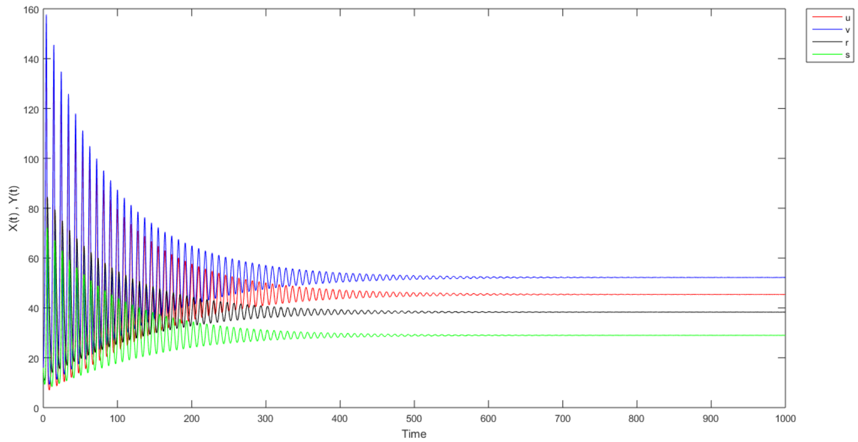


Figure (3.2.111): The solution of X(t) and Y(t) when $c = (0.395, 0.4, 0.6)$

From previous figures we conclude that if the distance for at least one support endpoints is long from the core then the solution will be asymptotically stable.

Now, we assume d a fuzzy number. First, using triangular fuzzy number. we let $d = (0.005, 0.01, 0.015)$ such that $[d]_\alpha = \left[0.005 + \frac{\alpha}{200}, 0.015 - \frac{\alpha}{200}\right]$. Then if $x(t)$ and $y(t)$ are (1)-differentiable, then we have the following model:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4s + (0.005 + \frac{\alpha}{200})ur$$
$$s' = -0.4r + (0.015 - \frac{\alpha}{200})vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (53)$$

The equilibrium points of (53) is (0,0,0,0) for any α –level but the model has another equilibrium point which varies according to the α–level, as in the following table (3.2.21).

Table (3.2.21): The equilibrium points of (53)

$\alpha - level$	u	v	r	s
0	38.46	55.4689	48.075	23.112
0.5	37.9402	44.9831	39.521	28.1144
1	40	40	33.3333	33.3333

We solve model (53) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is table (3.2.22) and figure (3.2.112):

Table (3.2.22): The solution of (53) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	15.6410	18.8640	12.6640	15.6980
0.5000	17.3390	22.5370	11.3440	15.7100
0.7500	18.9220	27.2870	9.9981	16.1200
1.0000	20.0520	33.4870	8.5708	17.0690
1.2500	20.0890	41.6510	6.9823	18.7960
1.5000	17.8000	52.4740	5.1148	21.7340
1.7500	10.7950	66.8510	2.7913	26.6920
2.0000	-5.8359	85.7520	-0.2573	35.3210
2.2500	-42.4270	109.6300	-4.4205	51.1180
2.5000	-121.8800	135.9400	-10.1660	81.9190
2.7500	-290.4000	151.4200	-17.5880	143.0400
3.0000	-599.3200	115.3900	-24.9260	245.4100
3.2500	-902.2300	-26.1100	-27.6720	303.6800
3.5000	-896.1600	-239.4600	-23.3530	187.8100
3.7500	-834.7100	-447.7200	-14.3160	52.4390
4.0000	-963.0500	-649.4200	-5.9358	7.0522
4.2500	-1220.2000	-865.6000	-1.6354	0.4969
4.5000	-1565.4000	-1121.0000	-0.2943	0.0245
4.7500	-2010.0000	-1441.2000	-0.0320	0.0013
5.0000	-2580.9000	-1850.8000	-0.0019	0.0001

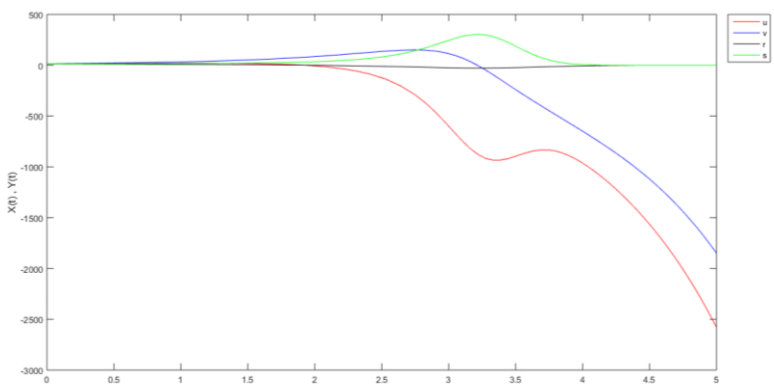


Figure (3.2.112): The solution of (53) at α=0

At α-level = 0.5, the solution is figure (3.2.113) in the appendix. At α-level = 1, the solution is figure (3.2.114):

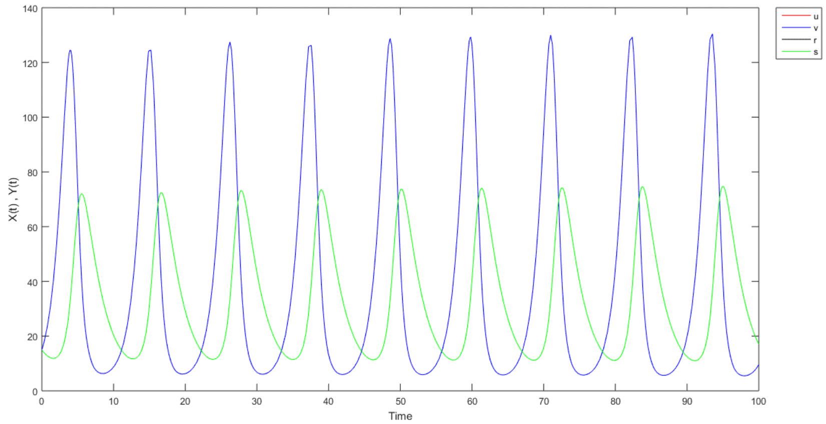


Figure (3.2.114): The solution of (53) at α=1

When $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, we have the following model:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4r + (0.015 - \frac{\alpha}{200})vs$$
$$s' = -0.4s + (0.005 + \frac{\alpha}{200})ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (54)$$

We solve (54) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.115):

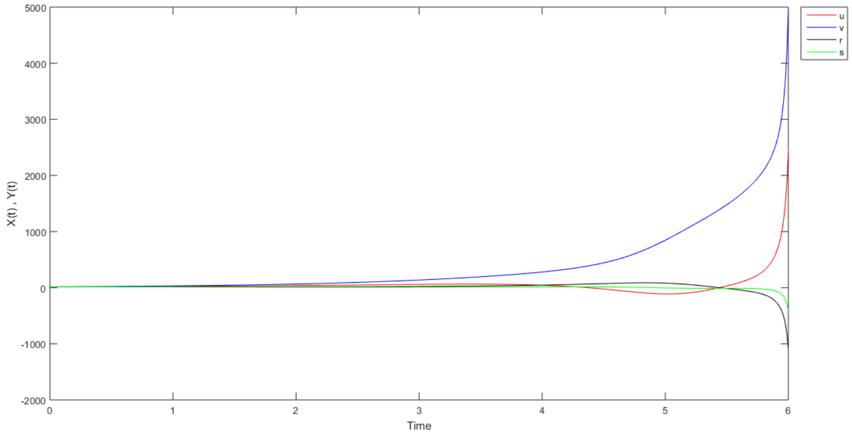


Figure (3.2.115): The solution of (54) at α=0

At α-level = 0.5, the solution is figure (3.2.116) in the appendix. At α-level = 1, the solution is figure (3.2.117):

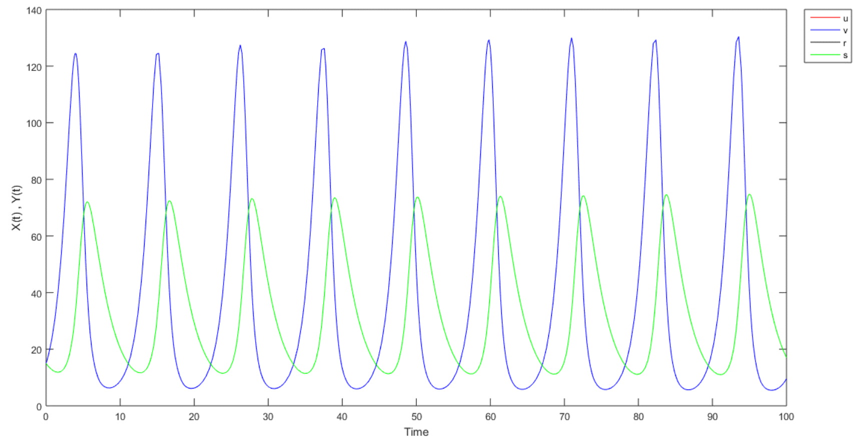


Figure (3.2.117): The solution of (54) at α=1

If $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable, then we have the following model:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4s + (0.005 + \frac{\alpha}{200})ur$$
$$s' = -0.4r + (0.015 - \frac{\alpha}{200})vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (55)$$

At α-level = 0, the solution is figure (3.2.118):

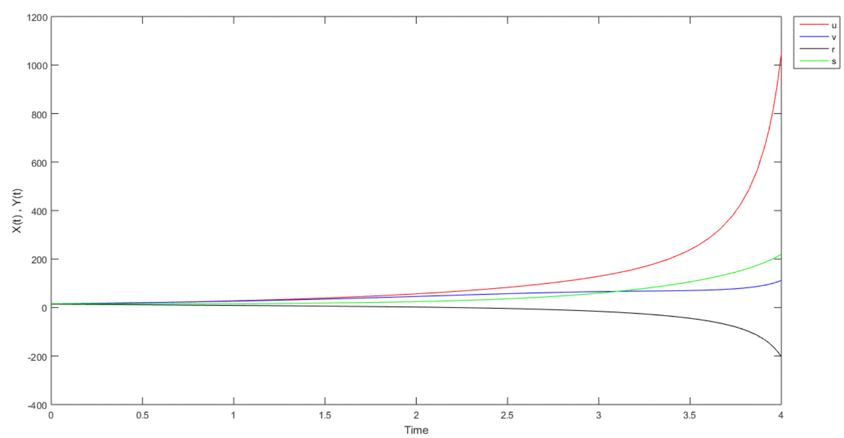


Figure (3.2.118): The solution of (55) at α=0

At α-level = 0.5, the solution is figure (3.2.119) in the appendix. At α-level = 1, the solution is figure (3.2.120):

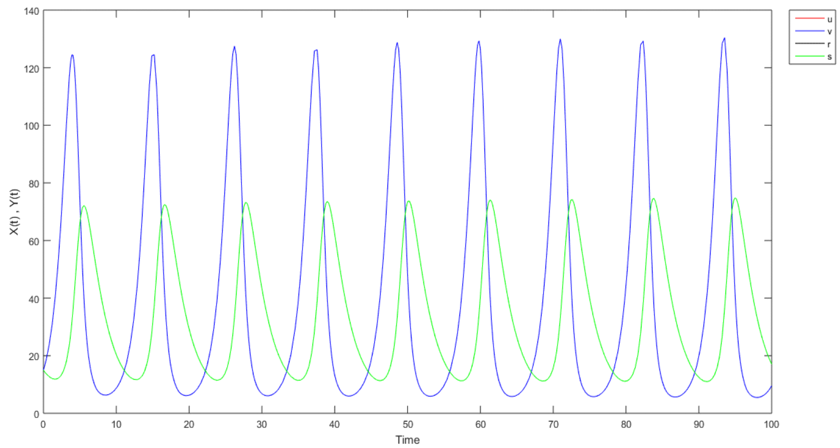


Figure (3.2.120): The solution of (55) at α=1

If $x(t)$ and $y(t)$ are (2)-differentiable, then we have the following model:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4r + (0.015 - \frac{\alpha}{200})vs$$
$$s' = -0.4s + (0.005 + \frac{\alpha}{200})ur$$

With the initial conditions:

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (56)$$

We solve this model by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution graphs are figure (3.2.121), figure (3.2.122) and figure (3.2.123):

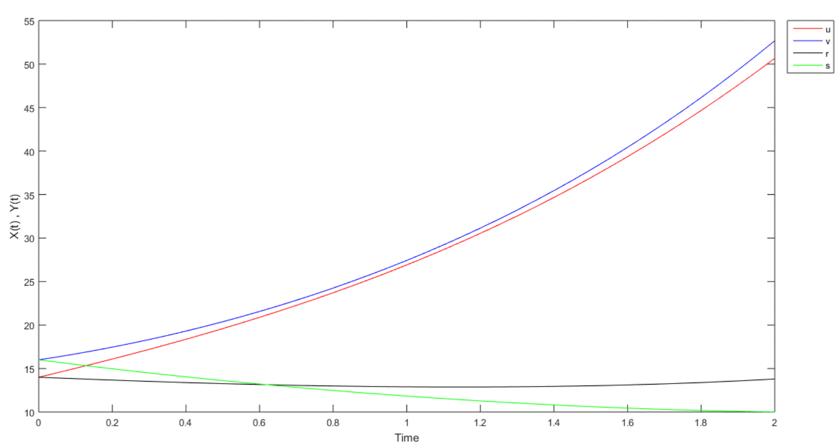
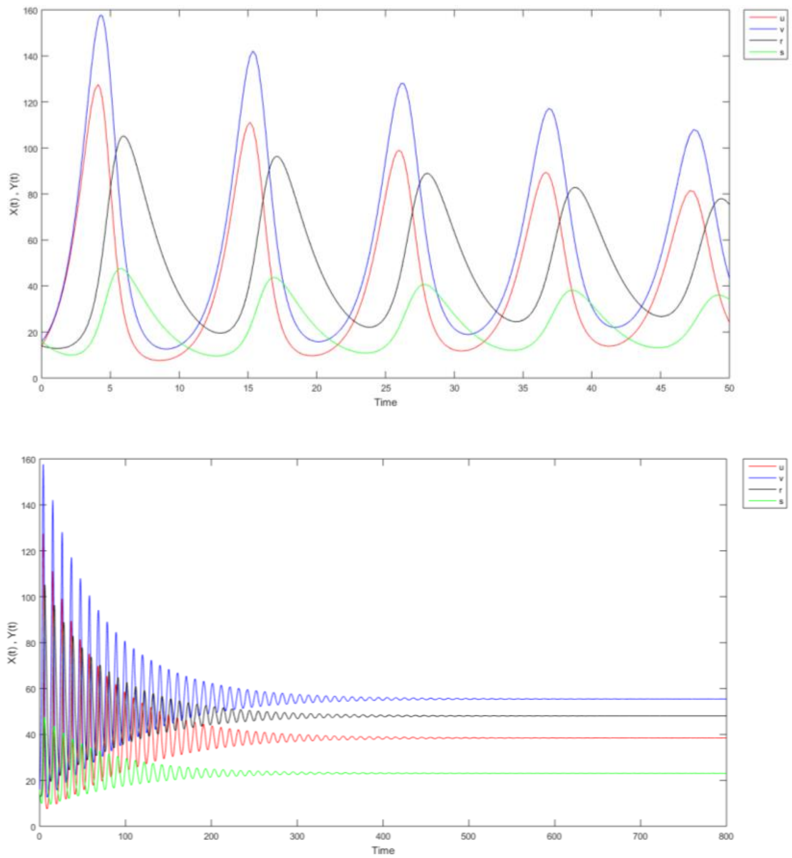


Figure (3.2.121): The solution of (56) at α=0 for short time period



Figures (3.2.122) and (3.2.123): The solution of (56) at α=0 as time increases

At α-level = 0, $as\ t \to \infty$, $u(t) \to 38.\ 46, v(t) \to 55.46, r(t) \to 48.07$, $s(t) \to 23.11$. So the solution is asymptotically stable but there is no fuzzy solution for $y(t)$ since $r(t) > s(t)$.

At α-level = 0.5, the solution graphs are figure (3.2.124), figure (3.2.125) and figure (3.2.126) in the appendix. At α-level = 1, the solution is figure (3.2.127):

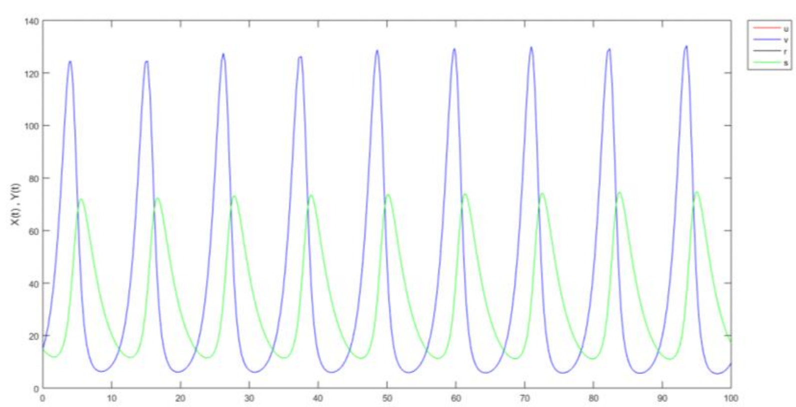


Figure (3.2.127): The solution of (56) at α=1

Now, we assume $d = (0.0025, 0.0075, 0.0125, 0.0175)$ such that $[d]_\alpha = \left[0.0025 + \frac{\alpha}{200},\ 0.0175 - \frac{\alpha}{200}\right]$ a trapezoidal fuzzy number. Then if $x(t)$ and $y(t)$ are (1)-differentiable, then we have the following model:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4s + (0.0025 + \frac{\alpha}{200})ur$$
$$s' = -0.4r + (0.0175 - \frac{\alpha}{200})vs$$

With the initial conditions:

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \tag{57}$$

The equilibrium points of (57) is (0,0,0,0) for any α –level but the model has another equilibrium point which varies according to the α–level, as in the following table (3.2.23).

Table (3.2.23): The equilibrium points of (57)

$\alpha - level$	u	v	r	s
0	43.7241	83.6413	63.7644	17.4253
0.5	38.46	55.4689	48.075	23.112
1	37.9402	44.9831	39.521	28.1144

We solve model (57) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is table (3.2.24) and figure (3.2.128):

Table (3.2.24): The solution of (56) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	15.6280	18.8720	12.5310	15.8830
0.5000	17.2660	22.5800	11.0540	16.1470
0.7500	18.6900	27.4130	9.5221	16.9140
1.0000	19.4590	33.7740	7.8682	18.3880
1.2500	18.7220	42.2160	5.9972	20.9320
1.5000	14.8030	53.4720	3.7609	25.2240
1.7500	4.3409	68.4190	0.9214	32.5950
2.0000	-19.8390	87.7440	-2.9275	45.8920
2.2500	-73.4110	110.5200	-8.4842	71.4030
2.5000	-190.4900	128.5600	-16.8940	122.7500
2.7500	-416.7000	108.0200	-29.1160	214.1800
3.0000	-648.6100	-28.2660	-42.1580	273.2700
3.2500	-579.5700	-282.3200	-46.2860	144.5400

3.5000	-497.5600	-557.8000	-39.4790	24.8990
3.7500	-585.5200	-871.6000	-29.0410	2.0055
4.0000	-743.8700	-1253.9000	-19.2860	0.4268
4.2500	-951.6500	-1718.4000	-11.4060	0.1728
4.5000	-1220.1000	-2283.5000	-5.8143	0.0650
4.7500	-1565.7000	-2978.7000	-2.4480	0.0208
5.0000	-2010.1000	-3847.5000	-0.8060	0.0053

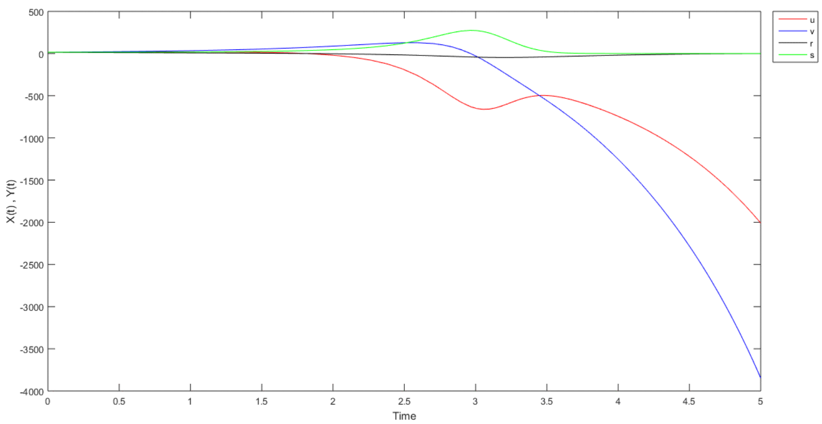


Figure (3.2.128): The solution of (57) at α=0

At α-level = 0.5, the solution is figure (3.2.129) in the appendix. At α-level = 1, the solution is table (3.2.25) and figure (3.2.130):

Table (3.2.25): The solution of (57) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	15.0000	15.0000	15.0000	15.0000
0.2500	17.2580	17.2790	13.9740	14.2880
0.5000	19.9440	20.0570	13.0430	13.7520
0.7500	23.1080	23.4500	12.1940	13.4110
1.0000	26.7890	27.6130	11.4110	13.2910
1.2500	30.9900	32.7490	10.6740	13.4420
1.5000	35.6390	39.1330	9.9509	13.9410
1.7500	40.5170	47.1460	9.1961	14.9170
2.0000	45.1020	57.3340	8.3326	16.5940
2.2500	48.2750	70.4970	7.2317	19.3740
2.5000	47.6250	87.8200	5.6722	24.0380
2.7500	37.9440	111.0200	3.2795	32.1880
3.0000	7.1398	142.1200	-0.5310	47.4880
3.2500	-74.5650	181.4500	-6.4158	79.0010
3.5000	-282.0000	218.4300	-14.0240	149.5100
3.7500	-763.1400	208.3800	-19.4110	300.3400
4.0000	-1502.9000	83.4790	-18.5530	492.8500
4.2500	-1851.5000	-140.0800	-14.6200	458.6200

4.5000	-1692.3000	-361.8200	-8.6507	207.9200
4.7500	-1732.8000	-545.5300	-2.8511	50.2590
5.0000	-2102.7000	-722.6000	-0.4281	7.0045

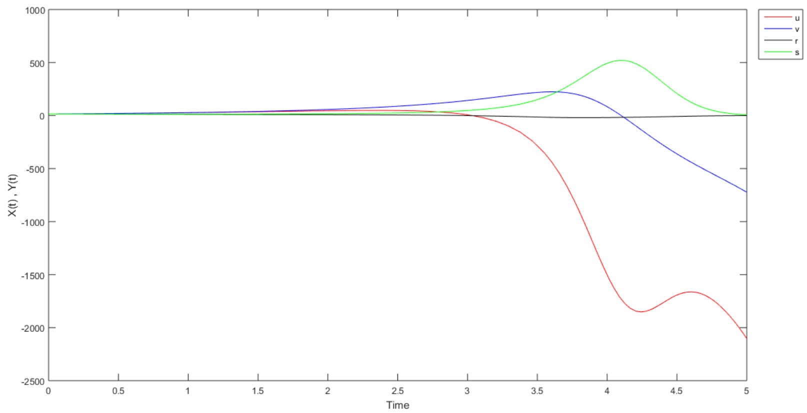


Figure (3.2.130): The solution of (57) at α=1

While $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, we have the following model:

$$u' = u - 0.03vs$$
$$v' = v - 0.03ur$$
$$r' = -0.4r + (0.0175 - \frac{\alpha}{200})vs$$
$$s' = -0.4s + (0.0025 + \frac{\alpha}{200})ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (58)$$

We solve (54) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.131):

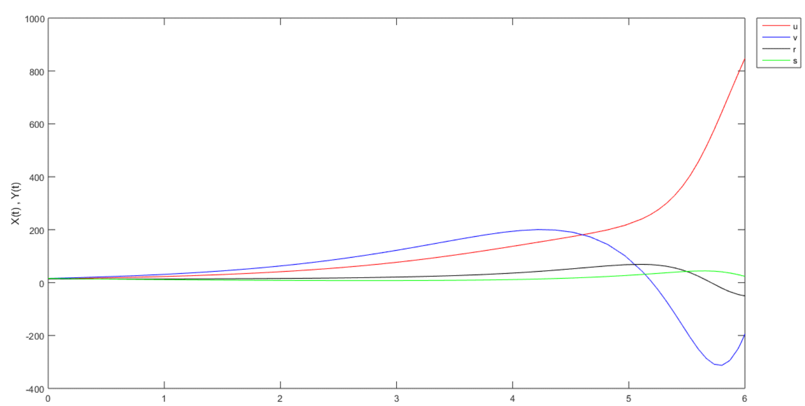


Figure (3.2.131): The solution of (58) at α=0

At α-level = 0.5, the solution figure (3.2.132) in the appendix. At α-level = 1, the solution is figure (3.2.133):

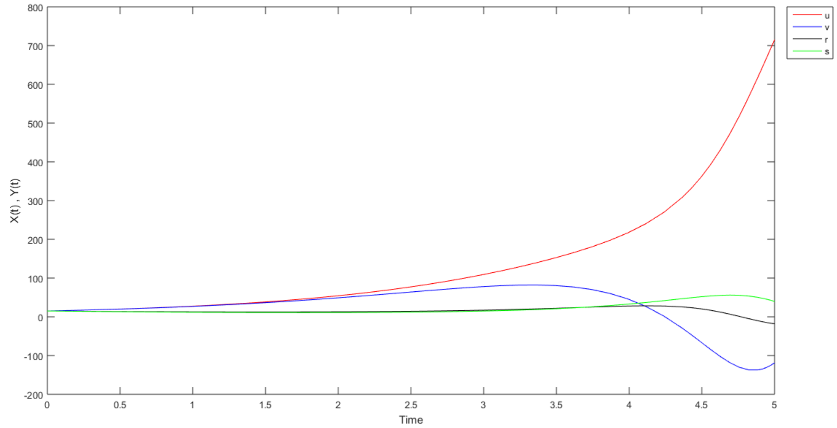


Figure (3.2.133): The solution of (58) at α=1

If $x(t)$ is (2)-differentiable and $y(t)$ is (1)-differentiable, then we have the following model:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4s + (0.005 + \frac{\alpha}{200})ur$$
$$s' = -0.4r + (0.015 - \frac{\alpha}{200})vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (59)$$

At α-level = 0, the solution is figure (3.2.134):

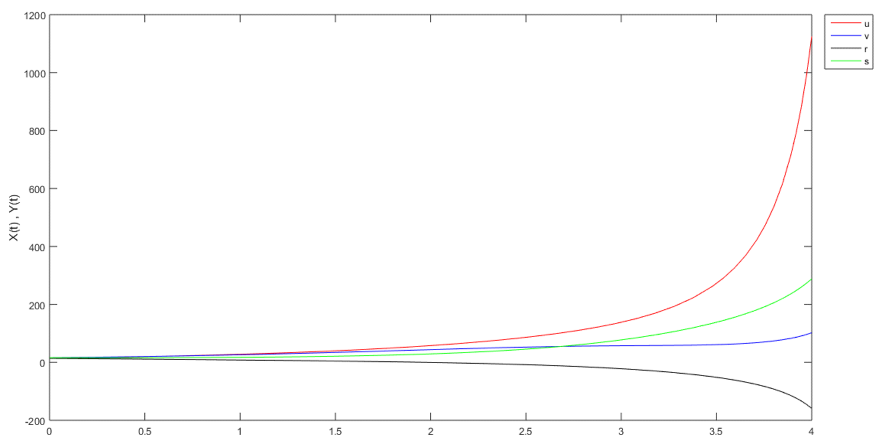


Figure (3.2.134): The solution of (59) at α=0

At α-level = 0.5, the solution is figure (3.2.135) in the appendix. At α-level = 1, the solution is figure (3.2.136):

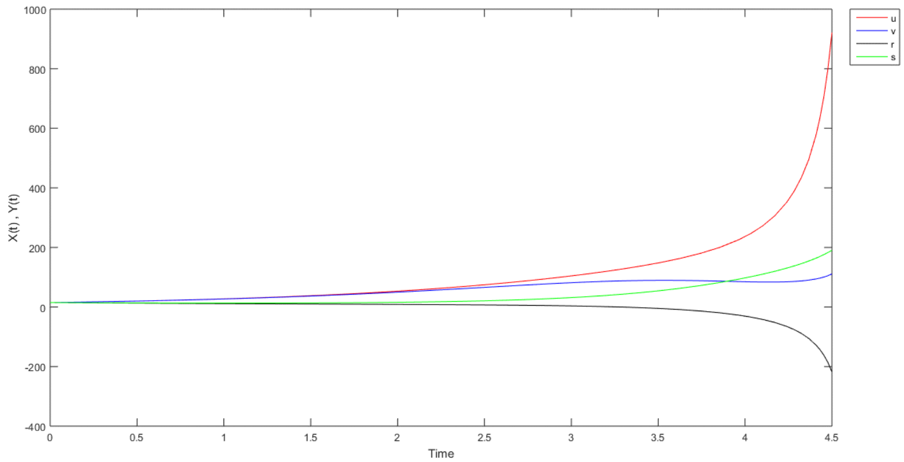


Figure (3.2.136): The solution of (59) at α=1

If $x(t)$ and $y(t)$ are (2)-differentiable, then we have the following model:

$$u' = v - 0.03ur$$
$$v' = u - 0.03vs$$
$$r' = -0.4r + (0.0175 - \frac{\alpha}{200})vs$$
$$s' = -0.4s + (0.0025 + \frac{\alpha}{200})ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (60)$$

We solve this model by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution graphs are figure (3.2.137), figure (3.2.138) and figure (3.2.139):

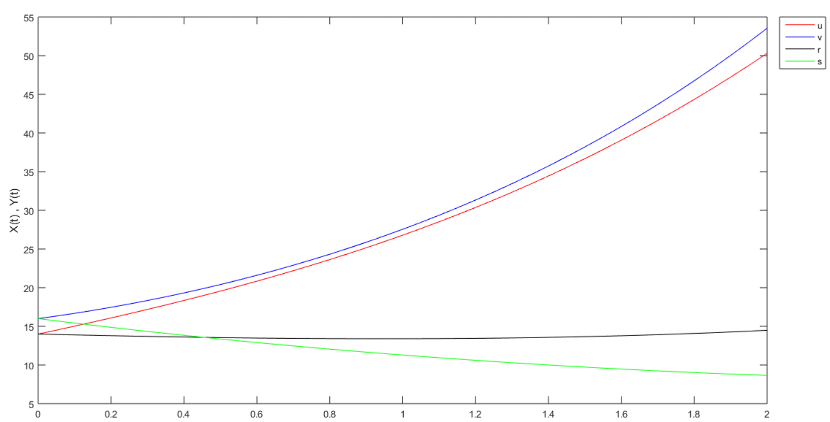
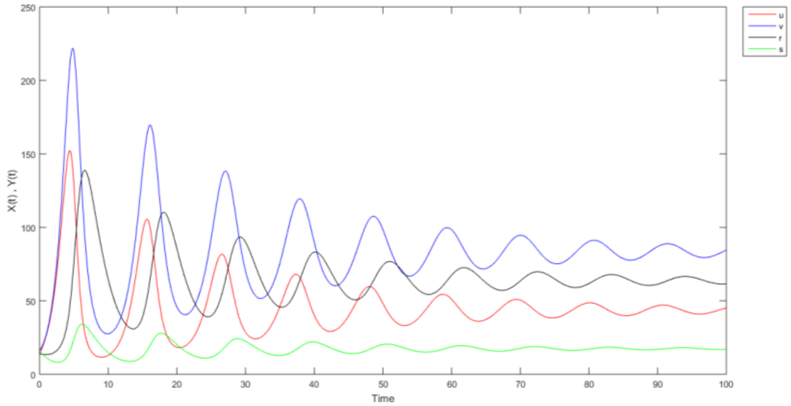
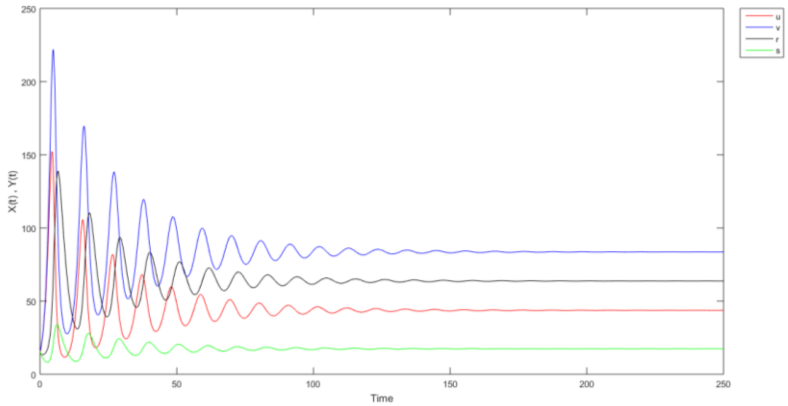


Figure (3.2.137): The solution of (60) at α=0 for short time period





Figures (3.2.138) and (3.2.139): The solution of (60) at α=0 as time increases

At $\alpha = 0$, $as\ t \to \infty$, $u(t) \to 43.72, v(t) \to 83.64,\ r(t) \to 63.76,\ s(t) \to 17.42$. So the solution is asymptotically stable but there is no fuzzy solution for $y(t)$ since $r(t) > s(t)$.

At α-level = 0.5, the solution graphs are figure (3.2.140), figure (3.2.141) and figure (3.2.142) in the appendix. At α-level = 1, the solution graphs are figure (3.2.143), figure (3.2.144) and figure (3.2.145):

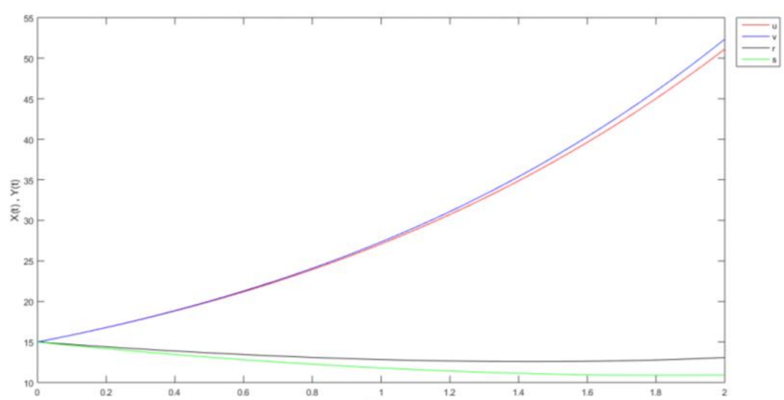
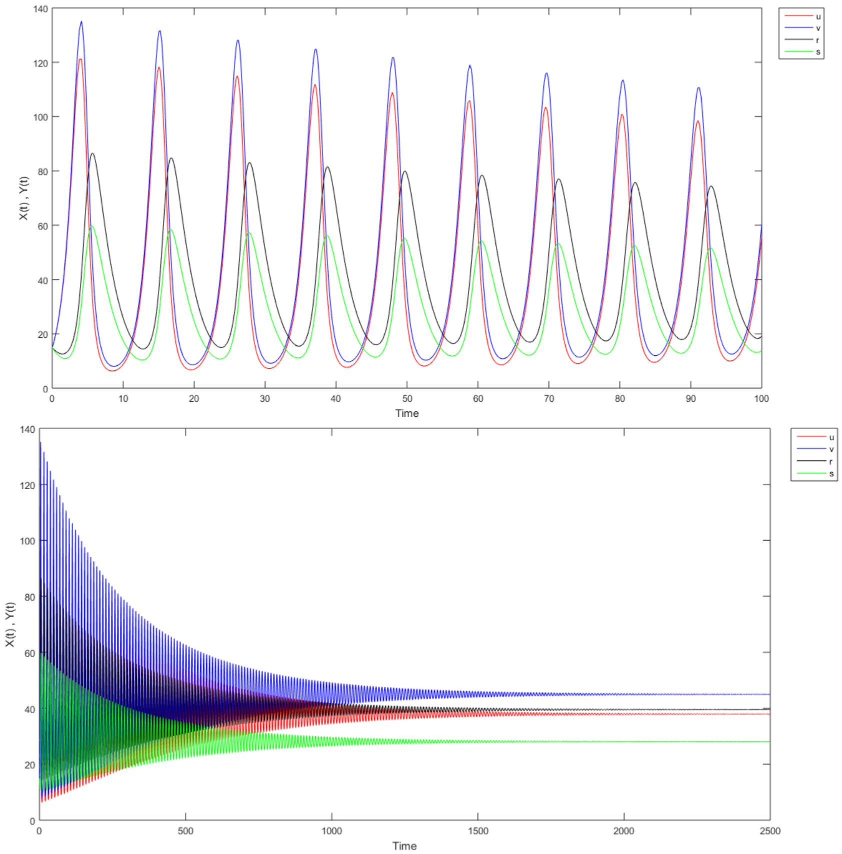


Figure (3.2.143): The solution of (60) at α=1 for short time period



Figures (3.2.144) and (3.2.145): The solution of (60) at α=1 as time increases

At $\alpha = 1$, the solution is asymptotically stable and there is no fuzzy solution for $Y(t)$ since $as\, t \to \infty$, $u(t) \to 37.94$, $v(t) \to 45.98$, $r(t) \to 39.52$, $s(t) \to 28.11$.

From previous work, at $\alpha < 1$ the solution is biologically acceptable only when $x(t)$ and $y(t)$ are (2)-differentiable. This solution is asymptotically stable but $r(t) > s(t)$ for $t \to \infty$, so there is no fuzzy solution for $y(t)$ whether for trapezoidal or triangular fuzzy number. At $\alpha = 1$ the solution is the same as the solution of the crisp case when d a triangular fuzzy number but it isn't when d a trapezoidal fuzzy number. So the triangular fuzzy number is better than the trapezoidal fuzzy number.

Therefore, if we fuzzify d by a triangular fuzzy number with small support , for example $d = (0.0095, 0.01, 0.0105)$ then the solution will be periodic and stable, but with large support, for example $d = (0.0001, 0.01, 0.0199)$ the solution will be asymptotically stable. As in figures (3.2.146) and (3.2.147) we plot the solution of $x(t)$ and $y(t)$ when they are (2)-differentiable at $\alpha = 0$.

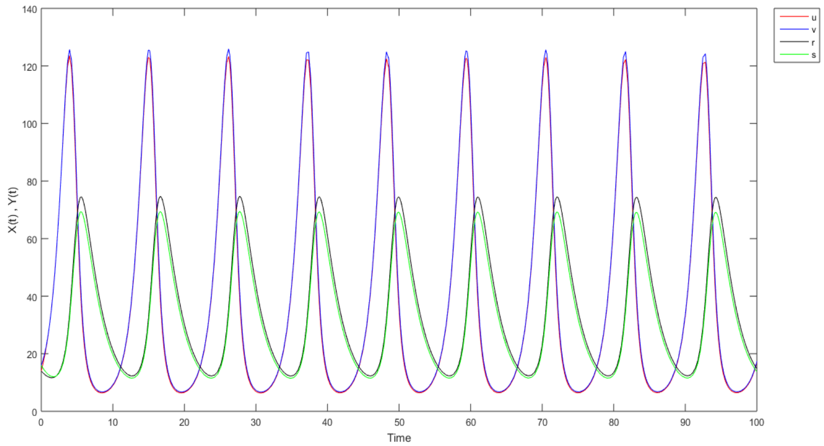


Figure (3.2.146): The solution of X(t) and Y(t) when $d = (0.0095, 0.01, 0.0105)$

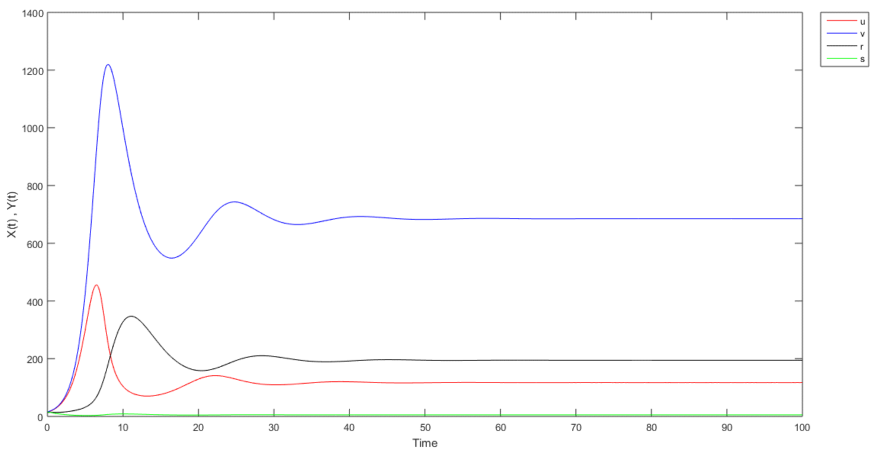


Figure (3.2.147): The solution of X(t) and Y(t) when $d = (0.0001, 0.01, 0.0199)$

In figure (3.2.146) the solution is stable. We try to solve this in Matlab for too long time period and then we note that $r(t) > s(t)$ with little clear difference but this difference is very large when d is a triangular fuzzy number with large support.

In addition, we assume d a triangular fuzzy number with support such that the distance between its endpoints and the core unequal. Figure (3.2.148) and figure (3.2.149) show the solution of $x(t)$ *and* $y(t)$ when they are (2)-differentiable at $\alpha - \text{level} = 0$ for $d = (0.0005, 0.01, 0.015)$ *and* $d = (0.0095, 0.01, 0.05)$, respectively.

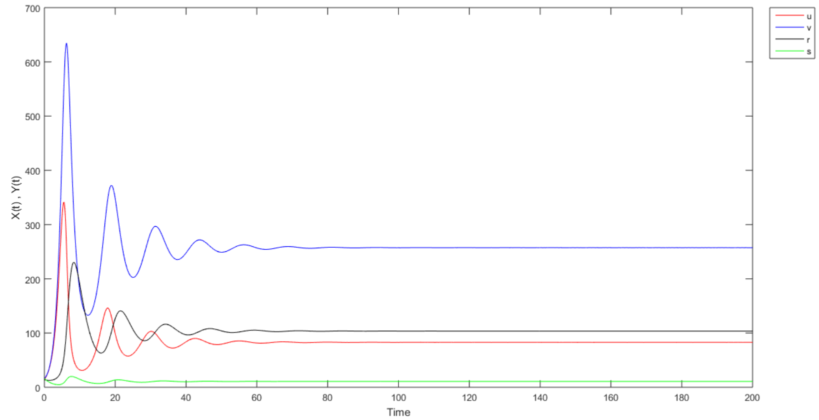


Figure (3.2.148): The solution of X(t) and Y(t) when $d = (0.0005, 0.01, 0.015)$

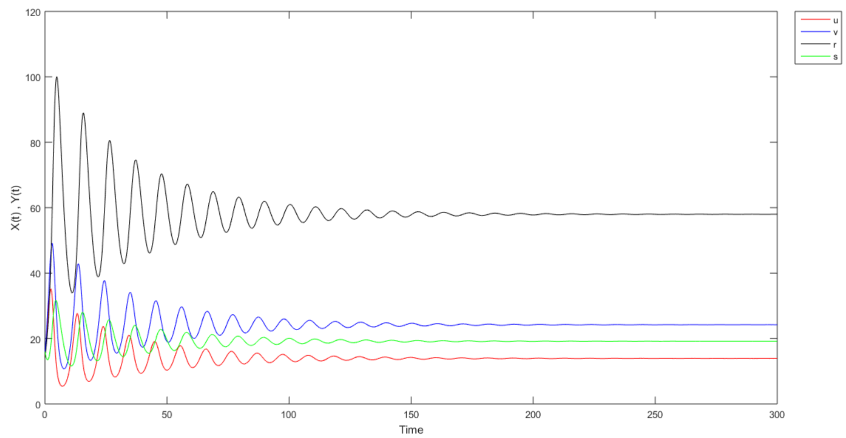


Figure (3.2.149): The solution of X(t) and Y(t) when $d = (0.0095, 0.01, 0.05)$

Also here we conclude that if the distance for at least one support endpoints is long from the core then the solution will be asymptotically stable.

Now, we consider all rates as fuzzy numbers at the same time. As we notice in previous works that the triangular fuzzy number is better than the trapezoidal one. Therefore, we use a triangular fuzzy numbers as follow:

We let $a = (0.5, 1, 1.5)$, $\mathrm{b} = (0.01, 0.03, 0.05)$, $\mathrm{c} = (0.3, 0.4, 0.5)$ and $\mathrm{d} = (0.005, 0.01, 0.015)$ with there $\alpha - levels$ $[a]_\alpha = \left[0.5 + \frac{\alpha}{2},\ 1.5 - \frac{\alpha}{2}\right], [b]_\alpha = \left[0.01 + \frac{\alpha}{50},\ 0.05 - \frac{\alpha}{50}\right], [c]_\alpha = \left[0.3 + \frac{\alpha}{10},\ 0.5 - \frac{\alpha}{10}\right]$ *and* $[d]_\alpha = \left[0.005 + \frac{\alpha}{200},\ 0.015 - \frac{\alpha}{200}\right]$. Then we have the following model:

$$x'(t) = (0.5, 1, 1.5)x \; - (0.01\ ,0.03\ , 0.05)xy$$
$$y'(t) = -(0.3\ ,0.4\ , 0.5)\ y + (0.005\ ,0.01\ , 0.015)xy$$

With fuzzy initial conditions:

$$[x_0]_\alpha = [14 + \alpha, 16 - \alpha],\ \ [y_0]_\alpha = [14 + \alpha, 16 - \alpha] \qquad (61)$$

This model has two equilibrium points. The first one is (0,0,0,0) and the second one varies according to the α–level, as in the following table (3.2.26).

Table (3.2.26): The equilibrium points of (61)

$\alpha - level$	u	v	r	s
0.25	65.4394	27.4508	38.4527	33.1095
0.5	53.924	31.155	36.1098	32.4531
1	40	40	33.3333	33.3333

If $x(t)$ and $y(t)$ are (1)-differentiable, then model (61) will be as follow:

$$u' = (0.5 + \frac{\alpha}{2})u - (0.05 - \frac{\alpha}{50})vs$$
$$v' = (1.5 + \frac{\alpha}{2})v - (0.01 - \frac{\alpha}{50})ur$$
$$r' = -(0.5 - \frac{\alpha}{10})s + (0.005 + \frac{\alpha}{200})ur$$
$$s' = -(0.3 - \frac{\alpha}{10})r + (0.015 + \frac{\alpha}{200})vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (62)$$

we solve (62) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is table (3.2.27), where its graph is figure (3.2.150):

Table (3.2.27): The solution of (62) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.2500	11.7840	22.7560	12.2090	16.1700
0.5000	7.3379	32.7660	10.2820	17.0270
0.7500	-1.1495	47.5630	8.0874	19.0140
1.0000	-17.2540	69.3570	5.4225	23.0240
1.2500	-49.5600	101.2400	1.9561	31.2520
1.5000	-120.4700	147.1900	-2.8655	49.5480
1.7500	-301.3700	210.4400	-9.8869	96.9350
2.0000	-862.8100	282.9300	-19.3970	246.7100
2.2500	-2791.5000	296.0300	-26.5810	765.5200

2.5000	-6364.5000	61.4670	-26.4250	1665.3000
2.7500	-4663.7000	-369.1300	-21.4820	930.4100
3.0000	-2369.5000	-707.0100	-9.3617	121.4500
3.2500	-2258.2000	-1061.6000	-1.4016	4.5910
3.5000	-2542.1000	-1548.5000	-0.0854	0.0389
3.7500	-2880.4000	-2253.2000	-0.0030	0.0001
4.0000	-3263.9000	-3278.4000	-0.0001	0.0000
4.2500	-3698.5000	-4770.1000	0.0000	0.0000
4.5000	-4190.9000	-6940.4000	0.0000	0.0000
4.7500	-4748.9000	-10098.0000	0.0000	0.0000
5.0000	-5381.2000	-14693.0000	0.0000	0.0000

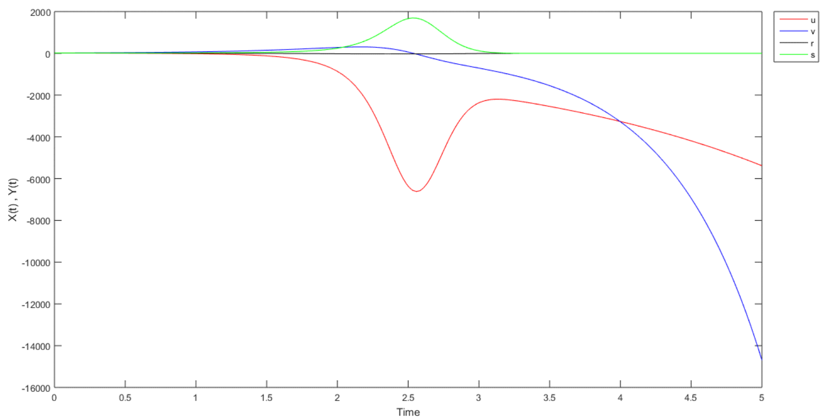


Figure (3.2.150): The solution of (62) at α=0

At α-level = 0.5, the solution is figure (2.151) in the appendix. At α-level = 1, the solution is figure (3.2.152):

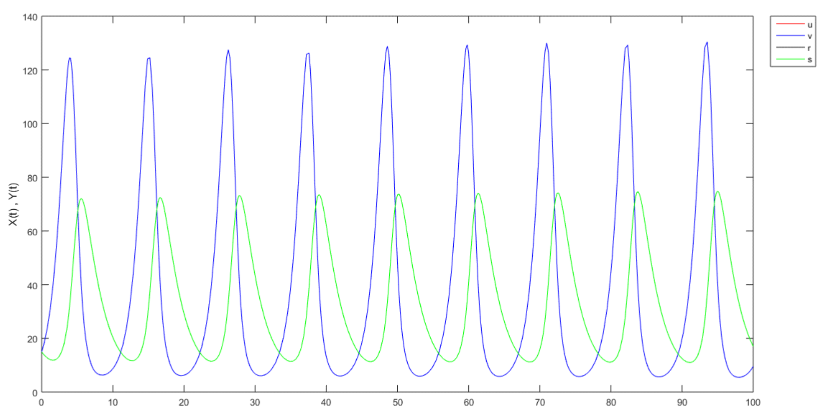


Figure (3.2.152): The solution of (62) at α=1

If $x(t)$ is (1)-differentiable and $y(t)$ is (2)-differentiable, then model (61) will be as follow:

$$u' = (0.5 + \frac{\alpha}{2})u - (0.05 - \frac{\alpha}{50})vs$$
$$v' = (1.5 + \frac{\alpha}{2})v - (0.01 - \frac{\alpha}{50})ur$$
$$r' = -(0.3 - \frac{\alpha}{10})r + (0.015 + \frac{\alpha}{200})vs$$
$$s' = -(0.5 - \frac{\alpha}{10})s + (0.005 + \frac{\alpha}{200})ur$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (63)$$

we solve (63) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is table (3.2.28) and figure (3.2.153):

Table (3.2.28): The solution of (63) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	14.0000	16.0000	14.0000	16.0000
0.1750	12.7360	20.4270	13.9970	14.8170
0.3500	10.8850	26.2240	14.1260	13.7150
0.5250	8.2900	33.8190	14.4100	12.6810
0.7000	4.7627	43.7750	14.8740	11.6990
0.8750	0.0821	56.8360	15.5470	10.7500
1.0500	-6.0033	73.9840	16.4590	9.8108
1.2250	-13.7590	96.5190	17.6360	8.8485
1.4000	-23.4270	126.1600	19.0900	7.8218
1.5750	-35.1420	165.1800	20.7940	6.6775
1.7500	-48.7650	216.5700	22.6440	5.3533
1.9250	-63.6060	284.2100	24.3870	3.7932
2.1000	-78.0050	373.0700	25.5300	1.9833
2.2750	-88.9570	489.3300	25.2700	0.0183
2.4500	-92.1650	640.6500	22.5630	-1.8329
2.6250	-83.2990	836.5200	16.5300	-3.1558
2.8000	-60.5130	1089.5000	7.1398	-3.6410
2.9750	-25.5010	1416.7000	-4.5870	-3.4140
3.1500	19.9980	1842.0000	-17.7570	-3.1367
3.3250	87.7570	2397.6000	-35.3990	-4.0670
3.5000	262.6900	3134.5000	-80.9730	-11.3330

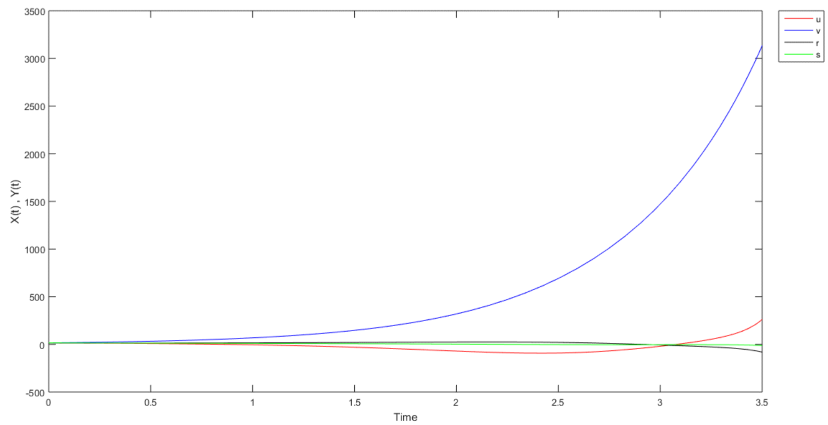


Figure (3.2.153): The solution of (63) at α=0

At α-level = 0.5, the solution is figure (3.2.154) in the appendix. At α-level = 1, the solution is figure (3.2.155):

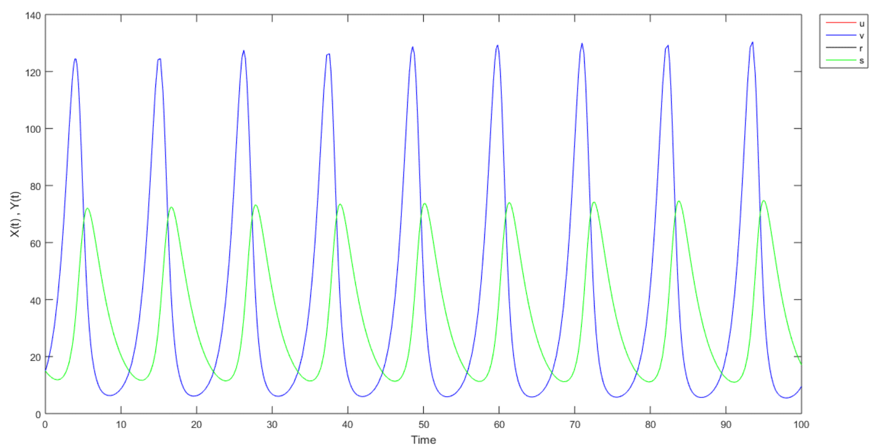


Figure (3.2.155): The solution of (63) at α=1

If $x(t)$is (2)-differentiable and $y(t)$ is (1)-differentiable, then (61) will be as follow:

$$u' = (1.5 + \frac{\alpha}{2})v - (0.01 - \frac{\alpha}{50})ur$$
$$v' = (0.5 + \frac{\alpha}{2})u - (0.05 - \frac{\alpha}{50})vs$$
$$r' = -(0.5 - \frac{\alpha}{10})s + (0.005 + \frac{\alpha}{200})ur$$
$$s' = -(0.3 - \frac{\alpha}{10})r + (0.015 + \frac{\alpha}{200})vs$$
$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \quad (64)$$

we solve (64) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution is figure (3.2.156):

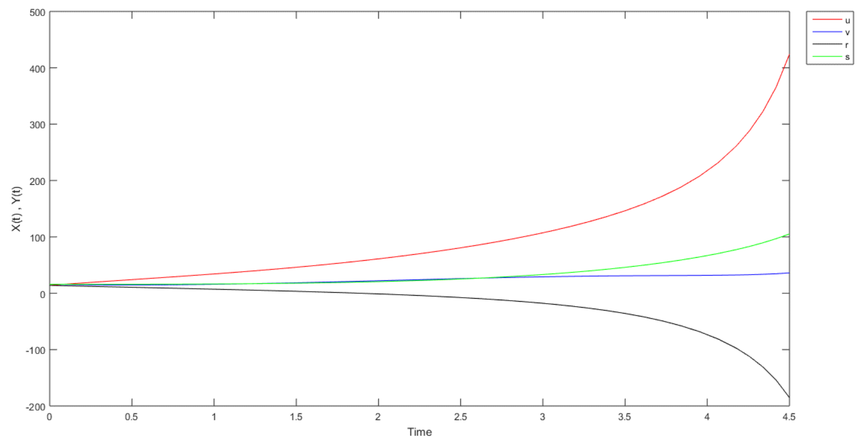


Figure (3.2.156): The solution of (64) at α=0

At α-level = 0.5, the solution is figure (3.2.157) in the appendix. At α-level = 1, the solution is figure (3.2.158):

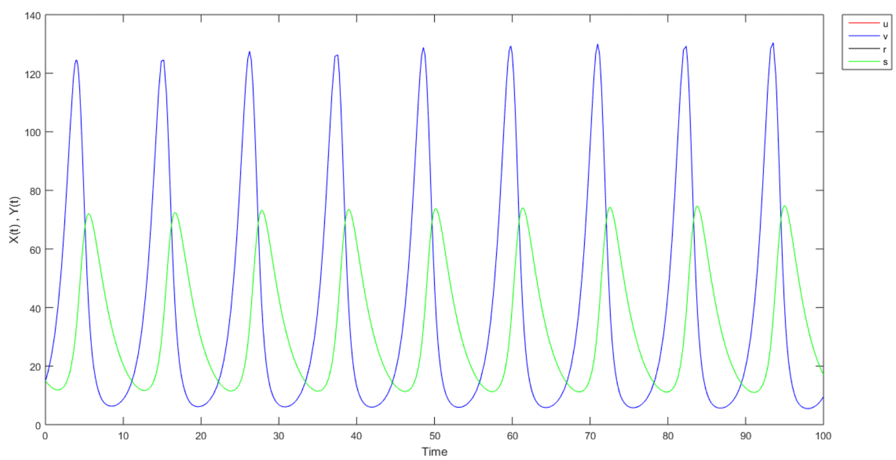


Figure (3.2.158): The solution of (64) at α=1

If $x(t)$and $y(t)$ are (2)-differentiable, then model will be as follow:

$$u' = (1.5 + \frac{\alpha}{2})v - (0.01 - \frac{\alpha}{50})ur$$
$$v' = (0.5 + \frac{\alpha}{2})u - (0.05 - \frac{\alpha}{50})vs$$
$$r' = -(0.3 - \frac{\alpha}{10})r + (0.015 + \frac{\alpha}{200})vs$$
$$s' = -(0.5 - \frac{\alpha}{10})s + (0.005 + \frac{\alpha}{200})ur$$

With the initial conditions:

$$u_0 = 14 + \alpha, v_0 = 16 - \alpha, r_0 = 14 + \alpha, s_0 = 16 - \alpha \qquad (65)$$

we solve (65) by Runge-Kutta method in Matlab at α-level=0, 0.5, 1. At α-level = 0, the solution graphs are figure (3.2.159) and figure (3.2.160):

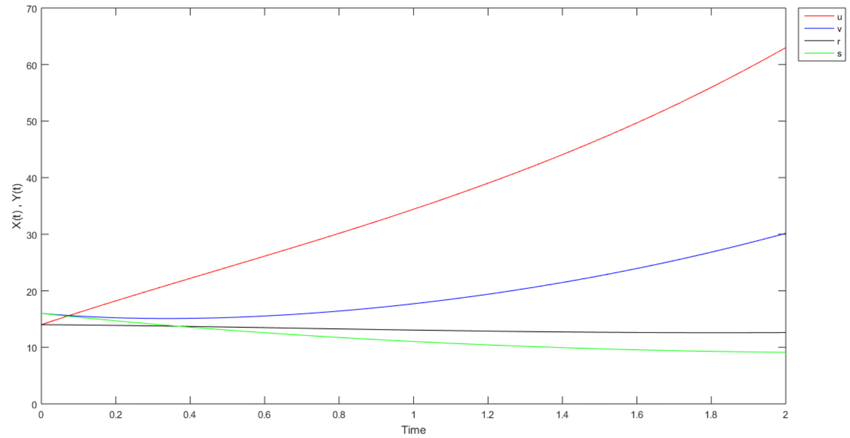


Figure (3.2.159): The solution of (65) at α=0 for short time period

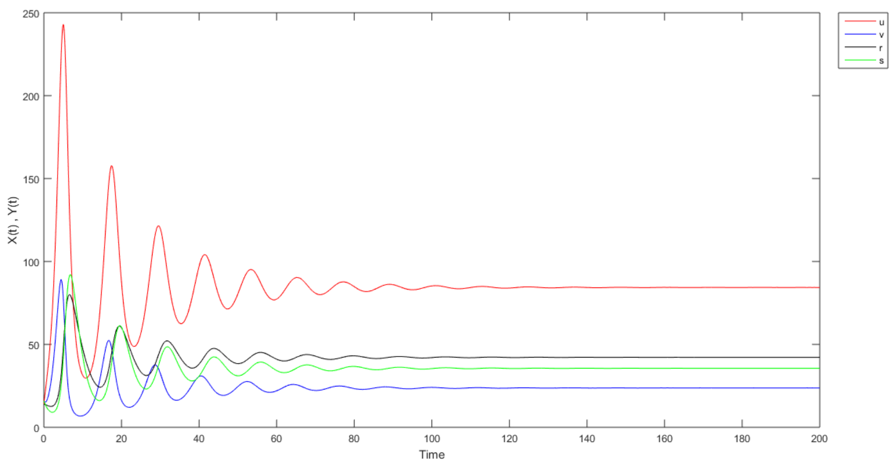


Figure (3.2.160): The solution of (65) at α=0 as time increases

At α-level = 0, $as\, t \to \infty$, $u(t) \to 84.34$, $v(t) \to 23.71$, $r(t) \to 42.17$, $s(t) \to 35.57$. So the solution is asymptotically stable but $u(t) > v(t)$ and $r(t) > s(t)$. Therefore, there are no fuzzy solutions for $x(t)$ and $y(t)$.

At α-level = 0.5, the solution graphs are figure (3.2.161), figure (3.2.162) and figure (3.2.163) in the appendix. At α-level = 1, the solution is figure (3.2.164):

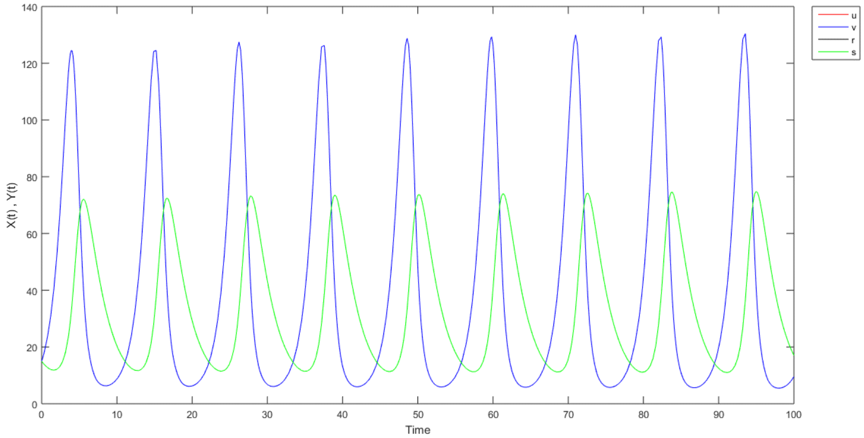


Figure (3.2.164): The solution of (65) at α=1

Now, we try to fuzzify the model using triangular fuzzy numbers of small supports. For example, we let $a = (0.9999,1.1.0001)$, $b = (0.0299, 0.03\ ,0.0301)$, $c = (0.3999, 0.4\ ,0.4001)$ and $d = (0.0099, 0.01\ ,0.0101)$. Since forms (1,1), (1,2), (2,1)-differentiable give unacceptable solutions for our model, we find the graphical solution of the new model when $x(t)$ and $y(t)$ are (2)-differentiable at $\alpha = 0$ as in figure (3.2.165) and compare it with the last model.

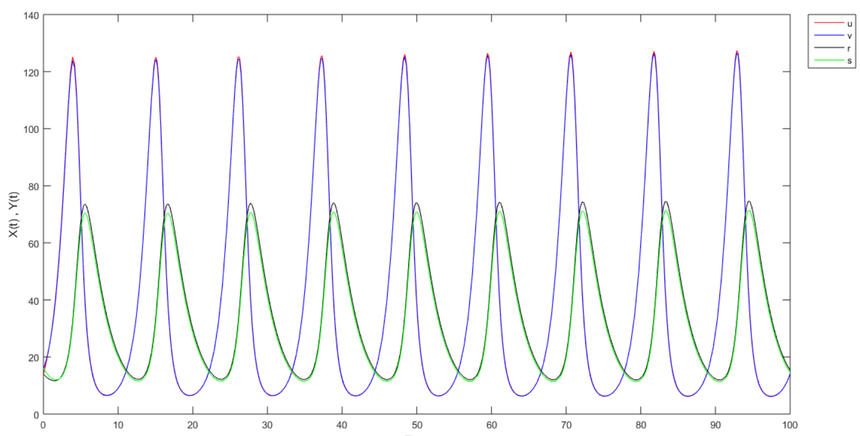


Figure (3.2.165): The solution at α=0 with small supports

Here, we make all rates triangular fuzzy numbers, when $x(t)$ and $y(t)$ are (1,1), (1,2) or (2,1)-differentiable, we obtain unacceptable solutions, but at $\alpha = 1$ the solution is the same as the solution of the crisp case. While, when $x(t)$ and $y(t)$ are (2)-differentiable, the solutions are asymptotically stable but as $t \rightarrow \infty$ we note that $r(t) > s(t)$ and $u(t) > v(t)$. So, there are no fuzzy solutions for

$x(t)$ and $y(t)$ but these solutions are acceptable biologically. At $\alpha = 1$, the solution is the same as the crisp solution. Finally, we try to take a triangular fuzzy number with very small supports, then the solutions are periodic about the equilibrium points but $r(t) > s(t)$ and $u(t) > v(t)$ with clear differences. Therefore, there are no fuzzy solutions for $x(t)$ and $y(t)$.

3.3: Summary

We reviewed an example of the simplest model of predation. We convert the model to a fuzzy one by fuzzifying the initial conditions and then by fuzzifying the parameters. We showed the simulations and graphical solutions of models under generalized Hukuhara derivative through Matlab program using Runge-Kutta method. We compared these solutions with the crisp one.

Chapter 4
Fuzzy Predator-Prey Model with a Functional Response of the Form Arctan(ax)

4.1: Fuzzy Predator-Prey Model with a Functional Response of the Form Arctan(ax) and Fuzzy Initial Conditions

In [4], the researchers dealt with the general predator prey model of the form

$$X'(t) = rX(1 - X) - Y\,tan^{-1}(aX)$$
$$Y'(t) = -DY + sY\,tan^{-1}(aX) \tag{66}$$

where $X\ and\ Y$ are the prey and the predator population sizes respectively, such that r, s ,a and D are positive parameters. Let (x^*, y^*) be the equilibrium point of (66), then $x^* = \frac{1}{a} tan\frac{D}{s}$ and $y^* = \frac{rsx^*(1-x^*)}{D}$, moreover (0,0) and (1,0). Where $D, s\ and\ a$ are chosen such that $0 < x^* < 1$. They established the necessary and sufficient condition for the nonexistence of limit cycles of (66). The system has no limit cycle if and only if $\tan(\frac{D}{s})[\frac{s\tan(\frac{D}{s})-2D[1+tan^2(\frac{D}{s})]}{s\tan(\frac{D}{s})-D[1+tan^2(\frac{D}{s})]}] \geq a$ [4].

So , if $\tan(\frac{D}{s})\left[\frac{s\tan\left(\frac{D}{s}\right)-2D\left[1+tan^2\left(\frac{D}{s}\right)\right]}{s\tan\left(\frac{D}{s}\right)-D\left[1+tan^2\left(\frac{D}{s}\right)\right]}\right] < a$ then there is a limit cycle. Depending on the existence condition we created the following model:

$$X'(t) = 2X(1 - X) - Y\,tan^{-1}(5X)$$
$$Y'(t) = -0.4\,Y + 0.6\,Y\,tan^{-1}(5X)$$
$$x_0 = 1\ and\ y_0 = 1 \tag{67}$$

The equilibrium points of (67) are $(0,0), (1,0)\ and\ (0.157369\,, 0.397811)$. By Matlab using Runge-Kutta method we find the solution of (67) as in table (4.1.1) and its graph is figure (4.1.1):

Table (4.1.1): The solution of (67)

Time	$X(t)$	$Y(t)$
0.0000	1.0000	1.0000
5.0000	0.0004	0.2800
10.0000	0.2536	0.0598
15.0000	0.7375	0.3892

20.0000	0.0123	0.3867
25.0000	0.5572	0.2186
30.0000	0.0359	0.6595
35.0000	0.1821	0.1628
40.0000	0.3046	0.7312
45.0000	0.0441	0.1979
50.0000	0.6131	0.4784
55.0000	0.0163	0.3125
60.0000	0.6503	0.2871
65.0000	0.0173	0.5135
70.0000	0.3583	0.1827
75.0000	0.1017	0.7595
80.0000	0.0953	0.1683
85.0000	0.4766	0.6195
90.0000	0.0258	0.2402
95.0000	0.6657	0.3817
100.0000	0.0140	0.3900

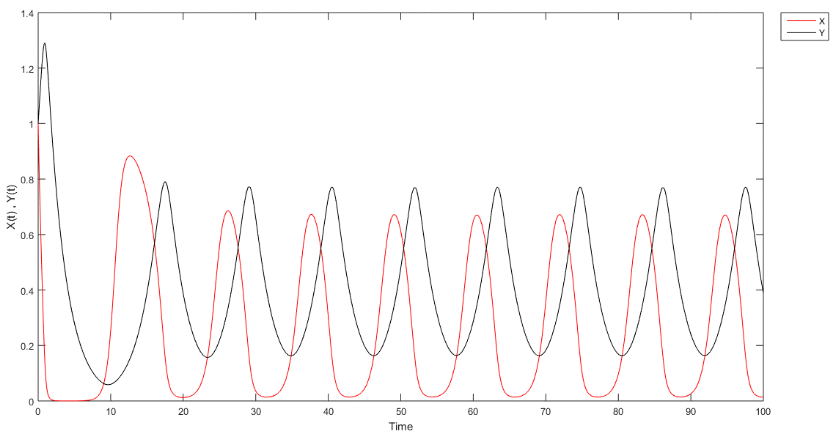


Figure (4.1.1): The solution of (67)

We can note that the curves of the solution of (67) are oscillated about the equilibrium point $(0.157369\,,0.397811)$. So, it is stable but the other points $(0,0)$ and $(1,0)$ are unstable.

Now, we want to explore a fuzzy model from model (67). Therefore, we assume that $X(t)$ *and* $Y(t)$ are fuzzy numbers with fuzzy initial conditions. Let $[X]_\alpha = [u,v]$ *and* $[Y]_\alpha = [r,s]$. And let $x_0 = y_0 = (0.5\,,1\,,1.5)$ a triangular fuzzy numbers then $[x_0]_\alpha = [y_0]_\alpha = [0.5 + \frac{\alpha}{2},\ 1.5 - \frac{\alpha}{2}]$.

As we did before using the generalized Hukuhara derivatives for X(t) and Y(t), we let $X(t)$ *and* $Y(t)$ are (1)-differentiable then $[x']_\alpha = [u', v']$ and $[y']_\alpha = [r', s']$. Then the model will be as follows:

$$u' = 2u - 2v^2 - s\,tan^{-1}(5\,v)$$

$$v' = 2v - 2u^2 - r\,tan^{-1}(5\,u)$$

$$r' = -0.4\,s + 0.6\,r\,tan^{-1}(5\,u)$$

$$s' = -0.4\,r + 0.6\,s\,tan^{-1}(5\,v)$$

$$u_0 = r_0 = 0.5 + \frac{\alpha}{2} \;\; and \;\; v_0 = s_0 = 1.5 - \frac{\alpha}{2} \qquad (68)$$

The equilibrium points of (68) are $\chi_{(0,0)}, \chi_{(1,0)}$ and $\chi_{(0.157369\,,0.397811)}$.We solve (68) by Runge-Kutta method in Matlab at $\alpha - levels = 0, 0.5, 1$. At α-level = 0, the solution is figure (4.1.2):

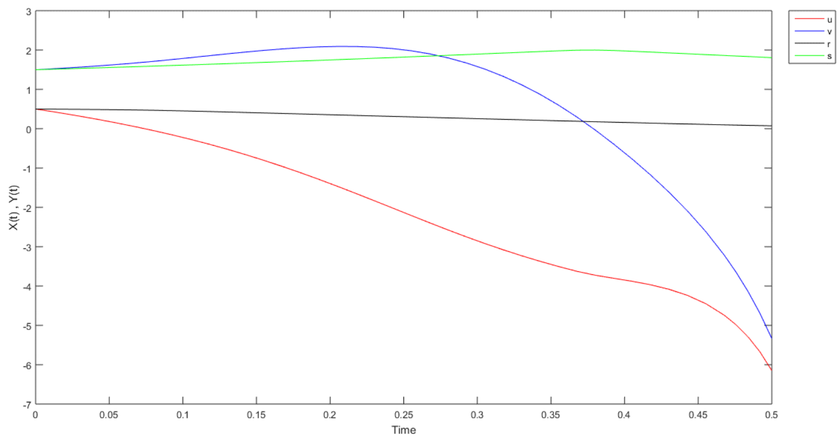


Figure (4.1.2): The solution of (68) at α=0

At α-level = 0.5, the solution is figure (4.1.3) in the appendix. At α-level = 1, the solution is table (4.1.2) and figure (4.1.4):

Table (4.1.2): The solution of (68) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	1.0000	1.0000	1.0000	1.0000
5.0000	0.0004	0.0004	0.2800	0.2800
10.0000	0.2536	0.2536	0.0598	0.0598
15.0000	0.7375	0.7375	0.3892	0.3892
20.0000	0.0123	0.0123	0.3867	0.3867
25.0000	0.5572	0.5572	0.2186	0.2186
30.0000	0.0359	0.0359	0.6595	0.6595
35.0000	0.1821	0.1821	0.1628	0.1628
40.0000	0.3046	0.3046	0.7312	0.7312
45.0000	0.0441	0.0441	0.1979	0.1979
50.0000	0.6131	0.6131	0.4784	0.4784
55.0000	0.0163	0.0163	0.3125	0.3125

60.0000	0.6503	0.6503	0.2871	0.2871
65.0000	0.0173	0.0173	0.5135	0.5135
70.0000	0.3583	0.3583	0.1827	0.1827
75.0000	0.1017	0.1017	0.7595	0.7595
80.0000	0.0953	0.0953	0.1683	0.1683
85.0000	0.4766	0.4766	0.6195	0.6195
90.0000	0.0258	0.0258	0.2402	0.2402
95.0000	0.6657	0.6657	0.3817	0.3817
100.0000	0.0140	0.0140	0.3900	0.3900

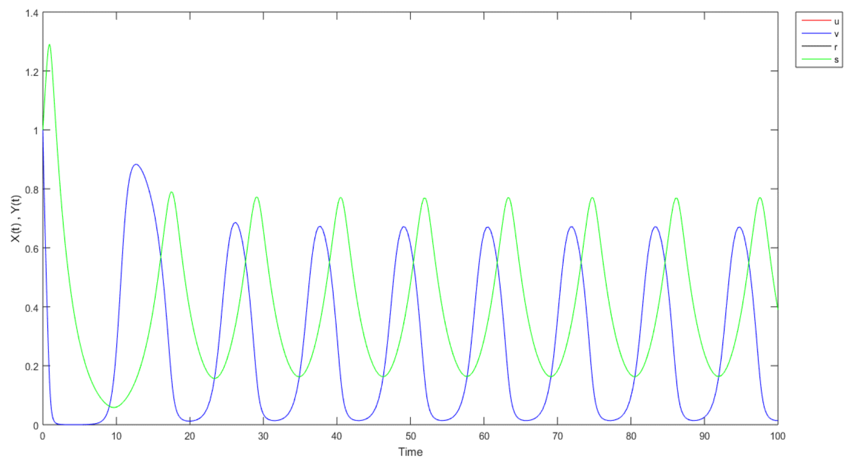


Figure (4.1.4): The solution of (68) at α=1

While if $X(t)$ is (1)-differentiable and $Y(t)$ is (2)-differentiable, then we have the following model:

$$u' = 2u - 2v^2 - s\,tan^{-1}(5\,v)$$

$$v' = 2v - 2u^2 - r\,tan^{-1}(5\,u)$$

$$r' = -0.4\,r + 0.6\,s\,tan^{-1}(5\,v)$$

$$s' = -0.4\,s + 0.6\,r\,tan^{-1}(5\,u)$$

$$u_0 = r_0 = 0.5 + \frac{\alpha}{2} \;\; and \;\; v_0 = s_0 = 1.5 - \frac{\alpha}{2} \qquad (69)$$

We solve (69) by Runge-Kutta method in Matlab at $\alpha - levels = 0, 0.5, 1$. At α-level = 0, the solution is figure (4.1.5):

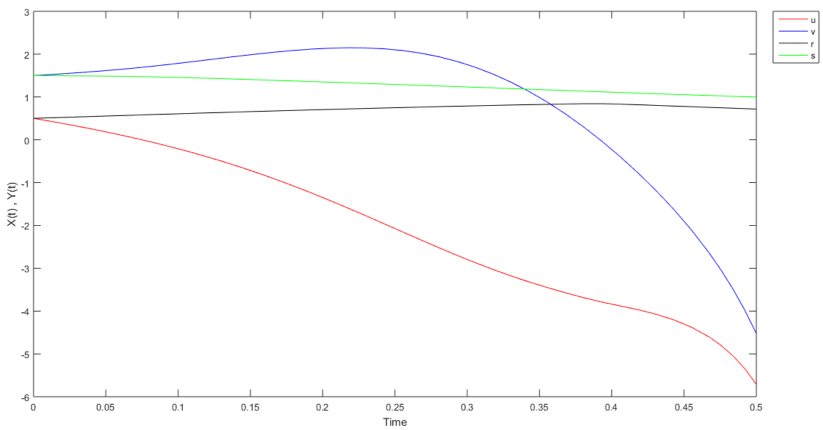


Figure (4.1.5): The solution of (69) at α=0

At α-level = 0.5, the solution is figure (4.1.6) in the appendix. At α-level = 1, the solution is figure (4.1.7):

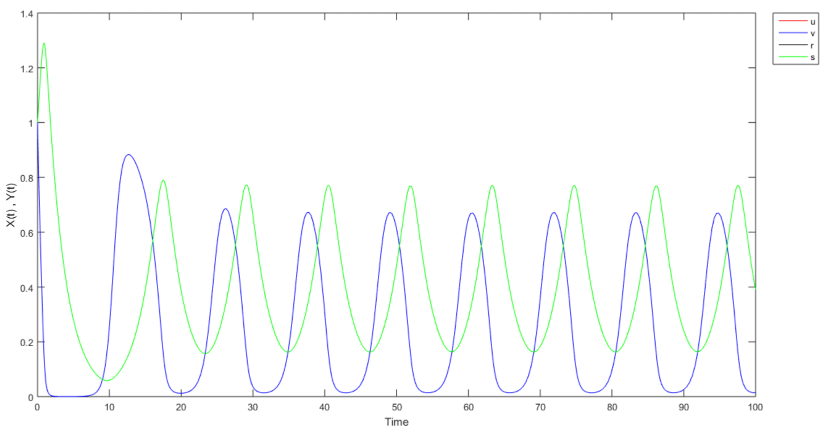


Figure (4.1.7): The solution of (69) at α=1

If $X(t)$ is (2)-differentiable and $Y(t)$ is (1)-differentiable, then we have the following model:

$$u' = 2v - 2u^2 - r\, tan^{-1}(5\, u)$$

$$v' = 2u - 2v^2 - s\, tan^{-1}(5\, v)$$

$$r' = -0.4\, s + 0.6\, r\, tan^{-1}(5\, u)$$

$$s' = -0.4\, r + 0.6\, s\, tan^{-1}(5\, v)$$

$$u_0 = r_0 = 0.5 + \frac{\alpha}{2} \;\; and \;\; v_0 = s_0 = 1.5 - \frac{\alpha}{2} \qquad (70)$$

We solve (70) by Runge-Kutta method in Matlab at $\alpha - levels = 0, 0.5, 1$. At α-level = 0, the solution is figure (4.1.8):

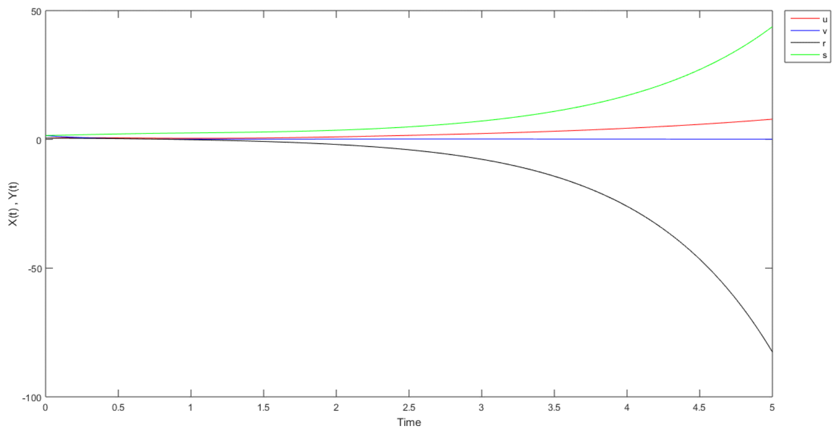


Figure (4.1.8): The solution of (70) at α=0

At α-level = 0.5, the solution is figure (4.1.9) in the appendix. At α-level = 1, the solution is figure (4.1.10):

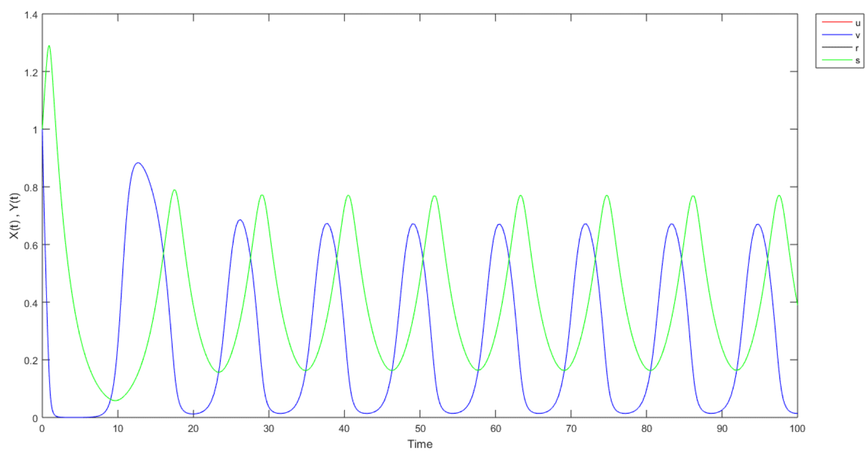


Figure (4.1.10): The solution of (70) at α=1

Now, if $X(t)$ and $Y(t)$ are (2)-differentiable, then we have the following model:

$$u' = 2v - 2u^2 - r\, tan^{-1}(5\,u)$$

$$v' = 2u - 2v^2 - s\, tan^{-1}(5\,v)$$

$$r' = -0.4\,r + 0.6\,s\, tan^{-1}(5\,v)$$

$$s' = -0.4\,s + 0.6\,r\, tan^{-1}(5\,u)$$

$$u_0 = r_0 = 0.5 + \frac{\alpha}{2} \;\; and \;\; v_0 = s_0 = 1.5 - \frac{\alpha}{2} \qquad (71)$$

We solve (71) by Runge-Kutta method in Matlab at $\alpha - levels = 0, 0.5, 1$. At α-level = 0, the solution graphs are figure (4.1.11) and figure (4.1.12):

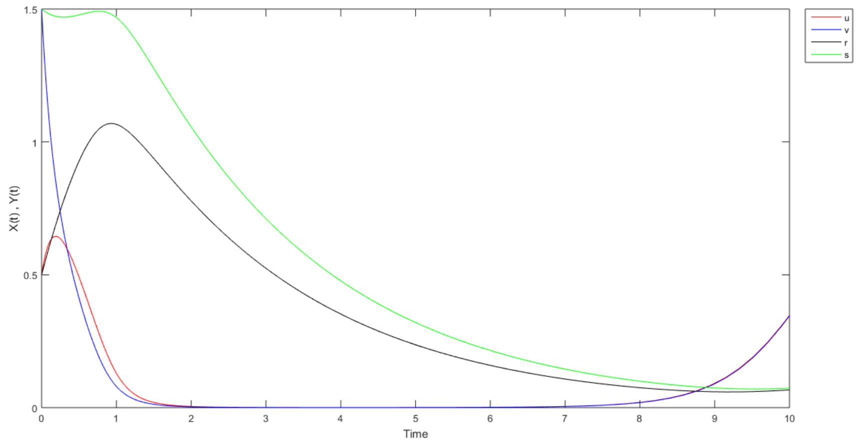


Figure (4.1.11): The solution of (71) at α=0 for short time period

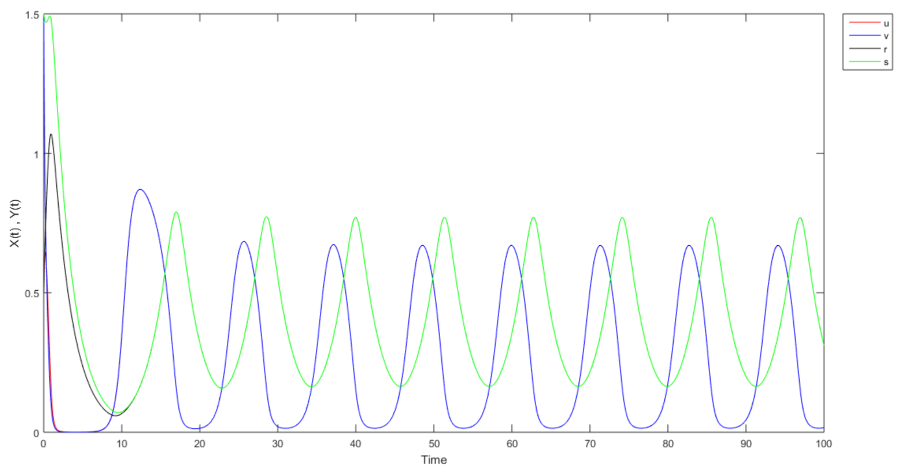


Figure (4.1.12): The solution of (71) at α=0 as time increases

At α-level = 0.5, the solution graphs are figure (4.1.13) and figure (4.1.14) in the appendix. At α-level = 1, the solution is figure (4.1.15):

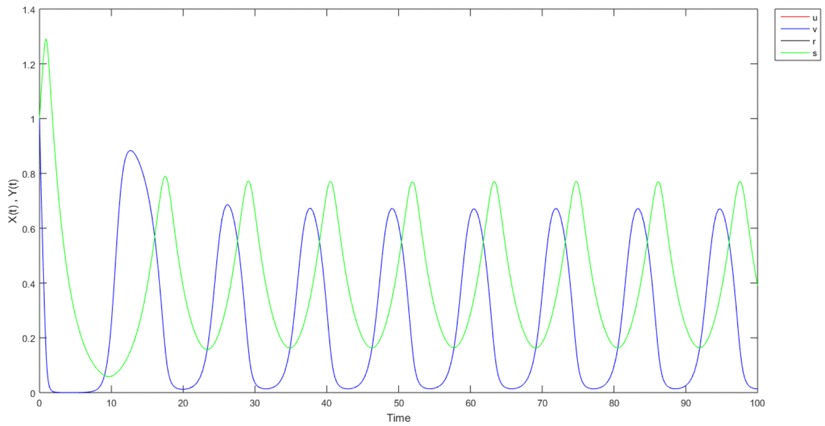


Figure (4.1.15): The solution of (71) at α=1

In this section, we create a new model. First, we find the crisp solution which periodic as $t \to \infty$ and stable about the equilibrium point $(0.157369\,, 0.397811)$, but the equilibrium points $(0{,}0)$ and $(1{,}0)$ are unstable. Second, we make the initial conditions triangular fuzzy numbers then we obtain biologically unacceptable and unstable solution when $X(t)$ and $Y(t)$ are (1,1), (1,2) and (2,1)-differentiable for $\alpha < 1$. At $\alpha = 1$ the solution is equivalent to the crisp case. While, when $X(t)$ and $Y(t)$ are (2)-differentiable, we note that $u(t) > v(t)$ for very short time period but as $t \to \infty$ the solution becomes periodic and stable.

Now, we try to use a triangular fuzzy numbers with small supports for the initial conditions. Let $x_0 = y_0 = (0.9999\,, 1\,, 1.0001)$ then $[x_0]_\alpha = [y_0]_\alpha$=$[0.9999 + \frac{\alpha}{10000}, 1.0001 - \frac{\alpha}{10000}]$. Since the model when $X(t)$ and $Y(t)$ are (2)-differentiable give a fuzzy solution which is biologically acceptable we find the solution of $X(t)\ and\ Y(t)$ when they are (2)-differentiable at $\alpha - level = 0$. Therefore, we have the following model:

$$u' = 2v - 2u^2 - r\,tan^{-1}(5\,u)$$

$$v' = 2u - 2v^2 - s\,tan^{-1}(5\,v)$$

$$r' = -0.4\,r + 0.6\,s\,tan^{-1}(5\,v)$$

$$s' = -0.4\,s + 0.6\,r\,tan^{-1}(5\,u)$$

$$u_0 = r_0 = 0.9999 + \frac{\alpha}{10000}\ \ and\ \ v_0 = s_0 = 1.0001 - \frac{\alpha}{10000} \qquad (72)$$

The solution graphs are figures (4.1.16) and (4.1.17):

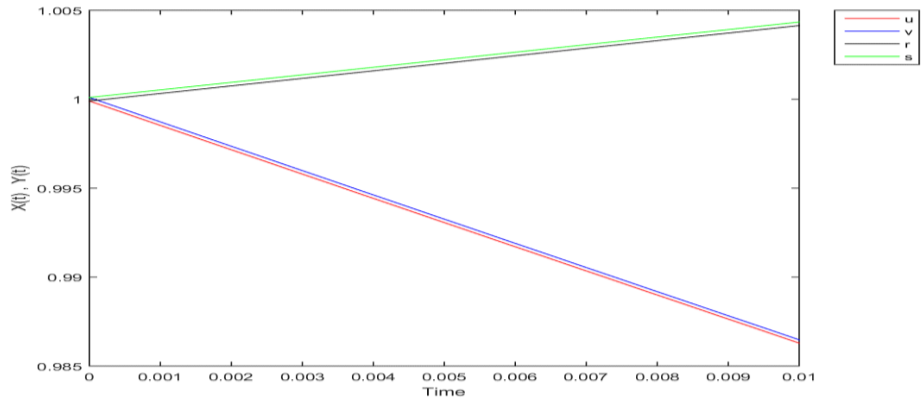


Figure (4.1.16): The solution of (72) at α=0 for short time period

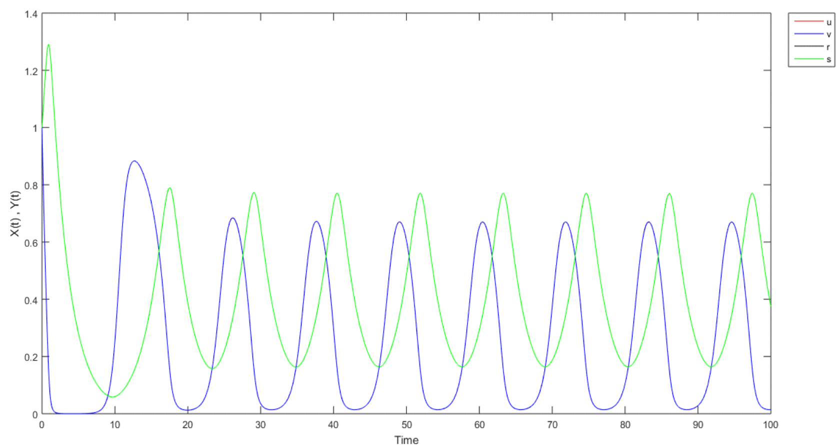


Figure (4.1.17): The solution of (72) at α=0 as time increases

We can note that initially $v(t) > u(t)$ and $s(t) > r(t)$ but as $t \to \infty$ the solution become periodic and stable with $v(t) = u(t)$ and $s(t) = r(t)$. So, the solution of (72) is better than the previous one using initial conditions with large supports.

4.2: Fuzzy Predator-Prey Model with a Functional Response of the Form Arctan(ax) and Fuzzy Parameters

For first time we want to make the parameters of the model (67) triangular fuzzy numbers. For example, we let $r = (1,2,3)$ with $[r]_\alpha = [1+\alpha, 3-\alpha], a = (4,5,6)$ with $[a]_\alpha = [4+\alpha, 6-\alpha], D = (0.2,0.4,0.6)$ with $[D]_\alpha = [0.2{+}\frac{\alpha}{5}, 0.6 - \frac{\alpha}{5}]$ and $s = (0.4,0.6,0.8)$ with $[s]_\alpha = \left[0.4 + \frac{\alpha}{5}, 0.8 - \frac{\alpha}{5}\right]$. Then (67) will be as follow:

$$X'(t) = (1,2,3)X(1-X) - Y\,tan^{-1}((4,5,6)X)$$

$$Y'(t) = -(0.2,0.4,0.6)\,Y + (0.4,0.6,0.8)\,Y\,tan^{-1}(5X)$$

$$x_0 = (0.5\,,1\,,1.5) \text{ and } y_0 = (0.5\,,1\,,1.5) \tag{73}$$

If $X(t)$ and $Y(t)$ are (1)-differentiable, then we have the following model:

$$u' = (1+\alpha)u - (3-\alpha)v^2 - s\,tan^{-1}((6-\alpha)v)$$

$$v' = (3-\alpha)v - (1+\alpha)u^2 - r\,tan^{-1}((4+\alpha)\,u)$$

$$r' = -(0.6 - \frac{\alpha}{5})s + (0.4 + \frac{\alpha}{5})r\,tan^{-1}((4+\alpha)u)$$

$$s' = -(0.2 + \frac{\alpha}{5})r + (0.8 - \frac{\alpha}{5})s\,tan^{-1}((6-\alpha)v)$$

$$u_0 = r_0 = 0.5 + \frac{\alpha}{2} \text{ and } v_0 = s_0 = 1.5 - \frac{\alpha}{2} \tag{74}$$

The equilibrium points of (74) are $\chi_{(0,0)}, \chi_{(1,0)}$.We solve this model numerically by Matlab at α=0,0.5,1. At α-level = 0, the solution is figure (4.2.1):

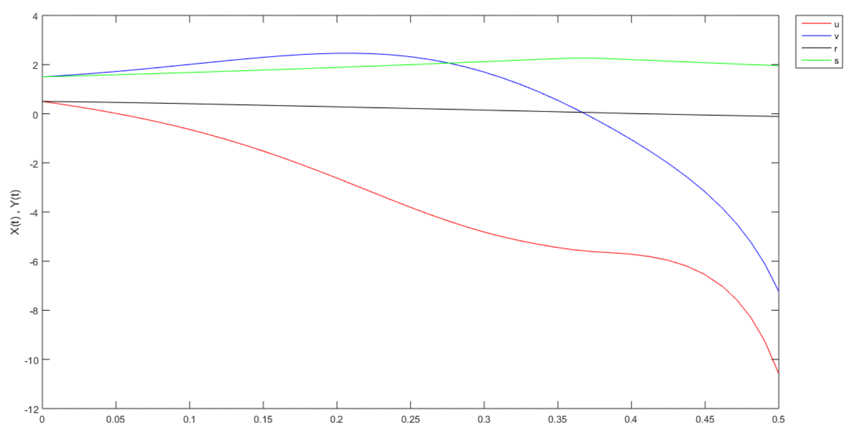


Figure (4.2.1): The solution of (74) at α=0

At α-level = 0.5, the solution is figure (4.2.2) in the appendix. At α-level = 1, the solution is table (4.2.1), where its graph is figure (4.2.3):

Table (4.2.1): The solution of (74) at α=1

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	1.0000	1.0000	1.0000	1.0000
5.0000	0.0004	0.0004	0.2800	0.2800
10.0000	0.2536	0.2536	0.0598	0.0598
15.0000	0.7375	0.7375	0.3892	0.3892
20.0000	0.0123	0.0123	0.3867	0.3867
25.0000	0.5572	0.5572	0.2186	0.2186
30.0000	0.0359	0.0359	0.6595	0.6595
35.0000	0.1821	0.1821	0.1628	0.1628
40.0000	0.3046	0.3046	0.7312	0.7312
45.0000	0.0441	0.0441	0.1979	0.1979
50.0000	0.6131	0.6131	0.4784	0.4784
55.0000	0.0163	0.0163	0.3125	0.3125
60.0000	0.6503	0.6503	0.2871	0.2871
65.0000	0.0173	0.0173	0.5135	0.5135
70.0000	0.3583	0.3583	0.1827	0.1827
75.0000	0.1017	0.1017	0.7595	0.7595
80.0000	0.0953	0.0953	0.1683	0.1683
85.0000	0.4766	0.4766	0.6195	0.6195
90.0000	0.0258	0.0258	0.2402	0.2402
95.0000	0.6657	0.6657	0.3817	0.3817
100.0000	0.0140	0.0140	0.3900	0.3900

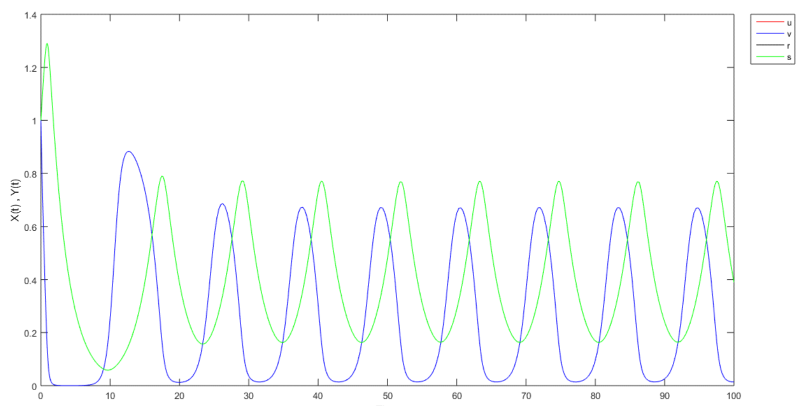


Figure (4.2.3): The solution of (74) at α=1

If $X(t)$ is (1)-differentiable and $Y(t)$ is (2)-differentiable, then we have the following model:

$$u' = (1+\alpha)u - (3-\alpha)v^2 - s\,tan^{-1}((6-\alpha)v)$$

$$v' = (3-\alpha)v - (1+\alpha)u^2 - r\,tan^{-1}((4+\alpha)\,u)$$

$$r' = -(0.2+\frac{\alpha}{5})r + (0.8-\frac{\alpha}{5})s\,tan^{-1}((6-\alpha)v)$$

$$s' = -(0.6 - \frac{\alpha}{5})s + (0.4 + \frac{\alpha}{5})r\ tan^{-1}((4+\alpha)u)$$

$$u_0 = r_0 = 0.5 + \frac{\alpha}{2} \ \ and \ \ v_0 = s_0 = 1.5 - \frac{\alpha}{2} \qquad (75)$$

We solve (75) by Matlab at $\alpha = 0,0.5,1$. At α-level = 0, the solution is figure (4.2.4):

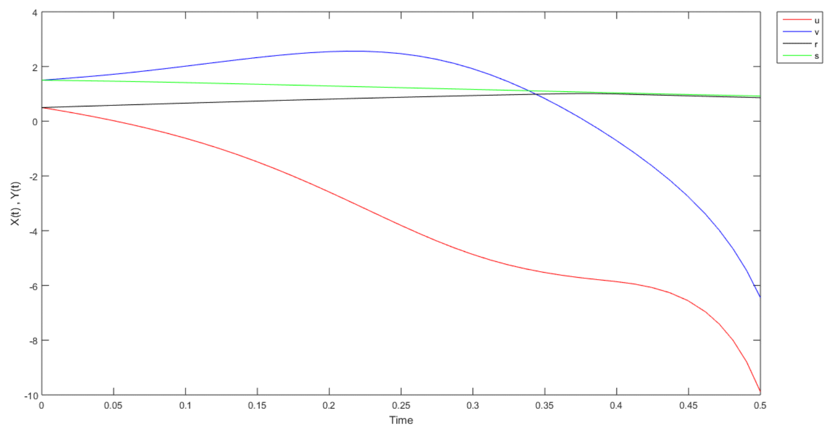


Figure (4.2.4): The solution of (75) at α=0

At α-level = 0.5, the solution is figure (4.2.5) in the appendix. At α-level = 1, the solution is figure (4.2.6):

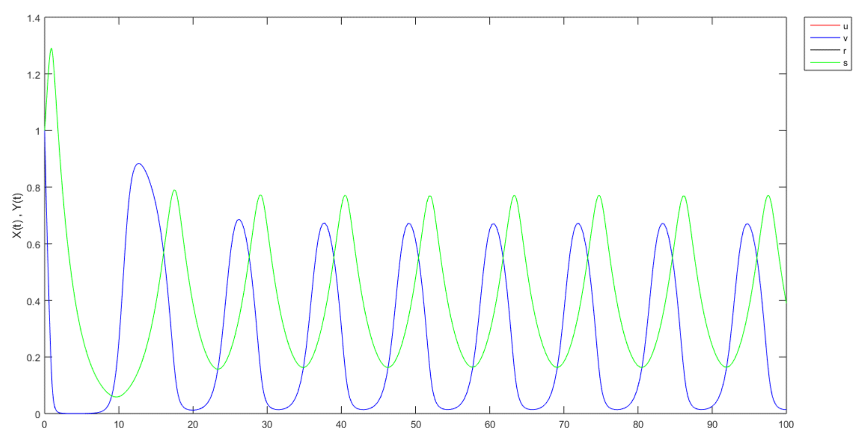


Figure (4.2.6): The solution of (75) at α=1

If $X(t)$ is (2)-differentiable and $Y(t)$ is (1)-differentiable, then we have the following model:

$$u' = (3-\alpha)v - (1+\alpha)u^2 - r\,tan^{-1}((4+\alpha)\,u)$$

$$v' = (1+\alpha)u - (3-\alpha)v^2 - s\,tan^{-1}((6-\alpha)v)$$

$$r' = -(0.6 - \frac{\alpha}{5})s + (0.4 + \frac{\alpha}{5})r\,tan^{-1}((4+\alpha)u)$$

$$s' = -(0.2 + \frac{\alpha}{5})r + (0.8 - \frac{\alpha}{5})s\,tan^{-1}((6-\alpha)v)$$

$$u_0 = r_0 = 0.5 + \frac{\alpha}{2} \;\; and \;\; v_0 = s_0 = 1.5 - \frac{\alpha}{2} \qquad (76)$$

We solve (76) using Matlab at $\alpha = 0{,}0.5{,}1$. At α-level = 0, the solution is table (4.2.2), where its graph is figure (4.2.7):

Table (4.2.2): The solution of (76) at α=0

Time	$u(t)$	$v(t)$	$r(t)$	$s(t)$
0.0000	0.5000	1.5000	0.5000	1.5000
0.5000	0.8420	0.1187	-0.0132	2.3504
1.0000	0.8739	0.0532	-0.8914	2.7674
1.5000	1.3734	0.0693	-2.2041	3.3358
2.0000	2.1710	0.0888	-4.2409	4.3304
2.5000	3.0778	0.0942	-7.4537	5.9350
3.0000	4.1125	0.0893	-12.5130	8.3406
3.5000	5.3515	0.0809	-20.4190	11.8410
4.0000	6.8573	0.0717	-32.6690	16.8830
4.5000	8.6924	0.0628	-51.5010	24.1250
5.0000	10.9300	0.0546	-80.2540	34.5240
5.5000	13.6560	0.0472	-123.9100	49.4840
6.0000	16.9780	0.0406	-189.8600	71.0550
6.5000	21.0220	0.0348	-289.1400	102.2500
7.0000	25.9450	0.0296	-438.0700	147.4700
7.5000	31.9350	0.0251	-660.9100	213.2100
8.0000	39.2230	0.0213	-993.6100	309.0200
8.5000	48.0880	0.0179	-1489.5000	448.9400
9.0000	58.8690	0.0151	-2227.4000	653.6700
9.5000	71.9810	0.0126	-3324.2000	953.7300
10.0000	87.9260	0.0105	-4953.0000	1394.1000

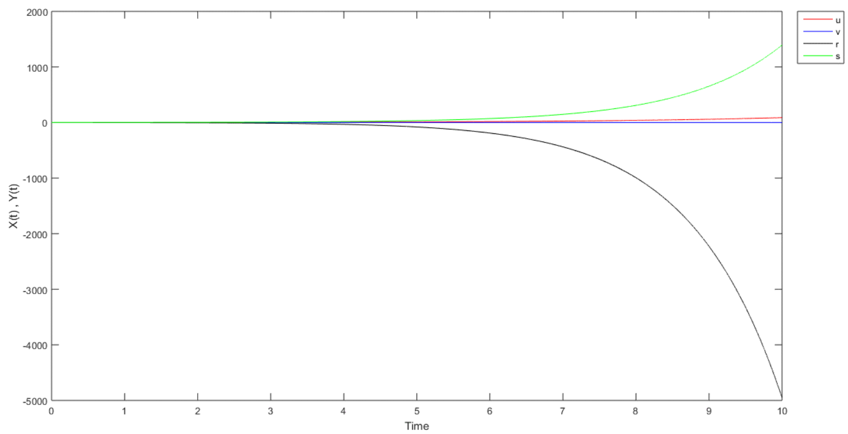


Figure (4.2.7): The solution of (76) at α=0

At α-level = 0.5, the solution is figure (4.2.8) in the appendix. At α-level = 1, the solution is figure (4.2.9):

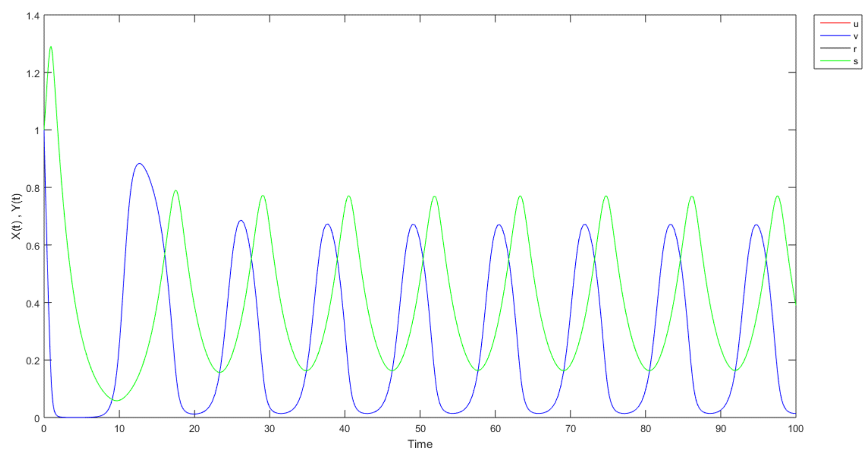


Figure (4.2.9): The solution of (76) at α=1

If $X(t)$ and $Y(t)$ are (2)-differentiable, then we have the following model:

$$u' = (3-\alpha)v - (1+\alpha)u^2 - r\, tan^{-1}((4+\alpha)\, u)$$

$$v' = (1+\alpha)u - (3-\alpha)v^2 - s\, tan^{-1}((6-\alpha)v)$$

$$r' = -(0.2 + \frac{\alpha}{5})r + (0.8 - \frac{\alpha}{5})s\, tan^{-1}((6-\alpha)v)$$

$$s' = -(0.6 - \frac{\alpha}{5})s + (0.4 + \frac{\alpha}{5})r\, tan^{-1}((4+\alpha)u)$$

$$u_0 = r_0 = 0.5 + \frac{\alpha}{2} \text{ and } v_0 = s_0 = 1.5 - \frac{\alpha}{2} \qquad (77)$$

We solve model (77) by Matlab at $\alpha = 0,0.5,1$. At α-level = 0, the solution graphs are figure (4.2.10), figure (4.2.11) and figure (4.2.12):

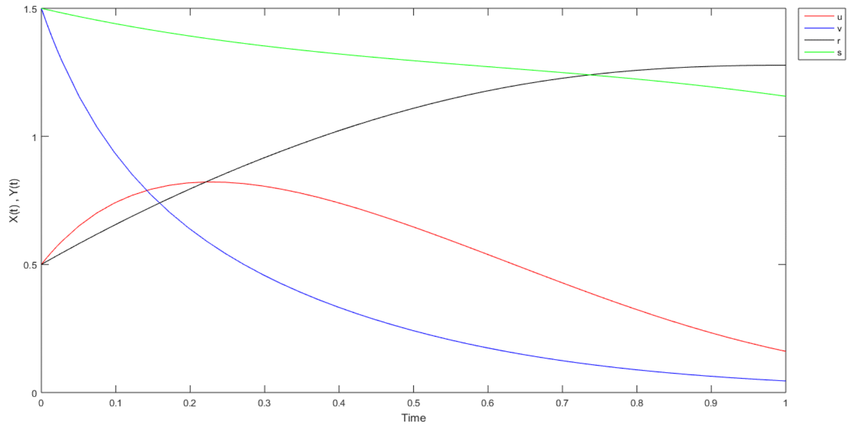
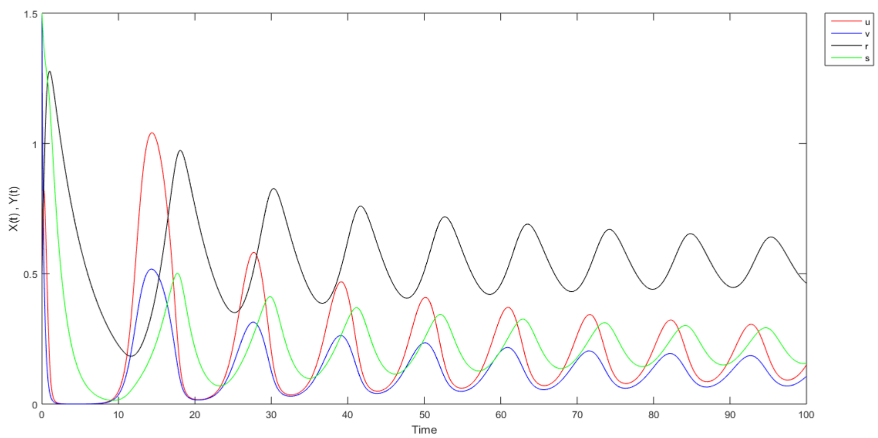
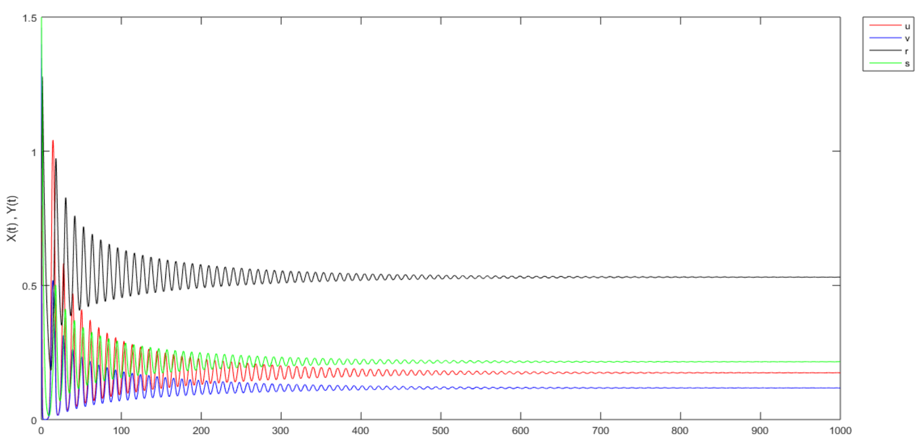


Figure (4.2.10): The solution of (77) at α=0 for short time period





Figures (4.2.11) and (4.2.12): The solution of (77) at α=0 as time increases

At α-level = 0 , the solution is unstable since $as\ t \to \infty$, $u(t) \to 0.1744$, $v(t) \to 0.1179$, $r(t) \to 0.5308$ and $s(t) \to 0.2155$.

At α-level = 0.5, the solution graphs are figure (4.2.13) and figure (4.2.14) in the appendix. At α-level = 1, the solution is figure (4.2.15):

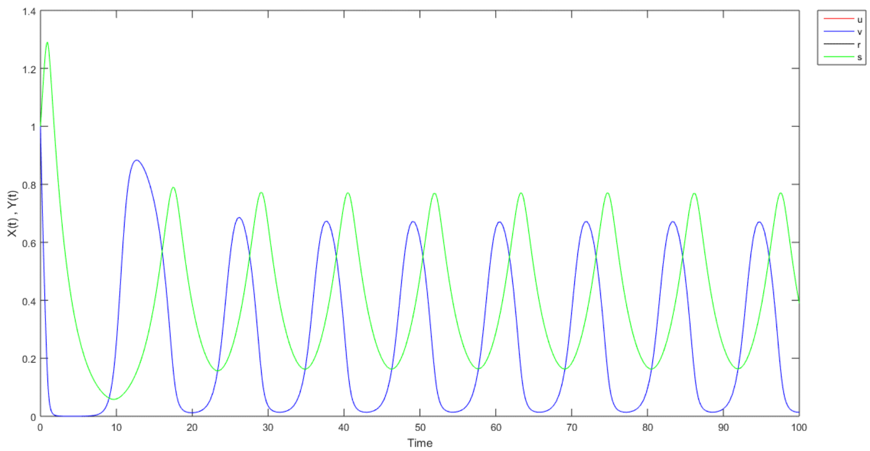


Figure (4.2.15): The solution of (77) at α=1

Now, we want to fuzzify the parameters of the model (67) using triangular fuzzy numbers with small support. As follow:

let $r = (1.9995,2,2.0005)$ with $[r]_\alpha = \left[1.9995 + \frac{\alpha}{2000}, 2.0005 - \frac{\alpha}{2000}\right]$,

$a = (4.9995,5,5.0005)$ with $[a]_\alpha = \left[4.9995 + \frac{\alpha}{2000}, 5.0005 - \frac{\alpha}{2000}\right]$,

$D = (0.3995,0.4,0.4005)$ with $[D]_\alpha = \left[0.3995 + \frac{\alpha}{2000}, 0.4005 - \frac{\alpha}{20000}\right]$,

$s = (0.5995,0.6,0.6005)$ with $[s]_\alpha = \left[0.5995 + \frac{\alpha}{2000}, 0.6005 - \frac{\alpha}{2000}\right]$,

$x_0 = (0.9995,1,1.0005)$ with $[x_0]_\alpha = \left[0.9995 + \frac{\alpha}{2000}, 1.0005 - \frac{\alpha}{2000}\right]$,

$y_0 = (0.9995,1,1.0005)$ with $[y_0]_\alpha = \left[0.9995 + \frac{\alpha}{2000}, 1.0005 - \frac{\alpha}{2000}\right]$,

Then we have the following model:

$$X'(t) = (1.9995,2,2.0005)X(1 - X) - Y\, tan^{-1}\big((4.9995,5,5.0005)X\big)$$

$$Y'(t) = -(0.3995,0.4,0.4005)\ Y + (0.5995,0.6,0.6005)\ Y\, tan^{-1}(5X)$$

$$x_0 = (0.9995\ ,1\ ,1.0005)\ and\ y_0 = (0.9995\ ,1\ ,1.0005) \qquad (78)$$

We solves model (78) when $X(t)$ and $Y(t)$ are (2)-differentiable, then it becomes as follow:

$$u' = (2.0005 - \frac{\alpha}{2000})v - (1.9995 + \frac{\alpha}{2000})u^2 - r\,tan^{-1}\left((4.9995 + \frac{\alpha}{2000})\,u\right)$$

$$v' = (1.9995 + \frac{\alpha}{2000})u - (2.0005 - \frac{\alpha}{2000})v^2 - s\,tan^{-1}((5.0005 - \frac{\alpha}{2000})v)$$

$$r' = -(0.3995 + \frac{\alpha}{2000})r + (0.6005 - \frac{\alpha}{2000})s\,tan^{-1}((5.0005 - \frac{\alpha}{2000})v)$$

$$s' = -(0.4005 - \frac{\alpha}{2000})s + (0.5995 + \frac{\alpha}{2000})r\,tan^{-1}((4.9995 + \frac{\alpha}{2000})u)$$

$$u_0 = r_0 = 0.9995 + \frac{\alpha}{2000} \quad and \quad v_0 = s_0 = 1.0005 - \frac{\alpha}{2000} \tag{79}$$

We solve (79) by Runge-Kutta method in Matlab at α-levels= 0. The solution graphs are figure (4.2.16) and figure (4.2.17):

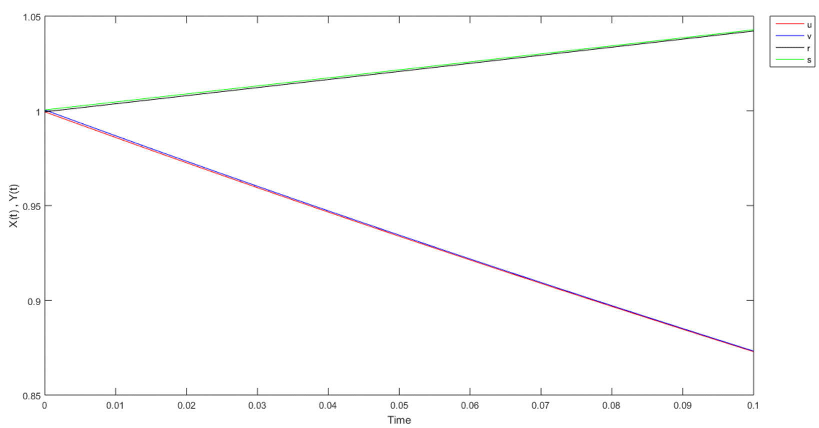


Figure (4.2.16): The solution of (79) at α=0 for short time period

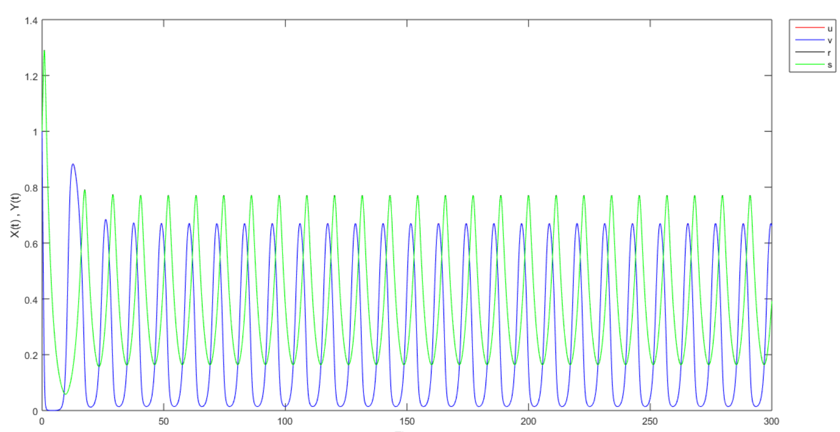


Figure (4.2.17): The solution of (79) at α=0 as time increases

When we make the parameters of (67) triangular fuzzy numbers and for $\alpha < 1$ we obtain unacceptable solution when $X(t)$ and $Y(t)$ are (1,1), (1,2) and (2,1)-differentiable. While, when $X(t)$ and $Y(t)$ are (2)-differentiable, the solution is unstable at $\alpha = 0$ but it becomes periodic as α increases for $\alpha < 1$ with $u(t) > v(t)\ and\ r(t) > s(t)$. So, there are no fuzzy solution for $X(t)$ and $Y(t)$. However, at $\alpha = 1$ the solution is equivalent to the crisp case for all derivatives forms of $X(t)$ and $Y(t)$. Then we use triangular fuzzy numbers of small supports to fuzzify the parameters and the initial conditions. Thereafter, we find the solution when $X(t)$ and $Y(t)$ are (2)-differentiable at $\alpha = 0$, then we obtain periodic and stable solution. Therefore, as $t \to \infty,\ r(t) > s(t)$ so there is no fuzzy solution for $Y(t)$.

4.3: summery

In this chapter, we created a new numerical model of predator-prey model with a functional response of the form $arctan(ax)$ and presented the solutions numerically and graphically. Then we converted the initial conditions to fuzzy numbers using triangular fuzzy numbers and triangular fuzzy numbers of small support. Thereafter, we explored a new fuzzy model with functional response $arctan$(x) with fuzzy parameters and initial conditions compared this model with another one of triangular fuzzy numbers with small supports.

Chapter 5
Conclusions and Comments

We covered the topic of predator prey model and solved it numerically using Runge-Kutta method and got periodic solutions and stable equilibrium points. As vagueness appears in problems which are analyzed, it is natural to use fuzzy differential equations. Therefore, using fuzzy sets is more realistic than the classical one. From the simulations and graphs of the solutions, we noted that the fuzzy solution is not always better than the crisp solution because the cases of derivatives of the forms (1,1), (1,2), (2,1) gave solutions that are incompatible with biological facts, while solutions obtained with (2,2) derivatives are biologically meaningful. In section 3.1 we had different initial populations of prey and predator using different cases of fuzzy numbers. We got different results at each time with derivatives of the form (2,2) but the solution with triangular and triangular shaped fuzzy numbers was better than the trapezoidal fuzzy numbers and as the initial populations of the prey and predator were closer to each other, the solution was better, that is the lower and upper bounds were equal and positive.

When we fuzzify the parameters of predator-prey model, in some cases we didn't get fuzzy solution, but these solutions were biologically acceptable only with derivatives form (2,2). However, the triangular and triangular shaped fuzzy numbers produced better solutions than the trapezoidal fuzzy numbers. Furthermore, as the endpoints of fuzzy numbers were closer to the core, the solution was closer to the crisp case and the equilibrium points were stable.

For the predator prey model with functional response $arctan(ax)$, we considered a numerical model that satisfies the existence condition then the solution was periodic as $t \to \infty$ and the equilibrium points were stable. When we converted the initial conditions to triangular fuzzy numbers we obtained the same results; that is, derivatives of the form (1,1), (1,2), (2,1) gave biologically unacceptable solutions but derivatives of the form (2,2) gave periodic solution and it was better with smaller supports of triangular fuzzy numbers. While, when we explored fuzzy model with fuzzy parameters, we didn't obtain a good solution and it wasn't acceptable with fuzzy logic.

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Appendix

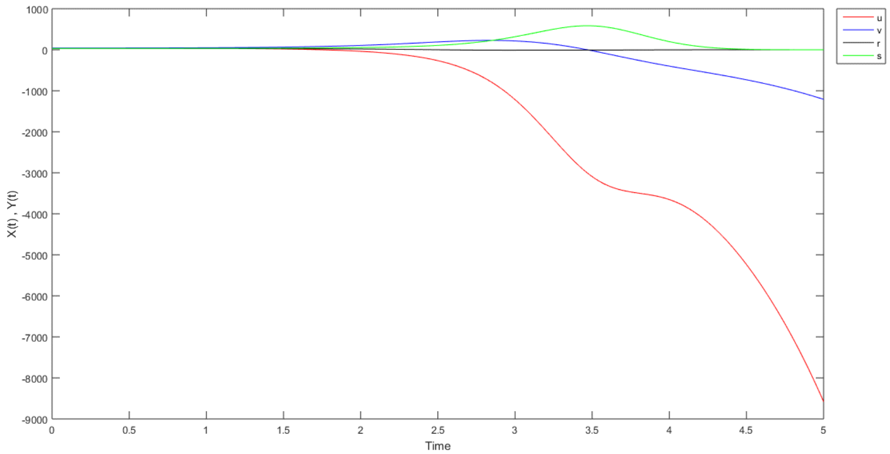


Figure (3.1.20): The solution of (11) at α=0.5

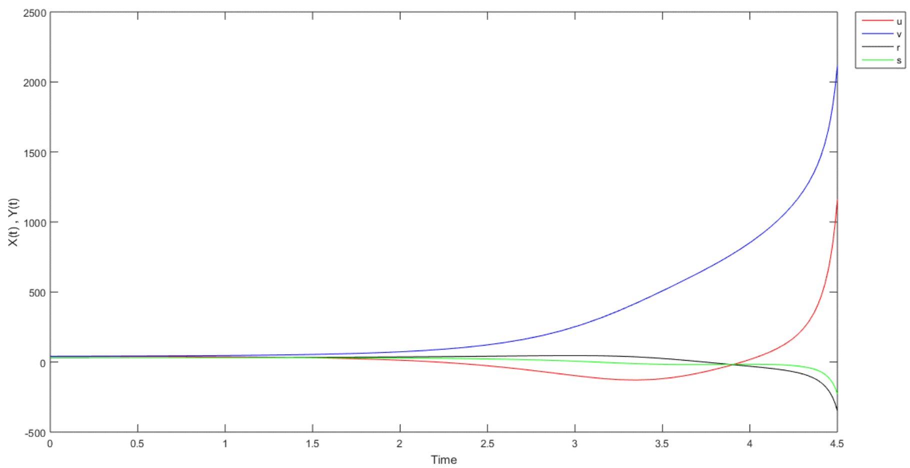


Figure (3.1.23): The solution of (12) at α=0.5

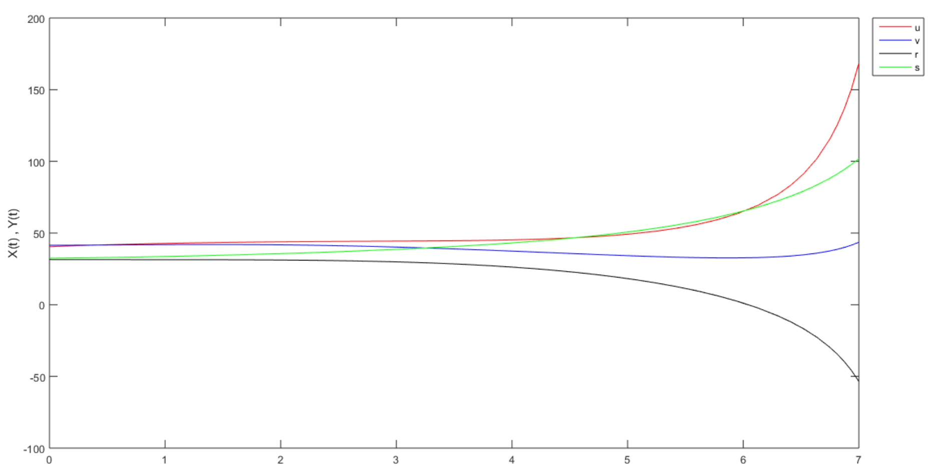


Figure (3.1.26): The solution of (13) at α=0.5

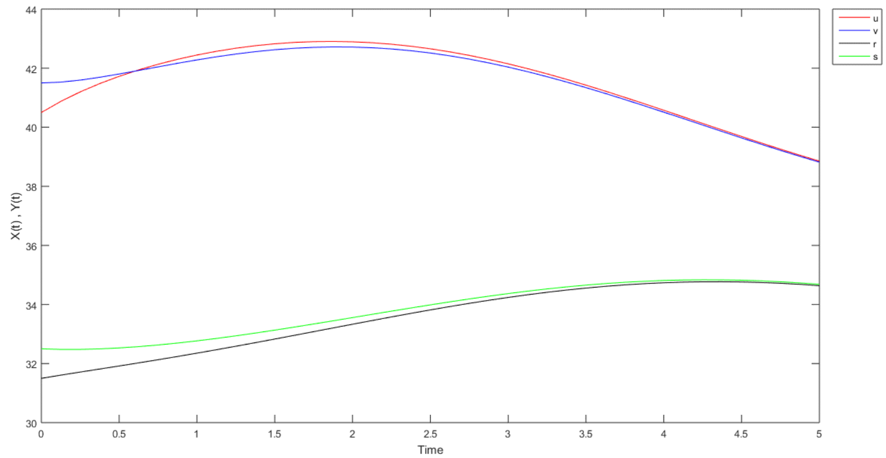


Figure (3.1.30): The solution of (14) at α=0.5 for short time period

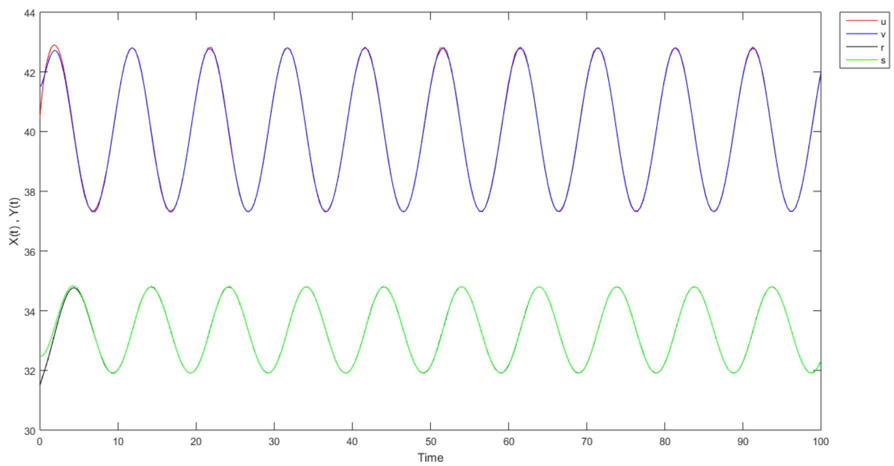


Figure (3.1.31): The solution of (14) at α=0.5 as time increases

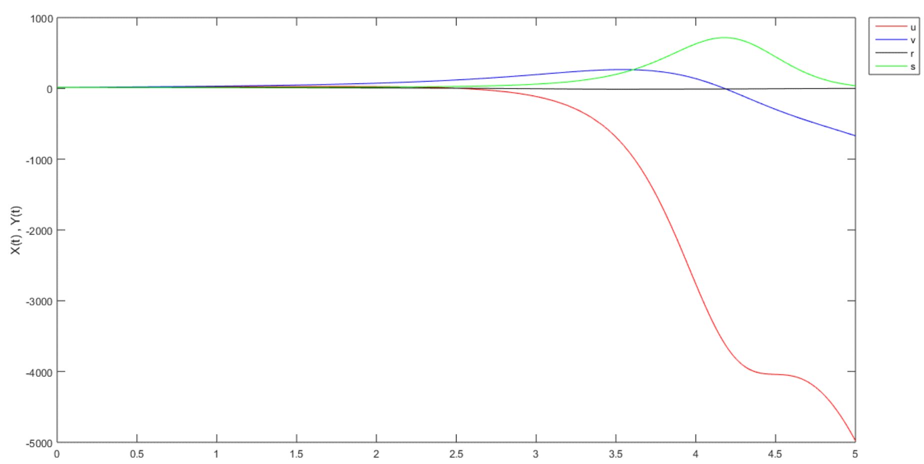


Figure (3.1.34): The solution of (15) at α=0.5

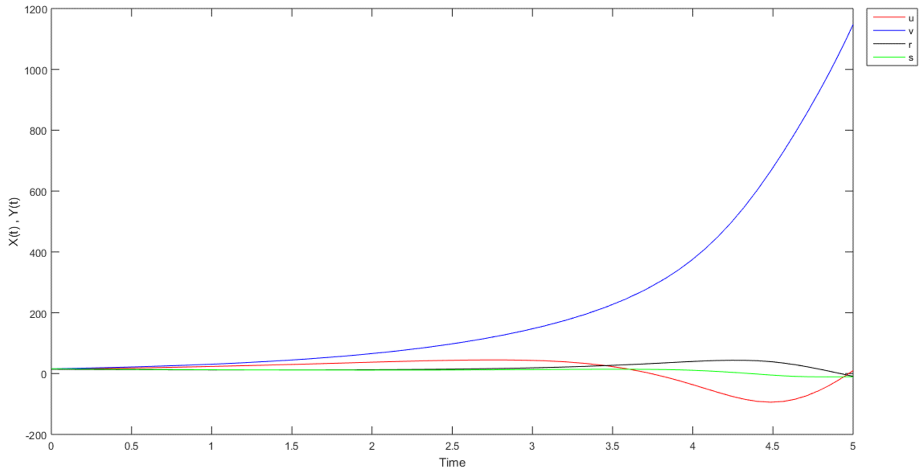


Figure (3.1.37): The solution of (16) at α=0.5

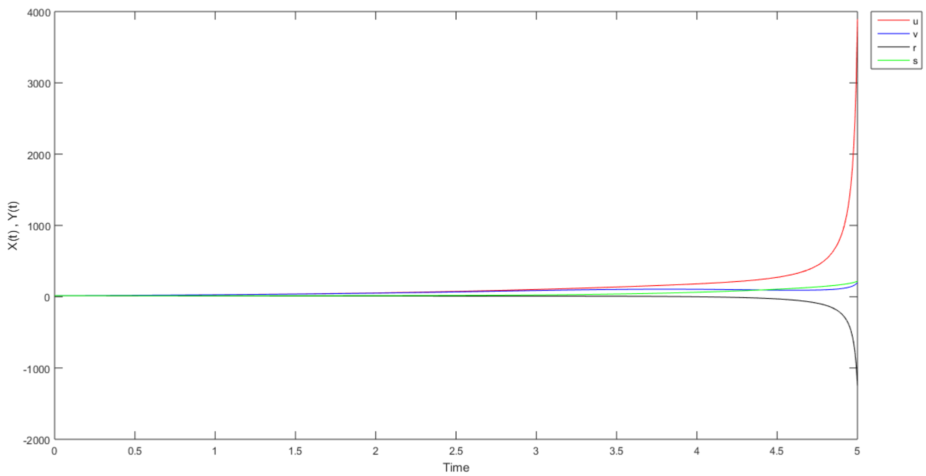


Figure (3.1.40): The solution of (17) at α=0.5

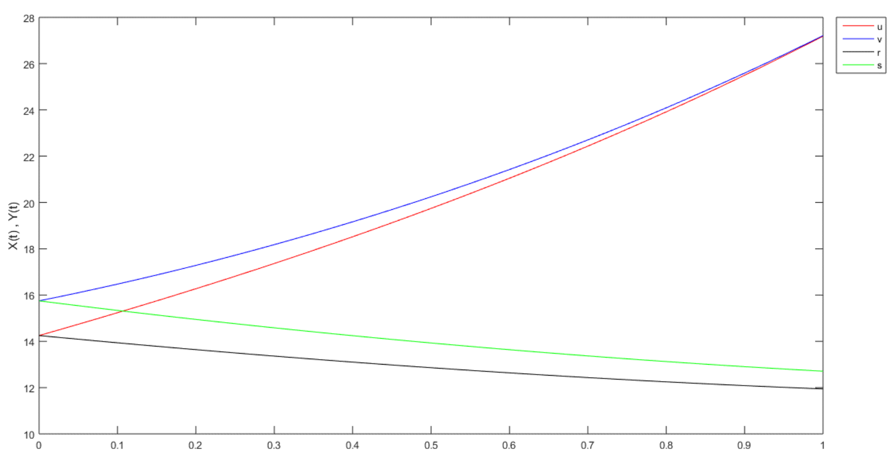


Figure (3.1.44): The solution of (18) at α=0.5 for short time period

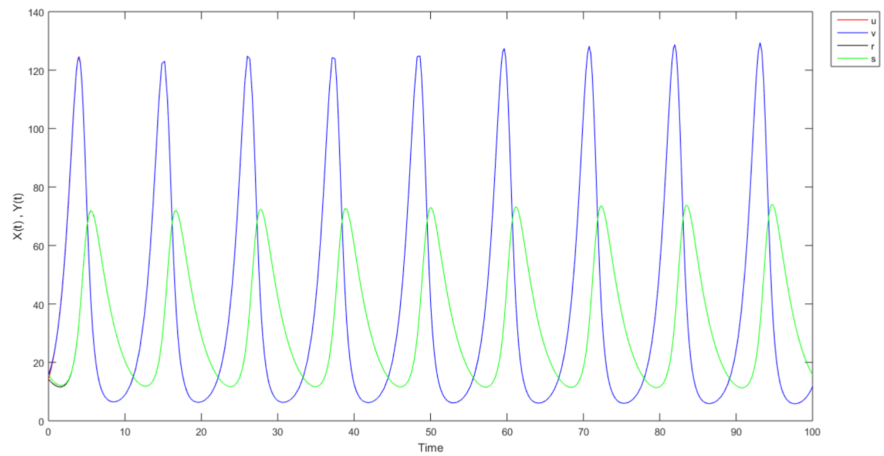


Figure (3.1.45): The solution of (18) at α=0.5 as time increases

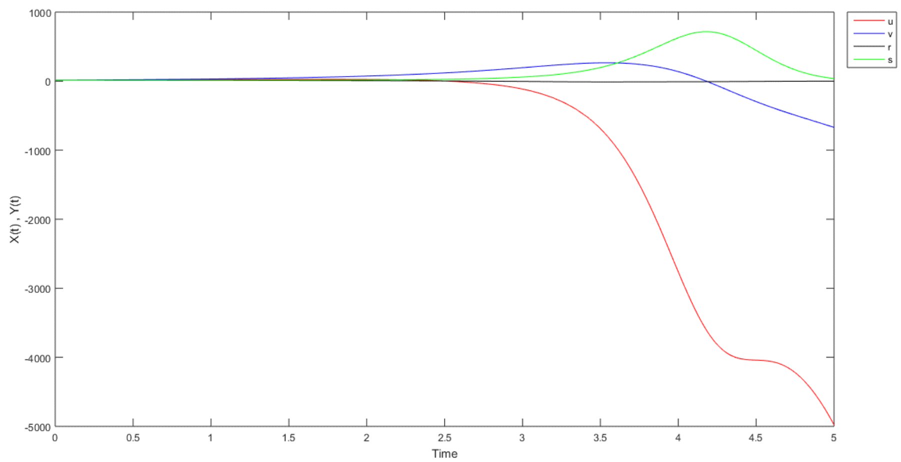


Figure (3.1.48): The solution of (19) at α=0.5

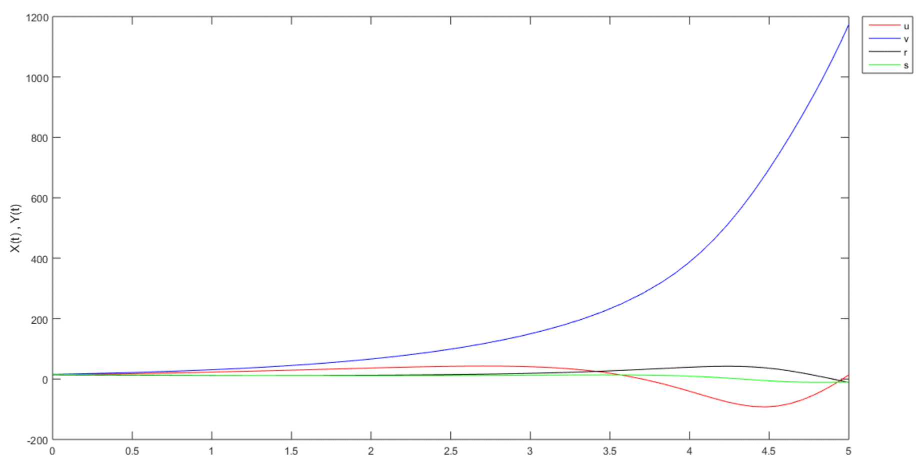


Figure (3.1.51): The solution of (20) at α=0.5

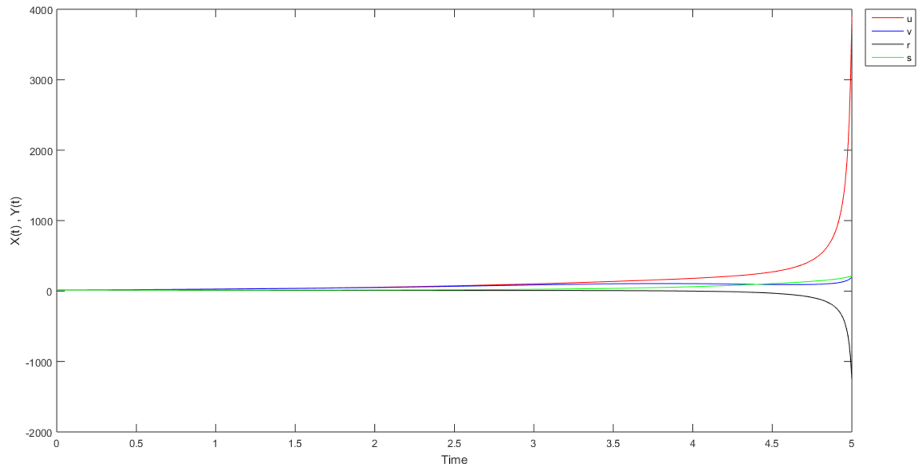


Figure (3.1.54): The solution of (21) at α=0.5

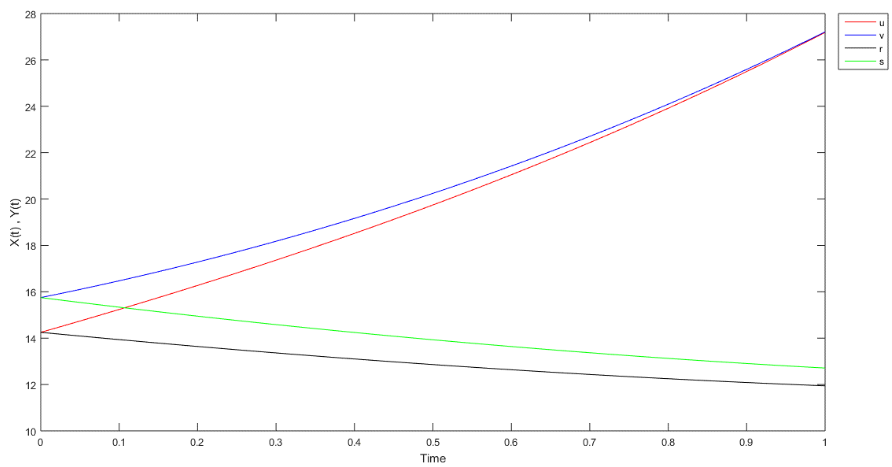


Figure (3.1.58): The solution of (22) at α=0.5 for short time period

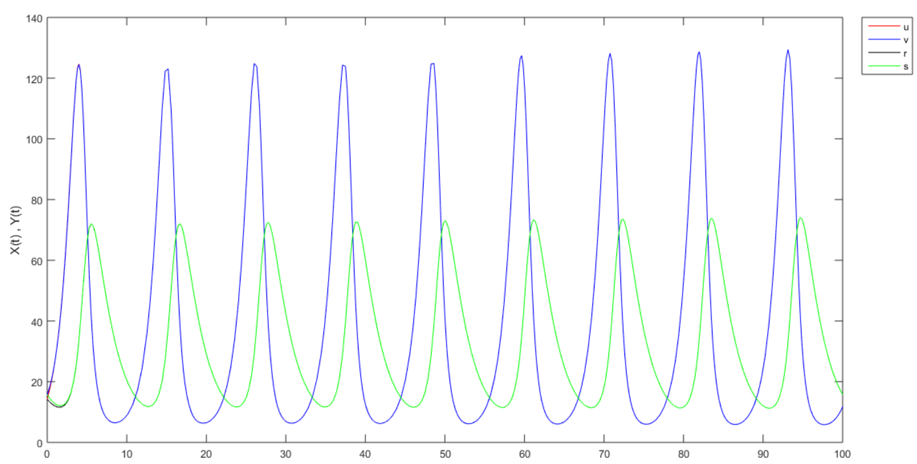


Figure (3.1.59): The solution of (22) at α=0.5 as time increases

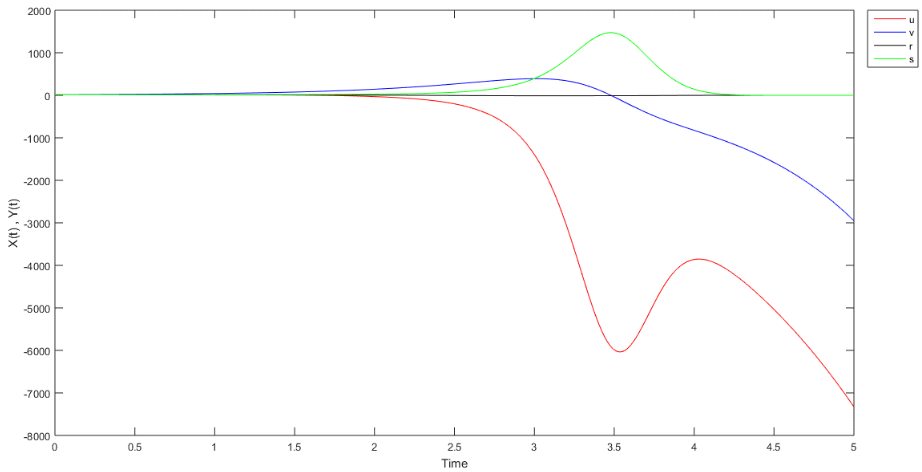


Figure (3.2.2): The solution of (23) at α=0.5

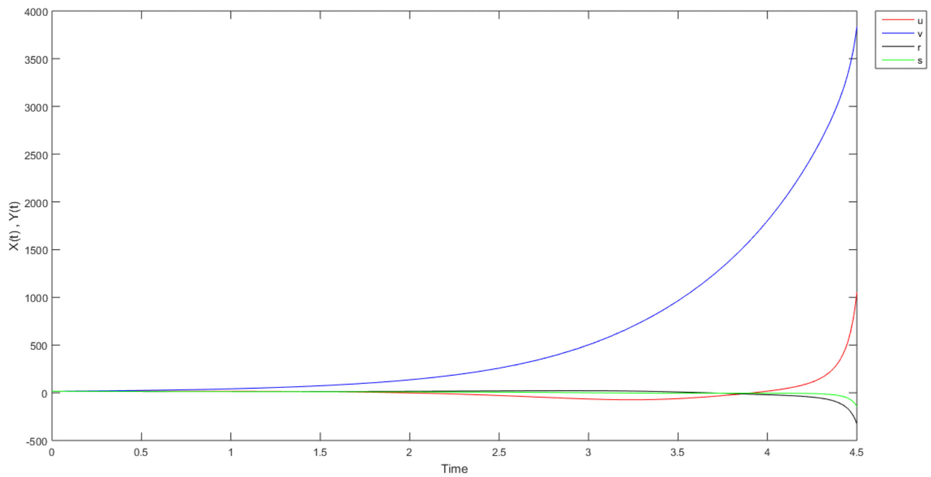


Figure (3.2.5): The solution of (24) at α=0.5

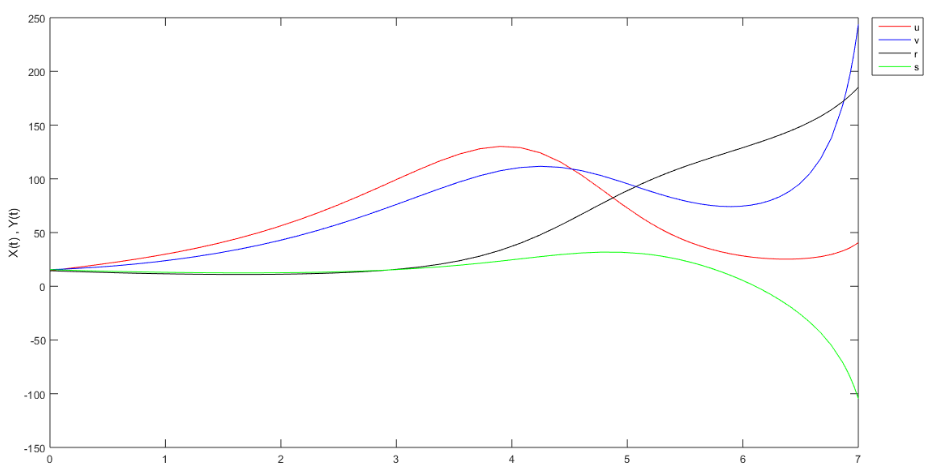


Figure (3.2.8): The solution of (25) at α=0.5

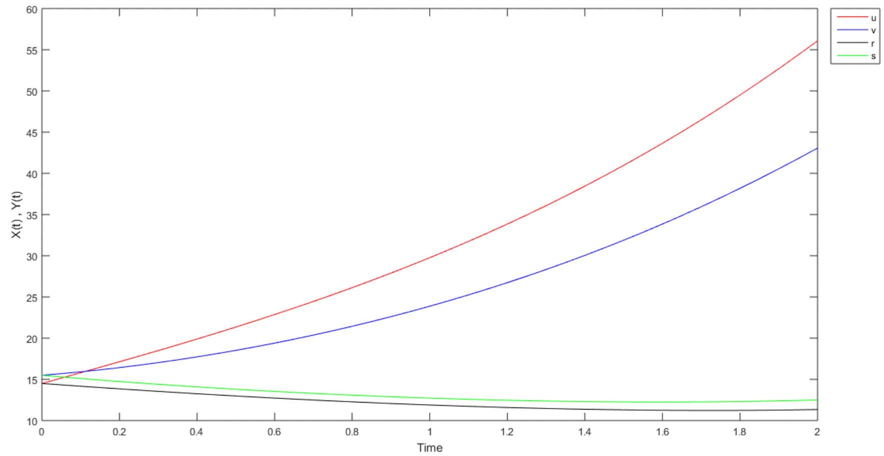
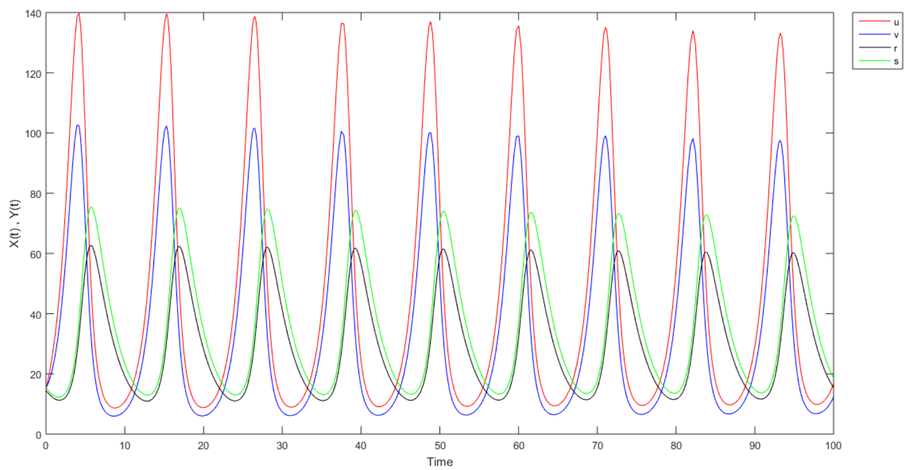
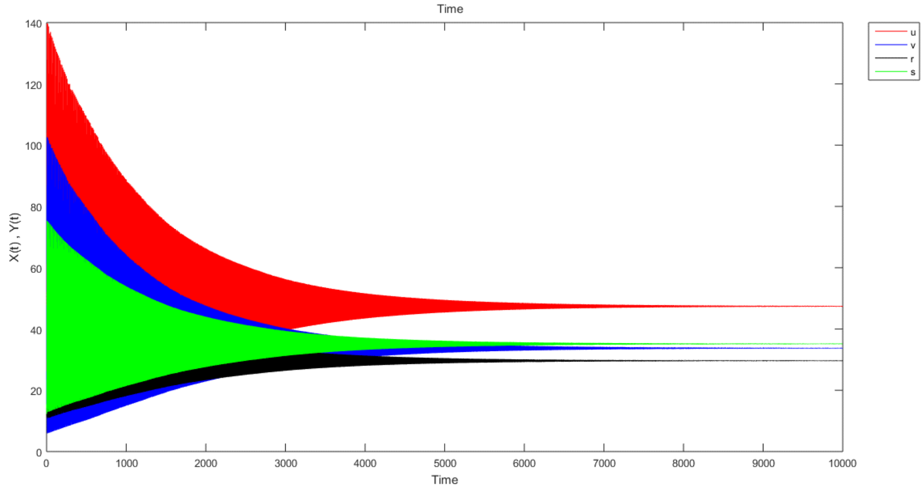


Figure (3.2.13): The solution of (26) at α=0.5 for short time period





Figures (3.2.14) and (3.2.15): The solution of (26) at α=0.5 as time increases

At α-level = 0.5, $as\ t \to \infty$, $u \to 47.42$, $v \to 33.74$, $r \to 29.64$, $s \to 35.14$. Therefore, the solution of $y(t)$ is asymptotically stable but there is no fuzzy solution for $x(t)$.

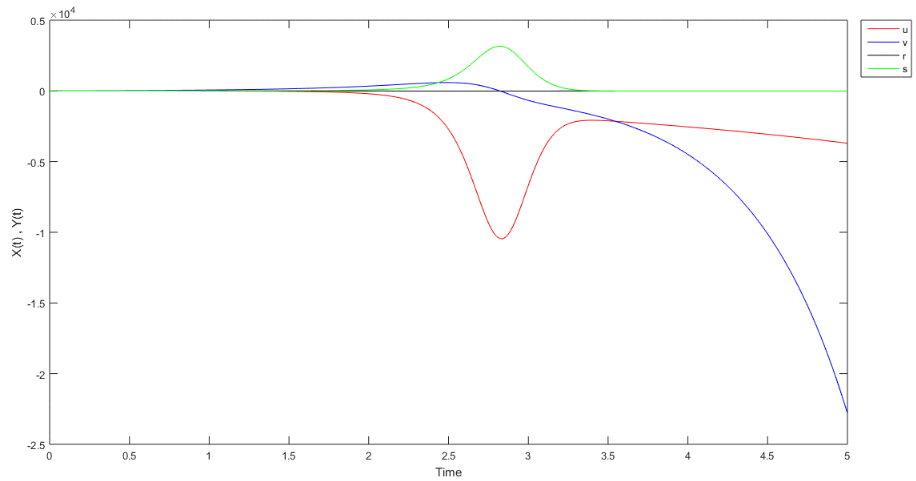


Figure (3.2.18): The solution of (27) at α=0.5

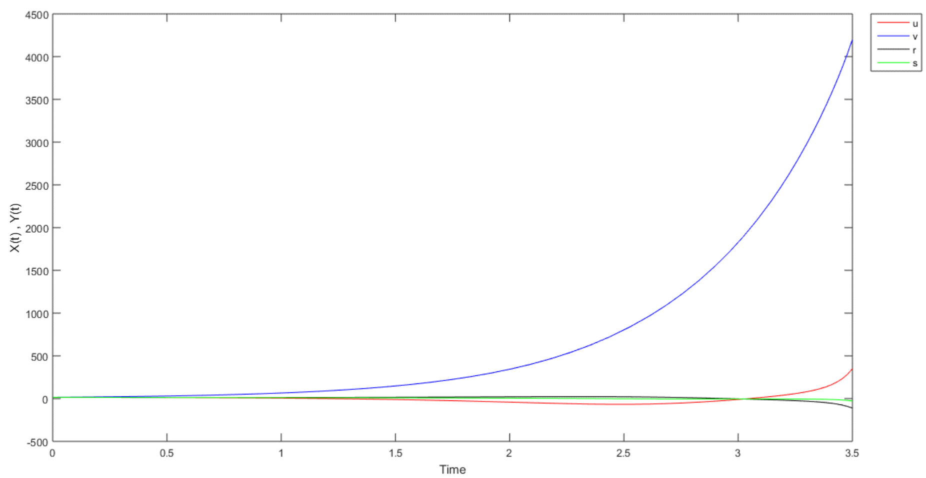


Figure (3.2.21): The solution of (28) at α=0.5

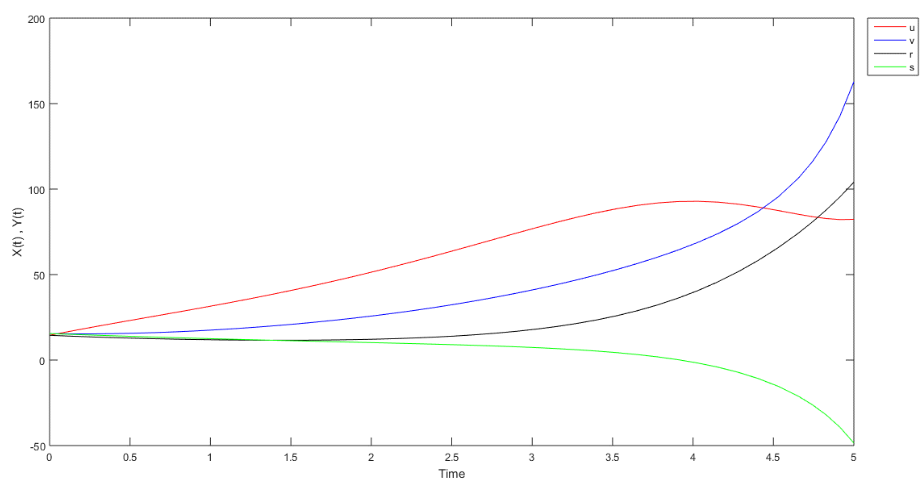


Figure (3.2.24): The solution of (29) at α=0.5

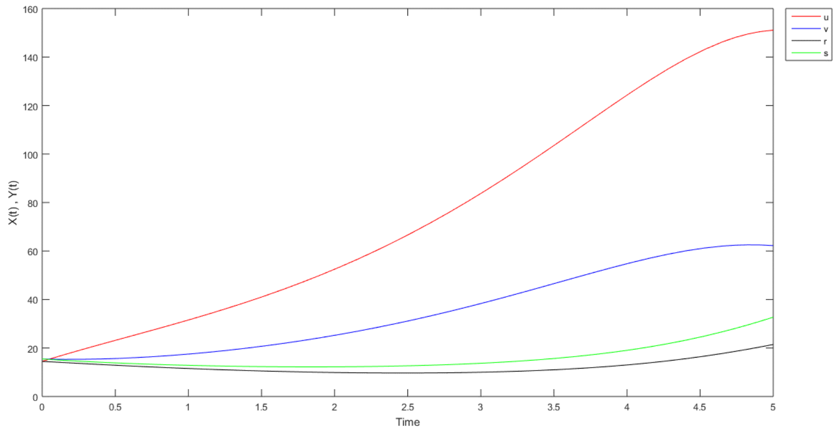
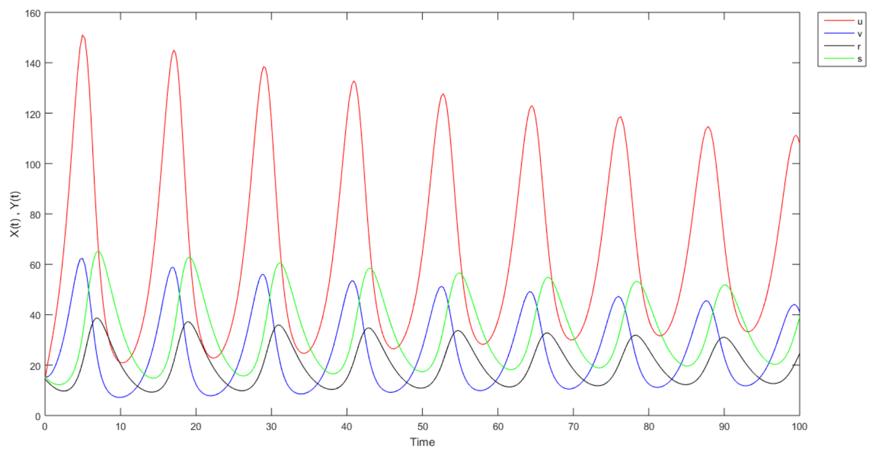
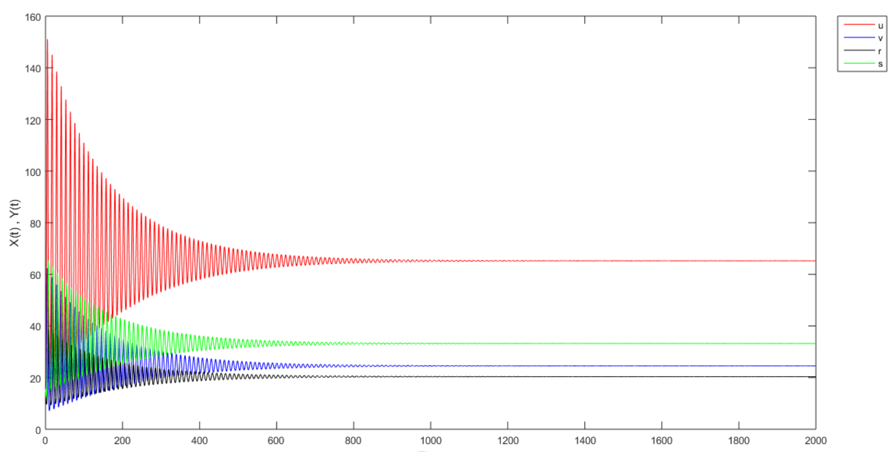


Figure (3.2.29): The solution of (30) at α=0.5 for short time period





Figures (3.2.30) and (3.2.31): The solution of (30) at α=0.5 as time increases

At α-level = 0.5, $as\ t \to \infty$, $u(t) \to 65.21$, $v(t) \to 24.53$, $r(t) \to 20.38$, $s(t) \to 33.22$. So the solution is asymptotically stable.

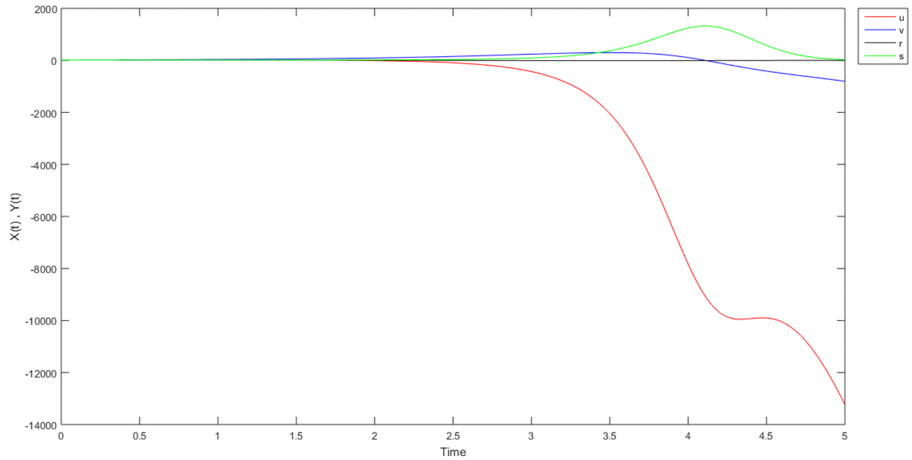


Figure (3.2.38): The solution of (35) at α=0.5

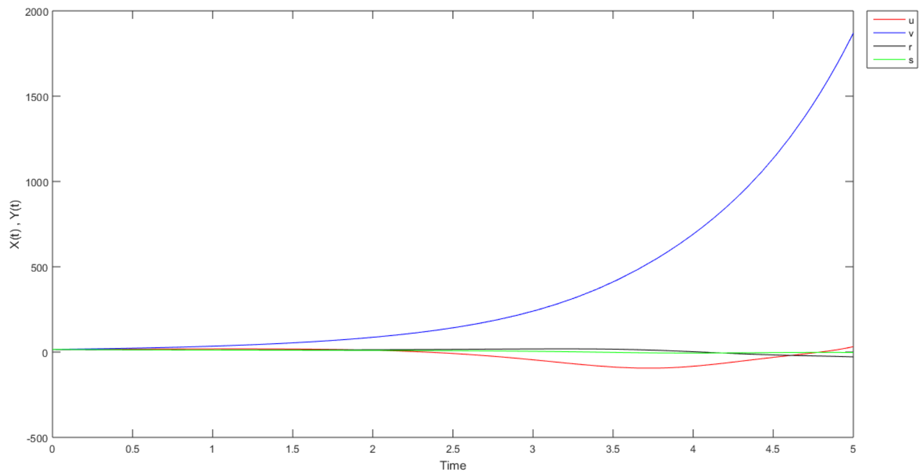


Figure (3.2.41): The solution of (36) at α=0.5

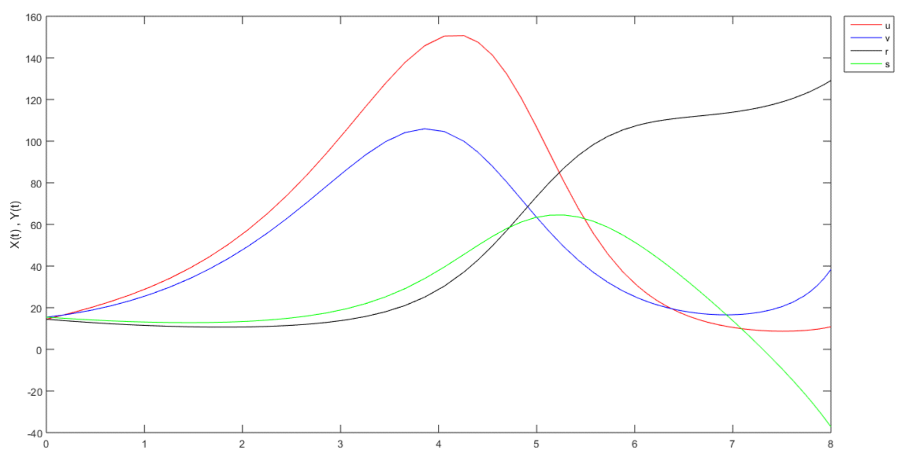


Figure (3.2.44): The solution of (37) at α=0.5

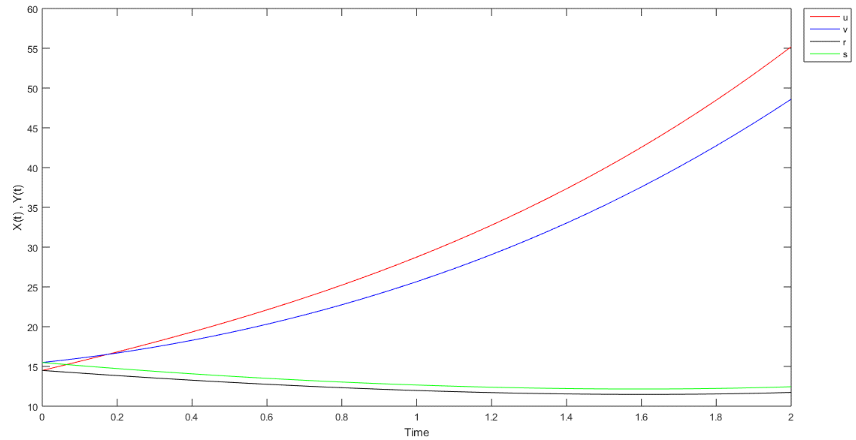
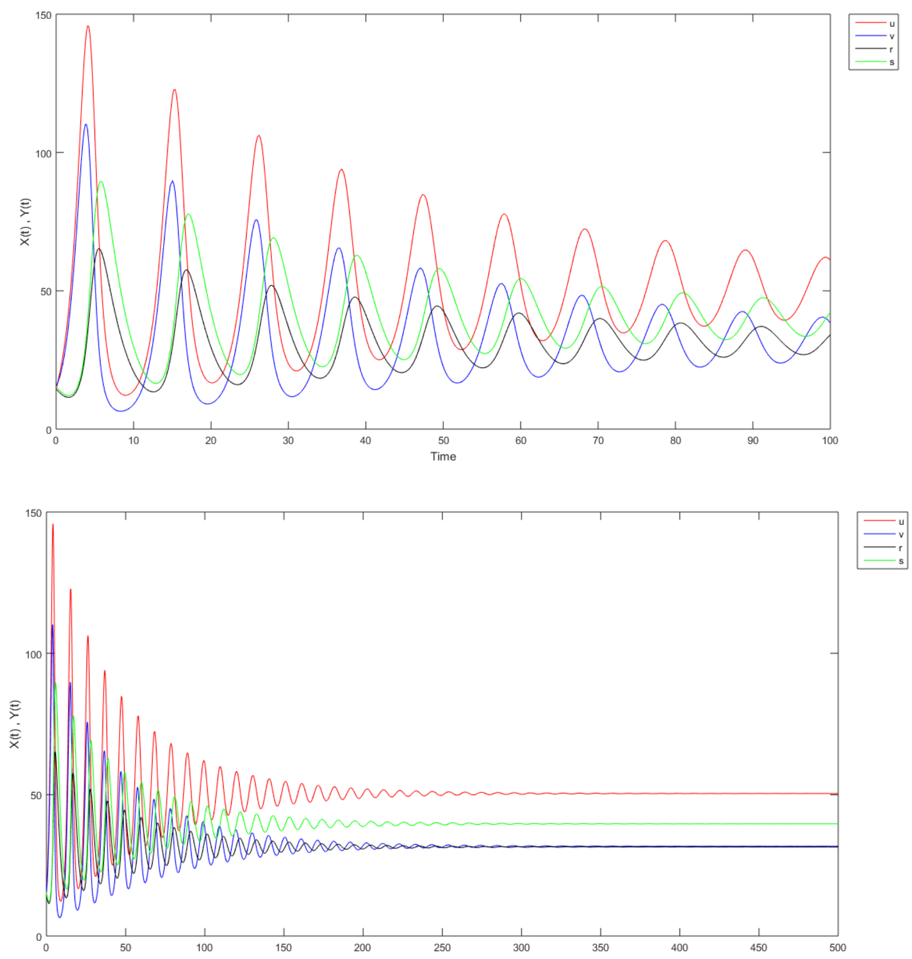


Figure (3.2.48): The solution of (38) at α=0.5 for short time period



Figures (3.2.49) and (3.2.50): The solution of (38) at α=0.5 as time increases

At α-level = 0.5, the solution is asymptotically stable for $Y(t)$ since $as\, t \to \infty$, $u(t) \to 50.40$, $v(t) \to 31.75$, $r(t) \to 31.50$, $s(t) \to 39.68$. As we see there is no fuzzy solution for $X(t)$.

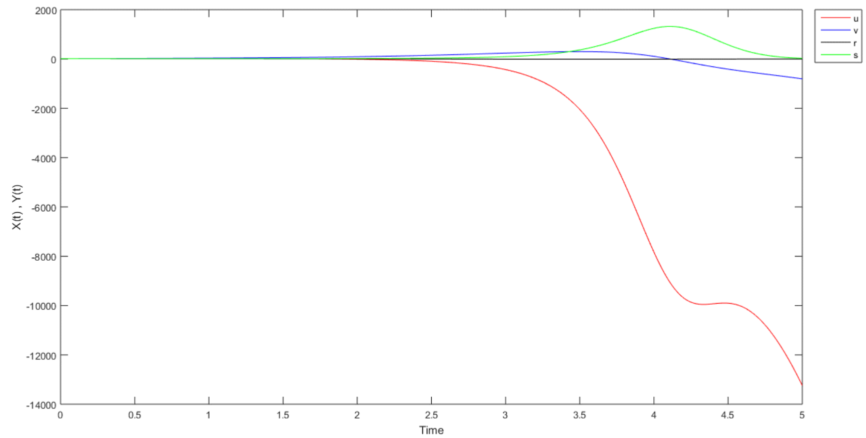


Figure (3.2.53): The solution of (39) at α=0.5

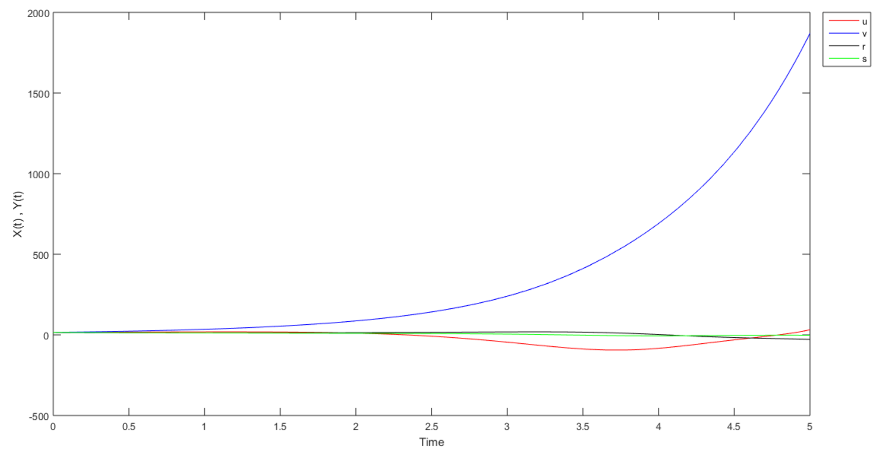


Figure (3.2.56): The solution of (40) at α=0.5

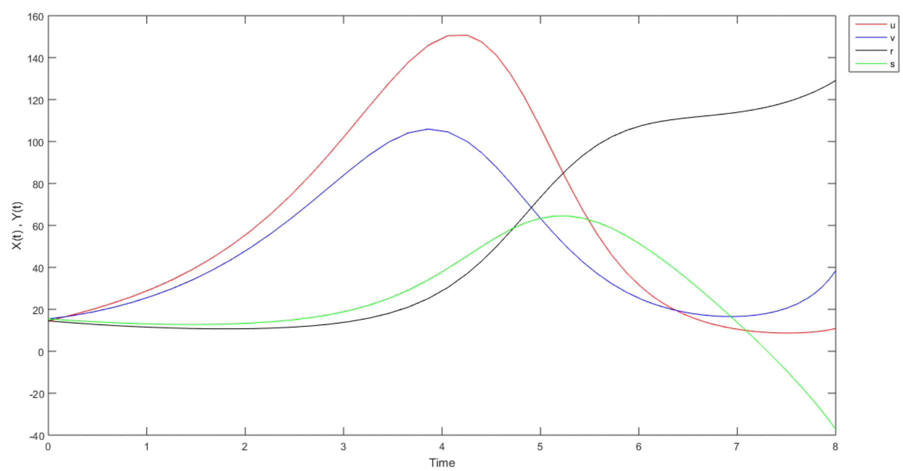


Figure (3.2.59): The solution of (41) at α=0.5

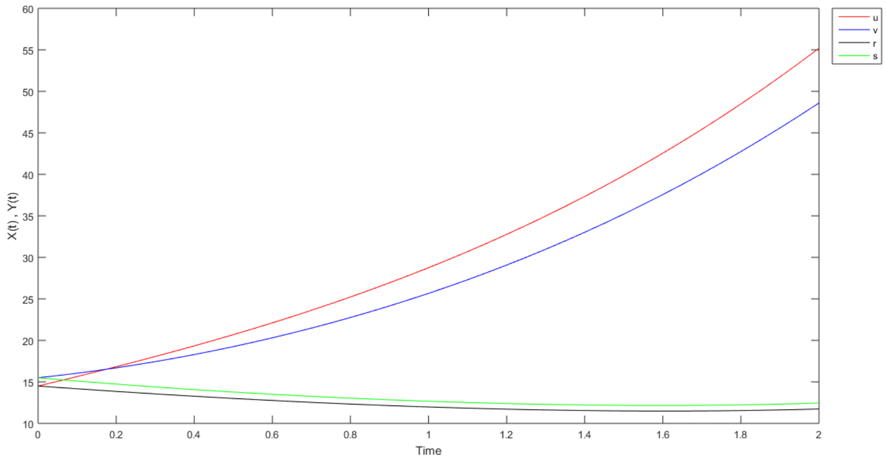
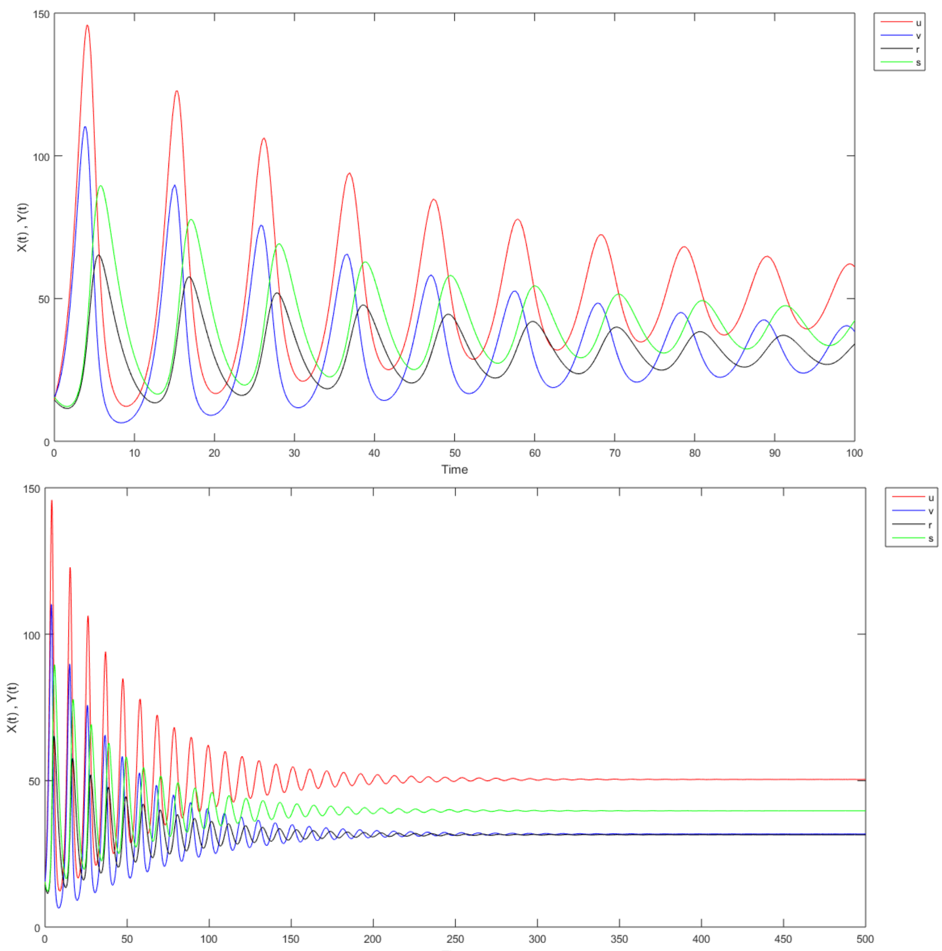


Figure (3.2.63): The solution of (42) at α=0.5 for short time period



Figures (3.2.64) and (3.2.65): The solution of (42) at α=0.5 as time increases

$as\ t \to \infty$, $u(t) \to 50.40$, $v(t) \to 31.76$, $r(t) \to 31.50$ and $s(t) \to 39.69$. So the solution is asymptotically stable but there is no fuzzy solution for $X(t)$.

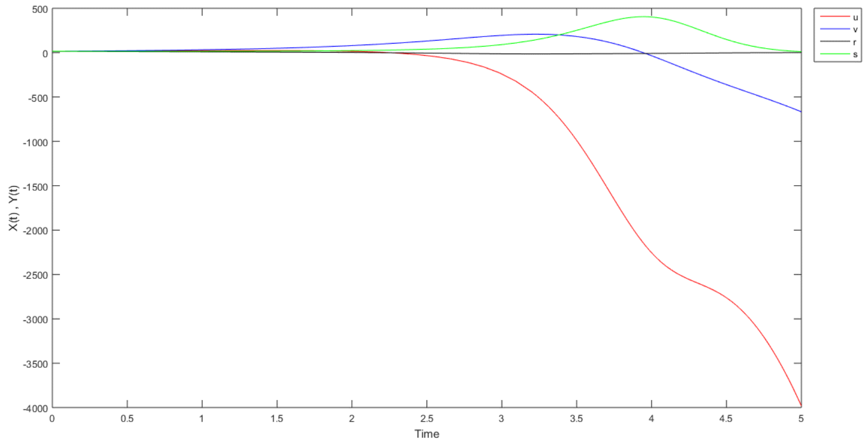


Figure (3.2.75): The solution of (45) at α=0.5

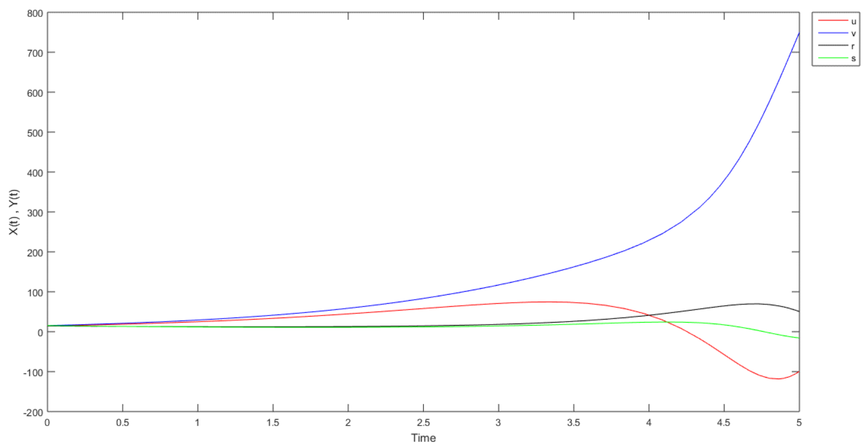


Figure (3.2.78): The solution of (46) at α=0.5

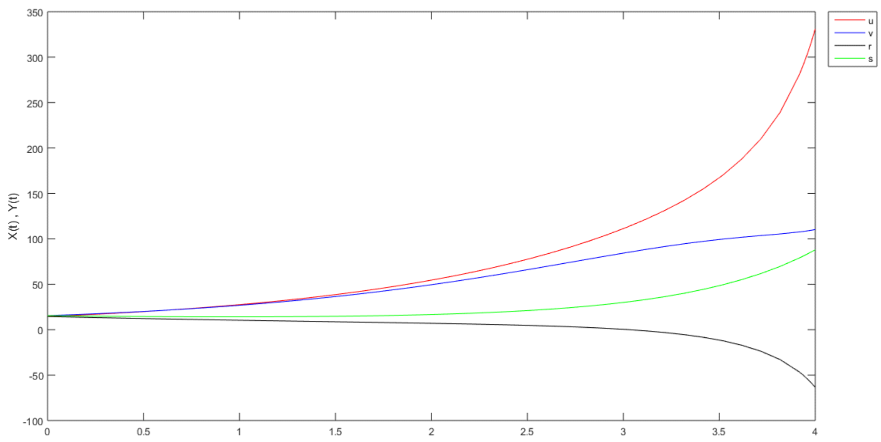


Figure (3.2.81): The solution of (47) at α=0.5

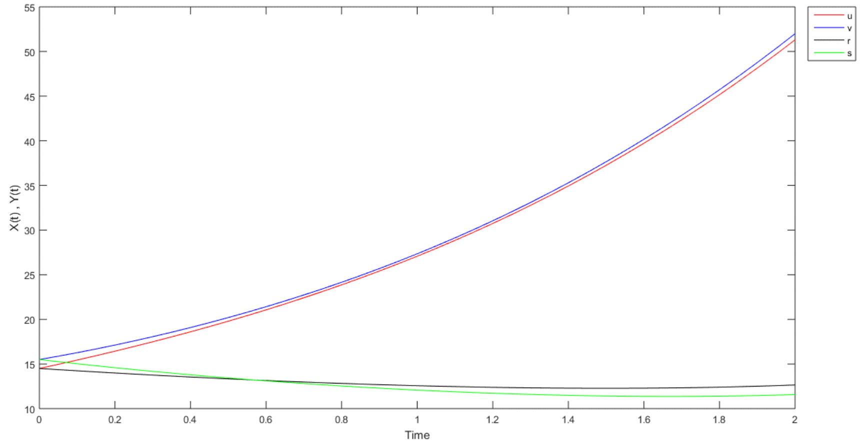
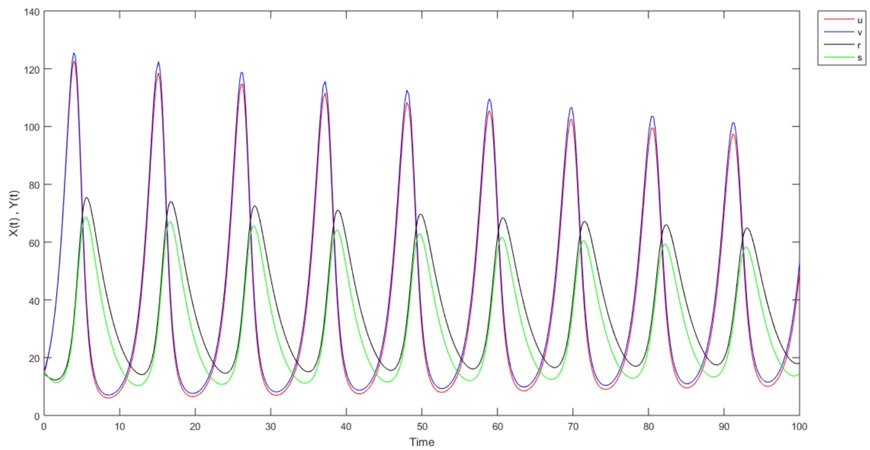
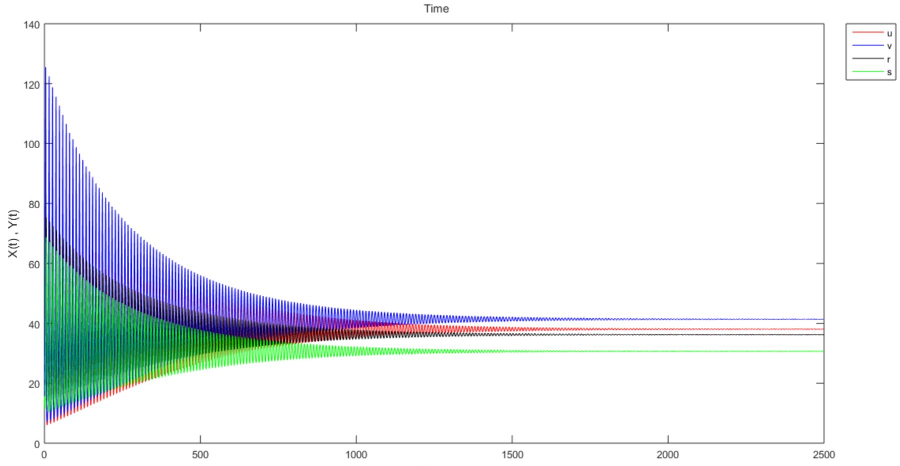


Figure (3.2.86): The solution of (48) at α=0.5 for short time period





Figures (3.2.87) and (3.2.88): The solution of (48) at α=0.5 as time increases

At α-level = 0.5, the solution is asymptotically stable since $as\, t \to \infty$, $u(t) \to 38.06$, $v(t) \to 41.38$, $r(t) \to 36.25$, $s(t) \to 30.66$. However, there is no fuzzy solution for $Y(t)$.

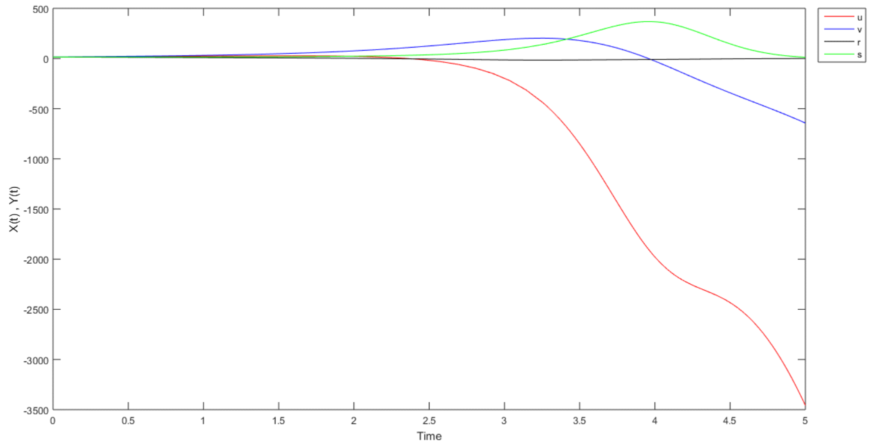


Figure (3.2.91): The solution of (49) at α=0.5

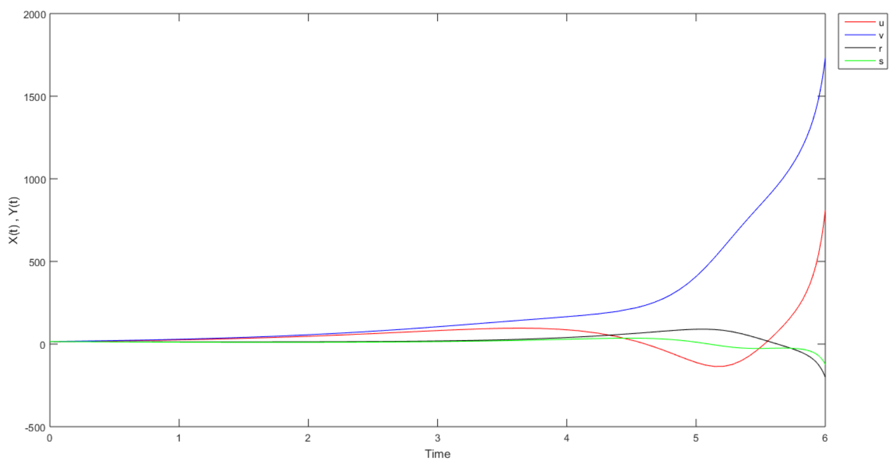


Figure (3.2.94): The solution of (50) at α=0.5

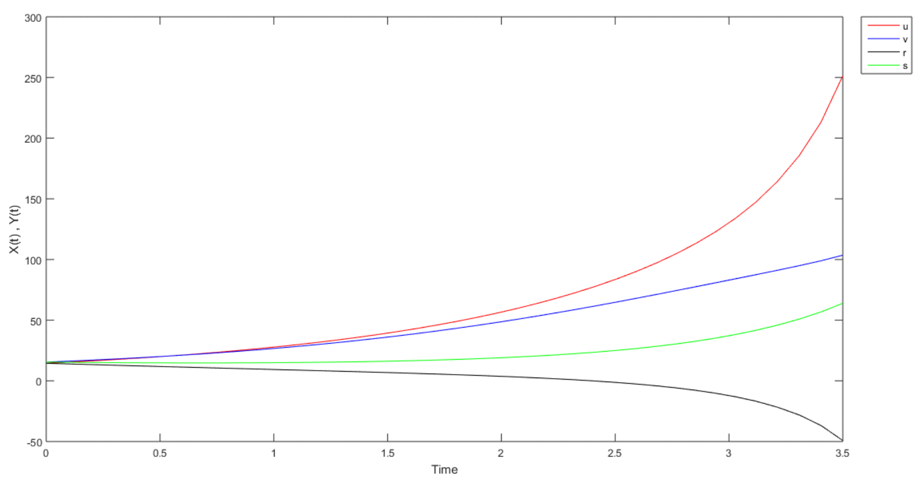


Figure (3.2.97): The solution of (51) at α=0.5

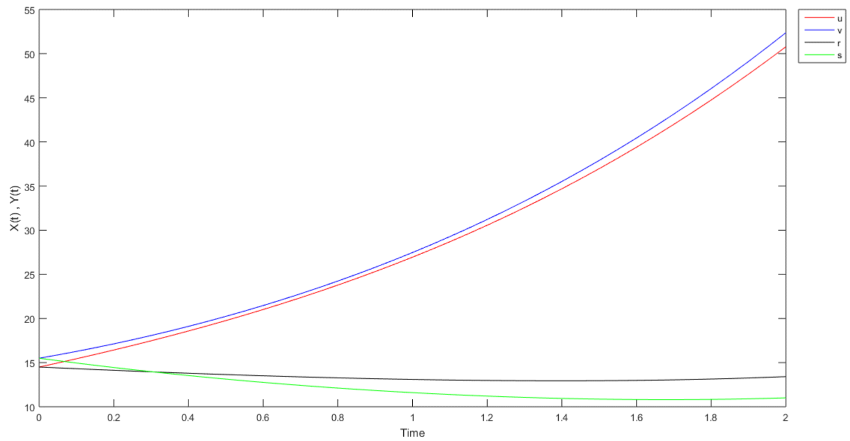
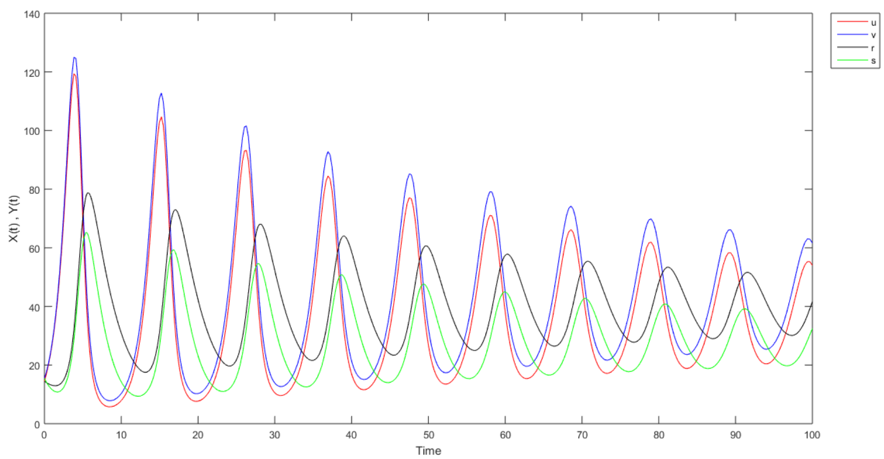
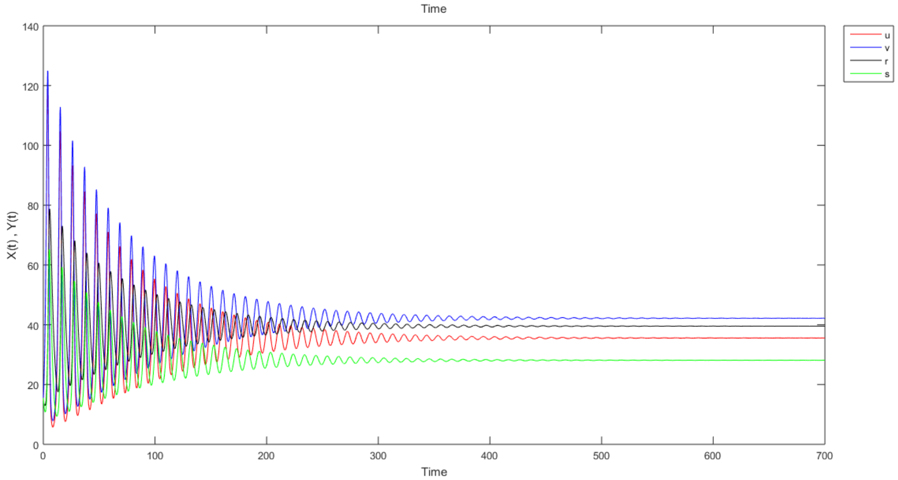


Figure (3.2.102): The solution of (52) at α=0.5 for short time period





Figures (3.2.103) and (3.2.104): The solution of (52) at α=0.5 as time increases

At α-level = 0.5, the solution is asymptotically stable and there is no fuzzy solution for $Y(t)$ since $as\ t \to \infty$, $u(t) \to 35.57$, $v(t) \to 42.17, r(t) \to 39.52\ and\ s(t) \to 28.11$.

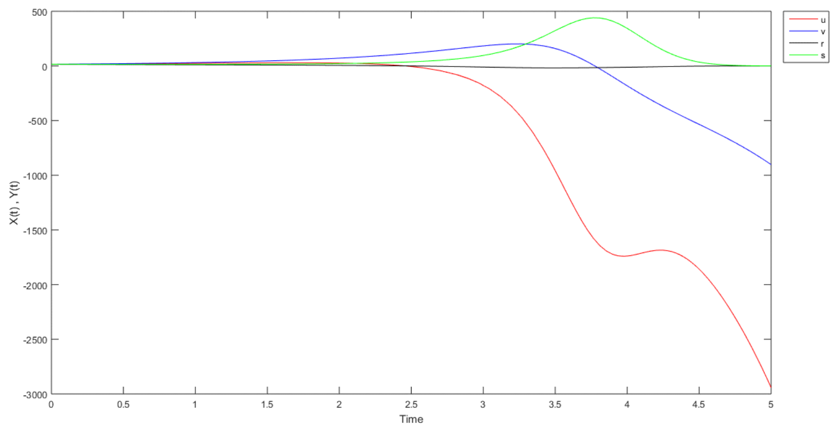


Figure (3.2.113): The solution of (53) at α=0.5

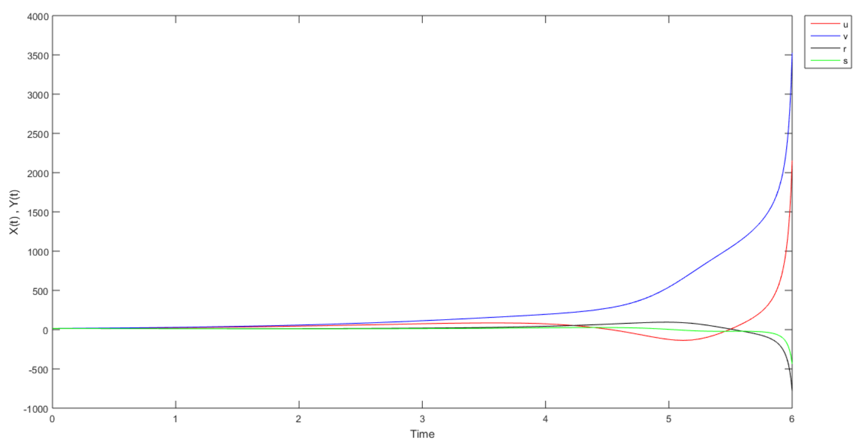


Figure (3.2.116): The solution of (54) at α=0.5

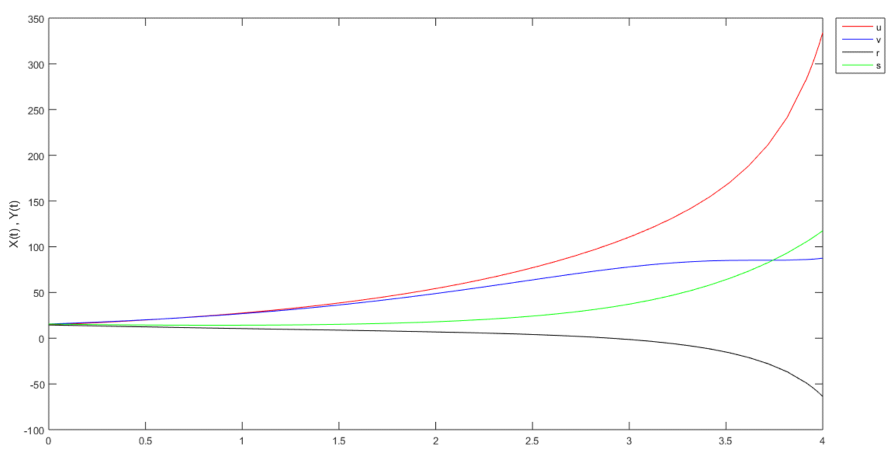


Figure (3.2.119): The solution of (55) at α=0.5

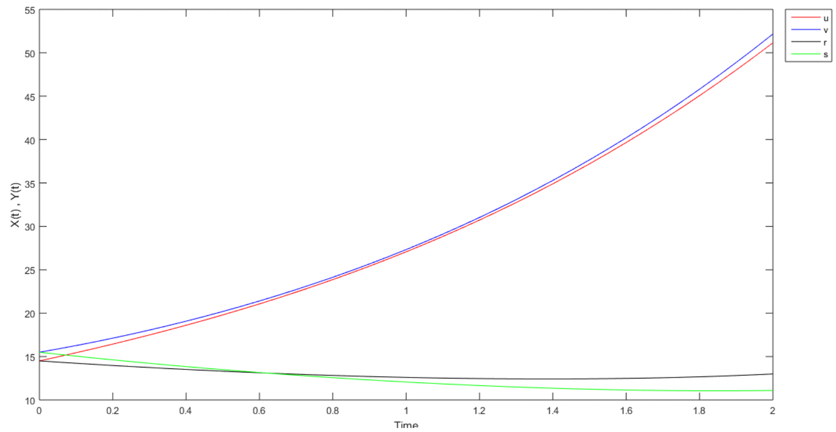
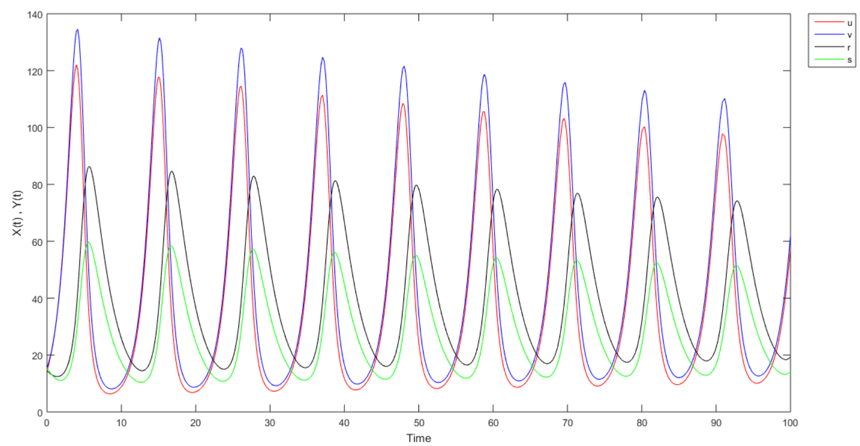
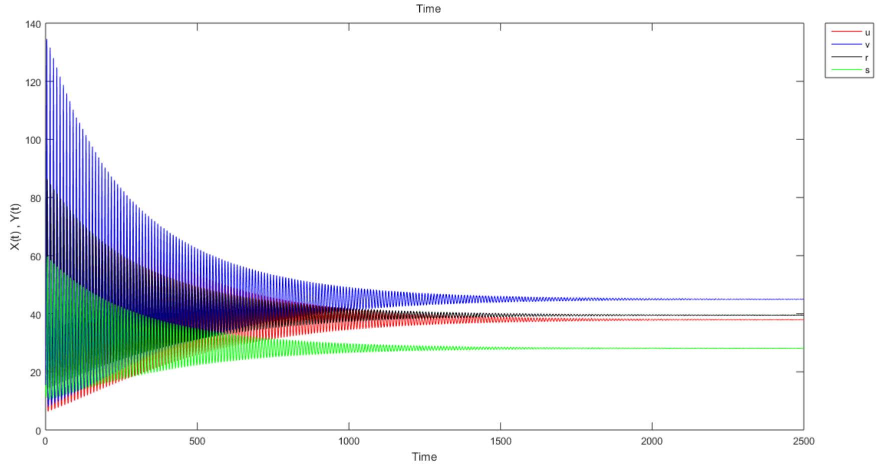


Figure (3.2.124): The solution of (56) at α=0.5 for short time period





Figures (3.2.125) and (3.2.126): The solution of (56) at α=0.5 as time increases

At α-level = 0.5, the solution is asymptotically stable and there is no fuzzy solution for $Y(t)$ since $as\, t \to \infty$, $u(t) \to 37.94$, $v(t) \to 44.98$, $r(t) \to 39.52$, $s(t) \to 28.11$.

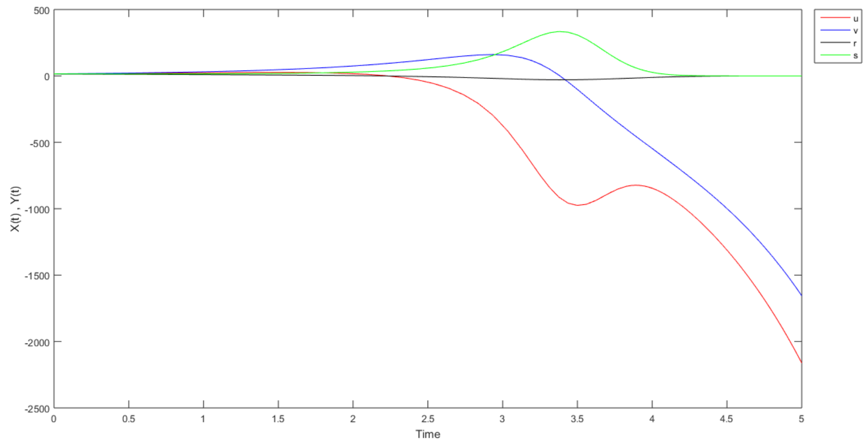


Figure (3.2.129): The solution of (57) at α=0.5

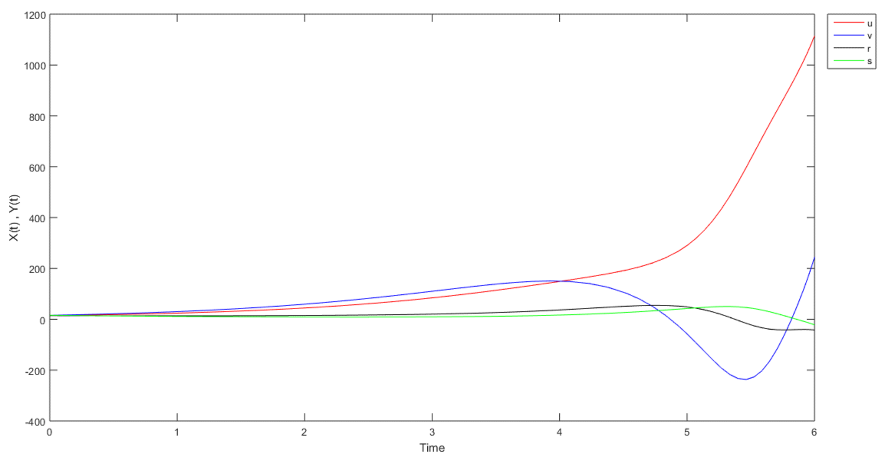


Figure (3.2.132): The solution of (58) at α=0.5

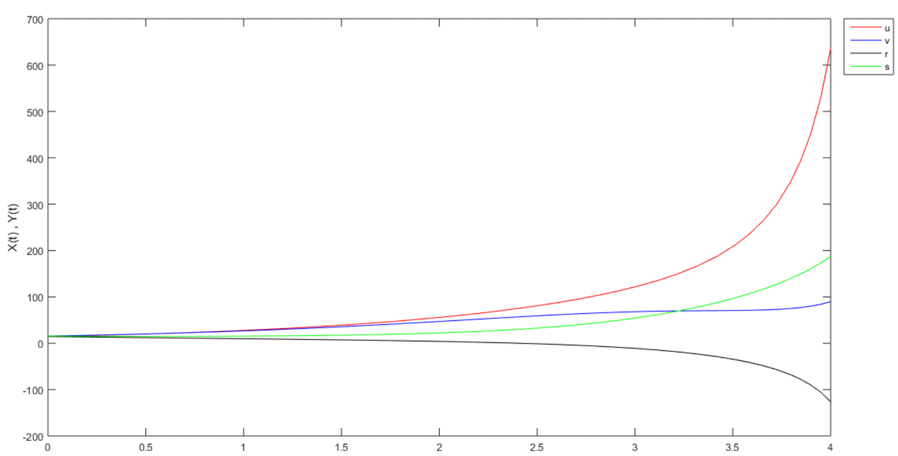


Figure (3.2.135): The solution of (59) at α=0.5

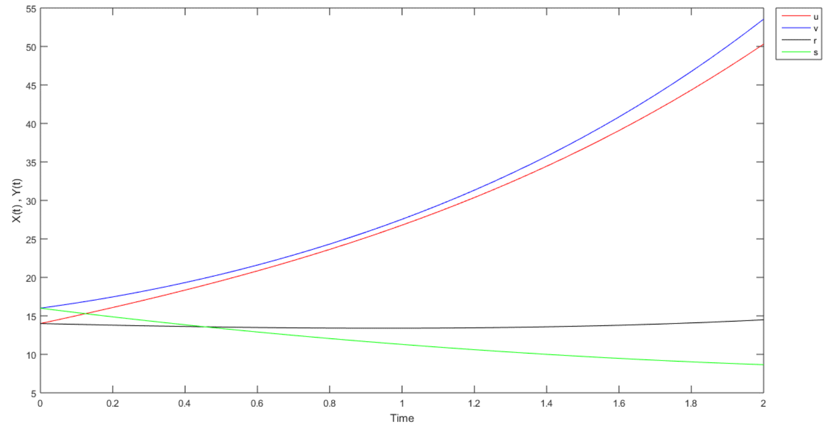
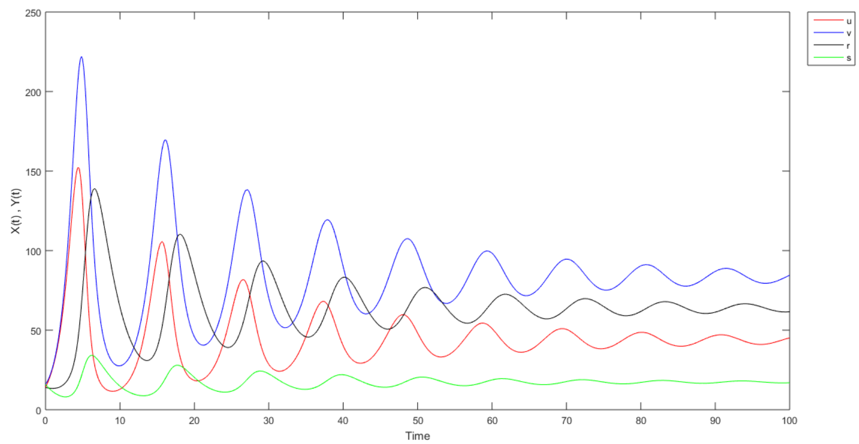
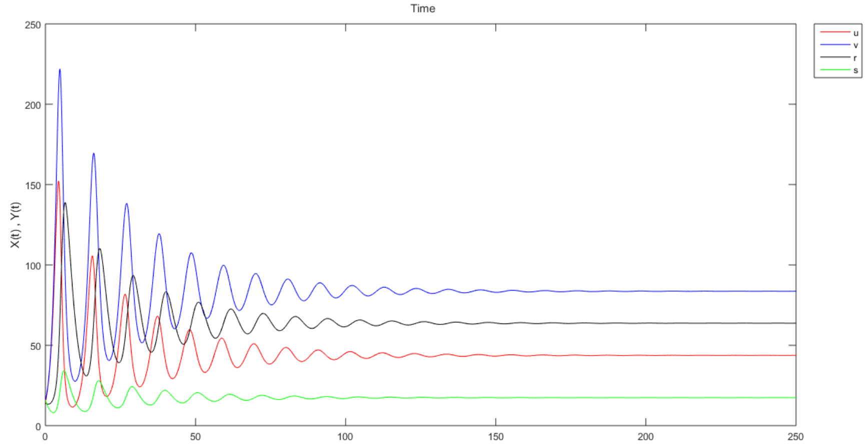


Figure (3.2.140): The solution of (60) at α=0.5 for short time period





Figures (3.2.141) and (3.2.142): The solution of (60) at α=0.5 as time increases

At α-level = 0.5, the solution is asymptotically stable and there is no fuzzy solution for $Y(t)$ since $as\ t \to \infty$, $u(t) \to 38.46$, $v(t) \to 55.47$, $r(t) \to 48.08$, $s(t) \to 23.11$.

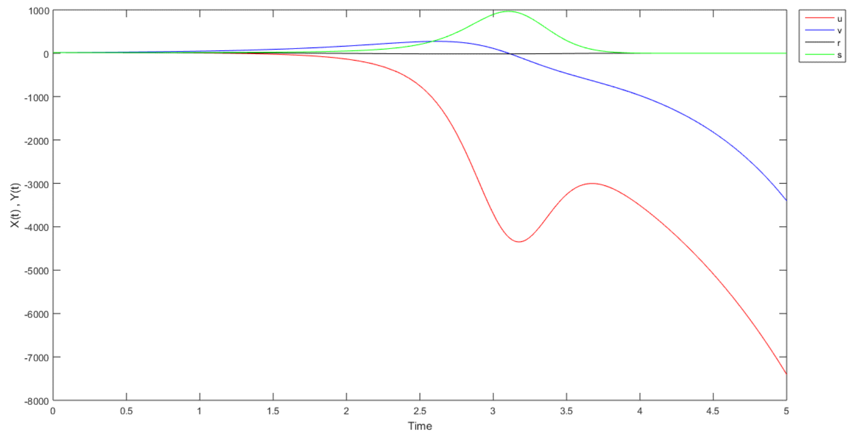


Figure (3.2.151): The solution of (62) at α=0.5

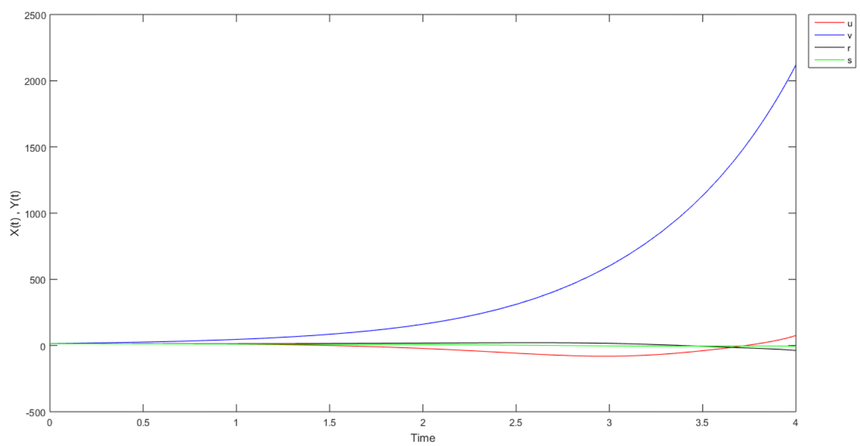


Figure (3.2.154): The solution of (63) at α=0.5

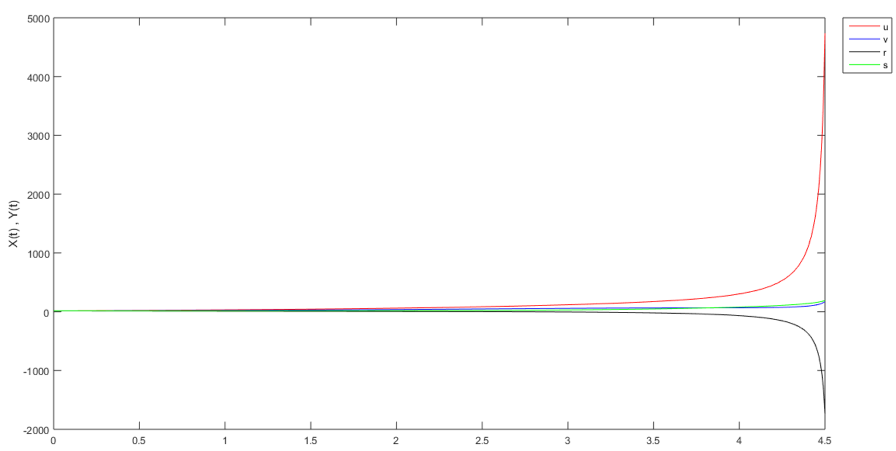


Figure (3.2.157): The solution of (64) at α=0.5

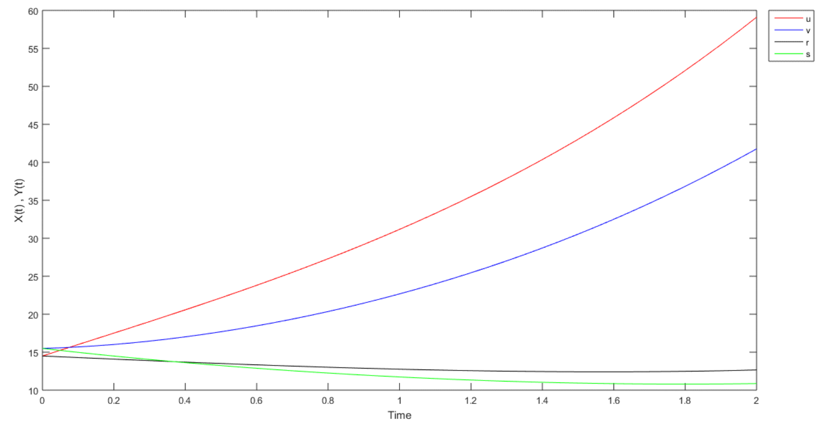
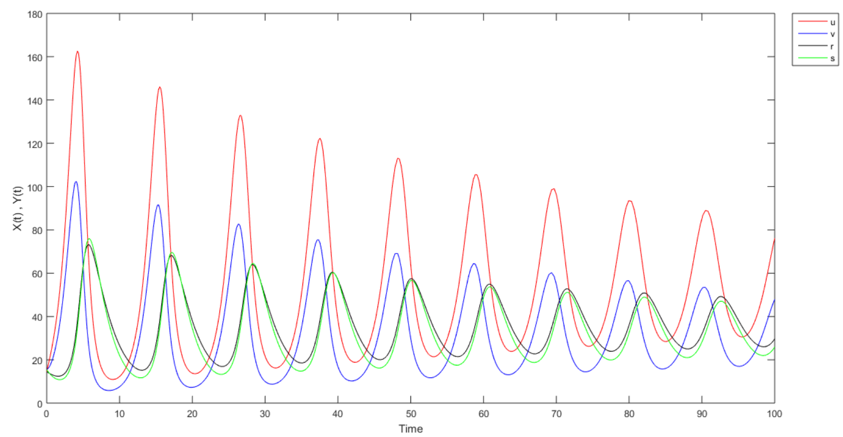


Figure (3.2.161): The solution of (65) at α=0.5 for short time period



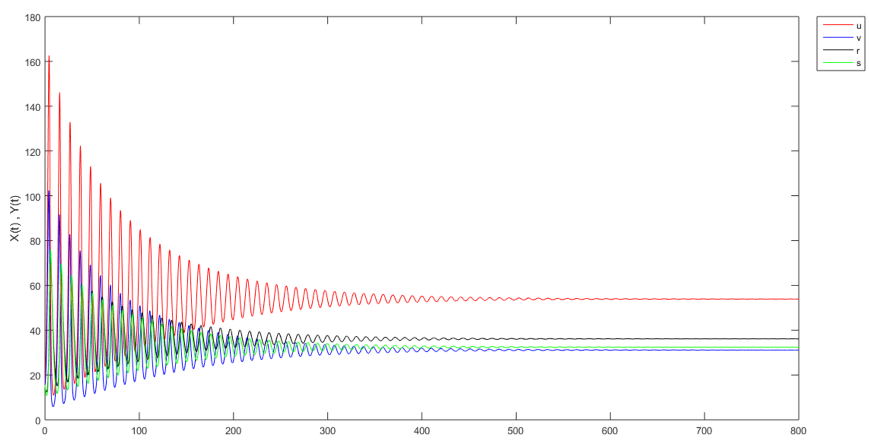


Figure (3.2.162) and figure (3.2.163): The solution of (65) at α=0.5 as time increases

At α-level = 0.5, The solution is asymptotically stable but there are no fuzzy solutions for $X(t)$ *and* $Y(t)$. Since *as* $t \to \infty$, $u(t) \to 53.92$, $v(t) \to 31.16, r(t) \to 36.11$, $s(t) \to 32.45$.

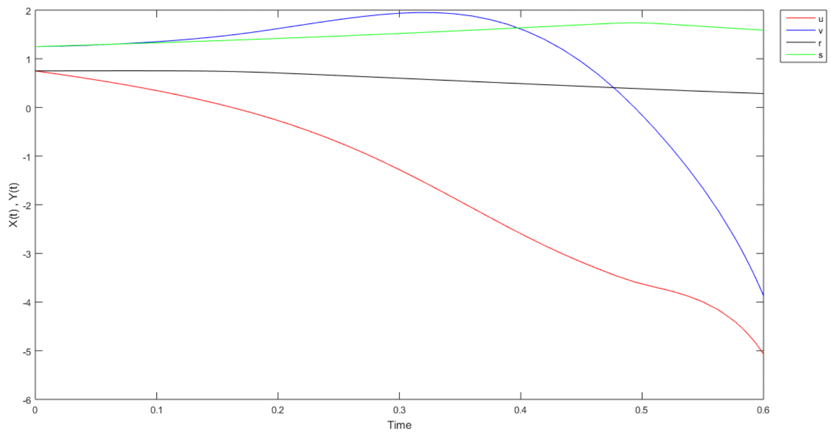


Figure (4.1.3): The solution of (68) at α=0.5

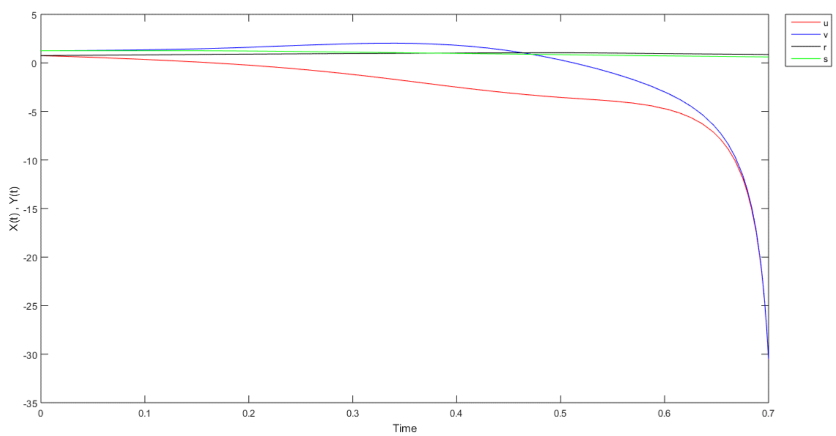


Figure (4.1.6): The solution of (69) at α=0.5

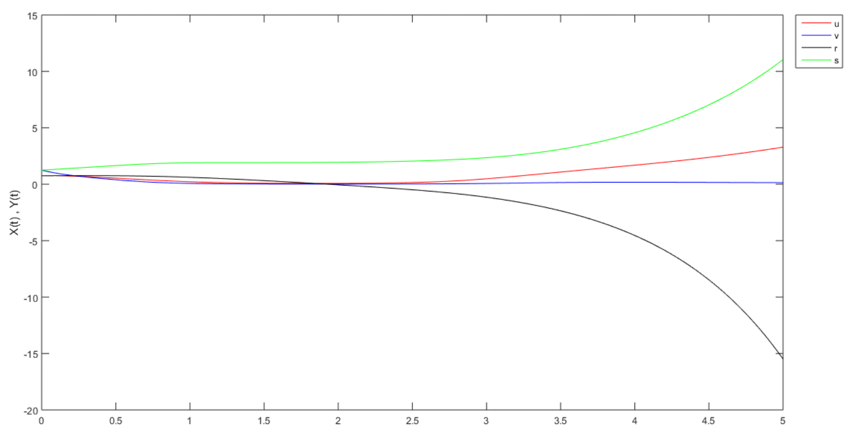


Figure (4.1.9): The solution of (70) at α=0.5

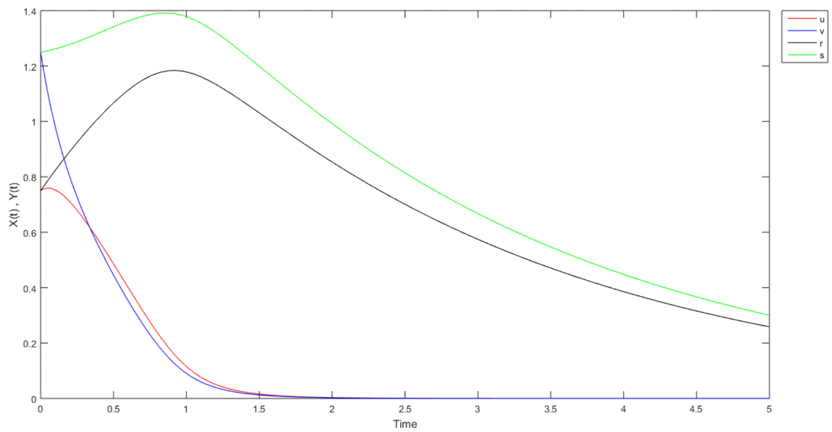


Figure (4.1.13): The solution of (71) at α=0.5 for short time period

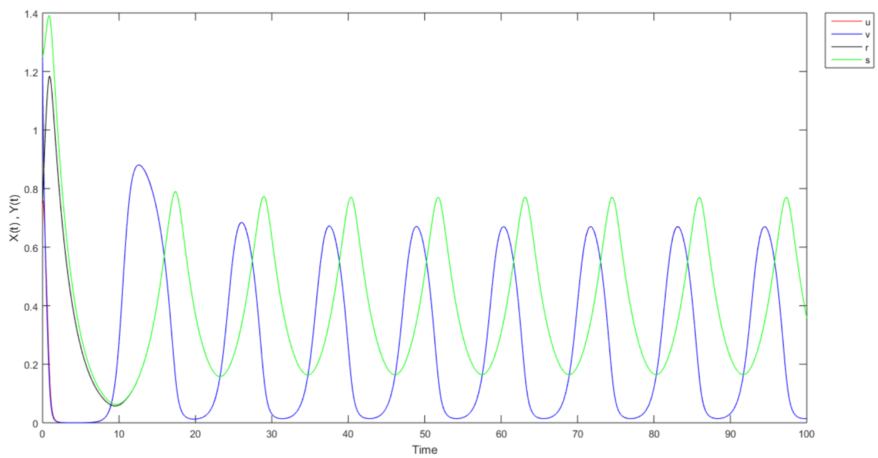


Figure (4.1.14): The solution of (71) at α=0.5 as time increase

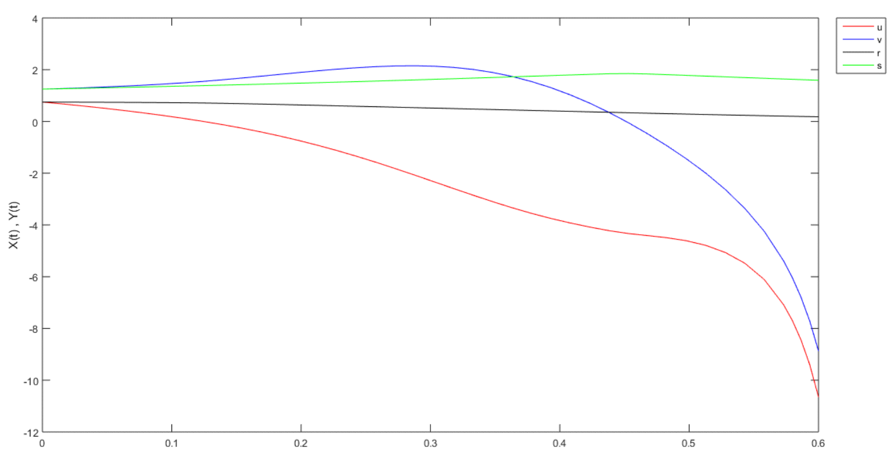


Figure (4.2.2): The solution of (74) at α=0.5

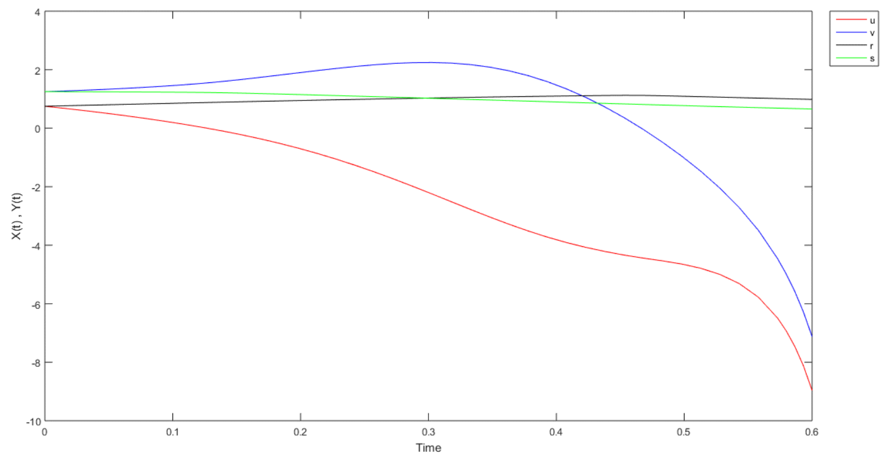


Figure (4.2.5): The solution of (75) at α=0.5

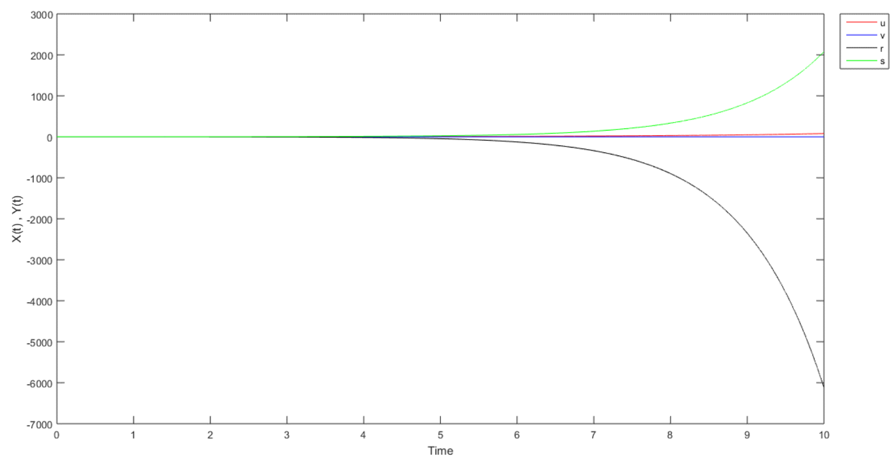


Figure (4.2.8): The solution of (76) at α=0.5

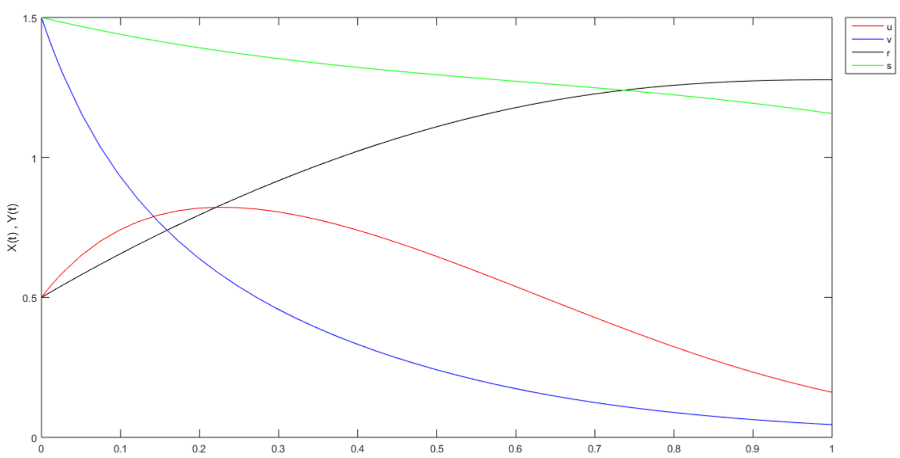


Figure (4.2.13): The solution of (77) at α=0.5 for short time period

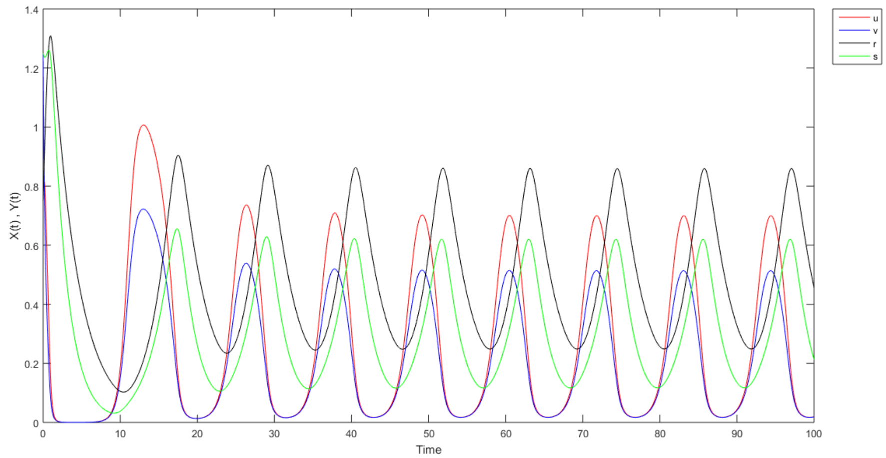


Figure (4.2.14): The solution of (77) at α=0.5 as time increases

The following code is the general code used to find the simulations and the graphical solutions for each system of ODE in this thesis.

```
%Here u=y(1), v=y(2), r=y(3), s=y(4)

f1 = @(t,y)[eq1;eq2;eq3; eq4];
%tf is the final time
tf=x;
%T is the time interval, and Y is the solutions matrix
[T,Y] = ode45(f1,[0 x],[u0, v0, r0, s0])

%We plot the solutions
plot(T,Y(:,1),'r',T,Y(:,2),'b',T,Y(:,3),'k',T,Y(:,4),'g
')
ylabel('X(t) , Y(t)')
xlabel('Time')
legend('u','v','r','s')

legend('Location','northeastoutside')
```