# On the oscillation of certain discrete fractional difference equations with delay 

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This Study Was Submitted in Partial Fulfillment of the Requirements for The Master's Degree of Science in Mathematical Modeling

Faculty of Graduate Studies

Palestine Technical University- Khadoorie (PTUK).

October - 2021

تذبذب بعض معادلات الفروق الكسرية المنفصلة ذات التأخير الوقتي

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## DEDICATION

To my father, who always urged me for more work.

To my mother, who taught me to believe in myself, whose prayers were with me all the way to success.

To my sisters and brothers, who support me in my life stages.

To my friends, who believe in me in all of these bad conditions.

To everyone, who always have inspired me.

## ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor, Dr. Mohammad Marabeh, for all of his excellent guidance and thoughtful suggestions, without which the completion of this thesis would not have been possible.

I would also like to extend my sincere appreciation to Prof. Thabit abdeljawad, for his serving as member of my thesis committee.

I would like to thank My fiancé, Jihad Kittaneh, for his support and encouragement

Lastly, I would like to thank the internal examiner, Dr. Basheer Abdullah, and the external examiner, Dr. Abdalhaleem Ziqan, for their valuable guidance to enrich this thesis.

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# (On the oscillation of certain discrete fractional <br> <br> difference equations with delay) 

 <br> <br> difference equations with delay)}


#### Abstract

In this thesis we employ the properties of left nabla Caputo difference operator and the discrete nabla Laplace transform on $h \mathbb{Z}$, to investigate the oscillatory behavior of solutions for two classes of fractional delay difference equations. The first one is with one positive coefficient and the second one is with two positive coefficients. Besides we give generalization for the two classes and study their oscillatory patterns. We show that the solutions act oscillatory if certain characteristic equations have not real roots. By applying the $Q$-operator, we prove the oscillatory for right nabla Caputo difference. Finally, two classical real life mathematical models on delay differential equations are applied to present our new results.


Keywords: Oscillatory solution, Fractional delay differential equation, Fractional difference operators, Fractional delay difference equation.

## (تذبذب بعض معادلات الفروق الكسرية المنفصلة ذات التأخير الوقتي)

## الملخص

في هذه الرسالة نقوم بتوظيف خصائص عامل الفرق الأيسر nabla Caputo وتحويل nabla المنفصل، للراسة السلوك التذبذبي لحلول فئيّين من معادلات الفرق الكسرية ذات التأخير الوقتي. اللئة الأولى تضم المعادلات بمعامل موجب والئئة الثانية تضم المعادلات بمعاملين موجبين. إلى جانب ذلك نعطي تعيما للفئتين ونقوم بدراسة أنماطهما التذبذبية. يظهر لنا بأن الحول تكون متذبذبة إذا لم يكن لبعض المعادلات الميزة جذور حقيقية. من خلال تطبيق عامل التثغيل Q، نثبت التنبذب لعامل الفرق الأيمن nabla Caputo . أخيرًا، نموذجان رياضيان كلاسيكيان للحياة الواقعية حول المعادلات التفاضلية ذات التأخير الوقتي تم تطبيقهما لعرض نتائجنا الجديدة. الكلمات المفتاحية: حل متذبذ، معادلة تغاضلية كسرية ذات تأخير وقتي، معاملات تثغيل فرق كسرية، معادلة فرق كسرية ذات تأخير وقتي.

## CHAPTER 1

## INTRODUCTION

### 1.1 Background

Differential equations with time delays are used to model real life phenomena related with the past states of these phenomena. The greatest use of delay differential equations (DDEs) is in the modeling of population dynamics. It is also used in chemical processes, economics, biosciences, technical processes and many other branches [17 and 25]. Many researchers focus on the study of oscillation for delay differential equations [20, 26 and 53].

Fractional calculus has received wide attention in the past decades, this is because of its importance in many branches of engineering and science [23, 24, 41] and in physics [32 and 49].

Fractional delay differential equations (FDDEs) are obtained by the combination of fractional derivative and time delay. There are many articles on the oscillation of FDDEs. Sufficient and necessary conditions for the oscillation of FDDEs with negative and positive coefficient were established [30, 47 and 52].

Discrete fractional differences and sums were developing in the last twenty years by many researchers in several fields of engineering and science. This kind of calculus has an important role in modeling many problems, this is due to the simplicity of the numerical algorithms [1, 3,4 and 8]. The researchers started
using discrete operators as nabla and delta investigation by employing the forward and backward jumping operators on the time scale $\mathbb{Z}[2,13,21,33$ and 34]. On the other hand, few researchers studied discrete delta and nabla fractional operators on $h \mathbb{Z}$ [14, 28, 38, and 44]. With best of our knowledge, there is no study on the oscillation for solutions of fractional delay difference equations.

### 1.2 Objectives of the Thesis

Our main aim in this thesis is to state and prove sufficient conditions for the oscillatory of two main classes of fractional delay difference equations on the left and right nabla Caputo $h$-fractional difference operators. Moreover, we provide numerical algorithms and examples to illustrate our main theoretical results.

### 1.3 Research Questions

Consider the fractional delay differential equation (FDDE) with positive coefficient of the form

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\alpha} z(t)+p z(t-\tau)=0 \tag{1.1}
\end{equation*}
$$

such that $0<\alpha<1$ and $p, \tau>0$
with the initial condition

$$
\begin{equation*}
z(t)=\varphi(t), \quad \text { for } t \in[-\tau, 0] \tag{1.2}
\end{equation*}
$$

where $\varphi(t) \in C[-\tau, 0], C$ is the space of all real-valued continuous functions on the interval $[-\tau, 0]$.

Also consider the FDDE with two positive coefficients of the form

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\alpha} z(t)+p z(t-\tau)+q z(t)=0, \quad t>0, \tag{1.3}
\end{equation*}
$$

such that $p, q \in \mathbb{R}^{+}$,
with the initial condition

$$
\begin{equation*}
z(t)=\varphi(t), \quad \text { for } t \in[-\tau, 0] \tag{1.4}
\end{equation*}
$$

where $\varphi(t) \in C[-\tau, 0]$
P. Zhu and Q. Xiang in 2018 provided sufficient conditions for the oscillation of solutions of (1.1) and (1.3), [54]. In our thesis we will answer three research questions.

1- How can we use recent difference operators to discretize the fractional derivative in equations (1.1) and (1.3) and generalize them by using nabla Caputo $h$-fractional difference operator?

2- How can we state and prove conditions for the oscillation of solutions of the resulted difference equations?

3- How we will apply our results in mathematical models related to fractional delay differential equations?

### 1.4 Structure of the thesis

This thesis is organized as follows. In the next chapter, we recall some basic concepts about differential equations with time delay and present some results related to the basic oscillation theory of the first order linear delay differential equations.

Chapter 3 contains definitions of The Riemann-Liouville ( $R L$ ) fractional derivative and fractional integral, together with some properties and examples. The relation between $R L$ and Caputo fractional derivative is also given.

In Chapter 4 we collect theorems and criteria on the oscillations of fractional delay differential equations (FDDEs).

In Chapter 5, we review nabla and delta differences and sums on the time scale $h \mathbb{Z}$, and their properties. The relation between Caputo fractional left nabla and delta and right ones is also given.

The core of the thesis is presented in Chapter 6. We establish sufficient conditions for the oscillation of two classes of fractional delay difference equations with Caputo fractional derivative. We provide numerical algorithms and examples to demonstrate our theoretical results.

In the last chapter, we apply our results on two classical realistic models.

### 1.5. Literature Review

A. Elbert and I. Stavroulak [26] established oscillation and non-oscillation criteria for the first-order delay differential equation of the form

$$
\begin{equation*}
z^{\prime}(t)+p(t) z(\tau(t))=0 \tag{1.5}
\end{equation*}
$$

such that $t>t_{0}$ and $\tau(t)<t$. In this case the oscillation condition is

$$
\int_{\tau(t)}^{t} p(s) d s \geq \frac{1}{e} \quad \text { and } \quad \lim _{t \rightarrow \infty} \int_{\tau(t)}^{t} p(s) d s=\frac{1}{e}
$$

G. Li introduced a new technique to investigate the generalized characteristic equations and obtain some infinite integral conditions for oscillatory of the nonautonomous delay differential equations [43].
P. Zhu and Q. Xiang studied the oscillation behavior of fractional delay differential equations of the form,

$$
\begin{equation*}
{ }_{0}^{c} D_{t}^{\alpha} z(t)+p z(t-\tau)=0 \tag{1.6}
\end{equation*}
$$

where $0<\alpha<1$ and $p, \tau>0$. They proved the oscillation for FDDEs when the associated characteristic equations have no zeros in $\mathbb{R}$. So using system parameters they gave direct oscillation rules [54].

The work of Q. Meng, Z. Jin, and G. Liu [46] is devoted to study the linear FDDEs of the form

$$
\begin{equation*}
{ }_{0}^{c} D_{t}^{\alpha} z(t)-p z(t-\tau)=0 \tag{1.7}
\end{equation*}
$$

such that $0<\alpha<1$ be the quotient of two odd natural numbers and $p, \tau>0$. They obtained the conclusion

$$
p^{\frac{1}{\alpha}} \tau>\frac{\alpha}{e}
$$

is a sufficient condition of the for all solutions of equation (1.7) to behave oscillatory.
B. Zhang and X. Deng [53] considered the delay differential equation of the form,

$$
\begin{equation*}
z^{\prime}(t)+p(t) z(\tau(t))=0 \tag{1.5}
\end{equation*}
$$

and they applied the theory of measure chains to analyze the oscillation and nonoscillation of equation (1.5) on the basis of some well-known results.
S. Grace, et al. initiated the oscillation theory for fractional differential equations [30]. They obtained oscillation criteria for a class of nonlinear fractional differential equations of the form

$$
\begin{equation*}
D_{a}^{\alpha} z+f_{1}(t, z)=v(t)+f_{2}(t, z) \tag{1.8}
\end{equation*}
$$

$\lim _{t \rightarrow a^{+}} J_{a}^{1-\alpha} z(t)=b_{1}$, where $D_{a}^{\alpha}$ denotes the Riemann-Liouville (RL) differential operator of $\operatorname{order} \alpha, 0<\alpha<1$ and $J_{a}^{1-\alpha}$ is the Riemann-Liouville fractional integral operator [30].
T. Abdeljawad studied fractional derivatives with singular kernels on the time scale $h \mathbb{Z}$. He defined fractional derivatives having nonsingular exponential kernels and Mittag-Leffler on the time scale $h \mathbb{Z}$ and investigated some of their properties. Some dual identities between left and right and delta and nabla, right and left $h$-fractional difference types were explored [9].
T. Abdeljawad, et al. explained the action of the discrete $h$-Laplace transform and its convolution theorem on the fractional proportional operators [11].
T. Abdeljawad, et al. presented a new type of fractional differences and sums called the discrete weighted fractional operators. Then they defined the weighted forward and backward difference operators on an isolated time scale with arbitrary step size and obeyed them the power rule [10].

## CHAPTER 2

## Preliminaries

### 2.1 Delay Differential Equation (DDE)

We study delay differential equations because so many of the processes, both manmade and natural, in chemistry, biology, physics, medicine, economics, engineering, etc., involve time delays. Time delays occur so often, in almost every situation, so if we ignore them that means we ignore the reality [42].

Our aim in this chapter is to present some basic definitions and results related to the basic oscillation theory of the first order linear delay differential equation with constant delays and with constant and variable coefficients.

Definition 2.1: [22] A differential equation in which the derivatives of some unknown function at present time is dependent on the values of the functions at past times is called Delay Differential Equation (DDE). Mathematically, a general DDE for $z(t) \in \mathbb{R}^{n}$ takes the form,

$$
z^{\prime}(t)=f(t, z(t-\tau))
$$

where $\tau>0$.

Definition 2.2: [31] The function $z(t)$ is called exponentially bounded, if there exist positive constants $L$ and $\beta$, where $|z(t)| \leq L e^{\beta t}$ for $t \geq 0$.

Theorem 2.1: [31] Consider the first order (DDE)

$$
\begin{equation*}
z^{\prime}(t)+p z(t-\tau)=0, \text { for } t>0 \text { and } p, \tau>0 \tag{2.1}
\end{equation*}
$$

where the initial function is

$$
\begin{equation*}
z(t)=\phi(t) \text { for } \phi(t):[-\tau, 0] \rightarrow \mathbb{R} \tag{2.2}
\end{equation*}
$$

If $z(t)$ is a solution of equation (2.1) on $[0, \infty)$ then there exists positive constants $L$ and $\beta$, where $|z(t)| \leq L e^{\beta t}$ for $t>0$
so, every solution of (2.1) is exponentially bounded.

Definition 2.3: [31] Let $z(t):[0, \infty) \rightarrow \mathbb{R}$ be a function. The Laplace transform of $z(t)$ is denoted by $L[z(t)]$ or $Z(s)$, and is given by

$$
\begin{equation*}
Z(s)=\int_{0}^{\infty} e^{-s t} z(t) d t \tag{2.4}
\end{equation*}
$$

The integral 2.4 behave in three ways as the following:
i. it converges for all $s \in \mathbb{R}$.
ii. it diverges for all $s \in \mathbb{R}$.
iii. there exists $\sigma_{0} \in \mathbb{R}$ where the integral in 2.4 is convergent for all $s$ with $\operatorname{Re}(s)>\sigma_{0}$ and diverges for all $s$ with $\operatorname{Re}(s)<\sigma_{0}$.

When (iii) is satisfied, the number $\sigma_{0}$ is called the abscissa of convergence of $Z(s)$. So, $\sigma_{0}=\inf \{\sigma \in \mathbb{R}: Z(\sigma)$ exists $\}$.

For example, the abscissa of convergence of the Laplace transforms of the functions $e^{-t^{2}}$ is $-\infty, e^{3 t}$ is 3 and $e^{t^{2}}$ is $\infty$.

Example 2.1: The abscissa of convergence of the Laplace transform of the function $f(t)=e^{3 t}$ is 3.

Solution: For $s \neq 3, L\left(e^{3 t}\right)=\int_{0}^{\infty} e^{-s t} e^{3 t} d t=\int_{0}^{\infty} e^{(-s+3) t} d t$

$$
\begin{aligned}
& =\int_{0}^{\infty} e^{-(s-3) t} d t \\
& =\lim _{b \rightarrow \infty} \int_{0}^{b} e^{(-s+3) t} d t \\
& =\left.\lim _{b \rightarrow \infty} \frac{e^{(-s+3) t}}{-(s-3)}\right|_{0} ^{b} \\
& =\frac{1}{s-3} \lim _{b \rightarrow \infty}\left[1-e^{-(s-3) b}\right] \\
& =\left\{\begin{array}{cc}
\frac{1}{s-3} & \text { for } s>3 \\
\text { Divergent } & \text { for } s<3
\end{array}\right.
\end{aligned}
$$

So the abscissa of convergence is 3 .

Lemma 2.1: [31] Let $z \in C[0, \infty)$ and $z(t)$ satisfies (2.3), the abscissa of convergence $\sigma_{0}$ of $Z(s)$ of $z(t)$ satisfies $\sigma_{0} \leq \beta$ and $Z(s)$ exists for $\operatorname{Re}(s)>\sigma_{0}$.

The following two lemmas shows that the Laplace transform of the shift function $z(t-\tau)$ and of the derivative has the same abscissa of convergence.

Lemma 2.2: [31] Let $z \in C[-\tau, \infty)$ and let $\sigma_{0}<\infty$, then the Laplace transform of the shift function $z(t-\tau)$ is given by

$$
\begin{equation*}
L[z(t-\tau)]=\mathrm{e}^{-s \tau} Z(s)+\mathrm{e}^{-s \tau} \int_{-\tau}^{0} \mathrm{e}^{-s t} z(t) d t \tag{2.5}
\end{equation*}
$$

for all $\operatorname{Re}(s)>\sigma_{0}$.

Lemma 2.3: [31] Let $z \in C^{1}[0, \infty)$, where $C^{1}[0, \infty)$ is the space of all realvalued continuously differentiable functions on $[0, \infty)$ and $\sigma_{0}<\infty$ then the Laplace transform of $z^{\prime}(t)$ is $L\left[z^{\prime}(t)\right]=s Z(s)-z(0)$ for all $\operatorname{Re}(s)>\sigma_{0}$.

Definition 2.4: [31] Let $z \in C[0, \infty)$, the function $z$ is oscillatory or oscillate if $z$ has arbitrary large many of zeros. Otherwise is called non-oscillatory.

Remark 2.1: As $z$ is continuous, if it is non-oscillatory then it must be eventually negative or eventually positive.

### 2.2 Oscillation Criteria for DDE

Consider the linear autonomous DDE

$$
\begin{equation*}
z^{\prime}(t)+\sum_{i=1}^{n} p_{i} z\left(t-\tau_{i}\right)=0 \tag{2.6}
\end{equation*}
$$

where $z \in C[-\tau, \infty), t>0, p_{i} \in \mathbb{R}$, and $\tau_{i} \in \mathbb{R}^{+}$for $i=1,2, \ldots, n$

Let $\tau=\max \left\{\tau_{1}, \tau_{2}, \ldots, \tau_{n}\right\}$,
where the initial function is

$$
\begin{equation*}
z(t)=\phi(t) \text { for } \phi(t) \in C[-\tau, 0] \tag{2.7}
\end{equation*}
$$

Theorem 2.2: [31] Assume $p_{i}$ is real number and $\tau_{i}$ is positive real number, for $i=1,2, \ldots, n$ then the equation,

$$
\begin{equation*}
F(\lambda)=\lambda+\sum_{i=1}^{n} p_{i} e^{-\lambda \tau_{i}}=0 \tag{2.8}
\end{equation*}
$$

has no zeros in $\mathbb{R}$ iff all solutions of equation (2.6) oscillate.

Theorem 2.3: [31] Let $p_{i}, \tau_{i}$ are positive real numbers for $i=1,2, \ldots, n$, then all solutions of equation (2.6) have oscillatory behavior iff
i. $\quad \sum_{i=1}^{n} p_{i} \tau_{i}>\frac{1}{e}$
ii. $\quad\left(\prod_{i=1}^{n} p_{i}\right)^{\frac{1}{n}}\left(\sum_{i=1}^{n} \tau_{i}\right)>\frac{1}{e}$

Theorem 2.4: [31] Let $p_{i}, \tau_{i}>0$ for $i=1,2, \ldots, n$, if

$$
\left(\sum_{i=1}^{n} p_{i}\right)\left(\min _{1 \leq i \leq n} \tau_{i}\right)>\frac{1}{e}
$$

then all solutions of equation (2.6) have oscillatory behavior.

Theorem 2.5: [31] Let $p, \tau \in \mathbb{R}$ then all solutions of equation (2.1) is oscillation iff the characteristic equation

$$
F(\lambda)=\lambda+p e^{-\lambda \tau}=0
$$

has no zeros in $\mathbb{R}$.

Theorem 2.6: [31] Let $p, \tau>0$ then all solution of equation (2.1) is oscillation iff $p \tau>\frac{1}{e}$.

Consider the linear non-autonomous DDE

$$
\begin{equation*}
z^{\prime}(t)+p(t) z(t-\tau)=0 \tag{2.9}
\end{equation*}
$$

such that $t>0, p(t)$ is a nonnegative continuous function on $[0, \infty)$ and $\tau \in \mathbb{R}^{+}$,
and the initial function is

$$
\begin{equation*}
z(t)=\phi(t) \text { for } \phi(t) \in C[-\tau, 0] \tag{2.10}
\end{equation*}
$$

Theorem 2.7: [31] Assume that $p(t)$ is a nonnegative continuous function on $[0, \infty)$ and $\tau$ is positive real number, if

$$
\liminf _{t \rightarrow+\infty} \int_{t-\tau}^{t} p(s) d s>\frac{1}{e}
$$

then all solutions of equation (2.9) is oscillate.

## CHAPTER 3

## Fractional Derivatives

Fractional Calculus is simply a non-integer order derivative or integral. There are several definitions of fractional derivatives and integrals. Riemann-Liouville (RL) definition is the most important among them.

In this chapter, we present the definition of the Riemann-Liouville (RL) fractional derivative and fractional integral together with some of their properties. Before we start we discuss some useful mathematical definitions that are related to fractional calculus.

### 3.1 Useful Mathematical Definitions

Definition 3.1: [29] The Gamma function is the generalization of the factorial for any $z \in \mathbb{R}$. It is given by

$$
\Gamma(z)=\int_{0}^{\infty} e^{-t} t^{z-1} d t
$$

Property 3.1: [29] The Gamma function has some properties:
i. $\quad \Gamma(z+1)=z \Gamma(z) \quad$ for $z \in \mathbb{R}^{+}$
ii. $\quad \Gamma(z)=(z-1)!\quad$ for $z \in \mathbb{N}$
iii. $\quad \lim _{z \rightarrow-n}|\Gamma(z)|=\infty \quad$ for $z \in \mathbb{N}$ and $n \in \mathbb{N}$

## Example 3.1:

i. $\quad \Gamma(1)=1$
ii. $\quad \Gamma(1 / 2)=\sqrt{\pi}$

Definition 3.2: The Beta function is defined by an integral as

$$
\beta(z, w)=\int_{0}^{1}(1-t)^{w-1} t^{z-1} d t \quad \text { for } z, w \in \mathbb{R}^{+}
$$

Remark 3.1: The following formula shows the well-known relation between Beta and Gamma functions

$$
\beta(z, w)=\frac{\Gamma(z) \Gamma(w)}{\Gamma(z+w)} \quad \text { for } z, w \in \mathbb{R}^{+}
$$

Definition 3.3: [40] The Mittag-Leffler function was defined and studied by Mittag-Leffler in the year 1903 and the generalization of it was studied by Wiman in 1905.
i. Mittag-Leffler function

$$
E_{\alpha}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha k+1)}, \quad \alpha>0
$$

ii. Generalized Mittag-Leffler function

$$
E_{\alpha, \beta}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha k+\beta)}, \quad \alpha>0 \text { and } \beta>0
$$

Mittag-Leffler function $E_{\alpha}(t)$ for $\alpha=1$ is the exponential function $E_{1}(t)=e^{t}$. It is known that exponential functions serve as solutions of linear ordinary differential equations with constant coefficients. In a similar manner MittagLeffler functions appear as solutions of fractional order differential equations, this will be shown later in Example 3.7.

## Example 3.2:

i. $\quad E_{1,1}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(k+1)}=\sum_{k=0}^{\infty} \frac{t^{k}}{k!}=e^{t}$
ii. $\quad E_{0}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(1)}=\frac{1}{1-t},|t|<1$
iii. $\quad E_{1,2}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(k+2)}=\sum_{k=0}^{\infty} \frac{t^{k}}{(k+1)!}\left(\frac{t}{t}\right)=\frac{1}{t} \sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)!}=\frac{e^{t}-1}{t}$
iv. $\quad E_{2,2}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(2 k+2)}=\sum_{k=0}^{\infty} \frac{(\sqrt{t})^{2 k}}{(2 k+1)!} \frac{\sqrt{t}}{\sqrt{t}}=\sum_{k=0}^{\infty} \frac{(\sqrt{t})^{2 k+1}}{(2 k+1)!} \frac{1}{\sqrt{t}}=\frac{\sinh (\sqrt{t})}{\sqrt{t}}$
v. $\quad E_{2,1}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(2 k+1)}=\sum_{k=0}^{\infty} \frac{(\sqrt{t})^{2 k}}{(2 k)!}=\cosh (\sqrt{t})$

### 3.2 Definition of RL Fractional Integral and Derivative

Definition 3.4: [40] Let $\alpha \in(0, \infty)$. Let $z$ be piecewise continuous on $k=(0, \infty)$ and integrable on any finite subinterval of $k$. Then the RL fractional integral of $z$ of order $\alpha$ for $t>0$ is defined by

$$
{ }_{0} D_{t}^{-\alpha} z(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} z(s) d s \quad \text { for } \alpha>0
$$

Definition 3.5: [40] Let $\alpha \in(0, \infty)$. Let $z$ be piecewise continuous on $k=(0, \infty)$ and differentiable on any finite subinterval of $k$. Then the RL fractional derivative of $z$ of order $\alpha$ for $t>0$ is defined by

$$
{ }_{0} D_{t}^{\alpha} z(t)=\frac{1}{\Gamma(1-\alpha)} \frac{d}{d t} \int_{0}^{t}(t-s)^{-\alpha} z(s) d s \quad \text { for } 0<\alpha<1
$$

Theorem 3.1: [40] Let $z \in C(0, \infty)$ and let $\alpha, \beta>0$. Then for all $t$ we have

$$
\begin{aligned}
{ }_{0} D_{t}^{-\beta}\left[{ }_{0} D_{t}^{-\alpha} z(t)\right] & ={ }_{0} D_{t}^{-(\alpha+\beta)} z(t) \\
& ={ }_{0} D_{t}^{-\alpha}\left[{ }_{0} D_{t}^{-\beta} z(t)\right]
\end{aligned}
$$

Corollary 3.1: [40] Let $z \in C(0, \infty)$ and let $0<\alpha<1$. Then for all $t$ we have
i. $\quad{ }_{0} D_{t}^{\alpha}{ }_{0} D_{t}^{-\alpha} z(t)=z(t)$
ii. $\quad{ }_{0} D_{t}^{-\alpha}{ }_{0} D_{t}^{\alpha} z(t)=z(t)-{ }_{0} D_{t}^{\alpha} \frac{z(0)}{\Gamma(\alpha)} t^{\alpha-1}$

## Corollary 3.2: [40]

i. $\quad D\left[{ }_{0} D_{t}^{-\alpha} z(t)\right] \neq{ }_{0} D_{t}^{-\alpha}[D z(t)]$
ii. $\quad D\left[{ }_{0} D_{t}^{-\alpha} z(t)\right]={ }_{0} D_{t}^{-\alpha}[D z(t)]+\frac{z(0)}{\Gamma(\alpha)} t^{\alpha-1}$

Lemma 3.1: [40] Relation between RL fractional derivative and RL fractional integral,

$$
{ }_{0} D_{t}^{\alpha} z(t)=\left(\frac{d}{d t}\right)^{n}\left({ }_{0} D_{t}^{-n+\alpha} z(t)\right)
$$

where $n=[\alpha]+1$

Remark 3.3: In particular for $0<\alpha<1$, then $n=1$ and

$$
{ }_{0} D_{t}^{\alpha} z(t)=D\left({ }_{0} D_{t}^{-(1-\alpha)} z(t)\right)
$$

### 3.3 Examples of RL Fractional Integral and Derivative

Example 3.3: [40] (The power rule) Let $\alpha>0, m>0$, and $t>0$, then

$$
\begin{aligned}
{ }_{0} D_{t}^{-\alpha} t^{m} & =\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} s^{m} d s \\
& =\frac{1}{\Gamma(\alpha)} \int_{0}^{t}\left(1-\frac{s}{t}\right)^{\alpha-1} t^{\alpha-1} s^{m} d s \\
& =\frac{1}{\Gamma(\alpha)} \int_{0}^{1}(1-u)^{\alpha-1} t^{\alpha-1}(t u)^{m} t d u \\
& =\frac{t^{m+\alpha}}{\Gamma(\alpha)} \int_{0}^{1}(1-u)^{\alpha-1} u^{m} d u \\
& =\frac{t^{m+\alpha}}{\Gamma(\alpha)} \beta(m+1, \alpha) \\
& =\frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} t^{m+\alpha}
\end{aligned}
$$

Remark 3.4: By using Remark (3.3) and Example (3.3) we have,

$$
{ }_{0} D_{t}^{\alpha} t^{m}=\frac{\Gamma(m+1) t^{m-\alpha}}{\Gamma(m-\alpha+1)} \quad \text { for } m>-1 \text { and } 0 \leq \alpha<1
$$

## Example 3.4:

$$
{ }_{0} D_{t}^{-\alpha} e^{a t}=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} e^{a s} d s
$$

Let $x=t-s$, hence

$$
{ }_{0} D_{t}^{-\alpha} e^{a t}=\frac{e^{a t}}{\Gamma(\alpha)} \int_{0}^{\infty} x^{\alpha-1} e^{-a x} d x=a^{-\alpha} e^{a t}
$$

## Example 3.5:

i. $\quad{ }_{0} D_{4}{ }^{-1 / 2} 3=\frac{3}{\Gamma(1 / 2)} 4^{1 / 2}=\frac{12}{\sqrt{\pi}}$
ii. $\quad{ }_{0} D_{4}{ }^{1 / 2} t=\frac{t^{1 / 2}}{\Gamma(3 / 2)}=2 \sqrt{\frac{t}{\pi}}$
iii. $\quad{ }_{0} D_{4}{ }^{1 / 2} 3=\frac{3}{\sqrt{4 \pi}}$
iv. $\quad{ }_{0} D_{t}^{-1 / 2} \sin (t)=\sqrt{2}\left(\sin (t) \int_{0}^{x} \cos \left(t^{2}\right) d t-\cos (t) \int_{0}^{x} \sin \left(t^{2}\right) d t\right)$, where $x=\sqrt{\frac{2 t}{\pi}}$.
v. $\quad{ }_{0} D_{t}^{-1 / 2} \cos (t)=\sqrt{2}\left(\cos (t) \int_{0}^{x} \cos \left(t^{2}\right) d t-\sin (t) \int_{0}^{x} \sin \left(t^{2}\right) d t\right)$, where $x=\sqrt{\frac{2 t}{\pi}}$.
vi. $\quad{ }_{0} D_{t}^{-1 / 2} t^{2}=\frac{\Gamma(3)}{\Gamma(7 / 2)} t^{5 / 2}=\frac{16}{15} \sqrt{\frac{t^{5}}{\pi}}$.

For a proof of parts iv and v see [29]

### 3.4 Laplace Transform of RL Integral and Derivative

Definition 3.6: [40] The fractional integral of $z(t)$ of order $\alpha$ is

$$
{ }_{0} D_{t}^{-\alpha} z(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} z(s) d s
$$

so the Laplace Transform of the fractional integral of $z(t)$ as

$$
L\left\{{ }_{0} D_{t}^{-\alpha} z(t)\right\}=\frac{1}{\Gamma(\alpha)} L\left\{t^{\alpha-1}\right\} L\{z(t)\}=s^{-\alpha} Z(s)
$$

Definition 3.7: [40] Let the fractional derivative of $z(t)$ of order $0<\alpha<1$ is

$$
{ }_{0} D_{t}^{\alpha} z(t)=\frac{1}{\Gamma(1-\alpha)} \frac{d}{d t} \int_{0}^{t}(t-s)^{-\alpha} z(s) d s
$$

The Laplace Transform of the fractional derivative of $z(t)$ is defined as

$$
L\left\{{ }_{0} D_{t}^{\alpha} z(t)\right\}=s^{\alpha} Z(s)-{ }_{0} D_{t}^{-(1-\alpha)} z(0)
$$

## Lemma 3.2:

i. $L\left\{t^{m}\right\}=\frac{\Gamma(m+1)}{s^{m+1}}$
ii. $\quad L\left\{{ }_{0} D_{t}^{-\alpha} t^{m}\right\}=\frac{\Gamma(m+1)}{s^{m+\alpha+1}}$
iii. $\quad L\left\{{ }_{0} D_{t}^{\alpha} t^{m}\right\}=\frac{\Gamma(m+1)}{s^{m-\alpha+1}}$
iv. $L\left\{e^{a t}\right\}=\frac{1}{s-a}$
v. $L\left\{{ }_{0} D_{t}^{-\alpha} e^{a t}\right\}=\frac{s^{-\alpha}}{s-a}$
vi. $\quad L\left\{t^{\alpha-1} E_{\alpha, \alpha}\left(a t^{\alpha}\right)\right\}=\frac{1}{s^{\alpha}-a}$

## Example 3.6:

i. $\quad L\left\{{ }_{0} D_{t}^{2 / 3} t^{3}\right\}=\frac{\Gamma(4)}{s^{4-\frac{2}{3}}}=\frac{6}{s^{10 / 3}}$
ii. $\quad L\left\{{ }_{0} D_{t}^{-\frac{1}{5}}\left(t^{2}+1\right)\right\}=\frac{\Gamma(3)}{s^{4+\frac{1}{5}}}+\frac{\Gamma(1)}{s^{1+\frac{1}{5}}}=\frac{2}{s^{\frac{21}{5}}}+\frac{1}{s^{\frac{6}{5}}}$

Example 3.7: Solve the fractional differential equation

$$
{ }_{0} D_{t}^{\frac{1}{3}} z(t)=a z(t)
$$

such that $a$ is a constant.

Solution: Inserting Laplace transform in the equation yields

$$
L\left\{{ }_{0} D_{t}^{\frac{1}{3}} z(t)\right\}=a L\{z(t)\}
$$

which implies that

$$
s^{\frac{1}{3}} Z(s)-{ }_{0} D_{t}^{-\left(1-\frac{1}{3}\right)} Z(0)=a Z(s)
$$

denote the constant quantity ${ }_{0} D_{t}^{-\left(1-\frac{1}{3}\right)} Z(0)=D^{-\frac{2}{3}} Z(0)$ by $c_{1}$, then the above equation turns to be

$$
s^{\frac{1}{3}} Z(s)-c_{1}=a Z(s)
$$

Hence,

$$
Z(s)=\frac{c_{1}}{s^{\frac{1}{3}}-a}
$$

Finally, Lemma 3.2 (vi) implies that

$$
z(t)=L^{-1}\left\{\frac{c_{1}}{s^{\frac{1}{3}}-a}\right\}=c_{1} t^{-\frac{2}{3}} E_{\frac{1}{3} \cdot \frac{1}{3}}\left(a t^{\frac{1}{3}}\right)
$$

### 3.5 Caputo Fractional Derivative

Definition 3.8: [40] The Caputo derivative of order $0<\alpha<1$ for a function $z \in C(0, \infty)$ is defined by

$$
{ }_{0}^{c} D_{t}^{\alpha} z(t)=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t}(t-s)^{-\alpha} \frac{d}{d s} z(s) d s \quad \text { for } t>0
$$

Lemma 3.3: [40] Relation between RL fractional derivative and Caputo fractional derivative where $0<\alpha<1$

$$
{ }_{0}^{c} D_{t}^{\alpha} z(t)={ }_{0} D_{t}^{\alpha}(z(t)-z(0))={ }_{0} D_{t}^{\alpha} z(t)-\frac{z(0) t^{-\alpha}}{\Gamma(1-\alpha)}
$$

Property 3.2: [40] Let $\lambda \in R, 0<\alpha<1$ and for any constant $c$ in $\mathbb{R}$, we have
i. $\quad{ }_{0}{ }^{c} D_{t}^{\alpha} 1=0$
ii. $\quad{ }_{0}{ }^{c} D_{t}^{\alpha} c=0$
iii. $\quad{ }_{0}^{C} D_{t}^{\alpha} e^{-\lambda t}=\lambda^{\alpha} e^{-\lambda t}$

Theorem 3.2: [40] Let $0<\alpha<1$ and $\beta>1$ then

$$
{ }_{0}^{C} D_{t}^{\alpha} t^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}
$$

Lemma 3.4: [40] For $0<\alpha<1$,

$$
\begin{gathered}
{ }_{0}^{C} D_{t}^{\alpha}{ }_{0} D_{t}^{-\alpha} z(t)=z(t)-{ }_{0} D_{t}^{-\alpha} z(0) \frac{t^{1-\alpha}}{\Gamma(1-\alpha)} \\
{ }_{0} D_{t}^{-\alpha}{ }_{0}^{C} D_{t}^{\alpha} z(t)=z(t)-z(0)
\end{gathered}
$$

Definition 3.9: [40] Let the Caputo fractional derivative of $z(t)$ of order $0<\alpha<1$ is

$$
{ }_{0}{ }^{C} D_{t}^{\alpha} z(t)=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t}(t-s)^{-\alpha} \frac{d}{d s} z(s) d s
$$

so the Laplace Transform of the Caputo fractional derivative of $z(t)$ as

$$
L\left\{{ }_{0}^{C} D_{t}^{\alpha} z(t)\right\}=s^{\alpha} Z(s)-s^{\alpha-1} z(0)
$$

## CHAPTER 4

## Fractional Delay Differential Equations

In the previous chapters we introduced basic notions and results on delay differential equations and on fractional derivatives. This chapters presents a combination of fractional derivatives and delay differential equations which yields fractional delay differential equations (FDDEs). Oscillations of a particular class of fractional delay differential equations are given.

### 4.1 Oscillation criteria for FDDEs with constant and variable coefficient

Consider the FDDE of the form

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\alpha} z(t)+p z(t-\tau)=0 \tag{4.1}
\end{equation*}
$$

such that $0<\alpha<1$ and $p, \tau>0$
with the initial condition

$$
\begin{equation*}
z(t)=\varphi(t), \quad \text { for } t \in[-\tau, 0] \tag{4.2}
\end{equation*}
$$

where $\varphi(t) \in C([-\tau, 0])$

Theorem 4.1: [54] Let $p, \tau>0$ and let $0<\alpha<1$ be the quotient of two odd natural numbers. If the characteristic equation

$$
F(\lambda)=\lambda^{\alpha}+p e^{-\lambda \tau}=0
$$

has no zeros in $\mathbb{R}$, then every solution of equations (4.1)-(4.2) has oscillatory behavior.

Theorem 4.2: [54] Let $p, \tau>0$ and let $0<\alpha<1$ be the quotient of two odd natural numbers.

If $\tau p^{\frac{1}{\alpha}}>\frac{1}{e}$ then every solution of equations (4.1)-(4.2) has oscillatory behavior.

Proposition 4.1: [54] Consider the FDDEs of the form

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\alpha} z(t)+p(t) z(t-\tau)=0 \tag{4.3}
\end{equation*}
$$

such that $0<\alpha<1$ be the quotient of two odd natural numbers and $\tau>0$
with the initial condition

$$
\begin{equation*}
z(t)=\varphi(t), \quad \text { for } t \in[-\tau, 0] \tag{4.4}
\end{equation*}
$$

where $p(t)$ is a nonnegative continuously function on $[0, \infty)$

If $\lim _{t \rightarrow+\infty} \inf p(t)=p>0$ and $\tau p^{\frac{1}{\alpha}}>\frac{1}{e}$, where $p>0$ then every solution of equations (4.3)-(4.4) oscillates.

### 4.2 Oscillation criteria for FDDEs with two constant coefficients

Consider the FDDEs of the form

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\alpha} z(t)+p z(t-\tau)+q z(t)=0, \quad t>0 \tag{4.5}
\end{equation*}
$$

such that $0<\alpha<1$ and $p, \tau>0$ and $q$ is a real number,
with the initial condition

$$
\begin{equation*}
z(t)=\varphi(t), \quad \text { for } t \in[-\tau, 0] \tag{4.6}
\end{equation*}
$$

where $\varphi(t) \in C[-\tau, 0]$

Proposition 4.2: [54] Let $p, \tau, q>0$ and let $0<\alpha<1$ be the quotient of two odd natural numbers. If the characteristic equation

$$
F(\lambda)=\lambda^{\alpha}+p e^{-\lambda \tau}+q=0
$$

has no zeros in $\mathbb{R}$, then every solution of equations (4.5)-(4.6) has oscillatory behavior.

Theorem 4.3: [54] Let $p, \tau, q>0$ and let $0<\alpha<1$ be the quotient of two odd natural numbers. If $p \leq q$ and $\frac{p \tau}{(q+p)^{\frac{\alpha-1}{\alpha}}+q \tau}>\frac{1}{e}$,
then all solutions of equations (4.5)-(4.6) oscillate.

Proposition 4.3: [54] Let $p, \tau, q>0$ and let $0<\alpha<1$ be the quotient of two odd natural numbers. If $p>q$, and $\frac{p \tau}{(q+p)^{\frac{\alpha-1}{\alpha}}+p \tau}>\frac{1}{e}$, then all solutions of equations (4.5)-(4.6) oscillate.

Corollary 4.1: [54] Let $p, \tau>0, q<0$ and let $0<\alpha<1$ be the quotient of two odd natural numbers. If $p>-q,(q+p)^{\frac{\alpha-1}{\alpha}}>-q \tau$ and $\frac{(p+q) \tau}{(q+p)^{\frac{\alpha-1}{\alpha}}+q \tau}>\frac{1}{e}$ then all solutions of equations (4.5)-(4.6) have oscillatory behavior.

### 4.3 Oscillation criteria for generalized FDDEs

Consider the FDDE of the form

$$
\begin{equation*}
{ }_{0}{ }^{c} D_{t}^{\alpha} z(t)-\sum_{i=1}^{n} p_{i} z\left(t-\tau_{i}\right)=0 \tag{4.7}
\end{equation*}
$$

where $p_{i}, \tau_{i} \geq 0$ for $i=1,2, \ldots, n$ and let $0<\alpha<1$ be the quotient of two odd natural numbers.

Theorem 4.4: [46] Let $p_{i}, \tau_{i} \geq 0$ for $i=1,2, \ldots, n$ and let $0<\alpha<1$ be the quotient of two odd natural numbers. If the characteristic equation

$$
F(\lambda)=\lambda^{\alpha}+\sum_{i=1}^{n} p_{i} e^{-\lambda \tau_{i}}=0
$$

has no negative roots in $\mathbb{R}$, then every solution of equation (4.7) has oscillatory behavior.

Theorem 4.5: [46] Let $p_{i}, \tau_{i} \geq 0$ for $i=1,2, \ldots, n$ and let $0<\alpha<1$ be the quotient of two odd natural numbers. If
i. $\quad \sum_{i=1}^{n} p_{i} \tau_{i}^{\alpha}>\left(\frac{\alpha}{e}\right)^{\alpha}$;
ii. $\prod_{i=1}^{n} p_{i}{ }^{\frac{1}{n}} \tau_{i}^{\frac{\alpha}{n}}>\left(\frac{\alpha}{e}\right)^{\alpha}$,
then all solutions of equation (4.7) have oscillatory behavior.

## CHAPTER 5

## Fractional Difference Operators on $h \mathbb{Z}$

Discrete mathematics is the opposite of continuous mathematics (which is characterized by real numbers) and it is dealing with distinct values which characterized by integers.

This branch of mathematics studies algorithms so it is the mathematical language of the computer science and comes up in many fields of medicine, science, physiology, engineering, biology and ecology.

Discrete calculus [18 and 19] and discrete fractional calculus were studied intensively on the time scale $\mathbb{Z}$, see $[6,35,36$ and 45]. In the last two decades many researchers started to develop discrete calculus and discrete fractional calculus on the time scale $h \mathbb{Z}$, where $0<h<1$, see [37, 47, and 51].

In this chapter we give the basic definitions about nabla and delta discrete fractional operators on time scale $h \mathbb{Z}$ and their properties.

A time scale is a nonempty closed subset of real numbers.

Example 5.1: These are some examples of time scales
i. The integers $\mathbb{Z}$
ii. The real numbers $\mathbb{R}$
iii. The natural numbers $\mathbb{N}$

Example 5.2: These are some examples of subsets that are not time scales
i. The irrational numbers $\mathbb{R} \backslash \mathbb{Q}$
ii. The rational numbers $\mathbb{Q}$
iii. Open interval $(0,1)$

Remark 5.1: Throughout this thesis, the time scale denoted by $h \mathbb{Z}$ is chosen to be
$h \mathbb{Z}:=\{\ldots,-2 h,-h, 0, h, 2 h, \ldots\}$, where $0<h \leq 1$.

Remark 5.2: The function $z(t)$ will be defined in this chapter on one of the following sets
i. $\quad \mathbb{N}_{a, h}:=\{a, a+h, a+2 h, \ldots\}$
ii. $\quad \mathbb{N}_{a, h}^{\mathrm{b}}:=\{a, a+h, a+2 h, \ldots, b\}$
iii. ${ }_{b, h} \mathbb{N}=\{b, b-h, b-2 h, \ldots\}$
where $a, b \in \mathbb{R}, b-a>0, \frac{b-a}{h} \in \mathbb{N}$.

Definition 5.1: [50] Assume $z(t): \mathbb{N}_{a, h} \rightarrow \mathbb{R}$,
i. the forward $h$-difference operator of $z(t)$ is given by

$$
\Delta_{h} z(t)=\frac{z(t+h)-z(t)}{h}
$$

ii. the backward $h$-difference operator of $z(t)$ is given by

$$
\nabla_{h} z(t)=\frac{z(t)-z(t-h)}{h}
$$

We state and prove the following properties which have an analogy on the time scale $\mathbb{Z}$

Proposition 5.1: Let $z, w: \mathbb{N}_{a, h} \rightarrow \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, then for all $t \in \mathbb{N}_{a, h}$
i. $\quad \Delta_{h} \alpha=0$
ii. $\quad \Delta_{h} \alpha z(t)=\alpha \Delta_{h} z(t)$
iii. $\quad \Delta_{h}(z+w)(t)=\Delta_{h} z(t)+\Delta_{h} w(t)$
iv. $\Delta_{h} \alpha^{t+\beta}=\frac{\left(\alpha^{h}-1\right) \alpha^{t+\beta}}{h}$

## Proof:

i. $\quad \Delta_{h} a=\frac{a-a}{h}=0$
ii. $\quad \Delta_{h} a z(t)=\frac{a z(t+h)-a z(t)}{h}=a\left(\frac{z(t+h)-z(t)}{h}\right)=a \Delta_{h} z(t)$
iii. $\quad \Delta_{h}(z+w)(t)=\frac{(z+w)(t+h)-(z+w)(t)}{h}=\frac{z(t+h)+w(t+h)-z(t)-w(t)}{h}=$

$$
\frac{z(t+h)-z(t)}{h}+\frac{w(t+h)-w(t)}{h}=\Delta_{h} z(t)+\Delta_{h} w(t)
$$

iv. $\Delta_{h} \alpha^{t+\beta}=\frac{\alpha^{t+h+\beta}-\alpha^{t+\beta}}{h}=\frac{\alpha^{t+\beta}}{h}\left(\alpha^{h}-1\right)$

Proposition 5.2: Let $z, w: \mathbb{N}_{a, h} \rightarrow \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, then for all $t \in \mathbb{N}_{a, h}$
i. $\quad \nabla_{h} \alpha=0$
ii. $\quad \nabla_{h} \alpha z(t)=\alpha \nabla_{h} z(t)$
iii. $\quad \nabla_{h}(z+w)(t)=\nabla_{h} z(t)+\nabla_{h} w(t)$
iv. $\quad \nabla_{h} \alpha^{t+\beta}=\frac{\left(1-\alpha^{-h}\right) \alpha^{t+\beta}}{h}$

## Proof:

i. $\quad \nabla_{h} \alpha=\frac{a-a}{h}=0$
ii. $\quad \nabla_{h} a z(t)=\frac{a z(t)-a z(t-h)}{h}=a \frac{z(t)-z(t-h)}{h}=a \nabla_{h} z(t)$
iii. $\quad \nabla_{h}(z+w)(t)=\frac{(z+w)(t)-(z+w)(t-h)}{h}=\frac{z(t)+w(t)-z(t-h)-w(t-h)}{h}=$

$$
\frac{z(t)-z(t-h)}{h}+\frac{w(t)-w(t-h)}{h}=\nabla_{h} z(t)+\nabla_{h} w(t)
$$

iv. $\quad \nabla_{h} \alpha^{t+\beta}=\frac{\alpha^{t+\beta}-\alpha^{t-h+\beta}}{h}=\left(1-\alpha^{-h}\right) \frac{\alpha^{t+\beta}}{h}$

Definition 5.2: [5] Let $t \in \mathbb{R}$ and $h>0$, then
i. The forward jump $h$-difference operator is

$$
\sigma_{h}(t)=t+h
$$

ii. The forward jump $h$-difference operator is

$$
\rho_{h}(t)=t-h
$$

Remark 5.3: For $h=1$ on time scale $h \mathbb{Z}$, yields the time scale of integers
i. The forward difference operator is

$$
\Delta z(t)=z(t+1)-z(t)
$$

ii. The backward difference operator is

$$
\nabla z(t)=z(t)-z(t-1)
$$

iii. The forward jump difference operator is

$$
\sigma(t)=t+1
$$

iv. The backward jump difference operator is

$$
\rho(t)=t-1
$$

Definition 5.3: [29] For any $v \in \mathbb{R}$, such that $t \in \mathbb{R} \backslash\{\ldots,-2,-1,0\}$
i. The falling function is

$$
\mathrm{t}^{(v)}=\frac{\Gamma(t+1)}{\Gamma(t-v+1)}
$$

ii. The rising function is

$$
t^{\bar{v}}=\frac{\Gamma(t+v)}{\Gamma(t)}
$$

iii. $\quad 0^{\bar{v}}=0$
iv. $\nabla\left(t^{\bar{v}}\right)=v t^{\overline{v-1}}$

### 5.1 Delta $h$ - Fractional Sums and Differences

Definition 5.4: [9] Let $t \in \mathbb{N}_{a, h}$ and $\alpha \in \mathbb{R}$, so the delta $h$-factorial function denoted by

$$
t_{h}^{(\alpha)}=h^{\alpha} \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\alpha\right)}
$$

Definition 5.5: [9] Let $z(t) \in \mathbb{N}_{a, h} \cap_{b, h} \mathbb{N} \rightarrow \mathbb{R}$, and $b=a+y h$ for some $y \in \mathbb{N}$. Then
i. The left delta $h$-discrete fractional sum of order $0<\alpha<1$ is:

$$
\begin{aligned}
\left(a \Delta_{h}^{-\alpha} z\right)(t) & =\frac{1}{\Gamma(\alpha)} \int_{a}^{\sigma(t-\alpha h)}(t-\sigma(s))_{h}^{(\alpha-1)} z(s) \Delta_{h} s \\
& =\frac{1}{\Gamma(\alpha)} \sum_{i=a / h}^{t / h-\alpha}(t-\sigma(i h))_{h}^{(\alpha-1)} z(i h) h, \quad t \in\left\{\tau+\alpha h: \tau \in \mathrm{N}_{a, h}\right\}
\end{aligned}
$$

ii. The right delta $h$-discrete fractional sum of order $0<\alpha<1$ is

$$
\begin{aligned}
& \left(_{h} \Delta_{b}^{-\alpha} z\right)(t)=\frac{1}{\Gamma(\alpha)} \int_{\rho(t+\alpha h)}^{b}(\rho(s)-t)_{h}^{(\alpha-1)} z(s) \nabla_{h} s . \\
& \quad=\frac{1}{\Gamma(\alpha)} \sum_{i=t / h+\alpha}^{b / h}(i h-\sigma(t))_{h}^{(\alpha-1)} z(i h) h, t \in{ }_{b-h, h} \mathbb{N}
\end{aligned}
$$

Definition 5.6: [9] Delta $h-R L$ discrete fractional difference
i. The delta left $h-R L$ discrete fractional difference of order $\alpha>0$, beginning at $a$, has the form:

$$
\left({ }_{a} \Delta_{h}^{\alpha} z\right)(t)=\left(\Delta_{h a}^{n} \Delta_{h}^{-(n-\alpha)} z\right)(t), \quad t \in \mathbb{N}_{a+(n-\alpha) h, h}
$$

where $n=[\alpha]+1$.
ii. The delta right $h-R L$ fractional difference of order $\alpha>0$, ending at $b$ has the form:

$$
\left({ }_{h} \Delta_{b}^{\alpha} z\right)(t)=(-1)^{n}\left(\nabla_{h}^{n}{ }_{h} \Delta_{b}^{-(n-\alpha)} z\right)(t) \quad t \in \mathbb{N}_{b-(n-\alpha) h, h}
$$

In particular, for $0<\alpha<1$ and $a=0$, we get
i. $\quad\left({ }_{0} \Delta_{h}^{\alpha} z\right)(t)=\left(\Delta_{h} 0_{0} \Delta_{h}^{-(1-\alpha)} z\right)(t)$
ii. $\quad\left({ }_{h} \Delta_{b}^{\alpha} z\right)(t)=(-1)\left(\nabla_{h}{ }_{h} \Delta_{b}^{-(1-\alpha)} z\right)(t)$

Definition 5.7: [9] Let $\alpha>0, n=[\alpha]+1$,
i. The delta left $h$ - Caputo fractional difference of order $\alpha>0$,

$$
\left({ }_{a}^{C} \Delta_{h}^{\alpha} z\right)(t)=\left({ }_{a} \Delta_{h}^{-(n-\alpha)} \Delta_{h}^{n} z\right)(t), \quad t \in \mathbb{N}_{a+(n-\alpha) h, h}
$$

ii. The delta right $h$ - Caputo fractional difference of order $\alpha>0$, ending at $b$

$$
\left({ }_{h}^{C} \Delta_{b}^{\alpha} z\right)(t)=\left({ }_{a} \Delta_{h}^{-(n-\alpha)}(-1)^{n} \Delta_{h}^{n} z\right)(t), \quad t \in \mathbb{N}_{b-(n-\alpha) h, h}
$$

Remark 5.4: In particular at $0<\alpha<1$, and $a=0$, then we have $n=1$, so,
i. $\quad\left({ }_{0}^{C} \Delta_{h}^{\alpha} Z\right)(t)=\left({ }_{0} \Delta_{h}^{-(1-\alpha)} \Delta_{h} Z\right)(t), \quad t \in \mathbb{N}_{(1-\alpha) h, h}$
ii. $\quad\left({ }_{h}^{C} \Delta_{b}^{\alpha} z\right)(t)=\left({ }_{h} \Delta_{b}^{-(1-\alpha)}\left(-\nabla_{h} z\right)(t), \quad t \in_{b-(1-\alpha) h, h} \mathbb{N}\right.$

Lemma 5.1: [9] The relation between delta $h-R L$ and delta $h$-Caputo discrete fractional difference for any $0<a<1$, is given by
i. $\quad\left({ }_{a}^{C} \Delta_{h}^{\alpha} z\right)(t)=\left({ }_{a} \Delta_{h}^{\alpha} z\right)(t)-\frac{(t-a)_{h}^{(-\alpha)}}{\Gamma(1-\alpha)} z(a)$
ii. $\quad\left({ }_{h}^{C} \Delta_{b}^{\alpha} z\right)(t)=\left({ }_{h} \Delta_{b}^{\alpha} z\right)(t)-\frac{(b-t){ }_{h}^{(-\alpha)}}{\Gamma(1-\alpha)} z(b)$

Proposition 5.3: [9] For $\alpha>0, h>0$ and $z$ defined on $\mathbb{N}_{a, h}$ we get for $t \in \mathbb{N}_{a+n h, h} \subset \mathbb{N}_{a, h}:$
i. $\quad\left(a+\alpha h \Delta_{h a}^{\alpha} \Delta_{h}^{-\alpha} z\right)(t)=z(t)$,
ii. $\left.\quad(a+(n-\alpha) h) \Delta_{h}^{-\alpha}{ }_{a} \Delta_{h}^{\alpha} z\right)(t)=z(t), \quad \alpha \notin \mathbb{N}$
iii. $\quad\left({ }_{a} \Delta_{h}^{-\alpha}{ }_{a} \Delta_{h}^{\alpha} z\right)(t)=\mathrm{z}(\mathrm{t})-\sum_{k=0}^{n-1} \frac{(t-a)_{h}^{(k)} \Delta_{h}^{k} z(a)}{k!}, \alpha \in \mathbb{N}$

Proposition 5.4: [9] For $\alpha>0, h>0$ and $z$ defined on $\mathbb{N}_{b, h}$ we have for $t \in \mathbb{N}_{b-n h, h} \subset \mathbb{N}_{b, h}:$
i. $\quad\left({ }_{h} \Delta_{b-\alpha h}^{\alpha}{ }_{h} \Delta_{b}^{-\alpha} z\right)(t)=z(t)$,
ii. $\quad\left({ }_{h} \Delta_{b-\alpha h ~}^{-\alpha} \Delta_{b}^{\alpha} z\right)(t)=z(t), \quad \alpha \notin \mathbb{N}$
iv. $\quad\left({ }_{h} \Delta_{b-\alpha h}^{-\alpha} \Delta_{b}^{\alpha} z\right)(t)=z(t)-\sum_{k=0}^{n-1} \frac{(b-t)_{n}^{(k)}(-1)^{k} \nabla_{h}^{k} z(b)}{k!}, \alpha \in \mathbb{N}$

### 5.2 Nabla $h$ - Fractional Differences and Sums

Definition 5.8: [2] Assume $t \in \mathbb{N}_{a, h}$ and $\alpha \in \mathbb{R}$, so the nabla $h$-factorial function is, for $h>0$, we have $t_{h}^{\bar{\alpha}}=h^{\alpha} \frac{\Gamma\left(\frac{t}{h}+\alpha\right)}{\Gamma\left(\frac{t}{h}\right)}$.

In particular, for $h=1$, we have $t^{\bar{\alpha}}=\frac{\Gamma(t+\alpha)}{\Gamma(t)}$.
Definition 5.9: [48] (Left and right nabla $h$-fractional sum).

Let $z(t): \mathbb{N}_{a, h} \rightarrow \mathbb{R}$ and $0<h \leq 1$,
i. The left nabla $h$-fractional sum of order $0<\alpha<1$ and $a=0$ is

$$
\begin{aligned}
\left({ }_{0} \nabla_{h}^{-\alpha} z\right)(t) & =\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-\rho(s))_{h}^{\overline{\alpha-1}} z(s) \nabla_{h} s \\
& =\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{t / h}(t-\rho(i h))_{h}^{\overline{\alpha-1}} z(i h) h, \text { for } t \in \mathbb{N}_{h, h}
\end{aligned}
$$

ii. The right nabla $h$-fractional sum of order $0<\alpha<1$ is

$$
\begin{aligned}
\left({ }_{h} \nabla_{b}^{-\alpha} z\right)(t) & =\frac{1}{\Gamma(\alpha)} \int_{t}^{b}(s-\rho(t))_{h}^{\overline{\alpha-1}} z(s) \nabla_{h} s . \\
& =\frac{1}{\Gamma(\alpha)} \sum_{i=t / h}^{b / h-1}(i h-\rho(t))_{h}^{\overline{\alpha-1}} z(i h) h, t \in_{b-h, h} \mathrm{~N}
\end{aligned}
$$

## Definition 5.10: [9]

i. The nabla left $h-R L$ discrete fractional difference of order $\alpha>0$, beginning at $a$, has the form:

$$
\left({ }_{a} \nabla_{h}^{\alpha} z\right)(t)=\left(\nabla_{h \mathrm{a}}^{n} \nabla_{h}^{-(n-\alpha)} z\right)(t) \quad \text { where } t \in \mathbb{N}_{a+h, h}, n=[\alpha]+1
$$

ii. The nabla righth - RL fractional difference of order $\alpha>0$, ending at $b$ has the form:

$$
\left({ }_{h} \nabla_{b}^{\alpha} z\right)(t)=(-1)^{n}\left(\Delta_{h}^{n} \nabla_{b}^{-(n-\alpha)} z\right)(t) \quad \text { where } t \in \mathbb{N}_{b-h, h}, \quad n=[\alpha]+1
$$

In particular, for $0<\alpha<1$ and $a=0$ we get,
i. $\quad\left({ }_{0} \nabla_{h}^{\alpha} z\right)(t)=\left(\nabla_{h}{ }_{0} \nabla_{h}^{-(1-\alpha)} z\right)(t)$
ii. $\quad\left({ }_{h} \nabla_{b}^{\alpha} z\right)(t)=(-1)\left(\Delta_{h}{ }_{h} \nabla_{b}^{-(1-\alpha)} z\right)(t)$

Definition 5.11: [9] Let $\alpha>0, n=[\alpha]+1$, then
i. The nabla left $h$ - Caputo discrete fractional difference of order $\alpha>0$, starting at $a_{h}(\alpha)=a+(n-1) h=a+[\alpha] h$, is

$$
\left(\begin{array}{l}
C \\
a_{h}(\alpha)
\end{array} \nabla^{\alpha} z\right)(t)=\left(a_{h}(\alpha) \nabla_{h}^{-(n-\alpha)} \nabla_{h}^{n} z\right)(t), \quad t \in \mathrm{~N}_{a+n h, h}
$$

ii. The nabla right $h$ - Caputo discrete fractional difference of order $\alpha>0$, ending at $b_{h}(\alpha)=b-(n-1) h=b-[\alpha] h$, is

$$
\left({ }_{h}^{C} \nabla_{b_{h}(\alpha)}^{\alpha} z\right)(t)=\left({ }_{h} \nabla_{b_{h}(\alpha) \ominus}^{-(n-\alpha)} \Delta_{h}^{n} z\right)(t), \quad t \in \mathbb{N}_{b-n h, h} \mathrm{~N}
$$

where $\ominus \Delta_{h}^{n}=(-1)^{n} \Delta_{h}^{n}$.

Remark 5.5: In Particular at $0<\alpha<1$, and $a=0$, then we have $n=1$,
$b_{h}(\alpha)=b$ and $a_{h}(\alpha)=0$, so,
i. $\quad\left({ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t)=\left({ }_{0} \nabla_{h}^{-(1-\alpha)} \nabla_{h} z\right)(t), \quad t \in \mathrm{~N}_{h, h}$
ii. $\quad\left({ }_{h}^{C} \nabla_{b}^{\alpha} z\right)(t)=\left({ }_{h} \nabla_{b}^{-(1-\alpha)}\left(-\Delta_{h}\right) z\right)(t), \quad t \in_{-h, h} \mathrm{~N}$

Lemma 5.2: [9] The relation between nabla left $h-R L$ and nabla left $h-$ Caputo fractional difference is:
i. For any $a \in \mathbb{R}$, we have

$$
\left(a \nabla_{h}^{-\alpha} \nabla_{h} z\right)(t)=\left(\nabla_{h} \nabla_{h}^{-\alpha} z\right)(t)-\frac{(t-a)_{h}^{\overline{\alpha-1}}}{\Gamma(\alpha)} z(a),
$$

ii. For $a=0$, and $n=1$, we have

$$
\begin{aligned}
& \left({ }_{0} \nabla_{h}^{-\alpha} \nabla_{h} z\right)(t)=\left(\nabla_{h}{ }_{0} \nabla_{h}^{-\alpha} z\right)(t)-\frac{(t)_{h}^{\overline{\alpha-1}}}{\Gamma(\alpha)} z(0) \\
& \begin{aligned}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t) & =\left({ }_{0} \nabla_{h}^{-(1-\alpha)} \nabla_{h} z\right)(t) \\
& =\left(\nabla_{h}{ }_{0} \nabla_{h}^{-(1-\alpha)} z\right)(t)-\frac{(t)_{h}^{\bar{\alpha}}}{\Gamma(1-\alpha)} z(0) \\
& =\left({ }_{0} \nabla_{h}^{\alpha} z\right)(t)-\frac{(t)_{h}^{\bar{\alpha}}}{\Gamma(1-\alpha)} z(0)
\end{aligned}
\end{aligned}
$$

Lemma 5.3: [9] For any $b \in \mathbb{R}, 0<\alpha<1$, and $n=1$, the relation between nabla right $h-R L$ and nabla right $h-C a p u t o$ fractional difference is

$$
\left({ }_{h} \nabla_{b}^{-\alpha}\left(-\Delta_{h}\right) z\right)(t)=\left(-\Delta_{h h} \nabla_{b}^{-\alpha} z\right)(t)-\frac{(b-t)_{h}^{\overline{\alpha-1}}}{\Gamma(\alpha)} z(b),
$$

hence,

$$
\left({ }_{h}^{C} \nabla_{b}^{\alpha} z\right)(t)=\left({ }_{h} \nabla_{b}^{\alpha} z\right)(t)-\frac{(b-t)_{h}^{\overline{1-\alpha}}}{\Gamma(\alpha)} z(b)
$$

Proposition 5.5: [9] For $\alpha>0, h>0$ and $z$ defined on $\mathbb{N}_{a, h}$

$$
\text { i. } \quad\left({ }_{a} \nabla_{h}^{-\alpha} \quad{ }_{a} \nabla_{h}^{\alpha} z\right)(t)=z(t),
$$

ii. $\quad\left({ }_{a} \nabla_{h}^{\alpha} \quad{ }_{a} \nabla_{h}^{-\alpha} z\right)(t)=z(t), \quad \alpha \notin \mathbb{N}$
iii. $\quad\left(\begin{array}{l}a \\ \nabla_{h}^{\alpha}\end{array} \quad{ }_{a} \nabla_{h}^{-\alpha} z\right)(t)=z(t)-\sum_{k=0}^{n-1} \frac{(t-a)_{n}^{(k)} \Delta^{k} z(a)}{k!}, \alpha \in \mathbb{N}$

Proposition 5.6: For $\alpha>0, h>0$ and $z$ defined on $\mathbb{N}_{b, h}$ we have for
i. $\quad\left({ }_{h} \nabla_{b}^{\alpha} \quad{ }_{h} \nabla_{b}^{-\alpha} z\right)(t)=z(t)$,
ii. $\quad\left({ }_{h} \nabla_{b}^{-\alpha} \quad{ }_{h} \nabla_{b}^{\alpha} z\right)(t)=z(t), \quad \alpha \notin \mathbb{N}$
iii. $\quad\left({ }_{h} \nabla_{b}^{-\alpha} \quad{ }_{h} \nabla_{b}^{\alpha} z\right)(t)=z(t)-\sum_{k=0}^{n-1} \frac{(b-t)_{n}^{(k)}(-1)^{k} \nabla_{h}^{k} z(b)}{k!}, \alpha \in \mathbb{N}$

Theorem 5.1: [9] Let $\alpha>0, h>0, \beta>-1$.Then,
i. $\quad{ }_{a} \nabla_{h}^{-\alpha}(t-a)_{h}^{\bar{\beta}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}(t-a)_{h}^{\overline{\alpha+\beta}}$
ii. In particular for $a=0$ and $0<\alpha<1$

$$
{ }_{0} \nabla_{h}^{-\alpha}(t)_{h}^{\overline{-\alpha}}=\frac{\Gamma(1-\alpha)}{\Gamma(-\alpha+1+\alpha)}(t)_{h}^{\overline{0}}=\Gamma(1-\alpha) .
$$

iii. $\quad a_{h} \nabla_{h}^{\alpha}(t-a)_{h}^{\bar{\beta}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)_{h}^{\overline{\beta-\alpha}}$
iv. $\quad\left({ }_{a} \nabla_{h}^{\alpha} 1\right)(t)=\frac{(t-a)_{h}^{\overline{-\alpha}}}{\Gamma(1-\alpha)}$
v. $\quad\left({ }_{a} \nabla_{h}^{-\alpha} 1\right)(t)=\frac{(t-a)_{h}^{\bar{\alpha}}}{\Gamma(1+\alpha)}$
vi. $\quad\left({ }_{h} \nabla_{b}^{-\alpha} 1\right)(t)=\frac{(b-t)_{h}^{\bar{\alpha}}}{\Gamma(1+\alpha)}$
vii.

$$
\left({ }_{h}^{C} \nabla_{b}^{-\alpha} 1\right)(t)=0
$$

Lemma 5.4: [27] For $\alpha<1$, then: $\quad \Gamma(1-\alpha)=\frac{\pi}{\sin (\pi \alpha) \Gamma(\alpha)}$

### 5.3 Nabla $h$-Discrete Laplace Transform

Definition 5.12: [9] The nabla $h$-discrete exponential kernel can be expressed as:

$$
{ }_{h} \widehat{\boldsymbol{e}}_{\ominus s}(t, a)=(1-h s)^{\frac{t a}{h}}
$$

where $\ominus s=\frac{-s}{1-h s}$ and $t \in \mathbb{N}_{a, h}$.

When $a=0$, then it will be

$$
{ }_{h} \widehat{e}_{\ominus s}(t, 0)=(1-h s)^{\frac{t}{h}} .
$$

Definition 5.13: [9] Suppose $z(t)$ is defined on $\mathbb{N}_{a, h},|1-h s|<1$, and
$0<\alpha<1$. Then the nabla $h$-discrete Laplace transform of $z$ is defined by

$$
\begin{aligned}
L_{a, h}\{z(t)\}(s) & =\int_{a}^{\infty} h \hat{e}_{\ominus s}^{\rho}(t, a) z(t) \nabla_{h} t \\
& =\int_{a}^{\infty} \frac{h \hat{e}_{\ominus s}(t, a)}{1-h s} z(t) \nabla_{h} t \\
& =\int_{a}^{\infty}(1-h s)^{\frac{t-a-h}{h}} z(t) \nabla_{h} t \\
& =h \sum_{i=a / h+1}^{\infty}(1-h s)^{i-a / h-1} z(i h)
\end{aligned}
$$

Lemma 5.5: [9] For any $\alpha>0$,

$$
L_{a, h}\left\{\left({ }_{a} \nabla_{h}^{-\alpha} z\right)(t)\right\}(s)=s^{-\alpha} Z(s)-z(0)
$$

where

$$
L_{a, h}\{z(t)\}(s)=Z(s)
$$

Lemma 5.6: [9] Let $z$ be a function defined on $\mathbb{N}_{a, h}$.

Then

$$
L_{a, h}\left\{\nabla_{h} z(t)\right\}(s)=s Z(s)-z(a)
$$

Lemma 5.7: [9] For any $\alpha \in \mathrm{R} \backslash\{\ldots,-2,-1,0\}$ and $|1-h s|<1$, we have:

$$
\begin{gathered}
\left.L_{a, h}\left\{(t-a)_{h}^{-\bar{\alpha}}\right)\right\}(s)=s^{-(1-\alpha)} \Gamma(1-\alpha) \\
\left.L_{0, h}\left\{(t)_{h}^{\overline{-\alpha}}\right)\right\}(s)=s^{-(1-\alpha)} \Gamma(1-\alpha)
\end{gathered}
$$

Lemma 5.8: The nabla $h$-delay discrete Laplace transform of $z(t-\tau)$ :

$$
L_{a, h}\{z(t-\tau)\}(s)=h(1-h s)^{\frac{\tau}{h}} \sum_{j=\frac{a}{h}+1-\tau / h}^{0}(1-h s)^{j-1-\frac{a}{h}} Z(j h)+h(1-h s)^{\frac{\tau}{h}} Z(s)
$$

## Proof:

$$
\begin{aligned}
L_{a, h}\{z(t-\tau)\}(s) & =h \sum_{i=a / h+1}^{\infty}(1-h s)^{i-1-a / h} z(i h-\tau) \\
& =h \sum_{j=a / h+1-\frac{\tau}{h}}^{\infty}(1-h s)^{j-1+\frac{\tau}{h}-a / h} z(j h) \\
& =h(1-h s)^{\frac{\tau}{h}} \sum_{j=a / h+1-\frac{\tau}{h}}^{\infty}(1-h s)^{j-1-a / h} z(j h) \\
& =h(1-h s)^{\frac{\tau}{h}}\left(\sum_{j=a / h+1-\frac{\tau}{h}}^{0}(1-h s)^{j-1-a / h} z(j h)+\sum_{j=a / h+1}^{\infty}(1-h s)^{j-1-a / h} z(j h)\right) \\
& =h(1-h s)^{\frac{\tau}{h}}\left(\sum_{j=a / h+1-\frac{\tau}{h}}^{0}(1-h s)^{j-1-a / h} z(j h)+Z(s)\right) \\
& =h(1-h s)^{\frac{\tau}{h}} \sum_{j=a / h+1-\frac{\tau}{h}}^{0}(1-h s)^{j-1-a / h} z(j h)+h(1-h s)^{\frac{\tau}{h}} Z(s)
\end{aligned}
$$

In particular, when $a=0$,
$L_{0, h}\{z(t-\tau)\}(s)=h(1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)+h(1-h s)^{\frac{\tau}{\hbar}} Z(s)$

Lemma 5.9: The nabla $h$-discrete Caputo fractional Laplace transform has the
form:

$$
\begin{aligned}
L_{0, h}\left\{{ }_{0}^{C} \nabla_{h}^{\alpha} z(t)\right\}(s) & =L_{0, h}\left\{{ }_{0} \nabla_{h}^{\alpha} z(t)\right\}(s) \\
& =s^{\alpha} Z(s)-s^{\alpha-1} z(0)
\end{aligned}
$$

Proof: By using Lemma 5.2 and Lemma 5.7, we have

$$
\begin{aligned}
L_{0, h} & \left\{{ }_{0}^{C} \nabla_{h}^{\alpha} z(t)\right\}(s)=L_{0, h}\left\{{ }_{0} \nabla_{h}^{\alpha} z(t)-\frac{(t)_{h}^{\overline{-\alpha}}}{\Gamma(1-\alpha)} z(0)\right\}(s) \\
& =L_{0, h}\left\{{ }_{0} \nabla_{h}^{\alpha} z(t)\right\}(s)-L_{0, h}\left\{\frac{(t)_{h}^{\overline{-\alpha}}}{\Gamma(1-\alpha)} z(0)\right\}(s) \\
& =L_{0, h}\left\{\nabla_{h_{0}} \nabla_{h}^{-(1-\alpha)} z(t)\right\}(s)-s^{-(1-\alpha)} \Gamma(1-\alpha) \frac{z(0)}{\Gamma(1-\alpha)} \\
& =s L_{0, h}\left\{\left({ }_{0} \nabla_{h}^{-(1-\alpha)} z\right)(t)\right\}-{ }_{0} \nabla_{h}^{-(1-\alpha)}(0)-s^{\alpha-1} z(0) \\
& =s s^{-(1-\alpha)} Z(s)-0-s^{\alpha-1} z(0) \\
& =s^{\alpha} Z(s)-s^{\alpha-1} z(0)
\end{aligned}
$$

Result 5.1: If $z(t)>0$, then $Z(s)>0$ for all $0<1-h s<1$.

### 5.4 The Discrete $Q$-Operator

Definition 5.14: [9] The $Q$-operator action, which denoted by
$(Q z)(t)=z(a+b-t)$, is a tool that is used dually to link right and left type fractional differences and sums and to transform left type $h$-fractional sums and differences equations to right ones and vice versa.

Lemma 5.10: [9] Assume $b=a+k h, h>0$, and $z(t) \in \mathbb{N}_{a, h} \cap \mathbb{N}_{b, h}$, then

$$
Q \Delta_{h} z(t)=-\nabla_{h} Q z(t)
$$

Proof: By using Definition (5.1), yields

$$
\begin{aligned}
\Delta_{h} z(t)= & \frac{z(t+h)-z(t)}{h} \\
Q \Delta_{h} z(t) & =\frac{Q z(t+h)-Q z(t)}{h} \\
& =\frac{z(a+b-t-h)-z(a+b-t)}{h} \\
& =-\left(\frac{z(a+b-t)-z(a+b-t-h)}{h}\right) \\
& =-\nabla_{h} z(a+b-t) \\
& =-\nabla_{h} Q z(t)
\end{aligned}
$$

Lemma 5.11: Assume $b=a+k h$, for some $k \in \mathbb{N}$. Then
i. $\quad Q Q z(t)=z(t)$
ii. $\quad Q z(a)=z(b)$

## Proof:

i. $\quad Q Q z(t)=Q z(a+b-t)$

$$
=z(t)
$$

ii. $\quad Q z(a)=z(a+b-a)$

$$
=z(b)
$$

Theorem 5.2: [9] For $\alpha>0, h>0, z(t) \in \mathbb{N}_{a, h} \cap \mathbb{N}_{b, h}$, and $b=a+k h$, for some $k \in \mathrm{~N}$. Then,
i. $\quad\left({ }_{a} \nabla_{h}^{-\alpha} Q z\right)(t)=Q\left({ }_{h} \nabla_{b}^{-\alpha} z\right)(t)=\left({ }_{h} \nabla_{b}^{-\alpha} z\right)(a+b-t)$
ii. $\quad\left({ }_{a} \nabla_{h}^{\alpha} Q z\right)(t)=Q\left({ }_{h} \nabla_{b}^{\alpha} z\right)(t)=\left({ }_{h} \nabla_{b}^{\alpha} z\right)(a+b-t)$
iii. $\quad\left({ }_{a}^{C} \nabla_{h}^{\alpha} Q z\right)(t)=Q\left({ }_{h}^{C} \nabla_{b}^{\alpha} z\right)(t)=\left({ }_{h}^{C} \nabla_{b}^{\alpha} z\right)(a+b-t)$.

Theorem 5.3: [9] For $\alpha>0, h>0, z(t) \in \mathbb{N}_{a, h} \cap \mathbb{N}_{b, h}$, and $b=a+k h$, for some $k \in \mathrm{~N}$. Then,
i. $\quad\left({ }_{a} \Delta_{h}^{-\alpha} Q z\right)(t)=Q\left({ }_{h} \Delta_{b}^{-\alpha} z\right)(t)=\left({ }_{h} \Delta_{b}^{-\alpha} z\right)(a+b-t)$
ii. $\quad\left({ }_{a} \Delta_{h}^{\alpha} Q z\right)(t)=Q\left({ }_{h} \Delta_{b}^{\alpha} z\right)(t)=\left({ }_{h} \Delta_{b}^{\alpha} z\right)(a+b-t)$.
iii. $\quad\left({ }_{a}^{C} \Delta_{h}^{\alpha} Q z\right)(t)=Q\left({ }_{h}^{C} \Delta_{b}^{\alpha} z\right)(t)=\left({ }_{h}^{C} \Delta_{b}^{\alpha} z\right)(a+b-t)$.

## CHAPTER 6

## Oscillation Criteria for Fractional Delay Difference Equations

A nonzero solution of a fractional delay difference equation is said to be oscillatory if it is neither negative nor positive, and this equation is said to be oscillatory if all solutions of it are oscillatory. The solution is called nonoscillatory if it is eventually negative or eventually positive.

We have reviewed in the former chapters many results related to the oscillation of delay differential equations and of fractional delay differential equations. However, the oscillation of fractional delay difference equations has yet to be investigated.

In this chapter we lay out the main results of our work. We state and prove sufficient conditions for the oscillation of two main classes of fractional delay difference equations. Numerical algorithms and examples are provided to illustrate our main results.

### 6.1 Oscillation Criteria for Fractional Delay Difference Equations with One Positive Coefficient

We study oscillation of Caputo $h$-fractional delay difference equation of the form

$$
\begin{equation*}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t)+p z(t-\tau)=0, \quad 0<\alpha<1 \tag{6.1}
\end{equation*}
$$

$t \in \mathbb{N}_{h, h}, p \in \mathbb{R}^{+}$and $\tau \in \mathbb{N}_{0, h}$ but fixed.

$$
\begin{equation*}
z(t)=\phi(t) \text { for } t \in \mathrm{~N}_{-\tau, h}^{0} . \tag{6.2}
\end{equation*}
$$

Later on we study oscillation of a generalizaion of (6.1) and (6.2) of the form

$$
\begin{equation*}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t)+\sum_{i=1}^{m} p_{i} z\left(t-\tau_{i}\right)=0, \quad 0<\alpha<1 \tag{6.3}
\end{equation*}
$$

where $t \in \mathrm{~N}_{h, h}, p_{i}>0, \tau_{i} \in \mathrm{~N}_{h, h}$ for all $i=1,2, \ldots, m$ and let $\tau=\max \left\{\tau_{1}, \tau_{2}, \ldots, \tau_{m}\right\}$.

$$
\begin{equation*}
z(t)=\phi(t) \text { on } t \in \mathrm{~N}_{-\tau, h}^{0} . \tag{6.4}
\end{equation*}
$$

Next we give sufficient conditions in terms of a characteristic equation for the oscillation of equations (6.1)-(6.2).

Theorem 6.1: Assume that $p \in \mathbb{R}^{+}, \tau \in \mathbb{N}_{h, h}$ and $t \in \mathbb{N}_{h, h}$. Let $\alpha \in(0,1)$ be the quotient of two odd natural numbers , $0<h<1$. If the equation

$$
\begin{aligned}
F(s) & =s^{\alpha}+p_{h} \hat{e}_{\ominus s}(\tau, 0) \\
& =s^{\alpha}+p h(1-h s)^{\tau / h}
\end{aligned}
$$

has no zeros in $\mathbb{R}$, then all solutions of equations (6.1)-(6.2) oscillate.

## Proof:

Assume on the contrary that there is a nonoscilatory solution $z(t)$ of equation (6.1). Without losing of generality, we suppose that $z(t)$ is an ultimately positive solution of equation (6.1) that is, there exists $T \in \mathbb{N}_{h, h}$, where $z(t)>0$ for $t \in \mathbb{N}_{-\tau, h}$.

Let $0<1-h s<1$, Then we insert the nabla $h$-discrete Laplace transform
starting at 0 to equation (6.1), so we get

$$
L_{0, h}\left\{{ }_{0}^{C} \nabla_{h}^{\alpha} z(t)\right\}(s)+L_{0, h}\{p z(t-\tau)\}=0
$$

By using Lemma 5.8 and Lemma 5.9,

$$
\begin{aligned}
& s^{\alpha} Z(s)-s^{\alpha-1} z(0)+p h(1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)+p h(1-h s)^{\frac{\tau}{h}} Z(s)=0 \\
& \left(s^{\alpha}+p h(1-h s)^{\frac{\tau}{h}}\right) Z(s)-s^{\alpha-1} z(0)+p h(1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)=0
\end{aligned}
$$

Let $\Phi(s)=s^{\alpha-1} z(0)-p h(1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)$ then,

$$
F(s) Z(s)-\Phi(s)=0
$$

$$
F(s) Z(s)=\Phi(s)
$$

By taking $s \rightarrow-\infty$ we reach an inconsistency because

$$
\begin{aligned}
\lim _{s \rightarrow-\infty} F(s) & =\lim _{s \rightarrow-\infty}\left(s^{\alpha}+p_{h} \hat{\mathrm{e}}_{\ominus s}(\tau, 0)\right) \\
& =\lim _{s \rightarrow-\infty}\left(s^{\alpha}+p h(1-h s)^{\frac{\tau}{h}}\right)>0
\end{aligned}
$$

and since $z(t)>0$ for all $t \in \mathbb{N}_{-\tau, h}$, we have $\lim _{s \rightarrow-\infty} Z(s)>0$,

$$
\begin{aligned}
\lim _{s \rightarrow-\infty} \Phi(s) & =\lim _{s \rightarrow-\infty}\left(s^{\alpha-1} z(0)-p(1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)\right) \\
& =\lim _{s \rightarrow-\infty}\left(s^{\alpha-1} z(0)\right)-p \lim _{s \rightarrow-\infty}\left((1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)\right) \\
& =0-p \lim _{s \rightarrow-\infty}\left((1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)\right) \\
& =-\infty<0
\end{aligned}
$$

Then $\Phi(s)$ becomes eventually negative which is a contradiction because both $F(s)$ and $Z(s)$ are positive for all real negative $s$.

## Lemma 6.1:

$$
\begin{aligned}
L_{a, h}\left\{\sum_{i=1}^{m} p_{i} z\left(t-\tau_{i}\right\}(s)\right. & =\sum_{i=1}^{m} p_{i} \sum_{j=\left(\frac{a}{h}\right)+1-\frac{\tau i}{h}}^{0} h(1-h s)^{j-1+\frac{\tau i}{h}-\frac{a}{h}} Z(j h) \\
& +\sum_{i=1}^{m} p_{i} h(1-h s)^{\tau i / h} Z(s)
\end{aligned}
$$

## Proof:

By using Lemma 5.8 the nabla $h$-delay discrete Laplace transform

$$
\begin{aligned}
L_{a, h}\left\{\sum_{i=1}^{m} p_{i} z\left(t-\tau_{i}\right\}(s)\right. & =\sum_{i=1}^{m} p_{i} \sum_{j=\frac{a}{h}+1}^{\infty} h(1-h s)^{i-1-\frac{a}{h}} Z(i h-\tau) \\
& =\sum_{i=1}^{m} p_{i} \sum_{j=\left(\frac{a}{h}\right)+1-\frac{\tau i}{h}}^{0} h(1-h s)^{j-1+\frac{\tau i}{h}-\frac{a}{h}} Z(j h) \\
& =\sum_{i=1}^{m} p_{i}(1-h s)^{\frac{\tau i}{h}} \sum_{j=\left(\frac{a}{h}\right)+1-\frac{\pi i}{h}}^{0} h(1-h s)^{j-1-\frac{a}{h}} Z(j h) \\
& +\sum_{i=1}^{m} p_{i}(1-h s)^{\frac{\tau i}{h}} \sum_{j=\left(\frac{a}{h}\right)+1}^{\infty} h(1-h s)^{j-1-\frac{a}{h}} Z(i h) \\
& =\sum_{i=1}^{m} p_{i} \sum_{j=\left(\frac{a}{h}\right)+1-\frac{\tau i}{h}}^{0} h(1-h s)^{j-1+\frac{\tau i}{h}-\frac{a}{h}} Z(j h) \\
& +\sum_{i=1}^{m} p_{i} h(1-h s)^{\tau i / h} Z(s)
\end{aligned}
$$

In particular, for $a=0$, we get

$$
L_{0, h}\left\{\sum_{i=1}^{m} p_{i} z\left(t-\tau_{i}\right\}(s)=\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau i}{h}}^{0} h(1-h s)^{j-1+\tau i / h} z(j h)+\sum_{i=1}^{m} p_{i} h(1-h s)^{\tau i / h} Z(s)\right.
$$

Theorem 6.2: Assume that $p_{i}>0, \tau_{i} \in \mathbb{N}_{h, h}$ for all $i=1,2, \ldots, m$ and let $\alpha \in(0,1)$ be the quotient of two odd natural numbers, $0<h<1$, If the equation

$$
F(s)=s^{\alpha}+\sum_{i=1}^{m} p_{i h} \hat{e}_{\ominus s}\left(\tau_{i}, 0\right)=s^{\alpha}+\sum_{i=1}^{m} p_{i} h(1-h s)^{\frac{\tau i}{h}}
$$

has no zeros in $\mathbb{R}$, then every solution of equations (6.3)-(6.4) oscillates.

## Proof:

Assume on the contrary that there is a nonoscilatory solution $z(t)$ of equation (6.3). Without losing of generality, we suppose that $z(t)$ is an ultimately positive solution of equation (6.3) that is, there exists $T \in \mathbb{N}_{h, h}$, where $z(t)>0$ for $t \in \mathbb{N}_{-\tau, h}$.

Let $0<1-h s<1$, Then we insert the nabla $h$-discrete Laplace transform beginning at 0 to (6.3), so we have

$$
L_{0, h}\left\{{ }_{0}^{C} \nabla_{h}^{\alpha} z(t)\right\}(s)+L_{0, h}\left\{\sum_{i=1}^{m} p_{i} z\left(t-\tau_{i}\right)\right\}(s)=0
$$

By using Lemma 5.9 and Lemma 6.1, we get
$\left(s^{\alpha}+\sum_{i=1}^{m} p_{i} h(1-h s)^{\frac{\pi i}{h}}\right) Z(s)-s^{\alpha-1} z(0)+\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{\tau i}{h}} z(j h)=0$

Let,

$$
\Phi(s)=s^{\alpha-1} z(0)-\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{t i}{h}} z(j h)
$$

Then,

$$
\begin{gathered}
F(s) Z(s)-\Phi(s)=0 \\
F(s) Z(s)=\Phi(s)
\end{gathered}
$$

By taking $s \rightarrow-\infty$ we reach an inconsistency because

$$
\begin{aligned}
\lim _{s \rightarrow-\infty} F(s) & =\lim _{s \rightarrow-\infty}\left(s^{\alpha}+\sum_{i=1}^{m} p_{i} h(1-h s)^{\frac{\pi i}{h}}\right) \\
& =\infty>0
\end{aligned}
$$

and since $z(t)>0$ for all $t \in \mathbb{N}_{-\tau, h}$, we have $\lim _{s \rightarrow-\infty} Z(s)>0$,

$$
\begin{aligned}
\lim _{s \rightarrow-\infty} \Phi(s) & =\lim _{s \rightarrow-\infty}\left(s^{\alpha-1} z(0)-\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{\pi i}{h}} z(j h)\right) \\
& =\lim _{s \rightarrow-\infty}\left(s^{\alpha-1} z(0)-\lim _{s \rightarrow-\infty}\left(\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{i i}{h}} z(j h)\right)\right. \\
& =0-\lim _{s \rightarrow-\infty}\left(\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{i i}{h}} z(j h)\right) \\
& =-\infty<0
\end{aligned}
$$

Then $\Phi(s)$ becomes ultimately negative which is a contradiction because both $F(s)$ and $Z(s)$ are positive for all real negative $s$.

### 6.2 Oscillation Criteria for Fractional Delay Difference Equations

 with Two Positive CoefficientsWe study the oscillation of the Caputo $h$-fractional delay difference equation with two positive coefficients of the form

$$
\begin{equation*}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t)+p z(t-\tau)+q z(t)=0, \quad 0<\alpha<1 \tag{6.5}
\end{equation*}
$$

$t \in \mathbb{N}_{h, h}, \quad p, q \in \mathbb{R}^{+}, \alpha \in(0,1)$ is the quotient of two odd natural numbers and $\tau \in \mathbb{N}_{h, h}$ but fixed.

$$
\begin{equation*}
z(t)=\phi(t) \text { on } t \in \mathbb{N}_{-\tau, h}^{0} . \tag{6.6}
\end{equation*}
$$

Theorem 6.3: Assume that $p, q \in \mathbb{R}^{+}, \tau \in \mathbb{N}_{h, h}$ and $t \in \mathbb{N}_{h, h}$, and $0<h<1$. If the equation

$$
\begin{aligned}
F(s) & =s^{\alpha}+p_{h} \hat{\mathrm{e}}_{\ominus s}(\tau, 0)+q \\
& =s^{\alpha}+p h(1-h s)^{\frac{\tau}{h}}+q
\end{aligned}
$$

has no zeros in $\mathbb{R}$, then every solution of equations (6.5)-(6.6) has oscillatory behavior.

## Proof:

Assume on the contrary that there is a nonoscilatory solution $z(t)$ of equation (6.5). Without losing of generality, we suppose that $z(t)$ is an ultimately positive solution of equation (6.5) that is, there exists $T \in \mathbb{N}_{h, h}$,
where $z(t)>0$ for $t \in \mathbb{N}_{-\tau, h}$.

Let $0<1-h s<1$, Then we insert the nabla $h$-discrete Laplace transform beginning at 0 to (6.5), so we get

$$
L_{0, h}\left\{{ }_{0}^{C} \nabla_{h}^{\alpha} z(t)\right\}(s)+L_{0, h}\{p z(t-\tau)\}+L_{0, h}\{q z(t)\}=0
$$

by using Lemmas 5.7, 5.8 and Lemma 5.9

$$
\begin{aligned}
& s^{\alpha} Z(s)-s^{\alpha-1} z(0)+p h(1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)+p h(1-h s)^{\frac{\tau}{h}} Z(s)+q Z(s)=0 \\
& \left(s^{\alpha}+p h(1-h s)^{\frac{\tau}{h}}+q\right) Z(s)-s^{\alpha-1} z(0)+p h(1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)=0
\end{aligned}
$$

Let,

$$
\Phi(s)=s^{\alpha-1} z(0)-p h(1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)
$$

Then,

$$
\begin{gathered}
F(s) Z(s)-\Phi(s)=0 \\
F(s) Z(s)=\Phi(s)
\end{gathered}
$$

by Taking $s \rightarrow-\infty$ we reach an inconsistency because:

$$
\begin{aligned}
\lim _{s \rightarrow-\infty} F(s) & =\lim _{s \rightarrow-\infty}\left(s^{\alpha}+p_{h} e_{\ominus s}(\tau, 0)+q\right) \\
& =\lim _{s \rightarrow-\infty}\left(s^{\alpha}+p(1-h s)^{\frac{\tau}{h}}+q\right) \\
& =\infty>0
\end{aligned}
$$

and since $z(t)>0$ for all $t \in \mathbb{N}_{-\tau, h}$, we have $\lim _{s \rightarrow-\infty} Z(s)>0$,

$$
\begin{aligned}
\lim _{s \rightarrow-\infty} \Phi(s) & =\lim _{s \rightarrow-\infty}\left(s^{\alpha-1} z(0)-p(1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)\right) \\
& =\lim _{s \rightarrow-\infty}\left(s^{\alpha-1} z(0)\right)-p \lim _{s \rightarrow-\infty}\left((1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)\right) \\
& =0-p \lim _{s \rightarrow-\infty}\left((1-h s)^{\frac{\tau}{h}} \sum_{j=1-\frac{\tau}{h}}^{0}(1-h s)^{j-1} z(j h)\right) \\
& =-\infty<0
\end{aligned}
$$

Then $\Phi(s)$ becomes ultimately negative which is a contradiction because both $F(s)$ and $Z(s)$ are positive for all real negative $s$.

In the following theorem we study oscillation behavior of a generalizaion of (6.5) and (6.6) of the form

$$
\begin{equation*}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} Z\right)(t)+\sum_{i=1}^{m} p_{i} Z\left(t-\tau_{i}\right)+q Z(t)=0, \quad 0<\alpha<1 \tag{6.7}
\end{equation*}
$$

where $t \in \mathbb{N}_{h, h}, q>0, p_{i}>0, \tau_{i} \in \mathbb{N}_{h, h}$ for all $i=1,2, \ldots, m,$.

$$
\begin{equation*}
z(t)=\phi(t) \text { on } t \in \mathbb{N}_{-\tau, h}^{0} . \tag{6.8}
\end{equation*}
$$

where $\tau=\max \left\{\tau_{1}, \tau_{2}, \ldots, \tau_{m}\right\}$.

Theorem 6.4: Assume that $p_{i}, q>0, \tau_{i} \in \mathbb{N}_{h, h}$ for all $i=1,2, \ldots, m$, and $t \in \mathbb{N}_{h, h}$, let $\alpha \in(0,1)$ be the quotient of two odd natural numbers, $0<h<1$, If the equation

$$
F(s)=s^{\alpha}+\sum_{i=1}^{m} p_{i h} \hat{e}_{\ominus s}\left(\tau_{i}, 0\right)+q
$$

$$
=s^{\alpha}+\sum_{i=1}^{m} p_{i}(1-h s)^{\frac{\tau i}{h}}+q
$$

has no zeros in $\mathbb{R}$, then every solution of equations (6.7)-(6.8) oscillates.

## Proof:

Assume on the contrary that there is a nonoscilatory solution $z(t)$ of equation (6.7). Without losing of generality, we suppose that $z(t)$ is an ultimately positive solution of equation (6.7) that is, there exists $T \in \mathbb{N}_{h, h}$, where $z(t)>0$ for $t \in \mathbb{N}_{-\tau, h}$.

Let $0<1-h s<1$, Then we insert the nabla $h$-discrete Laplace transform beginning at 0 to (6.7) so we have

$$
L_{0, h}\left\{{ }_{0}^{C} \nabla_{h}^{\alpha} z(t)\right\}(s)+L_{0, h}\left\{\sum_{i=1}^{m} p_{i} z\left(t-\tau_{i}\right)\right\}(s)+L_{0, h}\{q z(t)\}(s)=0
$$

by using Lemmas 5.7, 5.9 and 6.2,

$$
\begin{aligned}
& s^{\alpha} Z(s)-s^{\alpha-1} Z(0)+\sum_{i=1}^{m} p_{i h} \widehat{\mathrm{e}}_{\ominus \lambda}\left(\tau_{i}, 0\right) \int_{-\tau_{i}}^{0}(1-h s)^{\frac{t-h}{h}} Z(t) \nabla_{h} t \\
& \quad+\sum_{i=1}^{m} p_{i h} \hat{\mathrm{e}}_{\ominus \lambda}\left(\tau_{i}, 0\right) Z(s)+q Z(s)=0 \\
& \left(s^{\alpha}+\sum_{i=1}^{m} p_{i} h(1-h s)^{\frac{\tau i}{h}}+q\right) Z(s)-s^{\alpha-1} Z(0)+\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{\tau i}{h} Z(j h)=0}
\end{aligned}
$$

Let,

$$
\Phi(s)=s^{\alpha-1} z(0)-\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{\tau i}{h}} Z(j h)
$$

Then,

$$
\begin{gathered}
F(s) Z(s)-\Phi(s)=0 \\
F(s) Z(s)=\Phi(s)
\end{gathered}
$$

by taking $s \rightarrow-\infty$ we reach an inconsistency because:

$$
\lim _{s \rightarrow-\infty} F(s)=\lim _{s \rightarrow-\infty}\left(s^{\alpha}+\sum_{i=1}^{m} p_{i} h(1-h s)^{\frac{\tau i}{h}}+q\right)
$$

and since $z(t)>0$ for all $t \in \mathbb{N}_{-\tau, h}$, we have $\lim _{s \rightarrow-\infty} Z(s)>0$,

$$
\begin{aligned}
\lim _{s \rightarrow-\infty} \Phi(s) & =\lim _{s \rightarrow-\infty}\left(s^{\alpha-1} z(0)-\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{i i}{h}} z(j h)\right) \\
& =\lim _{s \rightarrow-\infty}\left(s^{\alpha-1} z(0)-\lim _{s \rightarrow-\infty}\left(\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{t i}{h}} z(j h)\right)\right. \\
& =0-\lim _{s \rightarrow-\infty}\left(\sum_{i=1}^{m} p_{i} \sum_{j=1-\frac{\tau}{h}}^{0} h(1-h s)^{j-1+\frac{\pi i}{h}} z(j h)\right) \\
& =-\infty<0
\end{aligned}
$$

Then $\Phi(s)$ becomes eventually negative which is a contradiction because both $F(s)$ and $Z(s)$ are positive for all real negative $s$.

### 6.3 The action of the $Q$-operator on Right Nabla Fractional

## Difference and Oscillation Criteria

Consider the Caputo right $h$-fractional delay difference equation in the form:

$$
\begin{equation*}
\left({ }_{h}^{C} \nabla_{b}^{\alpha} z\right)(t)+p z(t-\tau)=0, \quad 0<\alpha<1 \tag{6.9}
\end{equation*}
$$

$t \in \mathbb{N}_{0, h} \cap \mathbb{N}_{b . h}, p \in \mathbb{R}^{+}, b=k h$ for some $k \in \mathbb{N}$ and $\tau \in \mathbb{N}_{h, h}$ but fixed.

$$
\begin{equation*}
z(t)=\phi(t) \text { for } t \in \mathbb{N}_{-\tau, h}^{0} \text {. } \tag{6.10}
\end{equation*}
$$

Theorem 6.5: Assume that $p \in \mathbb{R}^{+}, \tau \in \mathbb{N}_{h, h}$ and $t \in \mathbb{N}_{0, h} \cap \mathbb{N}_{b . h}$. Let $\alpha \in(0,1)$ be the quotient of two odd natural numbers , $0<h<1$ and $b=k h$ for some $k \in \mathbb{N}$. If the equation

$$
\begin{aligned}
F(s) & =s^{\alpha}+p_{h} \hat{\mathrm{e}}_{\ominus s}(\tau, 0) \\
& =s^{\alpha}+p h(1-h s)^{\tau / h}
\end{aligned}
$$

has no zeros in $\mathbb{R}$, then every solution of equations (6.9) - (6.10) oscillates.

## Proof:

By applying the $Q$ - operator to (6.9)-(6.10) then,

$$
\begin{gathered}
Q\left({ }_{h}^{C} \nabla_{b}^{\alpha} z\right)(t)+Q(p z(t-\tau))=0, \quad 0<\alpha<1 \\
(Q z)(t)=(Q \phi)(t) \text { for } t \in \mathbb{N}_{-\tau, h}^{0}
\end{gathered}
$$

Then by using Theorem 5.10, we get

$$
\begin{gather*}
\left.\left({ }_{0}^{C} \nabla_{h}^{\alpha} Q z\right)(t)+p(Q z)(t-\tau)\right)=0, \quad 0<\alpha<1  \tag{6.11}\\
(Q z)(t)=(Q \phi)(t) \text { for } t \in \mathbb{N}_{-\tau, h}^{0} \tag{6.12}
\end{gather*}
$$

Let,
$y=(Q z)(t)$, so (6.10) - (6.11) become

$$
\begin{gather*}
\left.\left({ }_{0}^{C} \nabla_{h}^{\alpha} y\right)(t)+p(y)(t-\tau)\right)=0, \quad 0<\alpha<1  \tag{6.13}\\
(y)(t)=(\phi)(b-t) \text { for } t \in \mathbb{N}_{-\tau, h}^{0} \tag{6.14}
\end{gather*}
$$

As we proved that the equation (6.1) oscillates, the equation (6.13) also oscillates because it is just have a shifting to right to (6.1) and reflection around $y$-axis. So, we transform nabla right $h$-Caputo fractional difference of order $\alpha$ ending at $b$, to left one beginning at $a=0$, and prove the oscillatory behavior for it.

### 6.4 Numerical Results

In this section we give numerical examples to present the previous theorems. To achieve this goal we first rewrite each of the equations (6.1), (6.3), (6.5) and (6.7) in an equivalent form. Then we provide a numerical algorithm corresponding to each of the equivalent forms of equations (6.1), (6.3), (6.5) and (6.7).

### 6.4.1 Numerical results for equation (6.1)

In this subsection, a numerical example of theorem 6.1 that related to equation (6.1) will be studied. But first, the following remark should established.

Remark 6.1: Let $z(t)$ be the solution of equation (6.1)

$$
z(t)=z(0)-\frac{p h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} z(i h-\tau)
$$

Proof: Apply ${ }_{0} \nabla_{h}^{-\alpha}$ on equation (6.1)

$$
\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left({ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t)+p\left({ }_{0} \nabla_{h}^{-\alpha} z\right)(t-\tau)=0,
$$

by using Lemma 5.2, we get

$$
\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left({ }_{0} \nabla_{h}^{\alpha} z\right)(t)-\left({ }_{0} \nabla_{h}^{-\alpha}\right) \frac{(t)_{h}^{-\alpha}}{\Gamma(1-\alpha)} z(0)+p\left({ }_{0} \nabla_{h}^{-\alpha} z\right)(t-\tau)=0,
$$

from Proposition 5.3, and Theorem 5.3 we have

$$
z(t)=z(0)-p_{0} \nabla_{h}^{-\alpha} z(t-\tau)
$$

by using Definition 5.9 of left nabla, we obtain

$$
\begin{aligned}
z(t) & =z(0)-\frac{p}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-\rho(i h))_{h}^{\overline{\alpha-1}} z(i h-\tau) h \\
& =z(0)-\frac{p}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-i h+h)_{h}^{\overline{\alpha-1}} z(i h-\tau) h \\
& =z(0)-\frac{p}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha-1} \frac{\Gamma\left(\frac{t-i h+h}{h}+\alpha-1\right)}{\Gamma\left(\frac{t-i h+h}{h}\right)} z(i h-\tau) h \\
& =z(0)-\frac{p}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha} \frac{\Gamma\left(\frac{t}{h}-i+1+\alpha-1\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} z(i h-\tau) \\
& =z(0)-\frac{p h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} z(i h-\tau)
\end{aligned}
$$

## Numerical Algorithm for equation (6.1):

Given $p>0$.

Given $\alpha \in(0,1)$.

Given $h \in(0,1)$.

Given $k \in \mathbb{N}=\{1,2,3, \ldots\}$.

Given $N \in \mathbb{N}=\{1,2,3, \ldots\}$.

Take $\tau=k h$.

Given a function $\varphi$ on the discrete interval $[-\tau, 0]$.

Let $t_{j}=h(j-1)-k h$.

Let $x\left(t_{j}\right)=\varphi\left(t_{j}\right)$ for $j=1, \ldots, k+1$.

For $j=k+2, k+3, \ldots, N$, let

$$
\begin{aligned}
z\left(t_{j}\right) & =z\left(t_{k+1}\right)-\frac{p h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{j-k-1} \frac{\Gamma(j-k-1-i+\alpha)}{\Gamma(j-k-i)} z\left(t_{i+1}\right) \\
& =z\left(t_{k+1}\right)-\frac{p h^{\alpha}}{\Gamma(1-\alpha) \Gamma(\alpha)} \sum_{i=1}^{j-k-1} \frac{\Gamma(j-k-1-i+\alpha)}{\Gamma(j-k-i)} z\left(t_{i+1}\right) \\
& =z\left(t_{k+1}\right)-\frac{p h^{\alpha}}{\Gamma(1-\alpha) \Gamma(\alpha)} \sum_{i=1}^{j-k-1} B(j-k-1-i+\alpha, 1-\alpha) z\left(t_{i+1}\right) \\
& =z\left(t_{k+1}\right)-\frac{p h^{\alpha} \sin (\pi \alpha)}{\pi} \sum_{i=1}^{j-k-1} B(j-k-1-i+\alpha, 1-\alpha) z\left(t_{i+1}\right)
\end{aligned}
$$

where $B$ denotes the beta function. Note that $B$ is a function of two variables.

Remark 6.2: [12] There is no way of calculating the gamma function for values up to 172.5 without getting an infinity, so we replace it by an equivalent form in terms of beta function.

## Example 6.1:

i. Let $p=1.65, \alpha=5 / 11, h=0.5, k=2, N=100$, and $\varphi(t)=t^{2}$.


Figure 6.4.1: Characteristic equation of equation (6.1) with $p=1.65, \alpha=\frac{5}{11}, h=0.5$,

$$
F(s)=s^{0.45}+0.83(1-0.45 s)^{2}
$$



Figure 6.4.2: Oscillatory solution of equation (6.1) with $p=1.65, \alpha=\frac{5}{11}, h=0.5$,
ii. Let $p=0.4, \alpha=\frac{9}{11}, h=0.01, k=100, N=1000$, and $\varphi(t)=2$.


Figure 6.4.3: Characteristic equation of equation (6.1) with $\mathbf{p}=\mathbf{0 . 4}, \boldsymbol{\alpha}=\frac{\mathbf{9}}{\mathbf{1 1}}, \mathbf{h}=\mathbf{0 . 0 1}$,

$$
F(s)=s^{0.81}+0.04(1-0.01 s)^{100}
$$



Figure 6.4.4: Non-oscillatory solution of equation (6.1) with $\mathrm{p}=0.4, \alpha=\frac{9}{11}, \mathrm{~h}=0.01$

### 6.4.2 Numerical results for equation (6.3)

In this subsection, a numerical example of theorem 6.3 that related to equation (6.3) will be studied. But first, the following remark should established.

Remark 6.3: Let $z(t)$ be the solution of equation (6.3)

$$
z(t)=z(0)-\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} \sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right)
$$

Proof: Apply ${ }_{0} \nabla_{h}^{-\alpha}$ on equation (6.3) then we have that

$$
\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left({ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t)+\left({ }_{0} \nabla_{h}^{-\alpha}\right) \sum_{r=1}^{m} p_{r} z\left(t-\tau_{r}\right)=0
$$

by using Lemma 5.2, we get
$\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left({ }_{0} \nabla_{h}^{\alpha} z\right)(t)-\left({ }_{0} \nabla_{h}^{-\alpha}\right) \frac{(t)_{h}^{\overline{-\alpha}}}{\Gamma(1-\alpha)} z(0)+\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left(\sum_{r=1}^{m} p_{r} z\left(t-\tau_{r}\right)\right)=0$,
from Proposition 5.3, and Theorem 5.3 we have

$$
z(t)=z(0)-{ }_{0} \nabla_{h}^{-\alpha} \sum_{r=1}^{m} p_{r} z\left(t-\tau_{r}\right)
$$

by using Definition 5.9 of left nabla, we obtain

$$
\begin{aligned}
z(t) & =z(0)-\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-\rho(i h))_{h}^{\overline{\alpha-1}} \sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right) h \\
& =z(0)-\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-i h+h)_{h}^{\overline{\alpha-1}} \sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right) h \\
& =z(0)-\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha-1} \frac{\Gamma\left(\frac{t-i h+h}{h}+\alpha-1\right)}{\Gamma\left(\frac{t-i h+h}{h}\right)} \sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right) h \\
& =z(0)-\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha} \frac{\Gamma\left(\frac{t}{h}-i+1+\alpha-1\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} \sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right) \\
& =z(0)-\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} \sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right)
\end{aligned}
$$

## Numerical Algorithm for equation (6.3):

Given $p_{1}, p_{2}, \ldots, p_{m}>0$.

Given $\alpha \in(0,1)$.

Given $h \in(0,1)$.

Given $k_{1}, k_{2}, \ldots, k_{3} \in \mathbb{N}=\{1,2,3, \ldots\}$.

Given $N \in \mathbb{N}=\{1,2,3, \ldots\}$.

Take $\tau_{1}=k_{1} h, \tau_{2}=k_{2} h, \ldots, \tau_{m}=k_{m} h$.

Let $k=\max \left\{k_{1}, k_{2}, \ldots, k_{m}\right\}$.

Let $\tau=\max \left\{\tau_{1}, \tau_{2}, \ldots, \tau_{m}\right\}$.

Given a function $\varphi$ on the discrete interval $[-\tau, 0]$.

Let $t_{j}=h(j-1)-k h$

Let $z\left(t_{j}\right)=\varphi\left(t_{j}\right)$ for $j=1, \ldots, k+1$.

For $j=k+2, k+3, \ldots, N$, let

$$
\begin{aligned}
z\left(t_{j}\right) & =z\left(t_{k+1}\right)-\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{r=1}^{m} p_{r} \sum_{r=1}^{m} p_{r} \sum_{i=1}^{j-k_{r}-1} \frac{\Gamma\left(j-k_{r}-1-i+\alpha\right) \Gamma(1-\alpha)}{\Gamma\left(j-k_{r}-i\right)} z\left(t_{i+1}\right) \\
& =z\left(t_{k+1}\right)-\frac{h^{\alpha}}{\Gamma(1-\alpha) \Gamma(\alpha)} \sum_{r=1}^{m} p_{r} \sum_{i=1}^{j-k_{r}-1} \frac{\Gamma\left(j-k_{r}-1-i+\alpha\right) \Gamma(1-\alpha)}{\Gamma\left(j-k_{r}-i\right)} z\left(t_{i+1}\right) \\
& =z\left(t_{k+1}\right)-\frac{h^{\alpha}}{\Gamma(1-\alpha) \Gamma(\alpha)} \sum_{r=1}^{m} p_{r} \sum_{i=1}^{j-k_{r}-1} B\left(j-k_{r}-1-i+\alpha, 1-\alpha\right) z\left(t_{i+1}\right) \\
& =z\left(t_{k+1}\right)-\frac{h^{\alpha} \sin (\pi \alpha)}{\pi} \sum_{r=1}^{m} p_{r} \sum_{i=1}^{j-k_{r}-1} B\left(j-k_{r}-1-i+\alpha, 1-\alpha\right) z\left(t_{i+1}\right)
\end{aligned}
$$

## Example 6.2:

i. Let $p_{1}=0.5, p_{2}=1.1, p_{3}=0.6, \alpha=\frac{9}{11}, h=0.1$,

$$
k_{1}=3, k_{2}=20, k_{3}=100, k=100, N=1000, m=3, \text { and } \varphi(t)=t
$$



Figure 6.4.5: Characteristic equation of equation (6.3) with $p_{1}=0.5, p_{2}=1.1, p_{3}=0.6, \alpha=\frac{9}{11}, h=0.1$,

$$
F(s)=s^{0.81}+0.22(1-0.1 s)^{100}
$$



Figure 6.4.6: Oscillatory solution of equation (6.3) with $\mathbf{p}_{\mathbf{1}}=\mathbf{0 . 5}, \mathbf{p}_{2}=\mathbf{1 . 1}, \mathbf{p}_{\mathbf{3}}=\mathbf{0 . 6}, \boldsymbol{\alpha}=\frac{\mathbf{9}}{\mathbf{1 1}}, \mathbf{h}=\mathbf{0} . \mathbf{1}$,
ii. Let $p_{1}=0.4, p_{2}=0.5, p_{3}=0.3, \alpha=\frac{9}{11}, h=0.001$,

$$
k_{1}=100, k_{2}=200, k_{3}=150, k=200, N=1000, m=3, \text { and } \varphi(t)=5
$$



Figure 6.4.7: Characteristic equation of equation (6.3) with $p_{1}=0.4, \mathrm{p}_{2}=0.5, \mathrm{p}_{3}=0.3, \alpha=\frac{9}{11}, \mathrm{~h}=0.001$

$$
F(s)=s^{0.81}+0.0012(1-0.001 s)^{200}
$$



Figure 6.4.8: Non-oscillatory solution of equation (6.3) with $p_{1}=0.4, p_{2}=0.5, p_{3}=0.3, \alpha=\frac{9}{11}, \mathrm{~h}=0.001$,

### 6.4.3 Numerical results for equation (6.5)

In this subsection, a numerical example of theorem 6.5 that related to equation (6.5) will be studied. But first, the following remark should established.

Remark 6.4: Let $z(t)$ be the solution of equation (6.5)

$$
z(t)=z(0)-\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)}(p z(i h-\tau)+q z(i h)
$$

Proof: Apply ${ }_{0} \nabla_{h}^{-\alpha}$ on equation (6.5) then we have that

$$
\left.\left({ }_{0} \nabla_{h}^{-\alpha}\right){ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t)+p\left({ }_{0} \nabla_{h}^{-\alpha} z\right)(t-\tau)+q\left({ }_{0} \nabla_{h}^{-\alpha} z\right)(t)=0,
$$

by using Lemma 5.2, we get

$$
\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left({ }_{0} \nabla_{h}^{\alpha} z\right)(t)-\left({ }_{0} \nabla_{h}^{-\alpha}\right) \frac{(t)_{h}^{-\alpha}}{\Gamma(1-\alpha)} z(0)+p\left({ }_{0} \nabla_{h}^{-\alpha} z\right)(t-\tau)-q\left({ }_{0} \nabla_{h}^{-\alpha} z\right)(t)=0
$$

from Proposition 5.3, and Theorem 5.3 we have

$$
z(t)=z(0)-p_{0} \nabla_{h}^{-\alpha} z(t-\tau)+q_{0} \nabla_{h}^{-\alpha} z(t)
$$

by using Definition 5.9 of left nabla, we obtain

$$
\begin{aligned}
z(t) & =z(0)-\frac{p}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-\rho(i h))_{h}^{\overline{\alpha-1}} z(i h-\tau) h-\frac{q}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-\rho(i h))_{h}^{\overline{\alpha-1}} z(i h) h \\
& =z(0)-\frac{p}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-i h+h)_{h}^{\alpha-1} z(i h-\tau) h-\frac{q}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-i h+h)_{h}^{\alpha-1} z(i h) h \\
& =z(0)-\frac{p}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha-1} \frac{\Gamma\left(\frac{t-i h+h}{h}+\alpha-1\right)}{\Gamma\left(\frac{t-i h+h}{h}\right)} z(i h-\tau) h-\frac{q}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha-1} \frac{\Gamma\left(\frac{t-i h+h}{h}+\alpha-1\right)}{\Gamma\left(\frac{t-i h h}{h}\right)} z(i h) h \\
& =z(0)-\frac{p}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha} \frac{\Gamma\left(\frac{t}{h}-i+1+\alpha-1\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} z(i h-\tau)-\frac{q}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha} \frac{\Gamma\left(\frac{t}{h}-i+1+\alpha-1\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} z(i h) \\
& =z(0)-\frac{p h \alpha}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} z(i h-\tau)-\frac{q h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} z(i h) \\
& =z(0)-\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)}(p z(i h-\tau)+q z(i h))
\end{aligned}
$$

## Numerical Algorithm for equation (6.5):

Given $p>0$.

Given $q>0$.

Given $\alpha \in(0,1)$.

Given $h \in(0,1)$.

Given $k \in \mathbb{N}=\{1,2,3, \ldots\}$.

Given $N \in \mathbb{N}=\{1,2,3, \ldots\}$.

Take $\tau=k h$.

Given a function $\varphi$ on the discrete interval $[-\tau, 0]$.

Let $t_{j}=h(j-1)-k h$.

Let $z\left(t_{j}\right)=\phi\left(t_{j}\right)$ for $j=1, \ldots, k+1$.

For $j=k+2, k+3, \ldots, N$, let

$$
\begin{aligned}
z\left(t_{j}\right) & =z\left(t_{k+1}\right)-\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{j-k-1} \frac{\Gamma(j-k-1-i+\alpha)}{\Gamma(j-k-i)}\left(p z\left(t_{i+1}\right)+q z\left(t_{i+1+k}\right)\right) \\
& =z\left(t_{k+1}\right)-\frac{h^{\alpha}}{\Gamma(1-\alpha) \Gamma(\alpha)} \sum_{i=1}^{j-k-1} \frac{\Gamma(j-k-1-i+\alpha) \Gamma(1-\alpha)}{\Gamma(j-k-i)}\left(p z\left(t_{i+1}\right)+q z\left(t_{i+1+k}\right)\right) \\
& =z\left(t_{k+1}\right)-\frac{h^{\alpha}}{\Gamma(1-\alpha) \Gamma(\alpha)} \sum_{i=1}^{j-k-1} B(j-k-1-i+\alpha, 1-\alpha)\left(p z\left(t_{i+1}\right)+q z\left(t_{i+1+k}\right)\right) \\
& =\frac{\pi}{\pi+q h^{\alpha} \sin (\pi \alpha) \Gamma(\alpha)} z\left(t_{k+1}\right)-\frac{p h^{\alpha} \sin (\pi \alpha)}{\pi+q h^{\alpha} \sin (\pi \alpha) \Gamma(\alpha)} \sum_{i=1}^{j-k-1} B(j-k-1-i+\alpha, 1-\alpha) z\left(t_{i+1}\right) \\
& \left.-\frac{q h^{\alpha} \sin (\pi \alpha)}{\pi+q h^{\alpha} \sin (\pi \alpha) \Gamma(\alpha)} \sum_{i=1}^{j-k-2} B(j-k-1-i+\alpha, 1-\alpha) z\left(t_{i+1+k}\right)\right)
\end{aligned}
$$

## Example 6.3:

i. Let $p=1.3, q=0.8, \alpha=\frac{9}{11}, h=0.2, k=20, N=900$, and $\varphi(t)=2$.


Figure 6.4.9: Characteristic equation of equation (6.5) with $\mathbf{p}=\mathbf{1 . 3}, \mathbf{q}=\mathbf{0 . 8}, \boldsymbol{\alpha}=\frac{\mathbf{9}}{\mathbf{1 1}}, \mathbf{h}=\mathbf{0} .2$,

$$
F(s)=s^{0.81}+0.26(1-0.2 s)^{20}+0.8
$$



Figure 6.4.10: Oscillatory solution of equation (6.5) with

$$
p=1.3, q=0.8, \alpha=\frac{9}{11}, h=0.2
$$

ii. Let $p=0.3, q=1.6, \alpha=\frac{7}{11}, h=0.01, k=90, N=900$, and $\varphi(t)=2$


Figure 6.4.11: Characteristic equation of equation (6.5) with $\mathbf{p}=\mathbf{0 . 3}, \mathbf{q}=\mathbf{1 . 6}, \boldsymbol{\alpha}=\frac{7}{11}, \mathbf{h}=\mathbf{0 . 0 1}$,

$$
F(s)=s^{0.64}+0.003(1-0.01 s)^{90}+1.6
$$



Figure 6.4.12: Non-oscillatory solution of equation (6.5) with $\boldsymbol{p}=\mathbf{0} .3, \boldsymbol{q}=\mathbf{1 . 6}, \boldsymbol{\alpha}=\frac{\mathbf{7}}{\mathbf{1 1}}, \boldsymbol{h}=\mathbf{0} . \mathbf{0 1}$,

### 6.4.4 Numerical results for equation (6.7)

In this subsection, a numerical example of theorem 6.7 that related to equation (6.7) will be studied. But first, the following remark should established.

Remark 6.5: Let $z(t)$ be the solution of equation (6.7)

$$
z(t)=z(0)-\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} \sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right)+q z(t)
$$

Proof: Apply ${ }_{0} \nabla_{h}^{-\alpha}$ on equation (6.7) then we have that

$$
\begin{aligned}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t)+ & \sum_{r=1}^{m}\left({ }_{0} \nabla_{h}^{-\alpha}\right) p_{r} z\left(t-\tau_{r}\right)+\left({ }_{0} \nabla_{h}^{-\alpha}\right) q z(t)=0, \quad 0<\alpha<1 \\
\left({ }_{0}^{C} \nabla_{h}^{\alpha} z\right)(t) & =-\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left(\sum_{r=1}^{m} p_{r} z\left(t-\tau_{r}\right)+q z(t)\right) \\
& =-\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left(\sum_{r=1}^{m} p_{r} z\left(t-\tau_{r}\right)+q z(t)\right)
\end{aligned}
$$

by using Lemma 5.2, we get

$$
\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left({ }_{0} \nabla_{h}^{\alpha} z\right)(t)-\left({ }_{0} \nabla_{h}^{-\alpha}\right) \frac{(t)_{h}^{-\alpha}}{\Gamma(1-\alpha)} z(0)=-\left({ }_{0} \nabla_{h}^{-\alpha}\right)\left(\sum_{r=1}^{m} p_{r} z\left(t-\tau_{r}\right)+q z(t)\right)
$$

from Proposition 5.3, and Theorem 5.3 we have

$$
z(t)=z(0)-\left({ }_{0} \nabla_{h}^{-\alpha} \sum_{r=1}^{m} p_{r} z\left(t-\tau_{r}\right)+{ }_{0} \nabla_{h}^{-\alpha} q z(t)\right)
$$

by using Definition 5.9 of left nabla, we obtain

$$
\begin{aligned}
z(t) & =z(0)-\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-\rho(i h))_{h}^{\overline{\alpha-1}}\left(\sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right)+q z(t)\right) h \\
& =z(0)-\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-i h+h)_{h}^{\overline{\alpha-1}}\left(\sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right)+q z(t)\right) h \\
& =z(0)-\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha-1} \frac{\Gamma\left(\frac{t-i h+h}{h}+\alpha-1\right)}{\Gamma\left(\frac{t-i h+h}{h}\right)}\left(\sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right)+q z(t)\right) h \\
& =z(0)-\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha} \frac{\Gamma\left(\frac{t}{h}-i+1+\alpha-1\right)}{\Gamma\left(\frac{t}{h}-i+1\right)}\left(\sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right)+q z(t)\right) \\
& \left.=z(0)-\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} \sum_{r=1}^{m} p_{r} z\left(i h-\tau_{r}\right)+q z(t)\right)
\end{aligned}
$$

Numerical Algorithm for equation (6.7):

Given $p_{1}, p_{2}, \ldots, p_{m}>0$.

Given $q>0$.

Given $\alpha \in(0,1)$.

Given $h \in(0,1)$.

Given $k_{1}, k_{2}, \ldots, k_{3} \in \mathbb{N}=\{1,2,3, \ldots\}$.

Given $N \in \mathbb{N}=\{1,2,3, \ldots\}$.

Take $\tau_{1}=k_{1} h, \tau_{2}=k_{2} h, \ldots, \tau_{m}=k_{m} h$.

Let $k=\max \left\{k_{1}, k_{2}, \ldots, k_{m}\right\}$.

Let $\tau=\max \left\{\tau_{1}, \tau_{2}, \ldots, \tau_{m}\right\}$.

Given a function $\varphi$ on the discrete interval $[-\tau, 0]$.

Let $t_{j}=h(j-1)-k h$

Let $z\left(t_{j}\right)=\varphi\left(t_{j}\right)$ for $j=1, \ldots, k+1$.

For $j=k+2, k+3, \ldots, N$, let

$$
\begin{aligned}
z\left(t_{j}\right) & \left.=z(0)-\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{r=1}^{m} p_{r} \sum_{i=1}^{j-k_{r}-1} \frac{\Gamma\left(j-k_{r}-1-i+\alpha\right)}{\Gamma\left(j-k_{r}-i\right)} z\left(t_{i+1}\right)-q \sum_{i=1}^{j-k-1} \frac{\Gamma(j-k-1-i+\alpha)}{\Gamma(j-k-i)} z\left(t_{i+1-k}\right)\right) \\
& =z(0)-\frac{h^{\alpha}}{\Gamma(1-\alpha) \Gamma(\alpha)}\left(\sum_{r=1}^{m} p_{r} \sum_{i=1}^{j-k_{r}-1} \frac{\Gamma\left(j-k_{r}-1-i+\alpha\right) \Gamma(1-\alpha)}{\Gamma\left(j-k_{r}-i\right)} z\left(t_{i+1}\right)\right. \\
& \left.-q \sum_{i=1}^{j-k-1} \frac{\Gamma(j-k-1-i+\alpha) \Gamma(1-\alpha)}{\Gamma(j-k-i)} z\left(t_{i+1-k}\right)\right) \\
& =z(0)-\frac{h^{\alpha} \sin (\pi \alpha)}{\pi}\left(\sum_{r=1}^{m} p_{r} \sum_{i=1}^{j-k_{r}-1} \frac{\Gamma\left(j-k_{r}-1-i+\alpha\right) \Gamma(1-\alpha)}{\Gamma\left(j-k_{r}-i\right)} z\left(t_{i+1}\right)\right. \\
& \left.-q \sum_{i=1}^{j-k-1} \frac{\Gamma(j-k-1-i+\alpha) \Gamma(1-\alpha)}{\Gamma(j-k-i)} z\left(t_{i+1-k}\right)\right) \\
& =\frac{\pi}{\pi+q h^{\alpha} \sin (\pi \alpha) \Gamma(\alpha)} z\left(t_{k+1}\right) \\
& -\frac{h^{\alpha} \sin (\pi \alpha)}{\pi+q h^{\alpha} \sin (\pi \alpha) \Gamma(\alpha)}\left(\sum_{r=1}^{m} p_{r} \sum_{i=1}^{j-k_{r}-1} \frac{\Gamma\left(j-k_{r}-1-i+\alpha\right) \Gamma(1-\alpha)}{\Gamma\left(j-k_{r}-i\right)} z\left(t_{i+1}\right)\right) \\
& \left.-q \sum_{i=1}^{j-k-2} B(j-k-1-i+\alpha, 1-\alpha) z\left(t_{i+1+k}\right)\right)
\end{aligned}
$$

## Example 6.4:

i. Let $p_{1}=0.8, p_{2}=1.2, p_{3}=0.57, \alpha=\frac{9}{11}, h=0.1$,
$k_{1}=3, k_{2}=60, k_{3}=100, k=100, q=1.9 \mathrm{~N}=1100, m=3$, and $\varphi(t)=$ $t+2$


Figure 6.4.13: Characteristic equation of equation (6.7) with $p_{1}=0.8, \mathrm{p}_{2}=1.2, \mathrm{p}_{3}=0.57, \alpha=\frac{9}{11}, \mathrm{~h}=0.1, q=1.9$

$$
F(s)=s^{0.81}+0.257(1-0.01 s)^{100}+1.9
$$



Figure 6.4.14: Oscillatory solution of equation (6.3) with $p_{1}=0.8, \mathrm{p}_{2}=1.2, \mathrm{p}_{3}=0.57, \alpha=\frac{9}{11}, \mathrm{~h}=0.1, \mathrm{q}=1.9$
ii. Let $p_{1}=0.4, p_{2}=0.7, p_{3}=0.5, \alpha=\frac{9}{11}, h=0.001$, $k_{1}=3, k_{2}=60, k_{3}=100, k=100, q=1.9, N=1100, m=3$, and $\varphi(t)=$ $t+2$


Figure 6.4.15: Characteristic equation of equation (6.7) with $p_{1}=0.4, p_{2}=0.7, p_{3}=0.5, \alpha=\frac{9}{11}, \mathrm{~h}=0.001$,

$$
q=1.9, \quad F(s)=s^{0.81}+0.0016(1-0.001 s)^{100}+1.9
$$



Figure 6.4.16: Non-oscillatory solution of equation (6.7) with $p_{1}=0.4, \mathrm{p}_{2}=0.7, \mathrm{p}_{3}=0.5, \alpha=\frac{9}{11}, \mathrm{~h} 0.001, q=1.9$

Remark 6.6: P. Zhu and Q. Xiang [54] gave numerical examples to illustrate their results. In their numerical algorithm the product trapezoidal quadrature formula was used to evaluate the integral resulting from applying the RiemannLiouville integral operator. In our numerical algorithm we do not need to use approximation methods, because we are dealing with a prepared difference operators.

## CHAPTER 7

## Applications of Fractional Delay Difference Equations

Discrete mathematics and delay phenomena are developing rapidly and has important interrelations with many fields as engineering, biology, and the physical sciences, see [15, 16 and 44]

In this chapter we present two classical simple mathematical models on a discrete delay differential equation, which can be applied realistically and we write them as fractional difference delay equations. Then we apply the pervious results to prove the oscillatory of the two models.

Remark 7.1: [42] We say a function $z(t)$, where $t \geq b$, for some $b \in \mathbb{R}$ is oscillatory with respect to $z^{*}$ if there is a sequence $t_{n} \geq 0, \lim _{n \rightarrow \infty} t_{n}=\infty$, and $z\left(t_{n}\right)=z^{*}$. Otherwise, we say it is non-oscillatory. If $z^{*}=0$, we simply call it oscillatory or non-oscillatory.

### 7.1 Shower Temperature Dynamics

A familiar model of delay that it causes oscillatory patterns is when we want to regulate the shower heat recovery system. The water streams at a steady amount from the tap to the shower nozzle and we assume this time is $\tau$ seconds. We would never enter into the shower before modified the warmth. [12].

Consider $T(t)$ to be the temperature at the tap at time $t, t$ is time in seconds, $r$ is the human reaction rate to a non-wanted temperature, $T_{d}$ is the desired temperature.

Let $\tau$ be the time which the water takes it to flow from the tap to the shower head, $T_{0}$ an initial value of temperature [12].

The transition of the temperature can be characterized by the equation

$$
\begin{equation*}
\left.\frac{d T}{d t}=-r[T(t-\tau)]-T_{d}\right] \tag{7.1}
\end{equation*}
$$

The constant $r$ scales our reaction rate to an improper temperature. An active individual would rather have a big value of $r$, but a phlegmatic one would prefer a small value of $r$. But if $r$ is too small, the temperature will modify very slowly and if $r$ is too large, then oscillations may occur which leading to frostbite or burns [12].

Now we want to rewrite (7.1) by using nabla left $h$-Caputo fractional difference of order $0<\alpha<1$ be the ratio of two odd integers, $0<h<1$, so we have

$$
\begin{align*}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} T\right)(t) & =-r\left[T(t-\tau)-T_{d}\right] \\
& =-r T(t-\tau)+r T_{d} \\
\left({ }_{0}^{C} \nabla_{h}^{\alpha} T\right)(t) & +r\left[T(t-\tau)-T_{d}\right]=0 \tag{7.2}
\end{align*}
$$

then,

$$
\left({ }_{0}^{C} \nabla_{h}^{\alpha} y\right)(t)=\left({ }_{0}^{C} \nabla_{h}^{\alpha} T\right)(t)
$$

Where $\left({ }_{0}^{C} \nabla_{h}^{\alpha} y\right)\left(T_{d}\right)=0$

$$
y(0)=T(0)-T_{d}
$$

Then we can write (7.2) as

$$
\begin{gather*}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} y\right)(t)+r y(t-\tau)=0  \tag{7.4}\\
y_{0}=T_{0}-T_{d} \tag{7.5}
\end{gather*}
$$

Then by Theorem (6.1), If the equation

$$
F(s)=s^{\alpha}+r h(1-h s)^{\frac{\tau}{h}}
$$

has no real roots, then every solution of equations (7.4) - (7.5) oscillates.

If we apply ${ }_{0} \nabla_{h}^{-\alpha}$ on equation (7.4) then we have that

$$
\begin{aligned}
& y(t)=y(0)-\mathrm{r} \nabla_{h}^{-\alpha} y(t-\tau) \\
& =y(0)-\frac{r}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-\rho(i h))_{h}^{\overline{\alpha-1}} y(i h-\tau) h \\
& =y(0)-\frac{r}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-i h+h)_{h}^{\overline{\alpha-1}} y(i h-\tau) h \\
& =y(0)-\frac{r}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha-1} \frac{\Gamma\left(\frac{t-i h+h}{h}+\alpha-1\right)}{\Gamma\left(\frac{t-i h+h}{h}\right)} y(i h-\tau) h \\
& =y(0)-\frac{r}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha} \frac{\Gamma\left(\frac{t}{h}-i+1+\alpha-1\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} y(i h-\tau) \\
& =y(0)-\frac{r h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}\left(\frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} y(i h-\tau)\right.
\end{aligned}
$$

## Example 7.1:

i. Let $r=1.65, \alpha=\frac{5}{11}, h=0.5, k=2, N=100, y_{0}=0.5-T_{d}$ and $T_{d}=1$.


Figure 7.1.1: Characteristic equation of equation (7.4) with $r=1.65, \alpha=\frac{5}{11}, h=0.5$,

$$
\mathrm{F}(\mathrm{~s})=s^{0.45}+0.83(1-0.5 s)^{2}
$$



Figure 7.1.2: Oscillatory solution of equation (7.4) with $r=1.65, \alpha=\frac{5}{11}, h=0.5$,


Figure 7.1.3: Oscillatory solution of equation (7.3) with $r=1.65, \alpha=\frac{5}{11}, h=0.5$,
ii. Let $r=0.25, \alpha=\frac{5}{11}, h=0.5, k=2, N=100, y_{0}=0.5-T_{d}$ and $T_{d}=1$.


Figure 7.1.4: Characteristic equation of equation (7.4) with $r=0.25, \alpha=\frac{5}{11}, h=0.5$

$$
\mathrm{F}(\mathrm{~s})=s^{0.45}+0.31(1-0.5 s)^{2}
$$



Figure 7.1.5: Non-oscillatory solution of equation (7.4) with $r=0.25, \alpha=\frac{5}{11}, h=0.5$.


Figure 7.1.6: Non-oscillatory solution of equation (7.3) with $r=0.25, \alpha=\frac{5}{11}, h=0.5$,
iii. Let $r=1, \alpha=\frac{5}{11}, h=0.5, k=2, N=100, y_{0}=0.5-T_{d}$ and $T_{d}=1$.


Figure 7.1.7: Characteristic equation of equation (7.4) with $r=1, \alpha=\frac{5}{11}, h=0.5$,

$$
F(s)=s^{0.45}+0.5(1-0.5 s)^{2}
$$



Figure 7.1.8: Oscillatory solution of equation (7.3) with $r=1, \alpha=\frac{5}{11}, h=0.5$,


Figure 7.1.9: Oscillatory solution of equation (7.4) with $r=1, \alpha=\frac{5}{11}, h=0.5$,

Remark 7.2: Notice that we obtain oscillatory solution in Figures 7.1.3 and 7.1.9 around desired temperature $T_{d}=1$

### 7.2 Fluctuations of Professional Football

National Football League (NFL) teams select the new talents in opposite arrangement to the performance ranking of the former season. Usually the team needs a particular amount of time to turn around for the worse or better [16]. Of course, there are many factors that affect the success or failure of a squad (new coach, shift to a new arena or a new town, player trades, injuries of key players, etc.). The relation between the performance of a football squad and its level at a former time was monitored. If the performances are poor, squads plan to fund in new resources to have a better result in the next season. But if the performances are soaring, teams are not likely to invest and they turn to weaker regrading the competiveness which means that poor teams become better and good teams become worse, which causes oscillations and this is what distinguishes the delay time [12].

Consider $U$ is the decimal fraction of games won by a NFL team during a certain season, and it must be between zero and one, $a$ is a growth coefficient, 1/years, $t$ is time in years, and $U_{m}$ represents the League-wide average value of $U: U_{m}=0.5$.

Let $\tau=\frac{\pi}{a \sqrt{2}}$ be the number of years ago, $U_{0}$ is the value of $U$ at $t=0$.

The rate at which $U$ changes at the present time $t$ is proportional to the difference between $U_{m}$ and the value of $U$ at some previous time $t-\tau$, it is expressed by the equation

$$
\begin{equation*}
\left.\frac{d U}{d t}=-a[U(t-\tau)]-U_{m}\right] \tag{7.6}
\end{equation*}
$$

Now we want to rewrite (7.6) by using nabla left $h$-Caputo fractional difference of order $0<\alpha<1$ is the quotient of two odd natural numbers, $0<h<1$, so we have

$$
\begin{align*}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} U\right)(t) & =-a\left[U(t-\tau)-U_{m}\right] \\
& =-a U(t-\tau)+a U_{m} \\
\left({ }_{0}^{C} \nabla_{h}^{\alpha} U\right)(t) & +a\left[U(t-\tau)-U_{m}\right]=0 \tag{7.7}
\end{align*}
$$

Let,

$$
\begin{align*}
y(t) & =U(t)-U_{m}  \tag{7.8}\\
\left({ }_{0}^{C} \nabla_{h}^{\alpha} y\right)(t) & =\left({ }_{0}^{C} \nabla_{h}^{\alpha} U\right)(t)-{ }_{0}^{C} \nabla_{h}^{\alpha} U_{m} \\
y(0) & =U(0)-U_{m}
\end{align*}
$$

Then we can write (7.7) as

$$
\begin{gather*}
\left({ }_{0}^{C} \nabla_{h}^{\alpha} y\right)(t)+a y(t-\tau)=0  \tag{7.9}\\
y_{0}=U_{0}-U_{m} \tag{7.10}
\end{gather*}
$$

According to Theorem 6.1 if the equation

$$
F(s)=s^{\alpha}+a h(1-h s)^{\frac{\tau}{h}}
$$

has no real roots in $\mathbb{R}$, then every solution of equations (7.8)-(7.7) oscillates.

If we apply ${ }_{0} \nabla_{h}^{-\alpha}$ on equation (7.9)
then we have that

$$
\begin{aligned}
y(t) & =y(0)-a_{0} \nabla_{h}^{-\alpha} y(t-\tau) \\
& =y(0)-\frac{a}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-\rho(i h))_{h}^{\overline{\alpha-1}} y(i h-\tau) h \\
& =y(0)-\frac{a}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}}(t-i h+h)_{h}^{\overline{\alpha-1}} y(i h-\tau) h \\
& =y(0)-\frac{a}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha-1} \frac{\Gamma\left(\frac{t-i h+h}{h}+\alpha-1\right)}{\Gamma\left(\frac{t-i h+h}{h}\right)} y(i h-\tau) h \\
& =y(0)-\frac{a}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} h^{\alpha} \frac{\Gamma\left(\frac{t}{h}-i+1+\alpha-1\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} y(i h-\tau) \\
& =y(0)-\frac{a h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{\frac{t}{h}} \frac{\Gamma\left(\frac{t}{h}-i+\alpha\right)}{\Gamma\left(\frac{t}{h}-i+1\right)} y(i h-\tau)
\end{aligned}
$$

Example 7.4: Let $a=1, \alpha=\frac{3}{5}, h=0.5, k=6, N=80, y_{0}=U_{0}-U_{m}$ and $U_{m}=0.5, U_{0}=0.4$.


Figure 7.2.1: Characteristic equation of equation (7.9) with $a=1, \alpha=\frac{3}{5}, h=0.5$,

$$
F(s)=s^{0.6}+0.5(1-0.5 s)^{6}
$$



Figure 7.2. 2 : Oscillatory solution of equation (7.9) with $a=1, \alpha=\frac{3}{5}, h=0.5$


Figure 7.2. 3 : Oscillatory solution of equation (7.8) with $a=1, \alpha=\frac{3}{5}, h=0.5$

Example 7.5: Let $a=0.25, \alpha=\frac{9}{11}, h=\frac{1}{3}, k=6, N=80, y_{0}=U_{0}-U_{m}$ and $U_{m}=0.5, U_{0}=0.1$.


Figure 7.2. 4: Characteristic equation of equation (7.9) with $a=0.25, \alpha=\frac{9}{11}, h=\frac{1}{3}$.

$$
F(s)=s^{0.81}+0.205(1-0.33 s)^{6}
$$



Figure 7.2. 5: Oscillatory solution of equation (7.9) with $a=0.25, \alpha=\frac{9}{11}, h=\frac{1}{3}$.


Figure 7.2. 6: Oscillatory solution of equation (7.8) with $a=0.25, \alpha=\frac{9}{11}, h=\frac{1}{3}$.

Remark 7.3: Notice that oscillation in Figure 7.2.3 occur around $U_{m}=0.5$ which the League-wide average value of $U$.

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## Appendix

## Matlab Codes Of Equation (6.1)

## Oscillatory of equation (6.1)

clc
clear
close all
\%Parameters
$\mathrm{p}=1.65 ;$ Alpha=5/11; $\mathrm{h}=0.5 ; \mathrm{k}=2 ; \mathrm{N}=100$;
\%Time mesh
for $\mathrm{j}=1: \mathrm{N}$
$t(j)=h^{*}(j-1)-k^{*} h ;$
end
\%Here we define the function
$\mathrm{F}=@(\mathrm{t}) \mathrm{t}^{\wedge} 2 ;$
\%You can try different functions
$\% 1 / \log (3+t) \% \exp (t) \% \sin (t) \ldots$
\%Here we evaluate the function at $\mathrm{j}=1: \mathrm{k}+1$
for $\mathrm{j}=1: \mathrm{k}+1$
$Z(j)=F\left(h *(j-1)-k^{*} h\right) ;$
end
for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
\% Here we find the summation
$\mathrm{S}=0$;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k}-1$
S=S+beta(j-k-i-1+Alpha, 1-Alpha)*Z(i+1);
end
S1=0;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k}-2$
S1=S1+beta(j-k-i-1+Alpha,1-Alpha)*Z(i+1+k);
end
\% Here we calculuate $\mathrm{Z}(\mathrm{j})$
$\mathrm{Z}(\mathrm{j})=(\mathrm{pi} / \mathrm{d}) * \mathrm{Z}(\mathrm{k}+1)-\left(\left(\mathrm{p} *\left(\mathrm{~h}^{\wedge} \mathrm{Alpha}\right) * \sin (\mathrm{Alpha} * \mathrm{pi})\right) / \mathrm{d}\right) * \mathrm{~S}-$
$\left(\left(q^{*}\left(\mathrm{~h}^{\wedge}\right.\right.\right.$ Alpha $) * \sin ($ Alpha*pi) $\left.) / \mathrm{d}\right) * \mathrm{~S} 1$;
end
\%Here we plot the grph
figure(1)
set(gcf,'color','w');
plot(t,Z)
xlabel('t')
ylabel('Z(t)')

## Non-oscillatory of equation (6.1)

clc
clear
close all
\%Parameters
$\mathrm{p}=0.4 ;$ Alpha=9/11; $\mathrm{h}=0.01 ; \mathrm{k}=100 ; \mathrm{N}=1000$;
\%Time mesh
for $\mathrm{j}=1: \mathrm{N}$
$\mathrm{t}(\mathrm{j})=\mathrm{h}^{*}(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h}$;
end
\%Here we define the function
$\mathrm{F}=@(\mathrm{t}) 2 ;$
\%You can try different functions
$\% 1 / \log (3+\mathrm{t}) \% \exp (\mathrm{t}) \% \sin (\mathrm{t}) \ldots$.
\%Here we evaluate the function at $\mathrm{j}=1: \mathrm{k}+1$
for $\mathrm{j}=1: \mathrm{k}+1$
$\mathrm{Z}(\mathrm{j})=\mathrm{F}(\mathrm{h} *(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h}) ;$
end
\%Here we evaluate for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
\% Here we find the summation
$\mathrm{S}=0$;
for $i=1: j-k-1$
$\mathrm{S}=\mathrm{S}+\mathrm{beta}(\mathrm{j}-\mathrm{k}-\mathrm{i}-1+$ Alpha,1-Alpha)$* \mathrm{Z}(\mathrm{i}+1) ;$
end
\% Here we calculuate $\mathrm{Z}(\mathrm{j})$
$\mathrm{Z}(\mathrm{j})=\mathrm{Z}(\mathrm{k}+1)-\left(\mathrm{p}^{*} \mathrm{~h}^{\wedge} \mathrm{Alpha}{ }^{*} \sin \left(\mathrm{pi}{ }^{*} \mathrm{Alpha}\right) / \mathrm{pi}\right)^{*} \mathrm{~S} ;$
end
\%Here we plot the grph
figure(1)
set(gcf,'color','w');
$\operatorname{plot}(\mathrm{t}, \mathrm{Z})$
xlabel('t')
ylabel('Z(t)')

## Characteristic equation of equation (6.1)

clc
clear
close all
\%Parameters
$\mathrm{p}=1.65$; Alpha=5/11; h=0.5; $\mathrm{N}=100 ; \mathrm{k}=2$;
$\mathrm{s}=[-0.1: 1] ;$
\%Here we evaluate the function
$\mathrm{y}=(\mathrm{s} . \wedge \mathrm{Alpha})+((\mathrm{p} * \mathrm{~h}) *((1-(\mathrm{h} * \mathrm{~s})) . \wedge \mathrm{k}$
\%Here we plot the graph
figure(1)
set(gcf,'color','w');
plot( $\mathrm{s}, \mathrm{y}$ )
xlabel('s')
ylabel('F(s)')

## Matlab Codes Of Equation (6.3)

## Oscillatory of equation (6.3)

clear
close all
\% Parameters
$\mathrm{p} 1=0.5 ; \mathrm{p} 2=1.1 ; \mathrm{p} 3=0.6 ;$ Alpha=9/11; h=0.1; $\mathrm{k} 1=3 ; \mathrm{k} 2=20 ; \mathrm{k} 3=100 ; \mathrm{N}=1000$;
$\mathrm{k}=100$;
\% Time mesh
for $\mathrm{j}=1: \mathrm{N}$
$t(j)=h^{*}(j-1)-k^{*} h ;$
end
\% Here we define the function
$\mathrm{F}=@(\mathrm{t}) \mathrm{t} ;$
\% You can try different functions
$\% 1 / \log (3+\mathrm{t}) \% \exp (\mathrm{t}) \% \sin (\mathrm{t}) \ldots$
\%Here we evaluate the function at $\mathrm{j}=1: \mathrm{k}+1$
for $\mathrm{j}=1: \mathrm{k}+1$
$\mathrm{Z}(\mathrm{j})=\mathrm{F}(\mathrm{h} *(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h}) ;$
end
\% Here we evaluate for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
\% Here we find the summation
$\mathrm{S}=0$;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k} 1-1$
S=S+beta(j-k1-i-1+Alpha,1-Alpha)*Z(i+1);
end
S2=0;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k} 2-2$
S2=S2+beta(j-k2-i-1+Alpha,1-Alpha)*Z(i+1);
end
S3=0;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k} 3-2$
S3=S3+beta(j-k3-i-1+Alpha,1-Alpha)*Z(i+1);
end
\% Here we calculuate $\mathrm{X}(\mathrm{j})$
$\mathrm{Z}(\mathrm{j})=\mathrm{Z}(\mathrm{k}+1)-\left(\left(\mathrm{p} 1^{*}{ }^{\wedge}{ }^{\wedge}\right.\right.$ Alpha*sin(Alpha*pi))/pi)*S-
$((\mathrm{p} 2 * \mathrm{~h} \wedge$ Alpha*sin(Alpha*pi))/pi)*S2-((p3*h^Alpha*sin(Alpha*pi))/pi)*S3;
end
\% Here we plot the graph
figure(1)
set(gcf,'color','w');
$\operatorname{plot}(\mathrm{t}, \mathrm{Z})$
xlabel('t')
ylabel('Z (t)')

Non-oscillatory of equation (6.3)
clear
close all
\% Parameters
$\mathrm{p} 1=0.4 ; \mathrm{p} 2=0.5 ; \mathrm{p} 3=0.3 ;$ Alpha=9/11; h=0.001; $\mathrm{k} 1=100 ; \mathrm{k} 2=200 ; \mathrm{k} 3=150$;
$\mathrm{N}=1000$; $\mathrm{k}=200$;
\% Time mesh
for $\mathrm{j}=1: \mathrm{N}$
$\mathrm{t}(\mathrm{j})=\mathrm{h}^{*}(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h}$;
end
\% Here we define the function
$\mathrm{F}=$ @ (t) 5 ;
\% You can try different functions
$\% 1 / \log (3+t) \% \exp (t) \% \sin (t) \ldots$
$\%$ Here we evaluate the function at $\mathrm{j}=1: \mathrm{k}+1$
for $\mathrm{j}=1: \mathrm{k}+1$
$\mathrm{Z}(\mathrm{j})=\mathrm{F}\left(\mathrm{h}^{*}(\mathrm{j}-1)-\mathrm{k}^{*} \mathrm{~h}\right) ;$
end
\% Here we evaluate for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
\% Here we find the summation
$S=0 ;$
for $i=1: j-k 1-1$
$\mathrm{S}=\mathrm{S}+\mathrm{beta}(\mathrm{j}-\mathrm{k} 1-\mathrm{i}-1+\mathrm{Alpha}, 1-\mathrm{Alpha}) * \mathrm{Z}(\mathrm{i}+1) ;$
end
S2 $=0$;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k} 2-2$
$\mathrm{S} 2=\mathrm{S} 2+\mathrm{beta}\left(\mathrm{j}-\mathrm{k} 2-\mathrm{i}-1+\right.$ Alpha,1-Alpha) ${ }^{*} \mathrm{Z}(\mathrm{i}+1)$;
end

S3=0;
for $i=1: j-k 3-2$
$\mathrm{S} 3=\mathrm{S} 3+\mathrm{beta}(\mathrm{j}-\mathrm{k} 3-\mathrm{i}-1+$ Alpha, $1-\mathrm{Alpha}) * \mathrm{Z}(\mathrm{i}+1)$;
end
\% Here we calculuate $\mathrm{X}(\mathrm{j})$
$\mathrm{Z}(\mathrm{j})=\mathrm{Z}(\mathrm{k}+1)-\left(\left(\mathrm{p} 1^{*}{ }^{\wedge}{ }^{\wedge} \text { Alpha* } \sin (\text { Alpha*pi) }) / \mathrm{pi}\right)^{*} \mathrm{~S}-\right.$
$\left(\left(\mathrm{p} 2 *{ }^{\wedge} \wedge\right.\right.$ Alpha*sin(Alpha*pi))/pi)*S2-((p3*h^Alpha*sin(Alpha*pi))/pi)*S3;
end
\% Here we plot the grph
figure(1)
set(gcf,'color','w');
$\operatorname{plot}(\mathrm{t}, \mathrm{Z})$
xlabel('t')
ylabel('Z (t)')

## Characteristic equation of equation (6.3)

clc
clear
close all
\%Parameters
$\mathrm{p} 1=0.5 ; \mathrm{p} 2=1.1 ; \mathrm{p} 3=0.6 ;$ Alpha $=9 / 11 ; \mathrm{h}=0.1 ; \mathrm{N}=1000 ; \mathrm{q}=1 ; \mathrm{k} 1=3 ; \mathrm{k} 2=20 ;$
$k 3=100$;
$s=[-0.5: 20] ;$
$\%$ Here we evaluate the function
$\mathrm{y}=(\mathrm{s} . \wedge$ Alpha $)+\left((\mathrm{p} 1 * \mathrm{~h}) *\left((1-(\mathrm{h} * \mathrm{~s})) .{ }^{\wedge} \mathrm{k} 1\right)\right)+\left((\mathrm{p} 2 * \mathrm{~h}) *\left((1-(\mathrm{h} * \mathrm{~s})) .^{\wedge} \mathrm{k} 2\right)\right)+\left((\mathrm{p} 3 * \mathrm{~h})^{*}((1-\right.$ $\left.\left.\left(h^{*} s\right)\right) .{ }^{\wedge} 33\right)$ )
\%Here we plot the graph
figure(1)
set(gcf,'color','w');
$\operatorname{plot}(\mathrm{s}, \mathrm{y})$
xlabel('s')
ylabel('F(s)')

## MATLAB CODES OF EQUATION (6.5)

## Oscillatory of equation (6.5)

clc
clear
close all
\%Parameters
$\mathrm{p}=1.3$; Alpha $=9 / 11 ; \mathrm{h}=0.2 ; \mathrm{k}=20 ; \mathrm{N}=900 ; \mathrm{q}=0.8$;
$\mathrm{d}=\left(\mathrm{pi}+\left(\mathrm{q}^{*}\left(\mathrm{~h}^{\wedge} \text { Alpha) }\right)^{*} \sin \left(\mathrm{pi}{ }^{*} \text { Alpha)}\right)^{* g a m m a(A l p h a)))}\right.\right.$
\%Time mesh
for $\mathrm{j}=1: \mathrm{N}$
$\mathrm{t}(\mathrm{j})=\mathrm{h} *(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h}$;
end
$\%$ Here we define the function
$\mathrm{F}=@(\mathrm{t}) 2$;
\%You can try different functions
$\% 1 / \log (3+\mathrm{t}) \% \exp (\mathrm{t}) \% \sin (\mathrm{t}) . .$.
\%Here we evaluate the function at $\mathrm{j}=1: \mathrm{k}+1$
for $\mathrm{j}=1: \mathrm{k}+1$
$\mathrm{Z}(\mathrm{j})=\mathrm{F}(\mathrm{h} *(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h}) ;$
end
\%Here we evaluate for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
\% Here we find the summation
$\mathrm{S}=0$;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k}-1$
S=S+beta(j-k-i-1+Alpha,1-Alpha)*Z(i+1);
end
S1=0;
for $\mathrm{i}=1 \mathrm{j} \mathrm{j}-\mathrm{k}-2$
S1=S1+beta(j-k-i-1+Alpha,1-Alpha)*Z(i+1+k);
end
\% Here we calculuate $\mathrm{Z}(\mathrm{j})$
$\mathrm{Z}(\mathrm{j})=(\mathrm{pi} / \mathrm{d}) * \mathrm{Z}(\mathrm{k}+1)-\left(\left(\mathrm{p} *\left(\mathrm{~h}^{\wedge} \mathrm{Alpha}\right) * \sin (\right.\right.$ Alpha*pi) $\left.) / \mathrm{d}\right) * \mathrm{~S}-$
$\left(\left(q^{*}\left(\mathrm{~h}^{\wedge} \mathrm{Alpha}\right) * \sin (\right.\right.$ Alpha*pi) $\left.) / \mathrm{d}\right) * \mathrm{~S} 1$;
end
\%Here we plot the grph
figure(1)

```
set(gcf,'color','w');
plot(t,Z)
xlabel('t')
ylabel('Z(t)')
```

Non-oscillatory of equation (6.5)
clc
clear
close all
\%Parameters
p=0.3; Alpha=9/11; h=0.01; k=90; N=900; q=1.6;
$\mathrm{d}=\left(\mathrm{pi}+\left(\mathrm{q}^{*}\left(\mathrm{~h}^{\wedge} \mathrm{Alpha}\right) * \sin \left(\mathrm{pi}{ }^{*} \mathrm{Alpha}\right)^{*}\right.\right.$ gamma(Alpha) $\left.)\right)$
\%Time mesh
for $\mathrm{j}=1: \mathrm{N}$
$t(j)=h^{*}(j-1)-k^{*} h ;$
end
\%Here we define the function
$\mathrm{F}=@(\mathrm{t}) 2$;
\%You can try different functions
$\% 1 / \log (3+t) \% \exp (t) \% \sin (t) \ldots$
\%Here we evaluate the function at $\mathrm{j}=1: \mathrm{k}+1$
for $\mathrm{j}=1: \mathrm{k}+1$
$\mathrm{Z}(\mathrm{j})=\mathrm{F}(\mathrm{h} *(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h}) ;$
end
\%Here we evaluate for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$

$$
\text { for } \mathrm{j}=\mathrm{k}+2: \mathrm{N}
$$

\% Here we find the summation
$S=0 ;$
for $i=1: j-k-1$
$\mathrm{S}=\mathrm{S}+\mathrm{beta}(\mathrm{j}-\mathrm{k}-\mathrm{i}-1+$ Alpha, 1-Alpha) $* \mathrm{Z}(\mathrm{i}+1)$;
end

S1=0;
for $i=1: j-k-2$
$\mathrm{S} 1=\mathrm{S} 1+\mathrm{beta}(\mathrm{j}-\mathrm{k}-\mathrm{i}-1+\mathrm{Alpha}, 1-\mathrm{Alpha}) * \mathrm{Z}(\mathrm{i}+1+\mathrm{k}) ;$
end
\% Here we calculuate $\mathrm{Z}(\mathrm{j})$
$\mathrm{Z}(\mathrm{j})=(\mathrm{pi} / \mathrm{d}) * \mathrm{Z}(\mathrm{k}+1)-\left(\left(\mathrm{p}^{*}\left(\mathrm{~h}^{\wedge}\right.\right.\right.$ Alpha $) * \sin ($ Alpha*pi) $\left.) / \mathrm{d}\right) * \mathrm{~S}-$
$\left(\left(\mathrm{q}^{*}\left(\mathrm{~h}^{\wedge} \mathrm{Alpha}\right) * \sin (\right.\right.$ Alpha*pi $\left.\left.)\right) / \mathrm{d}\right) *$ S 1 ;
end
\%Here we plot the grph
figure(1)
set(gcf,'color','w');
$\operatorname{plot}(\mathrm{t}, \mathrm{Z})$
xlabel('t')
ylabel('Z(t)')

## Characteristic equation of equation (6.5)

clc
clear
close all
\%Parameters
$\mathrm{p} 1=1.3 ;$ Alpha=9/11; $\mathrm{h}=0.2 ; \mathrm{N}=1000 ; \mathrm{q}=0.8 ; \mathrm{k} 1=20 ; \mathrm{k} 2=200 ; \mathrm{k} 3=150$;
$s=[-0.5: 10] ;$
$\%$ Here we evaluate the function
$\mathrm{y}=(\mathrm{s} . \wedge$ Alpha $)+\left((\mathrm{p} 1 * \mathrm{~h})^{*}\left((1-(\mathrm{h} * \mathrm{~s})) .^{\wedge} \mathrm{k} 1\right)\right)+\mathrm{q}$
\%Here we plot the grphq
figure(1)
set(gcf,'color','w');
$\operatorname{plot}(\mathrm{s}, \mathrm{y})$
xlabel('s')
ylabel('F(s)')

Matlab Codes Of Equation (6.7)

Oscillatory of equation (6.7)
clear
close all
\% Parameters
$\mathrm{p} 1=0.8 ; \mathrm{p} 2=1.2 ; \mathrm{p} 3=0.57 ;$ Alpha $=9 / 11 ; \mathrm{h}=0.1 ; \mathrm{k} 1=3 ; \mathrm{k} 2=60 ; \mathrm{k} 3=100 ; \mathrm{N}=1100$;
$\mathrm{k}=100 ; \mathrm{d}=\left(\mathrm{pi}+\left(\mathrm{q}^{*}\left(\mathrm{~h}^{\wedge} \text { Alpha)}\right)^{*} \sin \left(\mathrm{pi}{ }^{*} \text { Alpha)}\right)^{*} \operatorname{gamma}(\right.\right.$ Alpha $\left.\left.)\right)\right)$
\% Time mesh
for $\mathrm{j}=1: \mathrm{N}$
$t(j)=h^{*}(j-1)-k^{*} h ;$
end
\% Here we define the function
$\mathrm{F}=@(\mathrm{t}) 2+\mathrm{t}$;
\% You can try different functions
$\% 1 / \log (3+\mathrm{t}) \% \exp (\mathrm{t}) \% \sin (\mathrm{t}) \ldots$
\%Here we evaluate the function at $\mathrm{j}=1: \mathrm{k}+1$
for $\mathrm{j}=1: \mathrm{k}+1$
$\mathrm{Z}(\mathrm{j})=\mathrm{F}(\mathrm{h} *(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h}) ;$
end
\% Here we evaluate for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
\% Here we find the summation
$S=0 ;$
for $i=1: j-k 1-1$

```
S=S+beta(j-k1-i-1+Alpha,1-Alpha)*Z(i+1);
end
S2=0;
for i=1:j-k2-2
S2=S2+beta(j-k2-i-1+Alpha,1-Alpha)*Z(i+1);
end
S3=0;
for i=1:j-k3-2
S3=S3+beta(j-k3-i-1+Alpha,1-Alpha)*Z(i+1);
end
S4=0;
for i=1:j-k-2
S4=S4+beta(j-k-i-1+Alpha,1-Alpha)*Z(i+1+k);
end
    % Here we calculate X(j)
    Z(j)=(pi/d)*Z(k+1)-((p1*(h^Alpha)*sin(Alpha*pi))/d)*S-
((p2*(h^Alpha)*sin(Alpha*pi))/d)*S2-((p3*(h^Alpha)*sin(Alpha*pi))/d)*S3;
    -((q*(h^Alpha)*sin(Alpha*pi))/d)*S4;
    end
    % Here we plot the graph
    figure(1)
    set(gcf,'color','w');
```

plot(t,Z)
xlabel('t')
ylabel('Z (t)')

## Non-oscillatory of equation (6.7)

clear
close all
\% Parameters
$\mathrm{p} 1=0.4 ; \mathrm{p} 2=0.7 ; \mathrm{p} 3=0.5 ;$ Alpha $=9 / 11 ; \mathrm{h}=0.001 ; \mathrm{k} 1=3 ; \mathrm{k} 2=60 ; \mathrm{k} 3=100 ; \mathrm{N}=1100$;
$\mathrm{k}=100$;
$\mathrm{q}=1.9 ; \mathrm{d}=\left(\mathrm{pi}+\left(\mathrm{q}^{*}\left(\mathrm{~h}^{\wedge} \mathrm{Alpha}\right)^{*} \sin \left(\mathrm{pi}^{*} \mathrm{Alpha}\right)^{*} \operatorname{gamma}(\right.\right.$ Alpha $\left.\left.)\right)\right)$
\% Time mesh
for $\mathrm{j}=1: \mathrm{N}$
$\mathrm{t}(\mathrm{j})=\mathrm{h} *(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h} ;$
end
\% Here we define the function
$\mathrm{F}=@(\mathrm{t}) 2+\mathrm{t}$;
\% You can try different functions
$\% 1 / \log (3+t) \% \exp (t) \% \sin (t) \ldots$
\%Here we evaluate the function at $\mathrm{j}=1: \mathrm{k}+1$
for $\mathrm{j}=1: \mathrm{k}+1$
$\mathrm{Z}(\mathrm{j})=\mathrm{F}\left(\mathrm{h} *(\mathrm{j}-1)-\mathrm{k}^{*} \mathrm{~h}\right) ;$
end
\% Here we evaluate for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
\% Here we find the summation
$S=0$;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k} 1-1$
$\mathrm{S}=\mathrm{S}+\mathrm{beta}(\mathrm{j}-\mathrm{k} 1-\mathrm{i}-1+$ Alpha,1-Alpha)$* \mathrm{Z}(\mathrm{i}+1) ;$
end

S2 $=0$;
for $i=1: j-k 2-2$
$\mathrm{S} 2=\mathrm{S} 2+\mathrm{beta}\left(\mathrm{j}-\mathrm{k} 2-\mathrm{i}-1+\right.$ Alpha, 1-Alpha) ${ }^{*} \mathrm{Z}(\mathrm{i}+1)$;
end
S3 $=0$;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k} 3-2$
$\mathrm{S} 3=\mathrm{S} 3+\mathrm{beta}(\mathrm{j}-\mathrm{k} 3-\mathrm{i}-1+$ Alpha, $1-\mathrm{Alpha}) * \mathrm{Z}(\mathrm{i}+1)$;
end

S4=0;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k}-2$
$\mathrm{S} 4=\mathrm{S} 4+\mathrm{beta}(\mathrm{j}-\mathrm{k}-\mathrm{i}-1+$ Alpha,1-Alpha) * $\mathrm{Z}(\mathrm{i}+1+\mathrm{k}) ;$
end
\% Here we calculuate $\mathrm{X}(\mathrm{j})$
$\mathrm{Z}(\mathrm{j})=(\mathrm{pi} / \mathrm{d}) * \mathrm{Z}(\mathrm{k}+1)-\left(\left(\mathrm{p} 1 *\left(\mathrm{~h}^{\wedge} \mathrm{Alpha}\right) * \sin (\mathrm{Alpha} * \mathrm{pi})\right) / \mathrm{d}\right) * \mathrm{~S}-$
$\left(\left(\mathrm{p} 2 *(\mathrm{~h} \wedge \mathrm{Alpha}) * \sin \left(\mathrm{Alpha}{ }^{*} \mathrm{pi}\right)\right) / \mathrm{d}\right) *$ S2-((p3*(h^Alpha)*sin(Alpha*pi))/d)*S3;
$-\left(\left(\mathrm{q}^{*}\left(\mathrm{~h}^{\wedge} \mathrm{Alpha}\right) * \sin \left(\mathrm{Alpha}{ }^{*} \mathrm{pi}\right)\right) / \mathrm{d}\right) * \mathrm{~S} 4 ;$
end
\% Here we plot the graph
figure(1)
set(gcf,'color','w');
plot(t,Z)
xlabel('t')
ylabel('Z (t)')

## Characteristic equation of equation (6.7)

clc
clear
close all
\%Parameters
$\mathrm{p} 1=0.4 ; \mathrm{p} 2=0.5 ; \mathrm{p} 3=0.3 ;$ Alpha $=9 / 11 ; \mathrm{h}=0.001 ; \mathrm{N}=1000 ; \mathrm{q}=1 ; \mathrm{k} 1=100 ; \mathrm{k} 2=200$;
$\mathrm{k} 3=150$;
$s=[-60: 50]$;
$\%$ Here we evaluate the function
$\mathrm{y}=(\mathrm{s} . \wedge \mathrm{Alpha})+\left((\mathrm{p} 1 * \mathrm{~h})^{*}((1-(\mathrm{h} * \mathrm{~s})) . \wedge \mathrm{k} 1)\right)+((\mathrm{p} 2 * \mathrm{~h}) *((1-(\mathrm{h} * \mathrm{~s})) . \wedge \mathrm{k} 2))+((\mathrm{p} 3 * \mathrm{~h}) *((1-$ $\left.\left.\left.\left(h^{*}\right)\right) .{ }^{\wedge} k 3\right)\right)+q$
\%Here we plot the graph
figure(1)
set(gcf,'color','w');
plot(s,y)
xlabel('s')
ylabel('F(s)')

## Matlab Codes Of Equation (7.3)

## Oscillatory of equation (7.3)

clc
clear
close all
\% Parameters
r=1.65; Alpha=5/11; h=1/2; k=2; N=100; d=1;
\% Time mesh
for $\mathrm{j}=\mathrm{k}+1: \mathrm{N}$
$\mathrm{t}(\mathrm{j})=\mathrm{h}^{*}(\mathrm{j}-1)-\mathrm{k} * \mathrm{~h}$;
end
\% Here we define the function
$\mathrm{T}(\mathrm{k}+1)=0.5$;
$y(k+1)=T(k+1)-d ;$
\% You can try different functions
$\% 1 / \log (3+\mathrm{t}) \% \exp (\mathrm{t}) \% \sin (\mathrm{t}) \ldots$
\%Here we evaluate for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$

$$
\text { for } \mathrm{j}=\mathrm{k}+2: \mathrm{N}
$$

\% Here we find the summation
$\mathrm{S}=0$;
for $\mathrm{i}=1: \mathrm{j}-\mathrm{k}-1$
S=S+beta(j-k-i-1+Alpha,1-Alpha)*y(i+1);
end
\% Here we calculuate $\mathrm{X}(\mathrm{j})$
$\mathrm{y}(\mathrm{j})=\mathrm{y}(\mathrm{k}+1)-\left(\mathrm{r} * \mathrm{~h}^{\wedge} \mathrm{Alpha} * \sin (\mathrm{pi} * A l p h a) / \mathrm{pi}\right)^{*} \mathrm{~S} ;$
$T(\mathrm{j})=\mathrm{y}(\mathrm{j})+\mathrm{d}$;
end
\% Here we plot the graph
figure(1)
set(gcf,'color','g');
$\operatorname{plot}(\mathrm{t}, \mathrm{y})$
xlabel('t')
ylabel('y (t)')
line (xlim, [d, d], 'color', 'r', 'linestyle', '--')
figure(2)
set(gcf,'color','w');
$\operatorname{plot}(\mathrm{t}, \mathrm{T})$
xlabel('t')
ylabel('T (t)')
line (xlim, [d, d], 'color', 'r', 'linestyle', '--')

## Characteristic equation of equation (7.3)

clc
clear
close all
\% Parameters
$\mathrm{r}=1$; Alpha $=5 / 11 ; \mathrm{h}=0.5 ; \mathrm{N}=100 ; \mathrm{k}=2$;
$s=[-1: 1] ;$
$\%$ Here we evaluate the function
$\mathrm{y}=\left(\mathrm{s} . \wedge^{\wedge}\right.$ Alpha $)+\left((\mathrm{p} 1 * \mathrm{~h}) *\left((1-(\mathrm{h} * \mathrm{~s})) .{ }^{\wedge} \mathrm{k}\right)\right)$
\%Here we plot the grphq
figure(1)
set(gcf,'color','w');
plot(s,y)
xlabel('s')
ylabel('F(s)')

Matlab Codes Of Equation (7.5)

Oscillatory of equation (7.5)
clc
clear
close all
\%Parameters
$a=1 ;$ Alpha $=3 / 5 ; \mathrm{h}=0.5 ; \mathrm{k}=6 ; \mathrm{N}=80 ; \mathrm{m}=0.5$;
\%Time mesh
for $\mathrm{j}=\mathrm{k}+1: \mathrm{N}$
$\mathrm{t}(\mathrm{j})=\mathrm{h}^{*}(\mathrm{j}-1)-\mathrm{k}^{*} \mathrm{~h} ;$
end
\%Here we define the function
$\mathrm{U}(\mathrm{k}+1)=0.4 ;$
$\mathrm{y}(\mathrm{k}+1)=\mathrm{U}(\mathrm{k}+1)-\mathrm{m} ;$
\%You can try different functions
$\% 1 / \log (3+t) \% \exp (t) \% \sin (t) \ldots$
\%Here we evaluate for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
for $\mathrm{j}=\mathrm{k}+2: \mathrm{N}$
\% Here we find the summation
$S=0$;
for $i=1: j-k-1$
$\mathrm{S}=\mathrm{S}+\mathrm{beta}(\mathrm{j}-\mathrm{k}-\mathrm{i}-1+$ Alpha, 1-Alpha) $* \mathrm{y}(\mathrm{i}+1) ;$
end
\% Here we calculuate $\mathrm{X}(\mathrm{j})$

```
y(j)=y(k+1)-(a*h^Alpha*sin(pi*Alpha)/pi)*S;
U(j)=y(j)+m;
end
%Here we plot the grph
figure(1)
set(gcf,'color','g');
plot(t,y)
xlabel('t')
ylabel('y(t)')
    figure(2)
set(gcf,'color','w');
plot(t,U)
xlabel('t')
ylabel('U(t)')
line (xlim, [0.5, 0.5], 'color', 'r', 'linestyle', '--')
```


## Characteristic equation of equation (7.5)

clc
clear
close all
\% Parameters
$\mathrm{a}=1 ;$ Alpha=3/5; $\mathrm{h}=1 / 5 ; \mathrm{N}=80 ; \mathrm{k}=6$;
$s=[-2: 2] ;$
\%Here we evaluate the function
$\mathrm{y}=\left(\mathrm{s} .{ }^{\wedge} \mathrm{Alpha}\right)+\left((\mathrm{p} 1 * \mathrm{~h}) *\left((1-(\mathrm{h} * \mathrm{~s})) .{ }^{\wedge} \mathrm{k}\right)\right)$
\%Here we plot the graph
figure(1)
set(gcf,'color','w');
$\operatorname{plot}(\mathrm{s}, \mathrm{y})$
xlabel('s')
ylabel('F(s)')

