Mathematical Modeling of the Global Positioning System GPS

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النمذجة الرياضية لنظام تحديد المواقع العالمي

Mathematical Modeling of the Global Positioning System GPS

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DEDICATION

I would like to dedicate my thesis :

To my parents, who always urged me to for more work.

To everyone who taught me in my school and university life and did not skimp on me in his knowledge and experience and helped me reach this prestigious scientific stage, they understood my way to achieve the most important goals and hopes of my life.

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List of Abbreviations

Abbreviations	The meaning
GPS	Global Positioning System
RF	Radio Frequency
PRN	Pseudo Random Noise
CDMA	Code Division Multiple Access
PLL	Phase Locked Loop
PD	Phase Detector
VCO	Voltage-Controlled Oscillator
DSSS	Direct Sequence Spread Spectrum
BPSK	Binary Phase Shift Keying
RFI	Radio frequency Interference
LFSR	Linear Feedback Shift Register
$\vec{B}$	Magnetic field
$\vec{E}$	Electric field
ρ	Charge density, number of electric charges in a given volume.
$\vec{J}$	Electric current, flow of electric charges in a specific direction
θ_i(t)	The input signal
θ_f(t)	The output signal from the voltage-controlled oscillator (VCO)
$\sum$	The phase comparator (detector), measures the phase difference of these two signal, and generates an error signal.
$\Upsilon(s)$	The error function
Q_0	The amplifier represents the gain of the phase comparator and the low-pass filter limits the noise in the loop.
Q_1	The gain of the VCO
V_0	The input voltage
$V_c(t)$	Voltage which proportional to the phase difference between the two inputs

w_0	The center angular frequency of the VCO
$u(t)$	The unit step function $u(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$
$\triangle w_i = \lvert w_i - w_0 \rvert$	The input frequency error
$\hbar$(s)	The transfer function
$\hbar_e$(s)	The error transfer function
Q	Loop gain
$F(s)$	The filter function
β_n	The equivalent noise bandwidth
w	The angular frequency
$w_n = \sqrt{\frac{k_0 k_1}{r_1}}$	The natural frequency
$\zeta = \frac{w_n r_2}{2}$	The damping factor
$\alpha : \alpha = \zeta \ w_n$	Damping factor
w: $w = w_n \sqrt{1-\zeta^2}$	Damped frequency
$t_s = \frac{4}{\zeta w_n}$	Settling time
A/D	Analog-to-Digital
D/A	Digital-to-Analog

Abstract

The Global Positioning System (GPS) is a space-based radio navigation system that provides consistent positioning, navigation and timing services to users on a continuous worldwide basis. The objective of this thesis research is to give the readers a working familiarity with both the basic theoretical and practical aspects of how the GPS works. In this research, we have addressed the most important issues related to GPS, such as the transmission of information through it, the mechanism used by GPS receiver to determine our position depending on satellite information, we also studied the factors that led to errors in determining the location. This thesis is divided into 5 sections. First, it introduces a condensed GPS program history, the main components of a GPS receiver (hardware) and some useful applications for the GPS. Second, the nature of the signal, the transmission of the signal, and the mechanisms involved in its transmission are discussed. It also shows how the receiver obtains the signal with the best accuracy, where the receiver basically corrects the frequency of the signal (the carrier plus the well-known PRN code). Third, the wave transmission in its three-dimensional form is discussed and how to deal with this shape mathematically. Fourth, we provide the basic mathematical methods used to calculate user's position with the help of Matlab program. Finally, we focused on how to model a mathematical expression for GPS signal tracking using a phase-locked loop filter receiver in order to obtain a zero steady state error. Then the Z-transform is used to build a phase-locked loop in software for digitized data.

key Words: GPS, Antenna, Pseudo Random Code/Noise (PRN), Code Division Multiple Access (CDMA), 3-D Wave Equation, Phase-Locked Loop, Transfer function, Z-Transform.

Chapter 1

Introduction

1.1 Overview of GPS

The Global Positioning System (GPS) is a system that aims to locate a specific object anywhere in the earth and sky and gives continuous information about its location updates and gives accurate information about the time related to the body's movements, as it depends on those calculations on information from satellites.

GPS was initially developed for military purposes by the US department of Defense in the 1970s. But later it was provided to civilians, as now the military and civilians benefit from it, and is a dual-use system. However, given that the GPS system can only be used by users to receive the signal, it is considered a one-way (passive) system.[28]

The GPS is made up of three parts: satellites orbiting the Earth; control and monitoring stations on Earth; and the GPS receivers owned by users. GPS satellites broadcast signals from space that are picked up and identified by GPS receivers. Each GPS receiver then provides a three-dimensional location (latitude, longitude,and altitude) plus the time. Now we will give a brief explanation of each part of GPS and how it works:

Space segment: This section consists of 24 satellites. These satellites send signals to help determine the location and time of the required GPS satellite .[18]

Control part: This section consists of control and monitoring stations that monitor the satellites in their regular orbits, as this helps in maintaining the regularity of the movement of the satellite group. Also, this section helps to control and adjust the satellite clock, which leads to preserving the transmitted navigational data and protecting it from noise.

User part: This section consists of receivers to receive GPS signals sent from satellites, so the exact location and time required is calculated.

1.2 Problems of Study

Because of the lack of knowledge and research in GPS for security reasons, we decide to make an almost self-contained reference in this topic. So, this can be considered as the first goal of our thesis.

Consider X to be a fixed point (location or position). Even though X is not moving, its determined position by GPS receivers always changes with each new computation. Accordingly, we address the following open problems:

Problem one: What is the mechanism used by GPS receiver to determine our position depending on satellite information?

Problem two: Determine the factors that cause such different output of determining a fixed position.

To remedy these errors of computations, we should search about the problems (3-6):

Problem three: Better formulations of some mathematical models that best describe phase locked loop necessary to determine approxamatively the GPS tracking signal.

Problem four: The best suitable numerical methods used to simulate systems of linear equations describing the exact position of a GPS signal receiver in the three dimension space.

Problem five: If there is a possibility that non-linear equations will come out of the GPS system, what are the best numerical methods to deal with them?

Problem six: Using available software mathematical packages to test the numerical solutions in the previous step. In case of a lack of software programs, then we hope to develop new algorithms and write new suitable codes for our methods.

It was necessary to study the transmission of signals (the carrier wave) for the GPS system in its three-dimensional form and to solve this system mathematically so that we can understand the GPS system more broadly, therefore we can put a problem to search for, which is :

Problem seven: How can derivation of the three-dimensional wave equation using Maxwell's physical equations?

Problem eight: How to avoid the problem of signal and information interference, as some satellites may send certain signals on the same frequency at the same time?

We can fulfill the objectives of this study by giving answers or partial answers to the above problems. Hence, the goal of this work is primarily to formulate some mathematical models that best describe phase locked loop necessary to determine approxamatively the GPS tracking signal. Then the next step will be to make a transform from continuous to discrete systems to build a phase-locked loop in software for digitized data, the continuous system must be changed into a discrete system. Numerical methods will be used to determine the position based on the information coming from the satellites and to determine which is better and more accurate. Finally some GPS applications area will be discussed.

It is expected to achieve other related objectives when solving some natural problems arising during the research process for the main objectives. In the other hand, we may not achieve all the objectives mentioned here, it depends on the direction the research process will take. However, these are the basic objectives which guide our study.

This study is significant for two reasons; short term and long term reasons. The short term is a scientific one, which we give some evidences for. The long term is not expected to fulfill in this study, at the very least we can obtain some suggestions to achieve it. In other words, our duty is to improve accuracy for GPS, which is very important in the scientific field. On the other hand, we hope our and previous results can be used in our society, because GPS cannot be avoided for civilian services nowadays. For this reason, we can formulate a scientific material that can be helpful for potential studies to make benefits from GPS to our people, such as indicating places and making precise graphs for our cities and villages.

1.3 Questions and Hypotheses the Study

How does a GPS receiver use satellite information to determine our position?

Why does the determined position change with each new computation, even though we are not moving?

The above couple of questions are expected to be easily answered, this guides us to think and make serious research in order to give solutions to the main question of this thesis; namely what is done to improve the accuracy of these varying positions?

It is clear that our answers will revolve around the accuracy, which force us to address the following final question:

What is the effect of different interference signals such as: continuous wave interference, narrow band interference, partial band interference, broadband interference , match spectrum interference and pulse interference on signal accuracy in global positioning system (GPS) receiver?

What is the three-dimensional wave equation and how can solve it?

What is the basic protocol in the signal transmission process that governs the shared data so that each receiver can access the desired signal without interference with information and frequencies?

1.4 Some Useful Applications for the GPS

- First, civilians and military can benefit from GPS service all over the world on an ongoing basis.

- Second, designing maps for various facilities. These maps help electricity, gas and water companies in planning, building and maintaining their assets. GPS assists in accurately locating features such as gas lines and if repairs are needed. GPS also helps provide information to locate buried facilities such as electrical cables or water pipes, providing accurate information on the location and depth of the buried utility.

- Third, civil engineering applications: Such as road construction, earth moving and fleet management. GPS allows contractors to deploy their equipment more efficiently, it can be easily accessed at any time and equipment location can be determined exactly.[24]

- Fourth, the Global Positioning System for Transportation Systems: GPS helps carriers to locate buses accurately around the clock. It helps in accurately calculating bus arrival times at bus stops, thus reducing waiting time at bus stops. This leads to better bus efficiency and customer service.[28]

- Finally, GPS Vehicle Navigation: When traveling through unfamiliar areas, the route can be guided electronically at the touch of a button using the GPS that helps to continuously determine the vehicle's location and road condition. GPS can also be used to determine the shortest distance and time to reach the destination, rush hour and road problems.[28]

1.5 GPS-Hardware

The main components of a GPS receiver are:
- Antenna with preamplifier.
- Radio Frequency (RF) with signal identification and signal processing.
- Micro-processor for receiver control, data sampling and data processing.
- Precision oscillator.
- Power supply.
- User interface, command and display panel.
- Memory, data storage.

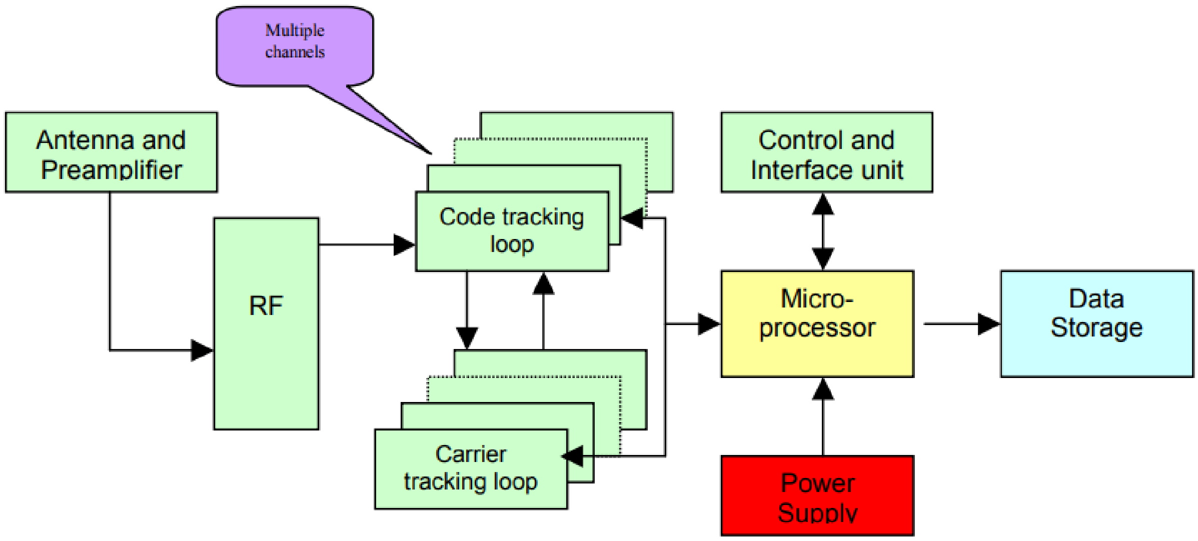


Figure (1.5.1): Basic conceptual architecture of a GPS receiver.

GPS signal reception flight.

The antenna collects the satellite's signals and convert signals from GPS satellites into measurements of position, velocity, and time. In addition, converts the incoming electromagnetic waves into electric currents sensible to the RF section of the receiver. There is a challenge in that the GPS signal has low power, a GPS satellite spreads a low power signal over a large area rather than directing a high power signal at a very specific area. The GPS signal itself would be completely obscured by the variety of electromagnetic noise that surrounds us.[8]

Therefore, the elements of a GPS receiver function cooperatively and iteratively to extract the signal and achieve unambiguous satellite tracking, that means that the data stream is repeatedly refined by the several components of the device working together as it makes its way through the receiver.

The preamplifier increases the signal's power, but it is important that the gain in the signal coming out of the preamplifier is considerably higher than the noise. And the electronic filter removes the unwanted high frequencies. It also eliminates some of the noise from the signal.[8]

Replica of the code is generated by the receiver's oscillator and that is correlated with the IF signal. The signals from several satellites enter the receiver simultaneously. But in the channels of the RF section, the undifferentiated signals are identified and segregated from one another.[8]

Now we will explain GPS elements in some details :

- Antenna:

Antenna is the first element of the receiver architecture, which is an electromagnetic device used to radiate or receive electromagnetic waves in free space, and it is a conductive metal in which an electric current flows in it when colliding with electro magnetic waves. The antenna mostly acts as an intermediary between the transmission line and the open space, and is a structure related to the transition region between the directed wave and the free space wave, or vice versa. It rejects any message operating outside the required operating frequency, and upon transmission, the antenna accepts energy from the transmission line and radiates it into space, and upon reception, it collects energy from the incident wave and sends it through the transmission line.[1]

The main role of the antenna is the satellite gives the position differently each time, so the external antenna improves the signal from the satellite and rejects any message operating outside the required operating frequency, which leads to better positioning.

The GPS receiving antenna detects an incoming electromagnetic signal from a satellite and the band-pass is filtered, providing appropriate selectivity for the filter to mitigate adjacent channel interference. Then the initial signal amplification process takes place, after which the signal is transmitted to the RF section for further processing by the receiver electronics.

The types of GPS Antennas can be internal, external, passive or active. For example, the most important antennas used in GPS are:

- Microstrip antenna (patch antennas): An antenna fabricated using photolithographic techniques on a printed circuit board, these are durable, compact, have a simple construction and a low profile. They are mostly used at microwave frequencies. An individual microstrip antenna consists of a patch of metal foil of various shapes on the surface of a printed circuit board.[32]

- Dipole antenna: Is the most commonly used antenna antenna that was used with the Macro meter, the first commercial GPS receiver. A dipole antenna has a stable phase center and simple construction, but needs a good ground plane.

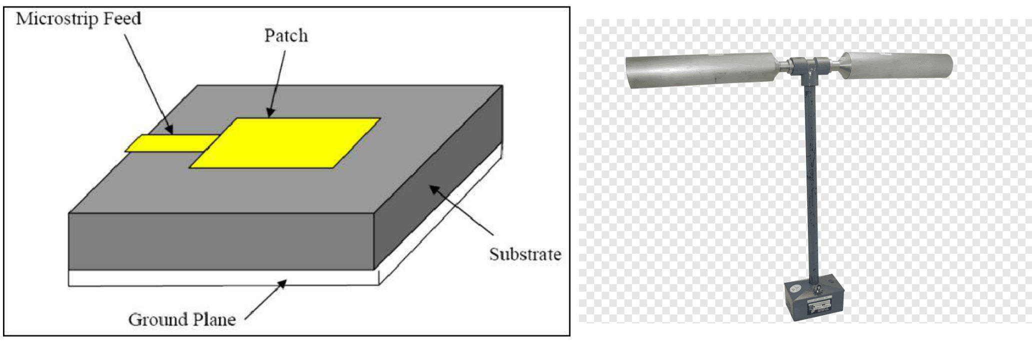


Figure (1.5.2) : Antenna species (Microstrip and Dipole).

- Quadrifilar antenna: Is a single frequency antenna that has two orthogonal bifilar helical loops on a common axis. Quadrifilar antennas perform better than a microstripon crafts, as they are used in many recreational handheld GPS receivers. Such antennas have a good gain pattern (ability to successfully receive a weak signal, orability to concentrate in a particular direction), do not require a ground plane, but are not azimuthally symmetric.[32]

- Helix antenna: The least common design. Helix is a dual frequency antenna, it has a good gain pattern, but a high profile.

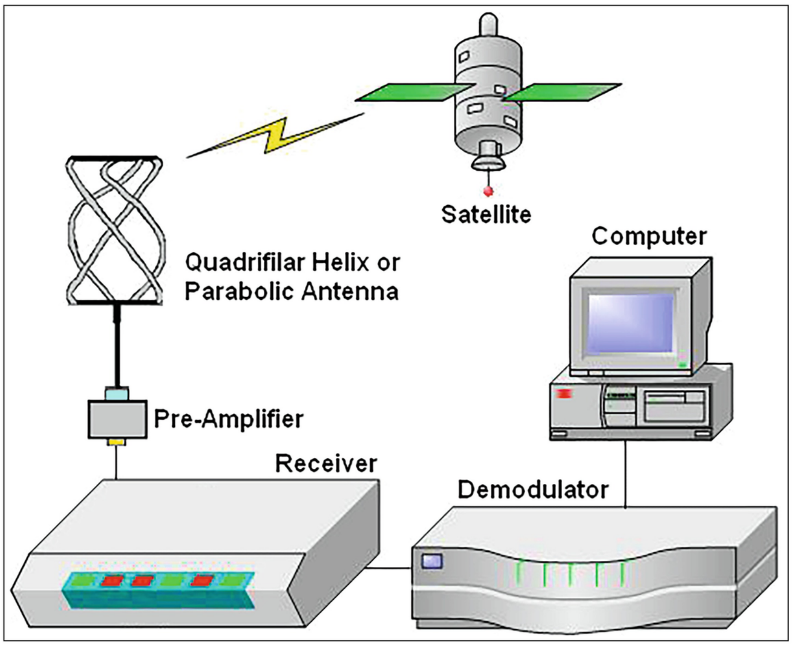


Figure (1.5.3): Antenna species (Quadrifilar and Helix).

- Smart Antennas-Integrated Receivers: Also known as adaptive antennas, these use an array of antennas in combination with smart signal processing algorithms that track the location of a mobile client device using techniques such as the direction of arrival of a signal. The location or angular direction is then used to calculate beam-forming vectors to focus more of the power of the antenna beam on the mobile target.

Smart antenna shave evolved further into a comprehensive survey packages and fully integrated GPS receivers, which are operates automatically and provide position, speed and time. Use in applications tracking weather buoys, time tagging of seismic or other events, navigation, or precise timing and synchronization of wireless voice and data networks.[8]

Figure (1.5.4): A six-channel GPS receiver and antenna.

Figure (1.5.5): Integrated Survey System

Figure (1.5.6): An integrated receiver.

GPS devices include internal antennas. But can add an external antenna to the some devices . External antennas used when there is an extreme obstruction between the sky and the GPS device . External antennas are also valuable to older GPS units that do not use the latest internal antennas. As a result, the external antenna is considered as an ideal process for GPS signal propagation data.[1]

As the most important factors that lead to errors in GPS position and signals, which antenna works to reduce as much as possible:

Ionosphere: The speed of the GPS signal slows down as it passes through this part of the upper atmosphere causing a propagation error.

Troposphere: The troposphere is the lowest portion of Earth's atmosphere. Radio reflections caused by dry atmosphere and water vapor within provoke GPS position error.

Multipath propagation: When the GPS signal hits the ground or solid objects, multiple reflections of those signals occur, causing GPS position errors.

- Preamplifier and filters.

Is the first active component that comes after the antenna. The main function of the preamplifier is to amplify the signal at the antenna's output, preamplifier boosts the signal level before feeding it to the receivers RF front-end section. Preamplifiers generally have three components:[8]

(1) A preselector filter that removes out-of-band interference and limits the noise bandwidth.

(2) Burnout protection that prevents possible high-power interference with the electronic components of the receiver.

(3) A low-noise amplifier (LNA).

The amplifier section should contain separate quality filters for each of the L1 and L2 bands to rejecting signals that are outside the GPS band required.[8]

Indeed, the filters amplify, down convert, and digitize the received signal, filtering is crucial for several reasons: it rejects out-of-band signals, reduces noise in the received signal, and lessens the impact of aliasing.

An amplified antenna can lead to an increased accuracy by allowing tracking of maximum number of satellites in view.

Figure (1.5.7): GPS preamplifiers, best combination of performance.

- RF front end .

After the received signal is amplified, reduced by conversion and digitized, the front end prepares the received signal for processing tasks, thus generating a clean signal ready for processing.[24]

The signal processing function of the receiver is one of the most important functions of the RF section. In addition, it performs many functions, including:[24]

• Extracting of carrier frequency measurements from the satellite signal.

• Satellite signal acquisition .

- Code and carrier tracking from multiple satellites .
- Estimating the relationship to GPS system.
- Doppler removal.
- Processing for each signal channel.[8]

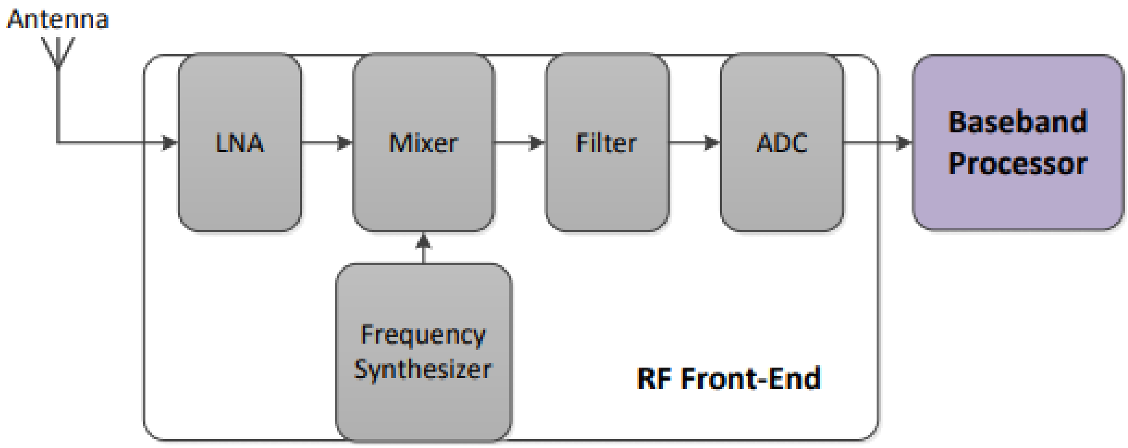


Figure (1.5.8): RF front-end architecture (main blocks).

- Micro-processor for receiver control, data sampling and data processing.

Is the computer that manages data collection and mitigate multi path, noise.

- Precision oscillator .

A good source for accurate timing, it consists of a GPS receiver and a high-quality fixed oscillator whose output is controlled to match the signals transmitted by GPS satellites.

- Power supply.

Most receivers contain an internal rechargeable nickel cadmium battery in addition to an external power input, as it was designed to consume the least amount of energy by focusing on the proper arrangement of the power source compared to the first generation that used high energy.

- GPS user interface, consists on:
* L-band radio receiver processors.
* Antennas which receive GPS signals.

- Memory, data storage.

After turning on the device, it automatically initializes the hardware file system to store the data from GPS receiver. Then GPS data will show on LCD.

What are the most important considerations when buying a GPS receiver?

1.What positioning performance is needed?

Having alot of satellites improves the reliability and speed of obtaining accurate positioning fixes. This strictly depends on the type of the signal and the number of satellites that the receiver can track. So dual frequency signals (L1-L2) eliminate the effect of the ionosphere.[8]

2.What is the battery life available?

Bigger screens and more features use more power. This could either add weight to the device or shorten battery life.

3.What is the best accuracy needed?

Accuracy tends to be higher in open areas with no adjacent tall buildings that can block signals. Taking into consideration some factors that can hinder GPS accuracy include:[22]

- Physical obstructions: There are a lot of physical obstacles that hinder the arrival of the signal and information in real time, such as mountains, tall buildings, forests and more.

- Atmospheric effects: Different storms can affect the accuracy of

GPS devices.

- Ephemera's: The orbital model within a satellite could be incorrect or out-of-date.

- Numerical miscalculations and jamming devices on GPS.[22]

4.Theft prevention?

The tracking system constantly updates the position of an body, vehicle or desired location, so it is easy to locate a missing object easily and accurately.

In addition to the above, it is necessary to inquire about some other important considerations before purchasing the device, such as:

5.What is the most durability device?

6.How much internal storage capacity is needed?

7.What is the nature of the transmitter system?

8.What are you expecting from the maps (the amount of information preloaded on the map)?

9.How important is the multi path and in-band interference rejection?

10.Is the price satisfactory compared to the performance and accuracy requirements?

Chapter 2

Signals

2.1 Introduction

GPS is a global navigation satellite system that provides location, velocity and time synchronization, where these satellites work to provide information as accurately as possible by sending radio signals that are an arrangement of multiple logics (ones and zeros). There are many satellites orbiting the earth today, estimated at 24 satellites.[36]

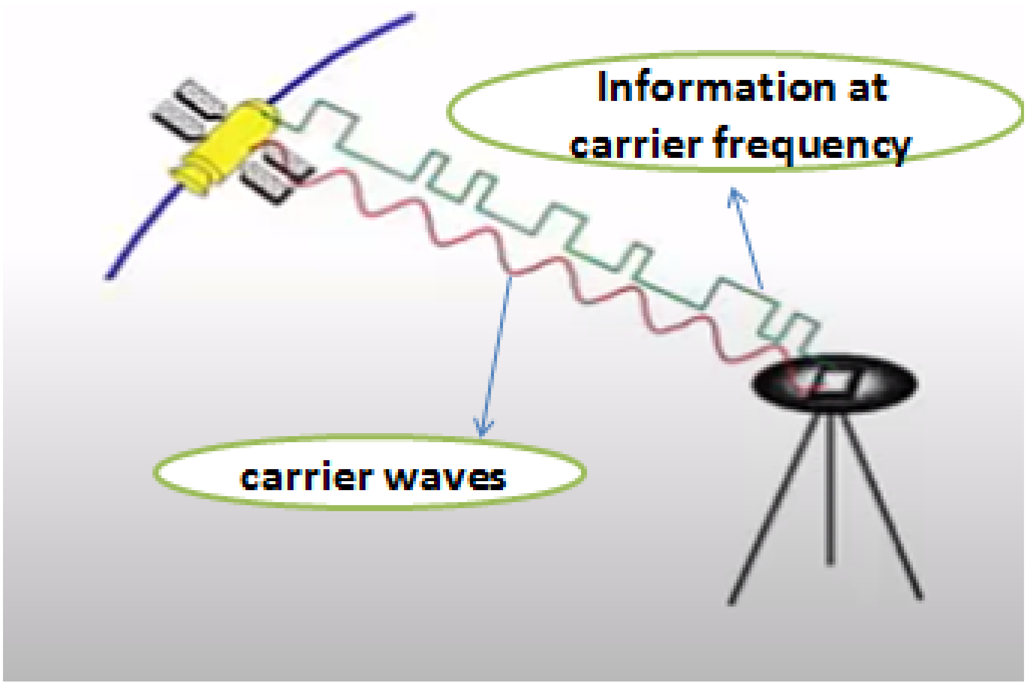


Figure (2.1.1): GPS signal structure.

To calculate the distance between satellite and receiver, the timing of the signal arriving at the receiver is compared with the timing of its transmission from satellite, so positions between latitude, longitude and altitude are determined based on the difference between time. The satellite sends a signal carrying information regarding its location and the timing of the signal and its transmission.[36]

The location of the receiver is determined through the intersection point of three signals coming from three different satellites, as the greater the number of satellites, the greater the accuracy in determining the location and reducing errors and calculations.[17] The principle of positioning and the accuracy of positioning depends on the nature of the transmitted signals. The information from the satellite signals is obtained by the receiver which repeats the signal containing the PRN code to distinguish it and it is represented by a specific carrier. Then the receiver tries to align the repeated signal with the incoming signal. Thus, navigation message, position estimation and clock bias estimation are obtained. For the purpose of restoring the bandwidth of the information and reducing the interference in the information and the resulting errors, the formation of the direct sequence is removed at the receiving side.

Direct Sequence Spread Spectrum (DSSS) technology is used to reduce overall signal interference, and DSSS provides a structure for transmitting range signals and basic navigation data such as coordinates, timing and health of satilites. As a result of using this technology, the signal bandwidth is improved, thus a wider bandwidth is obtained compared to the information bandwidth.[21]

GPS signal contains 3 different types of information each with a specific role. First, a pseudorandom code that carries information to determine which satellite is transmitting the information. Second, information that specifies the current date and time and information to locate the satellite where ephemeris data does this. Third, calendar data : shows the orbital information for each satellite.[36]

Some of the radio signals sent via GPS satellites may be low in power. Given that the signals will pass in the air through clouds, fog, storms and glass, but they will have difficulty penetrating buildings, mountains, cars and solid objects, this will lead to problems in the accuracy of the information and transmitted signals. To avoid these difficulties and problems, and so that the GPS units can receive the data in an ideal and accurate form and with the best accuracy under various conditions, the GPS signals must be sent in a wavelength range of appropriate frequency, and the best waves for that are the L-band of the frequency spectrum.[27]

There are three types of carrier signals that are sent by GPS satellite and at different frequencies, which are L1, L2 and L5. The frequency of (L1, L2, L5) is (1,575.42, 1,227.60, 1,176.45) MHz. Where these frequencies are modulated by P-code, C/A-code and navigation message.[27]

The next figure illustrates the GPS signal structure, and the way the PRN code and navigation data are modulated onto the carrier wave. The code bits and navigation data bits can be represented using ± 1, in which case they are combined using multiplication. Then, binary phase shift keying (BPSK) is used to modulate the code onto the carrier wave.

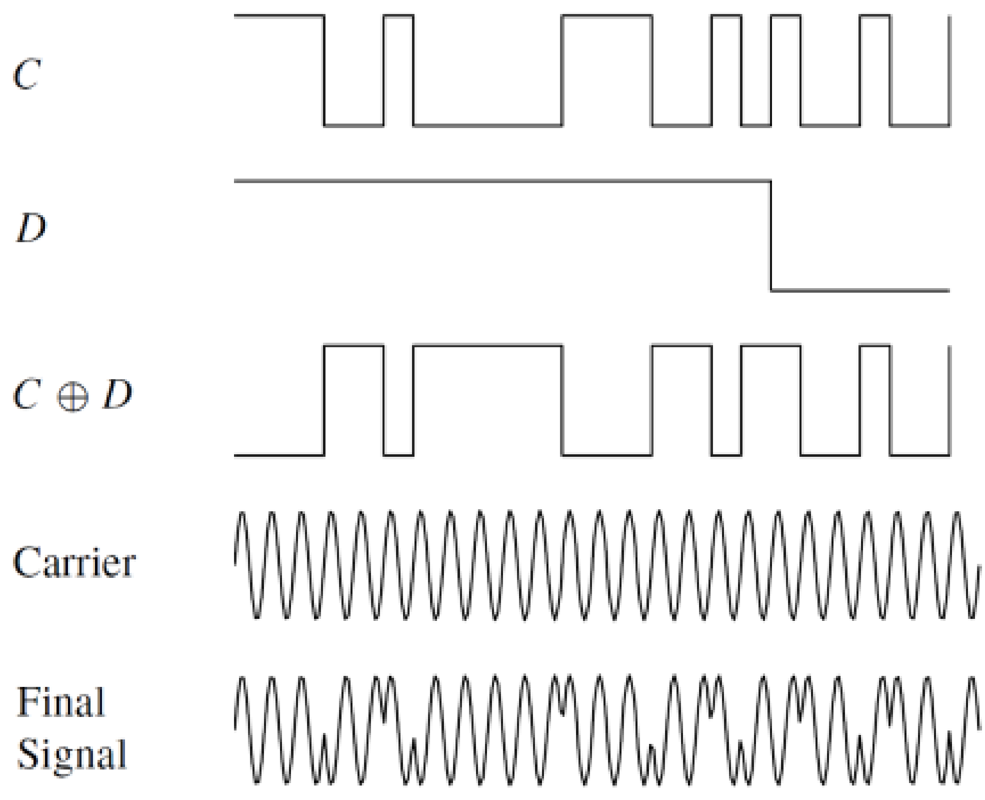


Figure (2.1.2): GPS signal structure.

Choosing an appropriate GPS bus frequency is of great importance in obtaining signals and information with the best possible accuracy. Therefore, to choose a suitable GPS transmitter frequency, some important requirements must be taken into account, the most important of which is: that these frequencies be less than 2 GHz, because frequencies above 2 GHz require beam antennas to receive the signal, and that these frequencies must be not affected by weather phenomena such as rain, snow or clouds, where the signal is within an appropriate range. It is also possible to choose a set of higher frequencies because some PRN codes require high bandwidth to modulate the code at the carrier frequency. Ionospheric delays must be considered for frequencies below 1000MHz and above 10GHz.[8]

2.2 C/A-Code and P-Code

L1 and L2 are the two main bands in GPS where each one contains certain codes, which are:

L1 band contains P-code, C/A-code and navigation data.
L2 band contains only P-code, it is encrypted by Y-code.

C/A code is a sequence of numeric data (1 and 0) where each satellite is distinguished from another by this data and this sequence will be repeated continuously one by one since the GPS includes 1023 consecutive patterns. The C/A codes are characterized as it has the best advantage of cross-linking. As this link will be less than the link of each symbol itself automatically.[5]

Precise code (P code) is a sequence of numeric data (1 and 0). It is carried on L1, L2 main bands in GPS. Each GPS satellite contains a portion of the P code to distinguish between data sent from a satellite or another satellite.[27]

There are some characteristics by which one can distinguish between the two codes and know the specifications of each code, which are:

(1) P-Code is transmitted at a chip rate 10.23 MHz greater than the transmit chip rate C/A-Code 1.023 MHz.
(2) C/A-Code is transmitted on a frequency L1, but P-Code is transmitted on frequencies Ll and L2.
(3) They can be compared by the price of each receiver, as the price of C/A-code receiver is cheaper.
(4) Wavelength of L1 is 0.19 meters. The distance covered by one chip of C/A code when sent is 293 meters will be greater than the distance covered by one chip of P code which is 29.3 meters.[2]

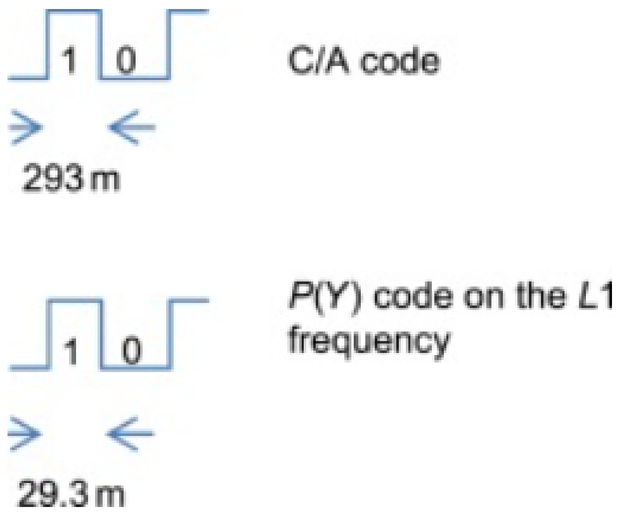


Figure (2.2.1): The distance which the codes occupies.

The accuracy of the transmission using P code will be better than the accuracy of C/A code due to the fact that the difference in the chip rate of P code is small as the difference in the energy levels of C/A code is large. Ionospheric corrections can be made by measuring the difference in transmission delay between frequencies Ll and L2.

2.3 Broadcast Radio Waves

Radio frequency interference (RFI) is any undesired electromagnetic radiation that causes a disturbance in an electrical system. Interference can be classified either narrow band or broadband. Narrow band interference has a bandwidth much smaller than the bandwidth of the receiver, while broadband interference has a bandwidth much larger than the receiver's bandwidth.

Radio waves are emitted from astronomical bodies and travel at the speed of light. They are transmitted through transmitters in space, and on its way to the receiver, it will pass through space and various obstacles, where the speed of these waves will decrease according to the objects, its hardness and its permeability.[36]

The oscillator in the sending part creates a radio frequency carrier through the use of certain electrical signals. Thus, the appropriate frequency for the data is configured in the sending part and then the transmission process takes place. At the receiving side, the modulated radio signal and its frequency are removed, thus the transmitted information is obtained at its original frequency. This transmission process is carried out using the antenna on the transmitter and receiver. As the transmitter antenna deals with the waves generated by electrical signals on the transmitter side and helps to transmit them as waves in space. The receiving antenna receives the radio waves coming from the satellite and then directs them into the receiver to enter the stages of technical treatment to reach the original message.[36]

2.4 Phenomenon The Doppler Effect

It is a phenomenon that affects waves and describes the change in frequency and wavelength of the waves. It describes this change when observed by an observer moving in relation to the wave source. In order for this phenomenon to be able to realize the changes that occur on the wave lengths of the waves that reach it from many and varied sound and light sources, it requires the necessity of the stability of the viewer, and thus it is possible to know the approach of the object or its distance by knowing the wavelengths and sounds.[10]

Let's take a life example of this phenomenon, which is an ambulance passing by us. We notice that the sound of the ambulance whistle coming at us will be louder than its sound when it is moving away from us. The reason for this is that in the event of the car coming, the wave frequency will be high and its length will be compressed, but when the car is moving away, the wavelength will increase, but the frequency will be low.

To clarify the theory, suppose there is a horizontally stationary ball that moves downwards and upwards to hit the surface of the water on a lake. Whenever the ball hits the water, it will be a small circle in the form of a wave that will expand until it disappears, each wave get away the same distance from the next wave, is called the (frequency of the wave).

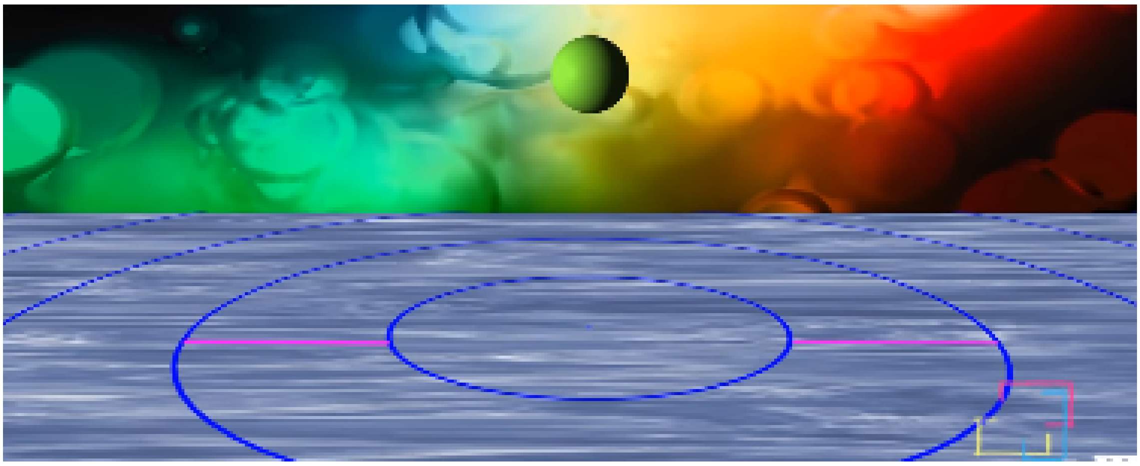


Figure (2.4.1): Doppler effect 1.

But, if the ball is not held horizontally and moves left and right as it moves down and up what happens to the wave's frequency? The distance between each wave and the next will be small in the direction the ball is moving (the frequency is small), and large in the opposite direction (the frequency is large).

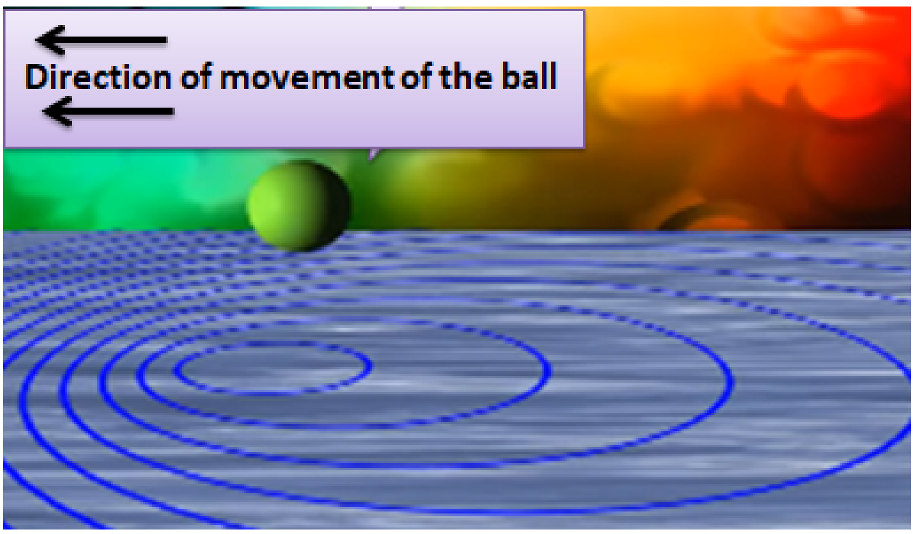


Figure (2.4.2): Doppler effect 2.

In the GPS system, the receiver tracking the signal results in several measurements and calculations that help clarify the effect of the Doppler shift. The most important of these measurements are:

1) Measurements of Pseudo Random Code/Noise (PRN).
2) Measurements of the carrier phase.
3) Measurements that help maintain signal locking by directing the frequency that is applied to replicas of the signal.

The Doppler phenomenon is represented in GPS through the change in the speed of sending various signals and the data carried on them, where this change is affected by changing the distance between the transmitter and the receiver of signal. The change in velocity of transmission is directly proportional to the shift in frequency of the transmitted signals, as the speed of the transmitted signal coming into the future differs from its speed while moving away from it, it gradually decreases when moving away.[10]

The Doppler effect will help to receive the signal sent at a specific frequency, for example L1 on the same frequency, and a signal transmitted at another frequency cannot be received on this frequency. Doppler also affects the receivers on the ground, where the change of the fixed receiver reaches 5kHz, and for a moving receivers device, the change in frequency is greater.[10]

This helped scientists to determine the position of the receiver by knowing the satellite's orbit by measuring the shift in the receiver frequency of the transmitted radio signals.

We are now focusing on some of the problems facing the GPS signal, which leads to problems in the accuracy of the information and the time of its arrival. The most important of these conditions and problems are:

(1) Errors resulting from the slowdown of satellite signals while passing through the atmosphere.
(2) Reflection of the GPS signal on buildings and solid objects will increase the signal transmission time.

(3) The clocks on the satellites will be atomic clocks with high accuracy and quality, which will lead to timing errors because the clocks on the receiver are less good than the atomic clocks.
(4) Some errors resulting from the information received from the satellite, therefore, may lead to determining the location of the moon with a not large degree of accuracy, which will lead to errors of orbitals.
(5) The greater the number of satellites that the receiver can access, the more accurate the transmitted data and signals, but the level and orbit of the transmitter may have a small number of satellites, which will reduce the accuracy of the signal.

2.5 The Participating Terms and Mechanisms in GPS Transmission

Security techniques and protocols are used to transmissions GPS signal as all kinds of data are carried out through a complex system.

We will discuss the main hubs of GPS signal transmission, as the mechanisms involved in GPS transmission: bits, chips, registers, ear-shaped feedback registers, modulo-2 add-on.

A. Bits and Chips

We will discuss some basic definition of GPS hardware.

BIT: It is defined that the minimum data unit that collectively forms a piece of information .
CHIP: The minimum data unit (generated by any means) representing together a sequence , it doesn't carry any piece of information.

For example, let's take the initial measurements of the length of a bridge in meters in the next table:[30]

This field represents only sequence-number. It doesn't carry any piece of information.

Observ. no.	Bridge length (meters)
1	110.25
2	111.49
3	110.00
4	109.97

A measurement is always a vital piece of information in surveying.

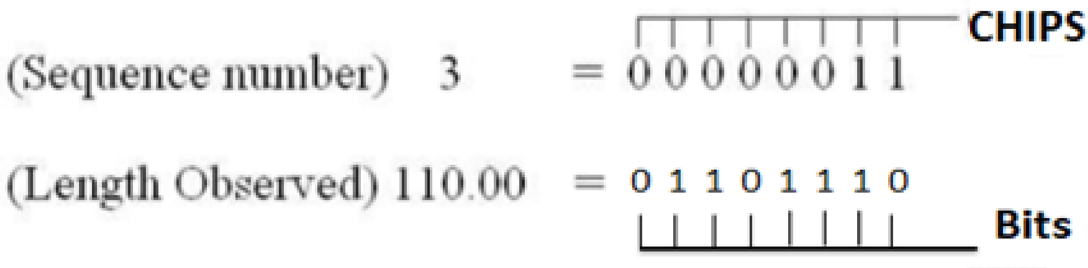
Figure (2.5.1): Observation Table .

Here we will take row three, we will convert two data-set i.e. observation number and length to corresponding binary values. The conversion in the next table.

CONVERTING TO BINARY

Data type	Data	Binary representation (8bit)
Obs. no.	3	00000011
Length	110.00	01101110

00000011 represents the sequence number and 01101110 represents the length information which is useful and unavoidable.



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B. Registers: An elementary memory storage unit

A computer supports representation of data in binary format only.

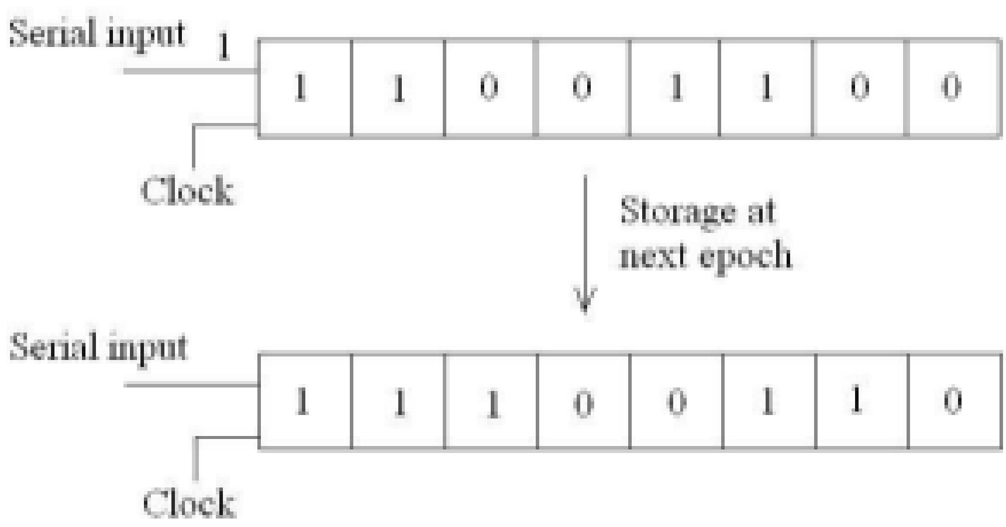
Shift Registers the are electronic circuits made for storage and offset, it stores a binary number consisting of a number of bits, and can be shifted left and right with each clock pulse.[30]

Register is a small unit of storing the data bits; it is designed to be containing sequence of bits as its containers. Based on their size or functionality registers are of different types. The data on the input is loaded and then transferred to the output once every hour cycle hence the name Shift Register.

Parallel input: Input to each bit-container is provided by 8 parallel input ports i.e. 8 inputs simultaneously, and a 'Clock' controls the epoch when the input will be taken in.

Serial input (consecutive): Input to each bit-container is provided by input each bits one after the other, at every click on the 'Clock'.

Let's take 8 entries: provide input to each bit container through 8 parallel input ports That is, 8 entries simultaneously, for example storing 11001100.[30]



To illustrate the work of this technique, we take the following figure

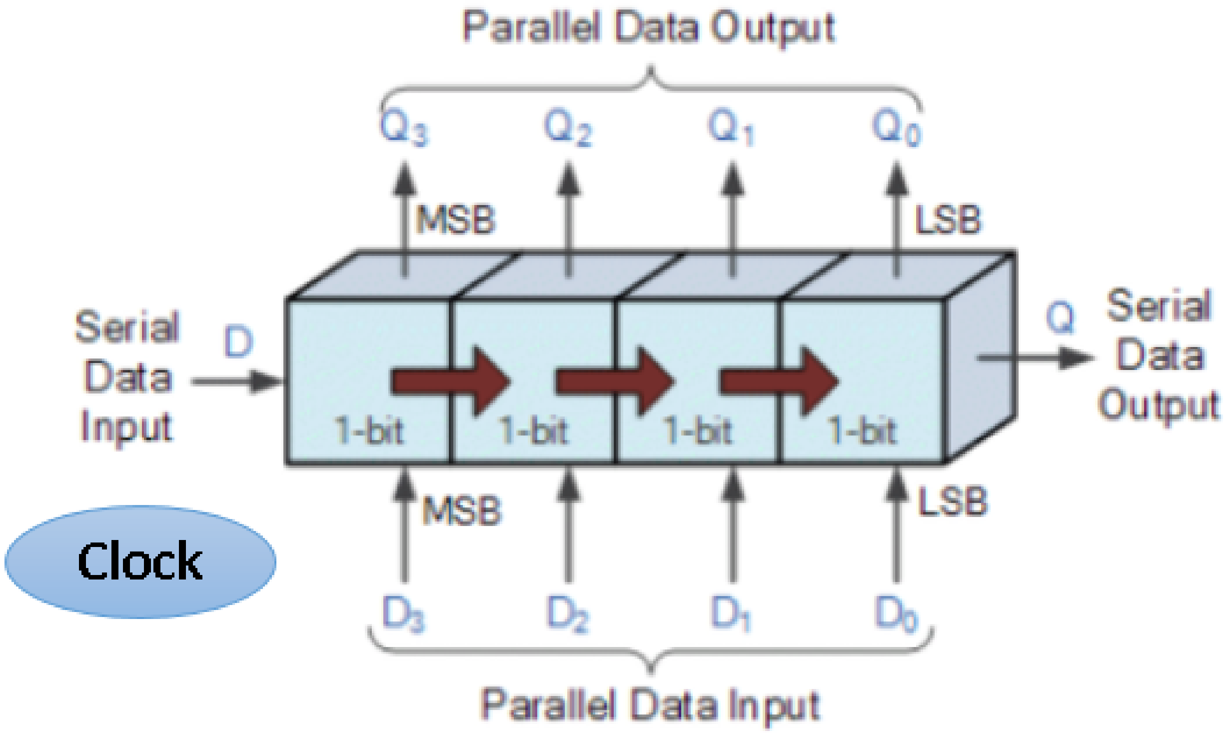


Figure (2.5.2): Shift Register .

Note that there are 4 bits, every bit is a represent memory stores one bit either 0 or 1, where these bits enter the system by serial input or parallel input.

This figure gives four ways in and out of bits in the system.

First: Serial Input to Serial Output (SISO), suppose we need to enter the bit (D) of the system by the consecutive input, to do this we need four pulses on the clock to reach the other end and exit (Q) because with each beat on the clock the bit moves from station to another station in succession and it move outside.[37]

Second: Parallel Input to Parallel Output (PIPO), suppose we need to enter the bits (D_0, D_1, D_2, D_3) of the system by the parallel input. To do this we need one pulse on the clock in order to enter the bits at once and store in the system, and another pulse on the clock comes out bits at once (Q_0, Q_1, Q_2, Q_3).[37]

Third: Parallel Input to Serial Output (PISO), suppose we need to enter the bits (D_0, D_1, D_2, D_3) of the system by the parallel input, to do this we need one pulse on the clock in order to enter the bits at once and store in the system, and press several times on the clock to extract the bits in succession, one by one.[37]

Fourth: Serial Input to Parallel Output (SIPO), suppose we need to enter the bit (D) of the system by the consecutive input, to do this we need four pulses on the clock until all the bits are entered and the stations are filled. To output the existing all bits at once in parallel, we need to do one click on the clock to do this.[37]

C. Linear Feedback Shift Register (LFSR)

Is a register of bits that performs discrete step operations that shifts the bits one position to the left and it replaces the vacated bit by the exclusive or of the bit shifted off and the bit previously at a given tap position in the register. Then, it applies logic (e.g. binary multiplication addition modulo 2 addition) on them to form an output fed back to the register's input serially.[30]

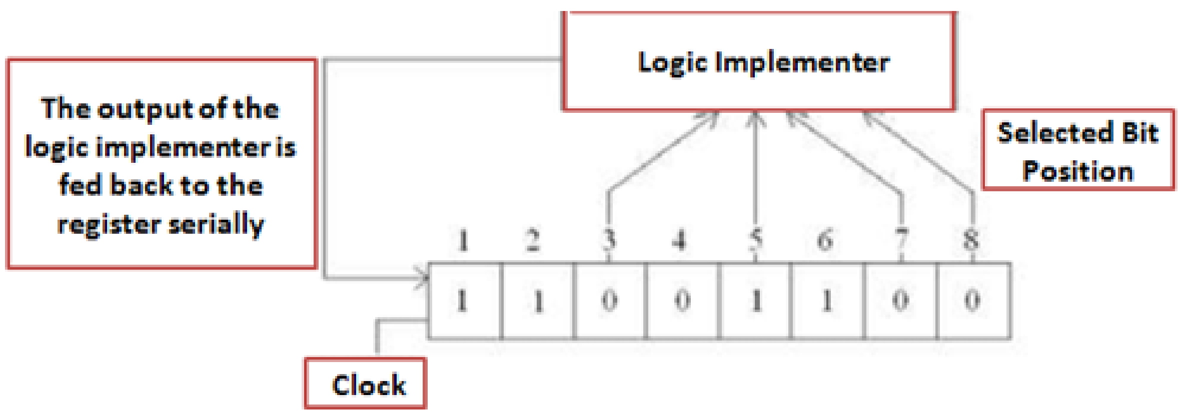


Figure (2.5.3): A linear feedback shift register

D. Modulo 2 Addition

For a particular number system, modulo 2 addition is defined to be the remainder value after division of the sum of two binary numbers by 2, bit-by-bit. The logic to get output by modulo 2 addition binary numbers, for example, take two 4 bit binary numbers 1010 and 0110.

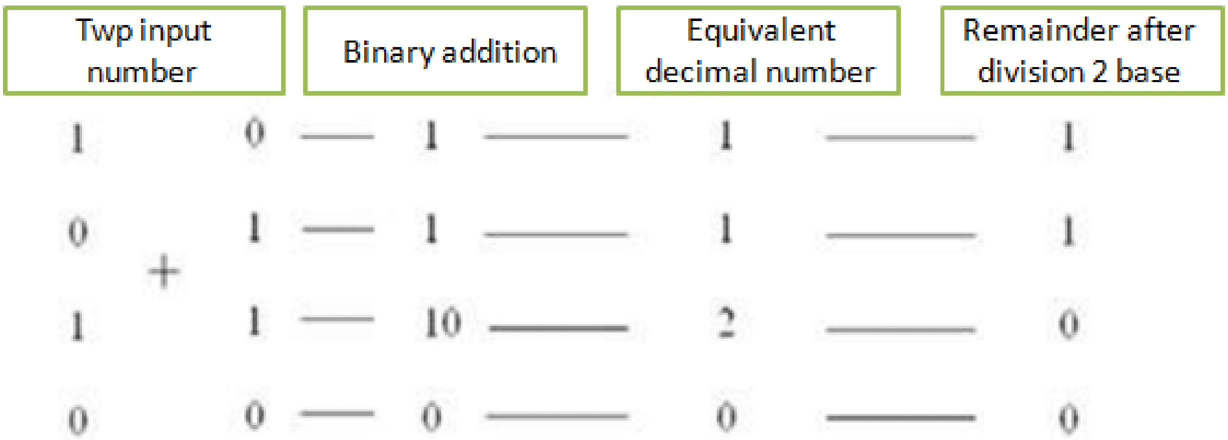


Figure (2.5.4): Modulo 2 Addition.

The implementation of modulo-2-addition is applied via electronic-circuiting by a logic gate called Exclusive-OR (or XOR).

Now, we will explain how the Exclusive-OR (or XOR) works in transmitting and receiving signals. The Logic of Exclusive-OR (or XOR) is shown in the following table.

Exclusive-OR (XOR)

Input		output
A	B	$C = A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

Figure (2.5.5): XOR.

In the transmission phase:

1) Represents the data input A, the information to be sent.

2) Some locally generated Pseudo Noise (PN) is added.

3) The transmitted signal is an encoding of the original signal by locally generated random numbers using Exclusive-OR (XOR) logic.

4) We note that the signal was amplified and expanded before it was sent, as each bit in the original signal was expanded to more than one bit in the transmitted signal.

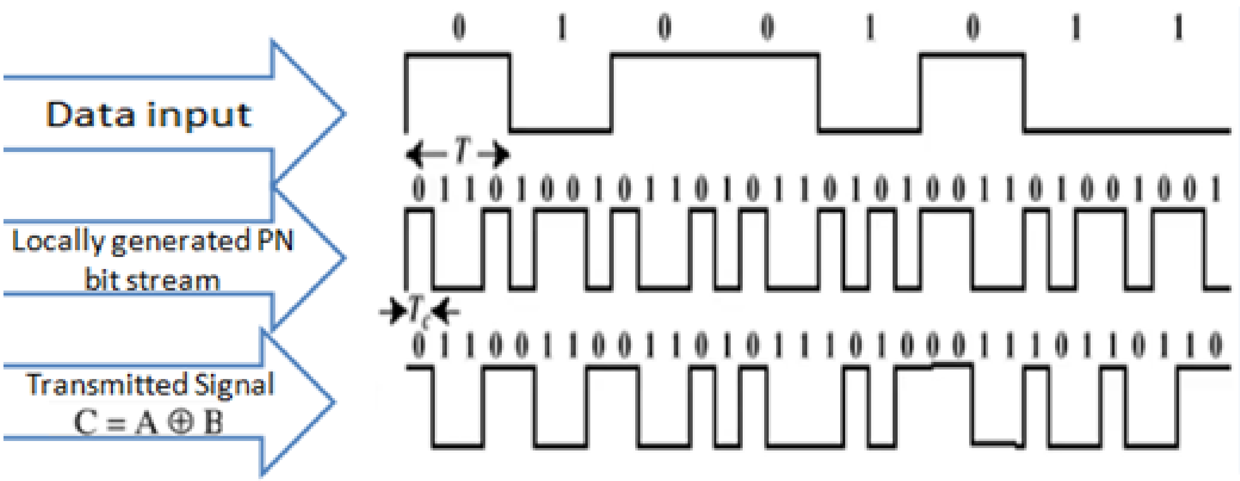


Figure (2.5.6): Transmitted signals.

In the receiver phase:

The Exclusive-OR (XOR) process is done to the received signal C process by using the same locally generated random numbers above. Which leads to the recovery of the original data sent.

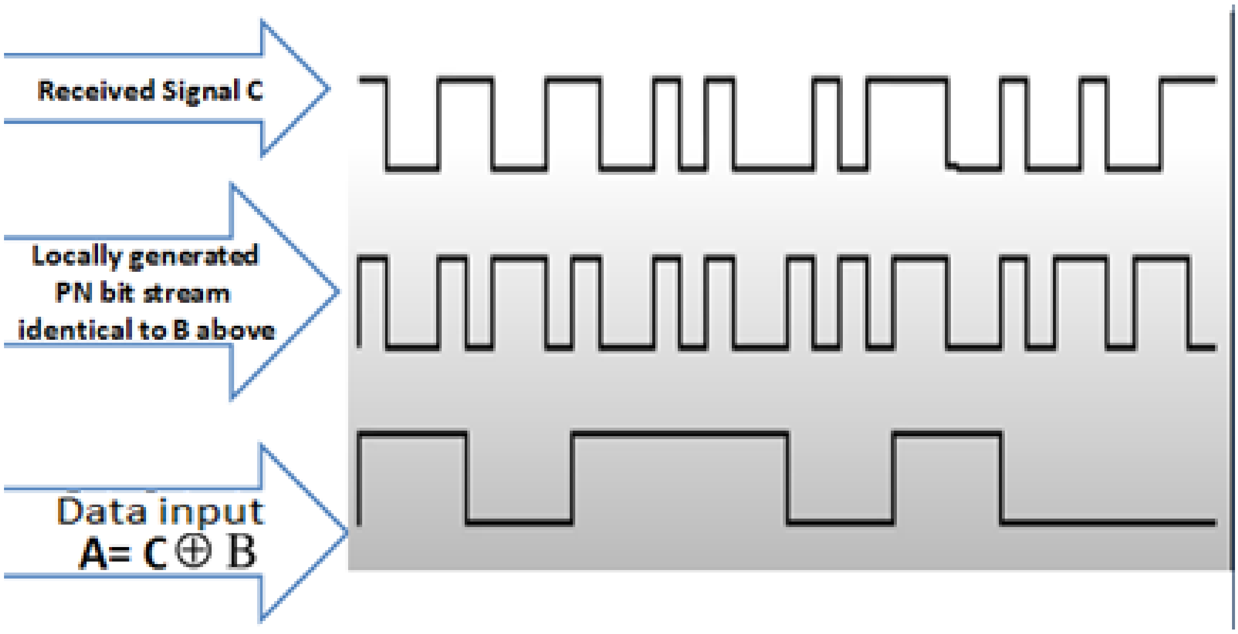


Figure (2.5.7): Received signals.

2.6 Data Generation

The data generation step, which presents the principle that form the basis of transmission medium mutilization and security measures.

The process of transmitting data to be sent to the future passes through a number of stages, as it is sent in the form of signals, the most important stages are:

1) For the purpose of protecting the signal, a pseudo-random code is created on the transmitter side for each data transmission process, and it varies from one satellite to another and differs from each information transmission process to another.
2) In this mode, the original information signal is available in the form of bits (1 and 0) and a pseudo-random code is available to protect it, where the process of merging the code with the original signal is done through a the logic of XOR.
3) After these two stages, the carrier signal has been modified.
4) Finally the carrier signal is amplified to be ready for the transmission process.

At the point of receiving the signal, a number of important steps are taken, namely:

1) The transmitted signal is received and amplified.
2) To restore the propagation of digital signals they are mixed with a local carrier.
3) The same pseudo-random code at the transmitter is used to be merged combined with the received signal.
4) Thus, the original information is obtained after performing the logic of XOR operations again.[30]

Data transmission in the global positioning system is subject to a set of main procedures, namely: securing the data by encrypting it and then transferring it through a specific medium as this data is encoded by binary representation (0 and 1). To transfer the field data stored in the total station's memory to a computer, the data must be sent and received simultaneously, where the field data is stored in a 3D formatted file .

The structure of overall process are shown in the next figure.

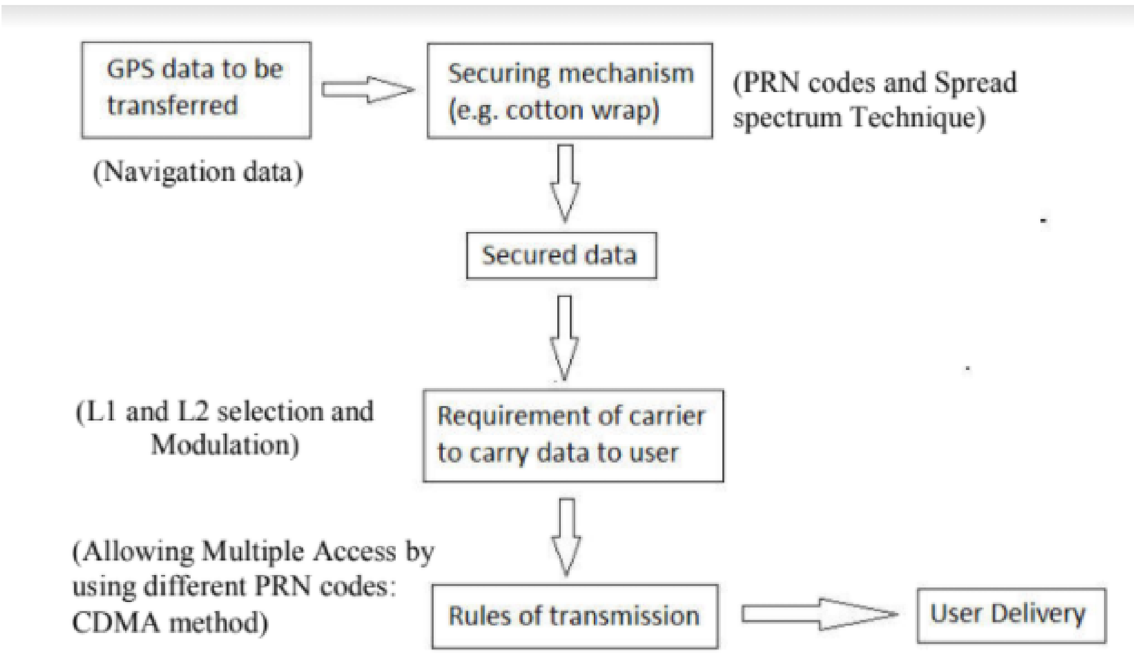


Figure (2.6.1): Overall data-transfer process.

1.Pseudo Random Code/Noise (PRN)

Pseudo-random noise (PRN) is a technology used in GPS systems that is sent as part of the C/A message that helps in determining the exact location, as the receiver helps in accurately identifying the satellite that carries the intended data, thus avoiding the wrong signal coming from each satellites, which leads to reducing interference between satellites . Each carrier signal is identified and modulated by the PRN code.[30]

In the transmission process, the data to be transmitted, which is in the form of an analog signal, is combined with a PRN(s) code. The PRN conforms to one of a number of ranging codes, such as the C/A code and the P(Y) code.[27]

The satellites broadcasts pilot signals. They are the unmodulated PN codes associated with each channel, used to synchronize and track the locally generated PN codes for despreading.

At the receivers, the PRN code is derived by combination of two code generators, which is linear feedback shift registers (LFSR). After derivation, the receiver must shift the phase of the replica code until it correlates with the satellite PRN code.

When the phase of the GPS receiver replica code matches the phase of the incoming satellite code, there is maximum correlation. When the phase of the replica code is offset by more than one chip either side of the incoming satellite code, there is minimum correlation. This is indeed the manner in which a GPS receiver detects the satellite signal when acquiring or tracking the satellite signal in code-phase dimension. It is important to understand that the GPS receiver must also detect the satellite in carrier phase dimension by replicating the carrier frequency.[30]

2.Spread Spectrum Technique

It is one of the main tools used in wireless transmitting and receiving. The logic behind spread spectrum technique:

1) The data to be sent is entered to channel encoder, which will produce analog signal.
2) Signal is further modulated using sequence of digits, which leads to increasing bandwidth of signal to be transmitted.
3) Pseudo noise generates the spreading sequence code.
4) On receiving end, the spread of the spectrum signal is demodulated using a number sequence.
5) Finally, to retrieve the original information, the signal is decoded by the decoding channel.

In GPS, a stream of 0 and 1 represent the data, The code (P and C/A) is used to replace 0 and 1 in a way that is controlled by logic of modulo-two- addition. The notion is to replace the data bit with a sequence of greater number of bit which are unique from their generation for this reason they are assigned to different satellites, (Bit 1 represents high voltage level and bit 0 represents zero level). The logic used in GPS is that, represent the PRN code by a digital signal as shown in the below figure. When data bit 1 or 0 arrives in the sequence of navigation message stream, it is simply modulo 2 summed with PRN code. Combining both whatever the output is generated, represents data bits in form of a continuous digital signal.

3.Modulation: Overloading the Carrier With Data.

To make the adjustment, three main characteristics of the wave are dealt with, any of which are to be modified: amplitude, frequency, and phase.

Phase shift is often chosen as the best modifiable property to represent a bit difference from 0 to 1 so that the level of noise distortion is secured. Because if different frequencies are chosen to be represented by different bits there must be a need for a mechanism to handle a large range of frequencies. Also, if the amplitude variation is chosen to be represented by bit (0 or 1), the GPS signal will be transmitted through many sources of error. Thus introducing an error in the carrier amplitude will directly lead to the bit widening, which can distort the reliability of the data to a large extent.[30]

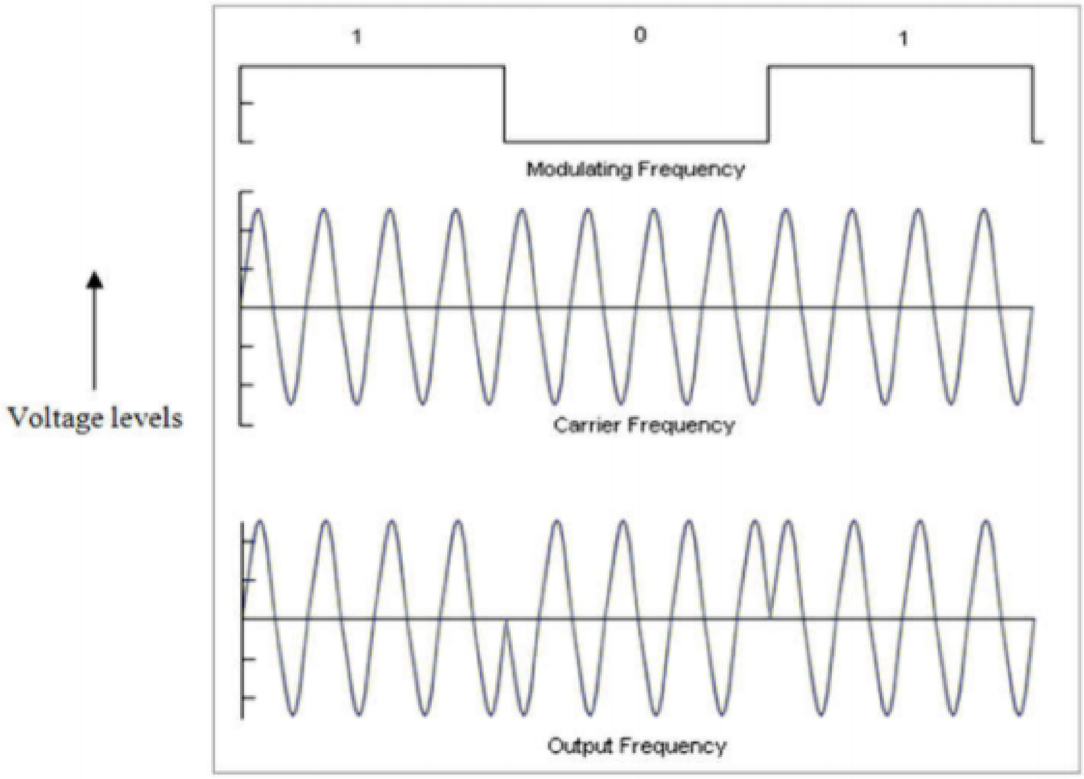


Figure (2.6.2): Process of modulation of sinusoidal wave by variation in phase, Binary phase shift keying (BPSK).

2.7 The Data Transmission Ethics and the Notion of CDMA (Code Division Multiple Access).

The GPS constellation contains many satellites that transmit digital data (information and signals) at various times and frequencies. But some satellites may send certain signals on the same frequency at the same time, so the transmission of the signal must contain a basic protocol governing data sharing so that each receiver can reach the required signal without interference in information and frequencies.[19]

Code division multiple access (CDMA) is one such type of algorithm that extracts the data of a particular satellite from all the data transmitted at the same time from all satellites over the same frequency.

In CDMA, each bit is subdivided into short periods (segments). Thus, the desired signal is obtained and received while any other signal is output as random noise.

CDMA Algorithm:

(1) Generation of PRN codes in the transmission parts.

(2) On CDMA each code to represent satellite that has very less correlation with any other satellite's code.

(3) These codes have an important property, it is any code specific to a particular satellite, if it is multiplied by a code specific to another satellite, the result is zero.[30]

Basic Principal of CDMA:

- Let data transfer rate as signals is D.
- Each bit is subdivided into an of N chips, these chips specific for each user.
- Chip data rate of new channel = ND.

To explanation of how the code works, in the transmission stage consider that :

- b(t) the information to be sent.
- c(t) cipher (PN) code
- y(t) = b(t) $\times$c(t) the information generated after multiplication.

To obtain the required information at the reception stage, the resulting information is multiplied by the code to get the original information.

b(t)= c(t) $\times$ y(t)

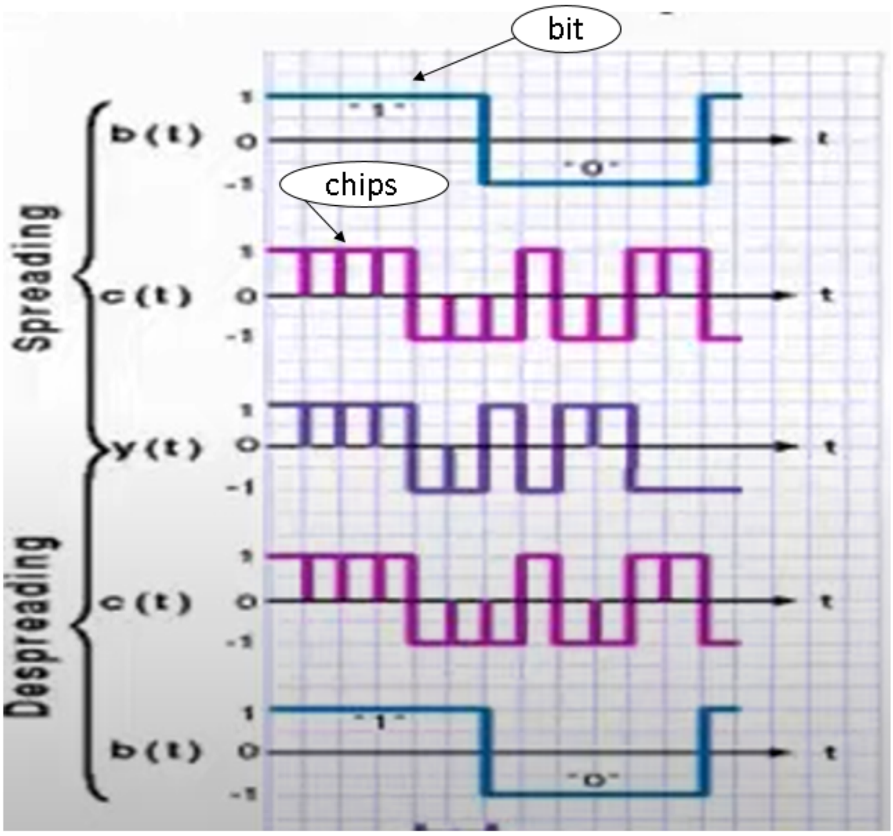


Figure (2.7.1): Spreading and despreading for signal.

Suppose that there is more than one satellite transmitting a signal (information) on the same frequency at the same time. The problem lies in how to distinguish the information for each satellite without overlapping information and signals?

To distinguish the information, we use a code for each satellite that is different from the other code. Where in the transmitter, before the information is sent, it is multiplied by a code and the information is sent on the same code.

Let:

f_1: data from user 1 , u_1: code to user 1

f_2: data from user 2 , u_2: code to user 2

f_3: data from user 3 , u_3: code to user 3

The signals are sent from each satellite in the form of a combination between the input data and the code for that satellite, as:

$$f_1 \times u_1$$

$$f_2 \times u_2$$

$$f_3 \times u_3$$

Since the signals are sent at the same time, this information will appear in the receiver as a single signal in the form of:

$$u_1 \times f_1 + u_2 \times f_2 + u_3 \times f_3$$

On the receiving side, the question is, what information do you want to obtain? Information coming from the first, second or third user.

If the information from the first user is what is required, then the receiver must owns the first code (u_1) with which the information of the first user is multiplied, and if the information from the second user is what is required, then the receiver must owns the second code (u_2) with which the information of the second user is multiplied, as well as for all receivers.

Suppose that the required information is related to the first user, the multiplication is carried out with the same code with which the first user was multiplied. To extract the data of satellite with PRN 1, will multiplying the total data by PRN code 1 .

$$(u_1 \times f_1 + u_2 \times f_2 + u_3 \times f_3) \times u_1$$

PRN codes have the property:

(1) $u_i \times u_i$= number of stations = 4 present case i=1 to 4 (Auto correlation)

(2) $u_i \times u_j$= 0 (Cross correlation), for i $\neq$ j

So ,

$$(u_1 \times f_1 + u_2 \times f_2 + u_3 \times f_3) \times u_1 = f_1 + 0 + 0 = f_1$$

This is how the required information is obtained and other information is discarded using (CDMA).

This algorithm extracts the data transmitted of a particular satellite (in present case (u_1), which corresponds to PRN 1), the separations among the satellites have been successfully established with the use of PRN codes, that uniquely identify their corresponding satellite vehicles.

Chapter 3

Wave Equation 3-Dimension

3.1 Three-Dimensional Wave Form

We will use Maxwell's equations to find 3d wave equation. Maxwell's equations contain four unknowns: ($\vec{E}$,$\vec{B}$,J,ρ).
$\vec{B}$: Magnetic field.
$\vec{E}$: Electric field.
ρ : Charge density, number of electric charges in a given volume.
$\vec{J}$: Electric current, flow of electric charges in a specific direction.[26]

First equation: Electric divergence.

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$$

where ϵ_0 is the constant permittivity of a vacuum for magnetic field lines to cross through it.

This equation is called the diffusion equation for the electric field. Since if there is an electric charge, this charge will generate electric field lines away from the charge.

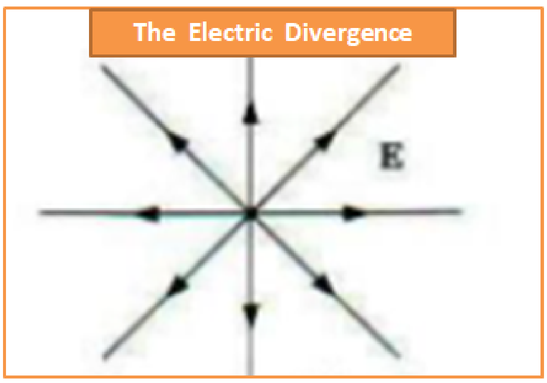


Figure (3.1.1): Electric Divergence.

Second equation: Magnetic Divergence.

$$\vec{\boldsymbol{\nabla}}.\vec{B} = 0$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

This equation represents the propagation of the magnetic field.[26]

Third equation: This equation describes the curl of electric field.

$$\vec{\boldsymbol{\nabla}} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Electric field wrap ($\vec{E}$) = change in magnetic potential($\vec{B}$) with time.[26]

Fourth equation: This equation describes the curl of magnetic field.

$$\vec{\boldsymbol{\nabla}} \times \vec{B} = \mu\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu\vec{J}$$

where μ is the permeability of free space.

Magnetic field deflection = change in electric field + electric current flowing in a specific direction.[26]

Now we will found Maxwell's wave equation .

From Maxwell's third equation:

$$\vec{\boldsymbol{\nabla}} \times \vec{E} = -\frac{\partial B}{\partial t}$$

Take the gradient to equation:[12]

$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{E}) = \vec{\boldsymbol{\nabla}} \times (-\frac{\partial B}{\partial t})$$

$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{E}) = -\frac{\partial}{\partial t}\Big(\vec{\boldsymbol{\nabla}} \times \vec{B}\Big)$$

But in the mathematics of rays there is a law to convert from vector multiplication (cross product) to standard multiplication (dot product), which is $\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{A}) = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}.\vec{A}) - \boldsymbol{\nabla}^2\vec{A}$.[12]

So, the equation $\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{E}) = -\frac{\partial}{\partial t}\Big(\vec{\boldsymbol{\nabla}} \times \vec{B}\Big)$ will be in the form :

$$\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}.\vec{E}) - \boldsymbol{\nabla}^2\vec{E} = -\frac{\partial}{\partial t}\Big(\vec{\boldsymbol{\nabla}} \times \vec{B}\Big)[12]$$

Using Maxwell's first and fourth equations and substituting them, we get:

$$\vec{\boldsymbol{\nabla}}(\frac{\rho}{\epsilon_0}) - \boldsymbol{\nabla}^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu\epsilon_0\frac{\partial E}{\partial t} + \mu\vec{J}\right)$$

Assuming that the electric current$(\vec{J})$ and the charge density constant(ρ) are equal to zero, we produce:

$$-\boldsymbol{\nabla}^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu\epsilon_0\frac{\partial E}{\partial t} + 0\right)$$

$$\boldsymbol{\nabla}^2\vec{E} = \frac{\partial}{\partial t}\mu\epsilon_0\frac{\partial E}{\partial t}$$

$$\boldsymbol{\nabla}^2\vec{E} = \mu\epsilon_0\frac{\partial^2 E}{\partial t^2}$$

This is the wave equation for electric fields using Maxwell's equations. In the same way, the wave equation for magnetic fields can be derived, which is:

$$\boldsymbol{\nabla}^2\vec{B} = \mu\epsilon_0\frac{\partial^2 B}{\partial t^2}$$

By comparing equation

$$\nabla^2\vec{E} = \mu\epsilon_0\frac{\partial^2 E}{\partial t^2}$$

with the general form of the wave equation

$$\nabla^2\vec{A} = \frac{1}{V^2}\frac{\partial^2 A}{\partial t^2}$$

It is clear that $\mu\epsilon_0$ which is the constant of the vacuum permittivity for the crossing of the electric field and the magnetic field is equal to $\frac{1}{V^2}$.Thus, we can calculate the speed of transmission of electric waves and magnetic waves through:

$$\mu\epsilon_0 = \frac{1}{V^2}$$

$$V = \sqrt{\frac{1}{\mu\epsilon_0}}$$

where, the permittivity of free space (vacuum) $\epsilon_0 = 8.8541878176\times 10^{-12}C^2s^2/kgm^3$. And the permeability of free space (vacuum) $\mu = 4\pi 10^{-7}mkg/C^2$.

Substituting the value of the constants the velocity is equal to :

$$V = 2.9979\times 10^8 m/s$$

From these equations, the shape of the traveling wave will be with respect to the three axes x,y,z is:

$$\frac{\partial^2 u}{\partial t^2} = p^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

where p^2 is constant.

So, three dimensional wave equation is:

$$\frac{\partial^2 u}{\partial t^2} = p^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

We will solve it, where X is a function of x only, Y is a function of y only, Z is a function of z only and T is a function of t only.[6]

To solve the equation:

let:

$$u(x, y, z, t) = XYZT$$

Now we substitute in the original equation, and we get:

$$XYZT'' = p^2(YZTX'' + XZTY'' + XYTZ'')$$

We divide by the assumption $(XYZT)$, and we get:

$$\frac{1}{p^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}$$

In order for this equation to be true, the right side must be equal to the left side equal to a negative constant because if the constant is greater or equal to zero, that doesn't give a result and makes the solution zero, so let:

$$\frac{X''}{X} = -k^2$$

$$\frac{Y''}{Y} = -m^2$$

$$\frac{Z''}{Z} = -s^2$$

$$\frac{1}{p^2}\frac{T''}{T} = -q^2$$

where

$$(-k^2) + (-m^2) + (-s^2) = (\frac{-q^2}{p^2})$$

$$k^2 + m^2 + s^2 = (\frac{q^2}{p^2})$$

Now we will solve the equations:

$$X'' + k^2X = 0, So \quad X = c_1cos(kx) + c_2sin(kx)$$

$$Y'' + m^2Y = 0, So \quad Y = c_3cos(my) + c_4sin(my)$$

$$Z'' + s^2Z = 0, So \quad Z = c_5cos(sz) + c_6sin(sz)$$

$$T'' + (pq)^2T = 0, So \quad T = c_7cos(pqt) + c_8sin(pqt)$$

We substitute in the assignment $u(x, y, z, t) = XYZT$:

$$u(x, y, z, t) = (c_1cos(kx) + c_2sin(kx)).(c_3cos(my) + c_4sin(my)).$$
$$(c_5cos(sz) + c_6sin(sz)).(c_7cos(pqt) + c_8sin(pqt)) \ .$$

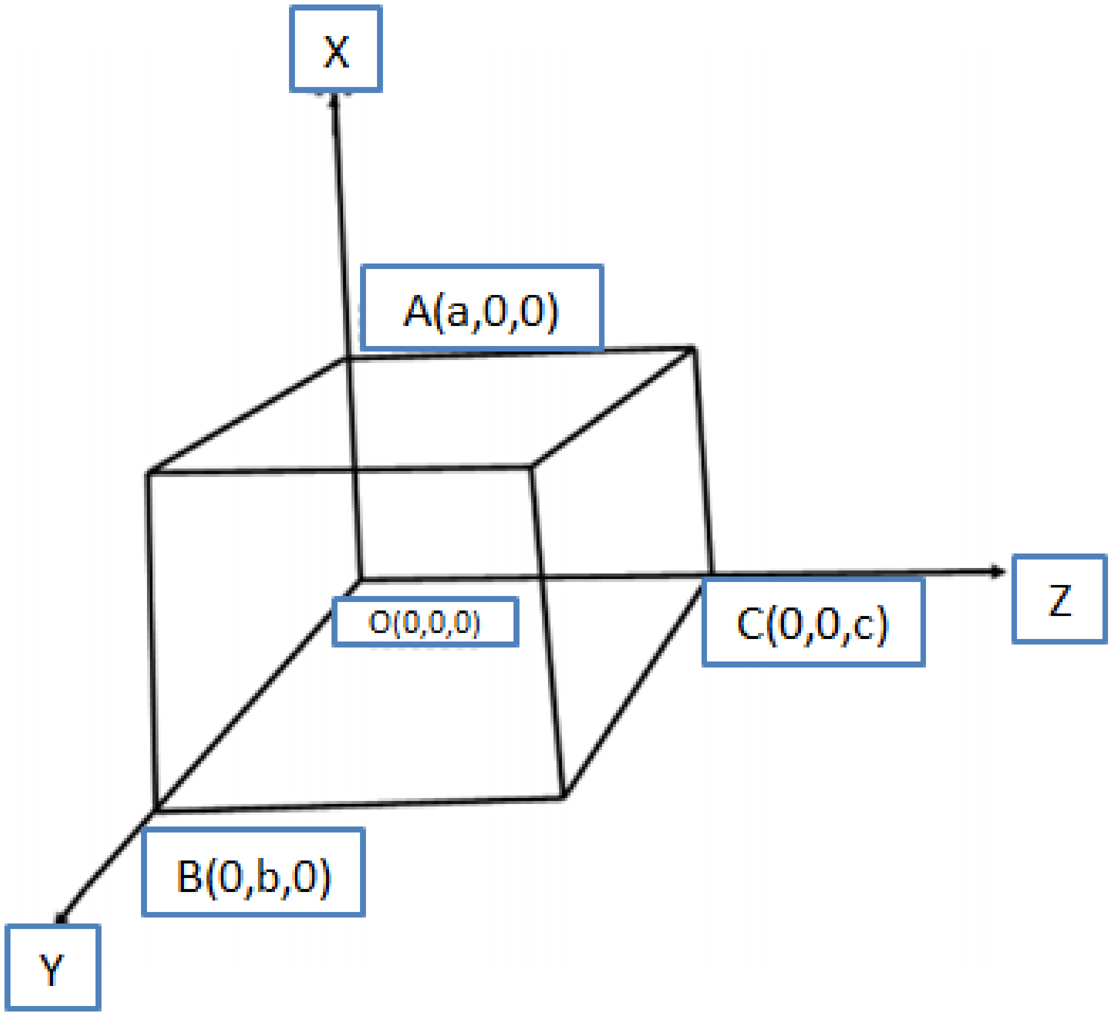


Figure (3.1.2): 3-dimensional wave form .

Boundary conditions:

$u(0, y, z, t) = 0$
$u(a, y, z, t) = 0$
$u(x, 0, z, t) = 0$
$u(x, b, z, t) = 0$
$u(x, y, 0, t) = 0$
$u(x, y, c, t) = 0$

Initial conditions:

- First $u_t(x, y, z, 0) = 0$
- Second $u(x, y, z, 0) = f(x, y, z)$ [29]

Use the boundary conditions, where we exclude any operation that makes $(u) =$ zero .

- $u(0, y, z, t) = 0$

$$X = c_1 cos(kx) + c_2 sin(kx)$$

$$0 = c_1(1) + c_2(0)$$

$$c_1 = 0$$

- $u(a, y, z, t) = 0$

$$X = c_1 cos(kx) + c_2 sin(kx)$$

$$0 = 0 + c_2 sin(ka), \quad c_2 \neq 0$$

$$sin(ka) = 0 \Rightarrow ka = n_x \pi \Rightarrow k = \frac{n_x \pi}{a}$$

So,

$$X = c_2(sin(x\frac{n_x \pi}{a})), n_x \in \mathbb{Z}$$

- $u(x, 0, z, t) = 0$

$$Y = c_3 cos(my) + c_4 sin(my)$$

$$0 = c_3(1) + c_4(0)$$

$$c_3 = 0$$

- $u(x, b, z, t) = 0$

$$Y = c_3 cos(my) + c_4 sin(my)$$

$$0 = 0 + c_4 sin(mb), \quad c_4 \neq 0$$

$$sin(mb) = 0 \Rightarrow mb = n_y \pi \Rightarrow m = \frac{n_y \pi}{b}$$

So,

$$Y = c_4(sin(y\frac{n_y \pi}{b})), n_y \in \mathbb{Z}$$

- $u(x,y,0,t)=0$

$$Y = c_5 cos(sz) + c_6 sin(sz)$$
$$0 = c_5(1) + c_6(0)$$
$$c_5 = 0$$

- $u(x,y,c,t)=0$

$$Y = c_5 cos(sz) + c_6 sin(sz)$$
$$0 = 0 + c_6 sin(sc), \quad c_6 \neq 0$$
$$sin(sc) = 0 \Rightarrow sc = n_z\pi \Rightarrow s = \frac{n_z\pi}{c}$$

So,

$$Z = c_6(sin(z\frac{n_z\pi}{c})), n_z \in \mathbb{Z}$$

So the general form position $u(x,y,z,t)$ becomes:

$$u(x,y,z,t) = c_2(sin(x\frac{n_x\pi}{a})).c_4(sin(y\frac{n_y\pi}{b})).c_6(sin(z\frac{n_z\pi}{c}))$$
$$.(c_7 cos(pqt) + c_8 sin(pqt))$$

Now use the initial conditions:

- First $u_t(x,y,z,0)=0$

where

$$u_t = -c_7(pq sin(pqt)) + c_8(pq cos(pqt))$$

$$0 = c_8(pq) \Rightarrow c_8 = 0$$

So ,

$$T = c_7(cos(pqt))$$

$$u(x,y,z,t) = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} c_9(sin(x\frac{n_x\pi}{a})).((sin(y\frac{n_y\pi}{b}))$$
$$.(sin(z\tfrac{n_z\pi}{c})).cos(pqt)$$

- Second initial condition

Suppose the members start from the rest from the initial position $u(x,y,z,0) = f(x,y,z)$

$$f(x,y,z) = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} c_9 sin(x\frac{n_x\pi}{a}).sin(y\frac{n_y\pi}{b}).sin(z\frac{n_z\pi}{c})$$

Because cos(0)=1

Now using Fourier series (A $= \frac{2}{L}\int_0^L sin(x)f(x)\,dx$) to find the constant c_9

$$c_9 = \frac{2}{a}\frac{2}{b}\frac{2}{c}\int_0^a\int_0^b\int_0^c sin(x\frac{n_x\pi}{a}).sin(y\frac{n_y\pi}{b}).sin(z\frac{n_z\pi}{c})f(x,y,z)\,dxdydz$$

$$c_9 = \frac{8}{abc}\int_0^a\int_0^b\int_0^c sin(x\frac{n_x\pi}{a}).sin(y\frac{n_y\pi}{b}).sin(z\frac{n_z\pi}{c})f(x,y,z)\,dxdydz$$

So, the solution of 3-D wave equation using Fourier series is:

$$u(x,y,z,t) = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} c_9 sin(x\frac{n_x\pi}{a}).sin(y\frac{n_y\pi}{b})$$

$$.sin(z\tfrac{n_z\pi}{c}).cos(pqt)$$

where

$$c_9 = \frac{8}{abc}\int_0^a\int_0^b\int_0^c sin(x\frac{n_x\pi}{a}).sin(y\frac{n_y\pi}{b}).sin(z\frac{n_z\pi}{c})f(x,y,z)\,dxdydz$$

Chapter 4

Numerical Methods for GPS

4.1 Introduction

In this chapter, we will discuss the mathematical methods behind GPS. We will solve different cases of GPS numerically. Focusing on the determining user's position by GPS, the basic mathematical methods behind GPS single point positioning will be explained using simplified geometric model. The single point positioning technique requires the satellites to have good geometry, four or more satellite ranges.

We will discuss various equation systems which are:
1) Underdetermined system of equations, three equations with four unknowns (x,y,z,t_r).
2) System of linear equations, three equations with three unknowns (x,y,z).
3) Over determined system of equations, three equations with two unknowns (x,y).
4) System of non linear equation.

GPS positioning techniques may be categorized as being predominantly based on code or carrier measurements .

In the following examples, we assume that there are no measurement errors, so that all satellite ranges intersect at one point. The purpose of the examples is to indicate to the reader how one might solve the relevant linear systems of equations.

4.2 The Hypotheses were Relied Upon

There are a set of important assumptions in dealing with systems of GPS, but it should be emphasized that some examples can be realistically resolved and others cannot be realistically applied.

1. Receiver and satellite clock are synchronized.
2. Signal delay (ionosphere and troposphere) does not exist.
3. Satellite orbit errors, receiver noise, and multi path ambiguities do not exist.
4. Satellite and receiver position are in the earth center, earth fixed coordinate system.
5. Since the satellites are in motion, the GPS receivers must then take into account Doppler data, which represents the relative speeds between the satellites and the receiver. However, for the purposes of this explanation, we will consider the system to be stationary.[36]

However, these assumptions are wrong, it is certain that our system of equations is not realistically consistent. This chapter will introduce how GPS receivers modify the mathematical range computations to accommodate the time errors and deal with inconsistent system of equations. Recall that the first assumption was that the clock in the GPS receiver was synchronized with the clocks in the satellites. Realistically, when a GPS receiver is switched on, its clock will, in general, not be synchronized with the satellite clocks. This is due to the fact that GPS receivers use inexpensive crystal clocks while GPS satellites use atomic clocks. The range measurements the GPS receiver makes are biased due to the satellite and receiver clock errors. Therefore, the ranges are referred to as pseudoranges. Now we will explain various examples of various satellite systems and how to determine the mathematically required position.

4.3 Linear Examples of GPS

Example 1 (Underdetermined system of equations)

Table (4.3.1) gives 3D user positions and times for the 4 satellites. Bearing in mind that the unit of time measurement is milliseconds, the speed of light is 0.047 in units of earth radii per millisecond.[36]

Table (4.3.1): Satelite data.

Satellite	Position	t_e
1	(1,2,0)	19.9
2	(2,0,2)	2.4
3	(1,1,1)	32.6
4	(2,1,0)	19.9

To locate the user's position two formulas are needed:

R $=(t_r - t_e)c$ and $R = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$ where:

Range= (reception time - emission time)$\times$ speed of light.

t_r= GPS signal reception time of the receiver.

t_e= GPS signal emission time of the satelite.

Earth's radius = 6371 kilometer

c= speed of light(3×10^8 meters per second) .

The speed of light (in units of earth radii per millisecond)

$= \frac{3\times10^8 meter/second}{6371 kilometer}$

$= \frac{0.00047\times10^8 meter/second}{kilometer}$

$= \frac{0.00047\times10^5 kilometer/second}{kilometer}$

$= \frac{0.00047\times10^2 kilometer/millisecond}{kilometer}$

= 0.047 earth radii per millisecond

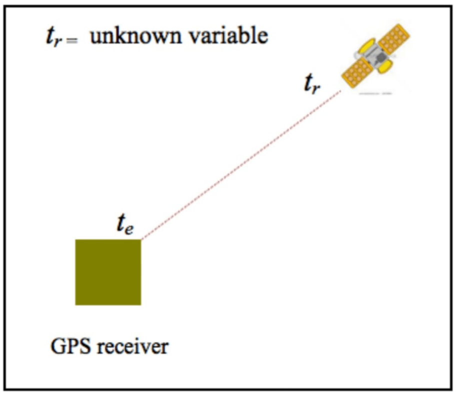


Figure (4.3.1): GPS reciver

For satellite 1, the signal was sent at time 19.9 and arrived at time t_r. Traveling at a speed of light, that makes the range $R = 0.047(t_r - 19.9)$. The satellite's position at (1, 2, 0) can also be expressed as

$$R = \sqrt{(x-1)^2 + (y-2)^2 + (z)^2}$$

$$(x-1)^2 + (y-2)^2 + (z)^2 = 0.047^2(t_r - 19.9)^2$$

We apply the same method to derive the position equations for the rest of the satellites.

Table (4.3.2) : Equation's of Satelites

Satellite	Position	t_e	Equations
1	(1,2,0)	19.9	$(x-1)^2 + (y-2)^2 + (z)^2 = 0.047^2(t_r - 19.9)^2$
2	(2,0,2)	2.4	$(x-2)^2 + (y)^2 + (z-2)^2 = 0.047^2(t_r - 2.4)^2$
3	(1,1,1)	32.6	$(x-1)^2 + (y-1)^2 + (z-1)^2 = 0.047^2(t_r - 32.6)^2$
4	(2,1,0)	19.9	$(x-2)^2 + (y-1)^2 + (z)^2 = 0.047^2(t_r - 19.9)^2$

We will simplify the system using the following steps :

1. We will simplify and arrange the four equations in the following form .

$$i)2x + 4y + 0z - 2(0.047^2)(19.9)t_r = 1^2 + 2^2 + 0^2 - 0.047^2(19.9^2)$$
$$+x^2 + y^2 + z^2 - 0.047^2(t_r)^2$$

$$ii)4x + 0y + 4z - 2(0.047^2)(2.4)t_r = 2^2 + 0^2 + 2^2 - 0.047^2(2.4^2)+$$
$$x^2 + y^2 + z^2 - 0.047^2(t_r)^2$$

$$iii)2x + 2y + 2z - 2(0.047^2)(32.6)t_r = 1^2 + 1^2 + 1^2 - 0.047^2(32.6^2)+$$
$$x^2 + y^2 + z^2 - 0.47^2(t_r)^2$$

$$iv)4x + 2y + 0z - 2(0.047^2)(19.9)t_r = 2^2 + 1^2 + 0^2 - 0.047^2(19.9^2)+$$
$$x^2 + y^2 + z^2 - 0.047^2(t_r)^2$$

2. Subtract the first equation from each of the other three.

i) Second equation minus the first equation

$$2x - 4y + 4z + 2(0.047^2)(17.5)t_r = 8 - 5 + 0.047^2(19.9^2 - 2.4^2)$$

ii) Third equation minus the first equation

$$0x - 2y + 2z + 2(0.047^2)(12.7)t_r = 3 - 5 + 0.047^2(19.9^2 - 32.6^2)$$

iii) Fourth equation minus the first equation

$$2x - 2y + 0z + 2(0.047^2)(0)t_r = 5 - 5 + 0.047^2(19.9^2 - 19.9^2)$$

3. End results.

i) Assembling the terms in the first equation

$$2x - 4y + 4z + 0.077t_r = 3.86$$

ii) Assembling the terms in the second equation

$$-2y + 2z - 0.056t_r = -3.47$$

iii) Assembling the terms in the third equation

$$2x - 2y = 0$$

Now we will put the resulting three linear equations in a matrix and solve them using the Gauss elimination method as the following:

$$\left(\begin{array}{cccc|c} 2 & -4 & 4 & 0.077 & 3.86 \\ 0 & -2 & 2 & -0.056 & -3.47 \\ 2 & -2 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0.095 & 5.41 \\ 0 & 1 & 0 & 0.095 & 5.41 \\ 0 & 0 & 1 & 0.067 & 3.67 \end{array}\right)$$

Assume that the satellite data is accurate, we will be able to find a linear consistent solution for this system.

The general solution is:

$$x = 5.41 - 0.095t_r$$

$$y = 5.41 - 0.095t_r$$

$$z = 3.67 - 0.067t_r$$

$$t_r = free$$

We substitute the values of x, y and z in the first equation, to become in one variable (t_r).

$$(5.41 - 0.095tr - 1)^2 + (5.41 - 0.095tr - 2)^2 + (3.67 - 0.067tr)^2$$
$$= 0.0472(t_r - 19.9)^2$$

Assembling the terms

$$0.02t_r^2 - 1.88t_r + 43.56 = 0$$

By solving the equations, the solution are 43.1 and 50.0. We can eliminate 43.1 since it will give us a point that lies 4,000 miles above the earths surface.

The user's position is :
x = 0.667
y = 0.667
z = 0.332

GDOP (geometric dilution of precision) or PDOP (position dilution of precision) describes the error caused by the relative position of the GPS satellites. Basically, the more signals a GPS receiver can "see" (spread apart versus close together), the more precise it can be. From the observer's point of view, if the satellites are spread apart in the sky, then the GPS receiver has a good GDOP.[11]

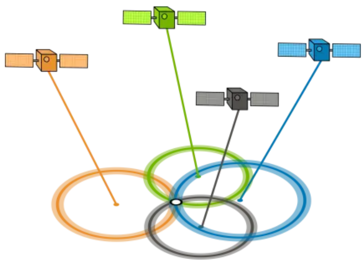


Figure (4.3.2): Good Satellite Geometry.

If the satellites are physically close together, then you have poor GDOP. This lowers the quality of your GPS positioning potentially by meters.[11]

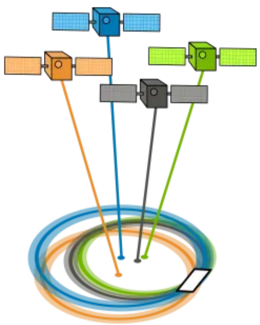


Figure (4.3.3): Poor Satellite Geometry.

Example 2 (System of linear equation, three equations with three unknowns(x,y,z)).

The position and ranges of four satellites are given in the following table. We calculate the user's position (x,y,z) three-dimensional using algebraic methods.

Table (4.3.3): Satellite data.

Satellite	Position	Range
1	(-0.6,5,1.1)	5.6
2	(-1.8,1,1)	6.1
3	(1,1,-3.3)	2.9
4	(-1.99851225,1.5,-5)	2.9

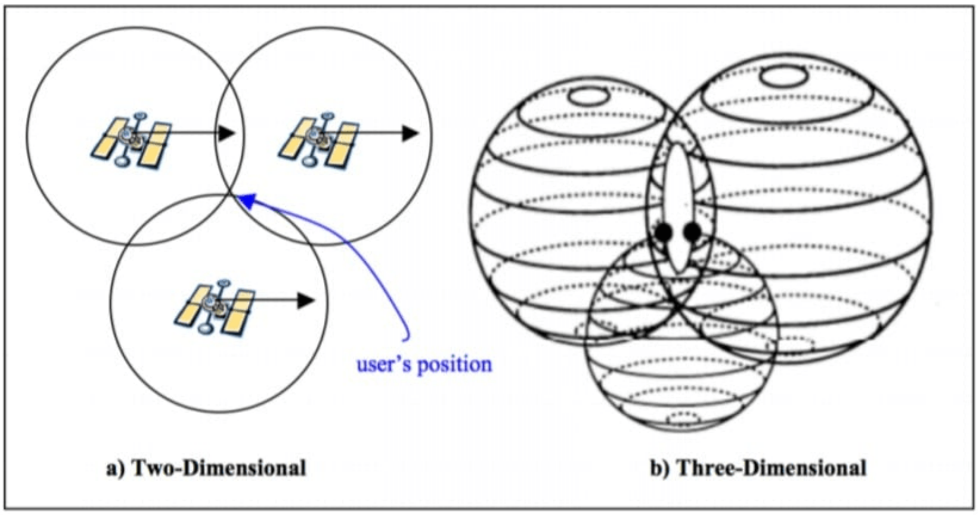


Figure (4.3.4): Position ambiguity removal by additional measurements.[36]

Since the range of each satellite is given, the formula $R = (t_r - t_e)c$ is not needed. To calculate the user's position, we will use the distance formula.

To locate the user's position two formulas are needed :

$$R = (t_r - t_e)c$$

$$R = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$$

The satellite's position at (-0.6,5,1.1) can also be expressed as

$$R = \sqrt{(x + .6)^2 + (y - 5)^2 + (z - 1.1)^2}$$

$$(x + .6)^2 + (y - 5)^2 + (z - 1.1)^2 = (5.6)^2$$

The positions of the other three satellites can be derived using the same logic. These are not linear equations, but can be linearized algebraically.

Table (4.3.4): Equation's of Satelites

Satellite	Position	Range	Equations
1	(-0.6,5,1.1)	5.6	$(x+.6)^2+(y-5)^2+(z-1.1)^2 = (5.6)^2$
2	(-1.8,1,1)	6.1	$(x+1.8)^2+(y-1)^2+(z-1)^2 = (6.1)^2$
3	(1,1,-3.3)	2.9	$(x-1)^2+(y-1)^2+(z+3.3)^2 = (2.9)^2$
4	(-1.99851225,1.5,-5)	2.9	$(x + 1.99851225)^2 + (y - 1.5)^2 + (z + 5)^2 = (2.9)^2$

Now we will follow the following steps to solve and simplify the system.

1. Simplify the four equations.

i) $x^2 + y^2 + z^2 + 1.2x - 10y - 2.2z = 4.79$

ii) $x^2 + y^2 + z^2 + 3.6x - 2y - 2z = 31.97$

iii) $x^2 + y^2 + z^2 - 2x - 2y + 6.62z = -4.48$

iv) $x^2+y^2+z^2+3.9970245x-3y+10z = -22.8340512$

2. Subtract the first equation from each of the other three.

i) First equation minus second equation

$$-2.4x - 8y - 0.2z = -27.18$$

ii) First equation minus third equation

$$3.2x - 8y - 8.8z = 9.27$$

iii) First equation minus fourth equation

$$-2.7970245x - 7y - 12.2z = 27.6240512$$

Now we subtract equation

$$-2.4x - 8y - 0.2z = -27.18$$

From equation

$$3.2x - 8y - 8.8z = 9.27$$

And we get

$$5.6x - 8.6z = 36.45$$

Multiply the equation ($3.2x - 8y - 8.8z = 9.27$) by 7, and multiply the equation ($-2.7970245x - 7y - 12.2z = 27.6240512$) by 8 . We subtract the two equations, and we get :

$$-44.776196x - 36z = 156.10241$$

By solving the 2 equations :

$$5.6x - 8.6z = 36.45$$

$$-44.776196x - 36z = 156.10241$$

We get the exact solution is :

$$(x, y, z) = (-0.05, 3.52, -4.27)$$

Now we will solve this system numerically by three method and Matlab.[33]

Iteration method to solving linear systems.

First, Gauss Jacobi Method.

Now we will solve the system :

$$-2.4x - 8y - 0.2z = -27.18$$

$$3.2x - 8y - 8.8z = 9.27$$

$$-2.79702x - 7y - 12.2z = 27.64051$$

With initial guess

$$(x, y, z) = (0, 0, -1)$$

By using Gauss Jacobi method .

Solution :
From equations :

$$x_{k+1} = \frac{1}{-2.4}(-27.18 + 8y_k + 0.2z_k)$$

$$y_{k+1} = \frac{1}{-8}(9.27 - 3.2x_k + 8.8z_k)$$

$$z_{k+1} = \frac{1}{-12.2}(27.6405 + 2.797x_k + 7y_k)$$

1^{st} Approximation

$$x_1 = \frac{1}{-2.4}[-27.18 + 8(0) + 0.2(-1)] = \frac{1}{-2.4}[-27.38] = 11.4083$$

$$y_1 = \frac{1}{-8}[9.27 - 3.2(0) + 8.8(-1)] = \frac{1}{-8}[0.47] = -0.0587$$

$$z_1 = \frac{1}{-12.2}[27.6405 + 2.797(0) + 7(0)] = \frac{1}{-12.2}[27.6405] = -2.2656$$

.

2^{nd} Approximation

$$x_2 = \frac{1}{-2.4}[-27.18 + 8(-0.0587) + 0.2(-2.2656)] = \frac{1}{-2.4}[-28.1031] = 11.7096$$

$$y_2 = \frac{1}{-8}[9.27 - 3.2(11.4083) + 8.8(-2.2656)] = \frac{1}{-8}[-47.1741] = 5.8968$$

$$z_2 = \frac{1}{-12.2}[27.6405 + 2.797(11.4083) + 7(-0.0587)] = \frac{1}{-12.2}[59.1386] = -4.8474$$

3^{rd} Approximation

$$x_3 = \frac{1}{-2.4}[-27.18 + 8(5.8968) + 0.2(-4.8474)] = \frac{1}{-2.4}[19.0246] = -7.9269$$

$$y_3 = \frac{1}{-8}[9.27 - 3.2(11.7096) + 8.8(-4.8474)] = \frac{1}{-8}[-70.8582] = 8.8573$$

$$z_3 = \frac{1}{-12.2}[27.6405 + 2.797(11.7096) + 7(5.8968)] = \frac{1}{-12.2}[101.6699] = -8.3336$$

4^{th} Approximation

$$x_4 = \frac{1}{-2.4}[-27.18 + 8(8.8573) + 0.2(-8.3336)] = \frac{1}{-2.4}[42.0115] = -17.5048$$

$$y_4 = \frac{1}{-8}[9.27 - 3.2(-7.9269) + 8.8(-8.3336)] = \frac{1}{-8}[-38.6995] = 4.8374$$

$$z_4 = \frac{1}{-12.2}[27.6405 + 2.797(-7.9269) + 7(8.8573)] = \frac{1}{-12.2}[67.4697] = -5.5303$$

5^{th} Approximation

$$x_5 = \frac{1}{-2.4}[-27.18 + 8(4.8374) + 0.2(-5.5303)] = \frac{1}{-2.4}[10.4135] = -4.339$$

$$y_5 = \frac{1}{-8}[9.27 - 3.2(-17.5048) + 8.8(-5.5303)] = \frac{1}{-8}[16.6186] = -2.0773$$

$$z_5 = \frac{1}{-12.2}[27.6405 + 2.797(-17.5048) + 7(4.8374)] = \frac{1}{-12.2}[12.5414] = -1.028$$

6^{th} Approximation

$$x_6 = \frac{1}{-2.4}[-27.18 + 8(-2.0773) + 0.2(-1.028)] = \frac{1}{-2.4}[-44.0042] = 18.3351$$

$$y_6 = \frac{1}{-8}[9.27 - 3.2(-4.339) + 8.8(-1.028)] = \frac{1}{-8}[14.1084] = -1.7635$$

$$z_6 = \frac{1}{-12.2}[27.6405 + 2.797(-4.339) + 7(-2.0773)] = \frac{1}{-12.2}[0.9631] = -0.0789$$

Iteration are tabulated as below :

Table (4.3.5): Solutions of Gauss Jacobi

Iteration	x	y	z
1	11.4083	-0.0587	-2.2656
2	11.7096	5.8968	-4.8474
3	-7.9269	8.8573	-8.3336
4	-17.5048	4.8374	-5.5303
5	-4.339	-2.0773	-1.028
6	18.3351	-1.7635	-0.0789

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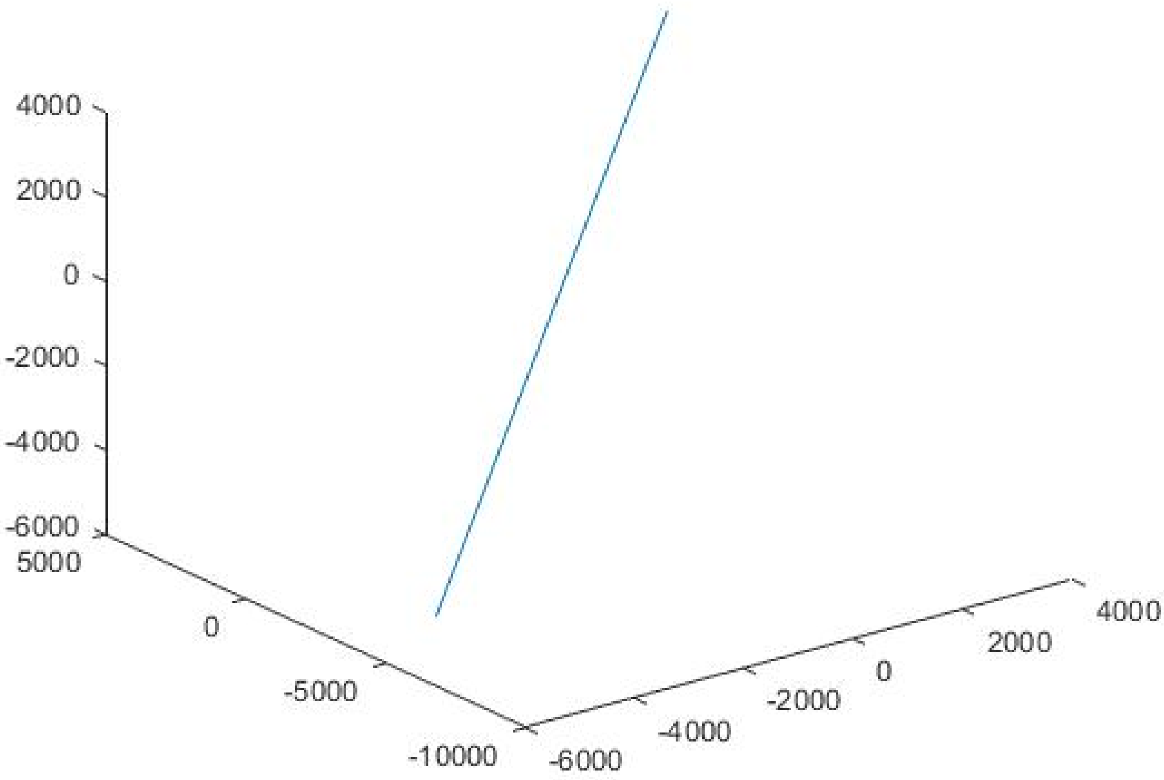

Figure (4.3.5): Solutions of Gauss Jacobi .

Second, Gauss Seidel method.

Now we will solve the system by using Gauss seidel method .

Total Equations are 3

$-2.4x - 8y - 0.2z = -27.18$

$3.2x - 8y - 8.8z = 9.27$

$-2.79702x - 7y - 12.2z = 27.64051$

From the above equations

$$x_{k+1} = \frac{1}{-2.4}\left(-27.18 + 8y_k + 0.2z_k\right)$$

$$y_{k+1} = \frac{1}{-8}\left(9.27 - 3.2x_{k+1} + 8.8z_k\right)$$

$$z_{k+1} = \frac{1}{-12.2}\left(27.64051 + 2.79702x_{k+1} + 7y_{k+1}\right)$$

Initial guess (x ,y , z)=(0,0,-1)

Solution steps are

1^{st} Approximation

$$x_1 = \frac{1}{-2.4}[-27.18 + 8(0) + 0.2(-1)] = \frac{1}{-2.4}[-27.38] = 11.408333333$$

$$y_1 = \frac{1}{-8}[9.27 - 3.2(11.408333333) + 8.8(-1)] = \frac{1}{-8}[-36.036666667] = 4.504583333$$

$$z_1 = \frac{1}{-12.2}[27.64051 + 2.79702(11.408333333) + 7(4.504583333)] = \frac{1}{-12.2}[91.081929833]$$

z_1= -7.465731954

2^{nd} Approximation

$$x_2 = \frac{1}{-2.4}[-27.18 + 8(4.504583333) + 0.2(-7.465731954)] = \frac{1}{-2.4}[7.363520276] = -3.068133448$$

$$y_2 = \frac{1}{-8}[9.27 - 3.2(-3.068133448) + 8.8(-7.465731954)] = \frac{1}{-8}[-46.610414157] = 5.82630177$$

$$z_2 = \frac{1}{-12.2}[27.64051 + 2.79702(-3.068133448) + 7(5.82630177)] = \frac{1}{-12.2}[59.842991769] = -4.90516326$$

3^{rd} Approximation

$$x_3 = \frac{1}{-2.4}[-27.18 + 8(5.82630177) + 0.2(-4.90516326)] = \frac{1}{-2.4}[18.449381505] = -7.687242294$$

$$y_3 = \frac{1}{-8}[9.27 - 3.2(-7.687242294) + 8.8(-4.90516326)] = \frac{1}{-8}[-9.296261347] = 1.162032668$$

$$z_3 = \frac{1}{-12.2}[27.64051 + 2.79702(-7.687242294) + 7(1.162032668)] = \frac{1}{-12.2}[14.273368238] = -1.169948216$$

4^{th} Approximation

$$x_4 = \frac{1}{-2.4}[-27.18 + 8(1.162032668) + 0.2(-1.169948216)] = \frac{1}{-2.4}[-18.117728297] = 7.549053457$$

$$y_4 = \frac{1}{-8}[9.27 - 3.2(7.549053457) + 8.8(-1.169948216)] = \frac{1}{-8}[-25.182515365] = 3.147814421$$

$$z_4 = \frac{1}{-12.2}[27.64051 + 2.79702(7.549053457) + 7(3.147814421)] = \frac{1}{-12.2}[70.790064445] = -5.802464299$$

5^{th} Approximation

$$x_5 = \frac{1}{-2.4}[-27.18 + 8(3.147814421) + 0.2(-5.802464299)] = \frac{1}{-2.4}[-3.157977495] = 1.315823956$$

$$y_5 = \frac{1}{-8}[9.27 - 3.2(1.315823956) + 8.8(-5.802464299)] = \frac{1}{-8}[-46.002322488] = 5.750290311$$

$$z_5 = \frac{1}{-12.2}[27.64051 + 2.79702(1.315823956) + 7(5.750290311)] = \frac{1}{-12.2}[71.572928099] = -5.866633451$$

.

6^{th} Approximation

$$x_6 = \frac{1}{-2.4}[-27.18 + 8(5.750290311) + 0.2(-5.866633451)] = \frac{1}{-2.4}[17.648995798] = -7.353748249$$

$$y_6 = \frac{1}{-8}[9.27 - 3.2(-7.353748249) + 8.8(-5.866633451)] = \frac{1}{-8}[-18.824379969] = 2.353047496$$

$$z_6 = \frac{1}{-12.2}[27.64051 + 2.79702(-7.353748249) + 7(2.353047496)] = \frac{1}{-12.2}[23.543261545] = -1.92977553$$

7^{th} Approximation

$$x_7 = \frac{1}{-2.4}[-27.18 + 8(2.353047496) + 0.2(-1.929775536)] = \frac{1}{-2.4}[-8.741575138] = 3.642322974$$

$$y_7 = \frac{1}{-8}[9.27 - 3.2(3.642322974) + 8.8(-1.929775536)] = \frac{1}{-8}[-19.367458239] = 2.42093228$$

$$z_7 = \frac{1}{-12.2}[27.64051 + 2.79702(3.642322974) + 7(2.42093228)] = \frac{1}{-12.2}[54.774686165] = -4.489728374$$

8^{th} Approximation

$$x_8 = \frac{1}{-2.4}[-27.18 + 8(2.42093228) + 0.2(-4.489728374)] = \frac{1}{-2.4}[-8.710487436] = 3.629369765$$

$$y_8 = \frac{1}{-8}[9.27 - 3.2(3.629369765) + 8.8(-4.489728374)] = \frac{1}{-8}[-41.853592941] = 5.231699118$$

$$z_8 = \frac{1}{-12.2}[27.64051 + 2.79702(3.629369765) + 7(5.231699118)] = \frac{1}{-12.2}[74.413823643] = -6.099493741$$

9^{th} Approximation

$$x_9 = \frac{1}{-2.4}[-27.18 + 8(5.231699118) + 0.2(-6.099493741)] = \frac{1}{-2.4}[13.453694192] = -5.605705914$$

$$y_9 = \frac{1}{-8}[9.27 - 3.2(-5.605705914) + 8.8(-6.099493741)] = \frac{1}{-8}[-26.467286] = 3.30841075$$

$$z_9 = \frac{1}{-12.2}[27.64051 + 2.79702(-5.605705914) + 7(3.30841075)] = \frac{1}{-12.2}[35.120113696] = -2.878697844$$

.

10^{th} Approximation

$$x_{10} = \frac{1}{-2.4}[-27.18 + 8(3.30841075) + 0.2(-2.878697844)] = \frac{1}{-2.4}[-1.288453569] = 0.536855654$$

$$y_{10} = \frac{1}{-8}[9.27 - 3.2(0.536855654) + 8.8(-2.878697844)] = \frac{1}{-8}[-17.780479118] = 2.22255989$$

$$z_{10} = \frac{1}{-12.2}[27.64051 + 2.79702(0.536855654) + 7(2.22255989)] = \frac{1}{-12.2}[44.700025229] = -3.663936494$$

11^{th} Approximation

$$x_{11} = \frac{1}{-2.4}[-27.18 + 8(2.22255989) + 0.2(-3.663936494)] = \frac{1}{-2.4}[-10.132308181] = 4.221795075$$

$$y_{11} = \frac{1}{-8}[9.27 - 3.2(4.221795075) + 8.8(-3.663936494)] = \frac{1}{-8}[-36.48238539] = 4.560298174$$

$$z_{11} = \frac{1}{-12.2}[27.64051 + 2.79702(4.221795075) + 7(4.560298174)] = \frac{1}{-12.2}[71.371042477] = -5.850085449$$

12^{th} Approximation

$$x_{12} = \frac{1}{-2.4}[-27.18 + 8(4.560298174) + 0.2(-5.850085449)] = \frac{1}{-2.4}[8.1323683] = -3.388486792$$

$$y_{12} = \frac{1}{-8}[9.27 - 3.2(-3.388486792) + 8.8(-5.850085449)] = \frac{1}{-8}[-31.367594218] = 3.920949277$$

$$z_{12} = \frac{1}{-12.2}[27.64051 + 2.79702(-3.388486792) + 7(3.920949277)] = \frac{1}{-12.2}[45.609489615] = -3.738482755$$

13^{th} Approximation

$$x_{13} = \frac{1}{-2.4}[-27.18 + 8(3.920949277) + 0.2(-3.738482755)] = \frac{1}{-2.4}[3.439897667] = -1.433290694$$

$$y_{13} = \frac{1}{-8}[9.27 - 3.2(-1.433290694) + 8.8(-3.738482755)] = \frac{1}{-8}[-19.042118024] = 2.380264753$$

$$z_{13} = \frac{1}{-12.2}[27.64051 + 2.79702(-1.433290694) + 7(2.380264753)] = \frac{1}{-12.2}[40.293420533] = -3.302739388$$

.

14^{th} Approximation

$$x_{14} = \frac{1}{-2.4}[-27.18 + 8(2.380264753) + 0.2(-3.302739388)] = \frac{1}{-2.4}[-8.798429853] = 3.666012439$$

$$y_{14} = \frac{1}{-8}[9.27 - 3.2(3.666012439) + 8.8(-3.302739388)] = \frac{1}{-8}[-31.525346418] = 3.940668302$$

$$z_{14} = \frac{1}{-12.2}[27.64051 + 2.79702(3.666012439) + 7(3.940668302)] = \frac{1}{-12.2}[65.479098228] = -5.367139199$$

15^{th} Approximation

$$x_{15} = \frac{1}{-2.4}[-27.18 + 8(3.940668302) + 0.2(-5.367139199)] = \frac{1}{-2.4}[3.271918579] = -1.363299408$$

$$y_{15} = \frac{1}{-8}[9.27 - 3.2(-1.363299408) + 8.8(-5.367139199)] = \frac{1}{-8}[-33.598266846] = 4.199783356$$

$$z_{15} = \frac{1}{-12.2}[27.64051 + 2.79702(-1.363299408) + 7(4.199783356)] = \frac{1}{-12.2}[53.225817781] = -4.362771949$$

16^{th} Approximation

$$x_{16} = \frac{1}{-2.4}[-27.18 + 8(4.199783356) + 0.2(-4.362771949)] = \frac{1}{-2.4}[5.545712457] = -2.310713524$$

$$y_{16} = \frac{1}{-8}[9.27 - 3.2(-2.310713524) + 8.8(-4.362771949)] = \frac{1}{-8}[-21.728109878] = 2.716013735$$

$$z_{16} = \frac{1}{-12.2}[27.64051 + 2.79702(-2.310713524) + 7(2.716013735)] = \frac{1}{-12.2}[40.189494204] = -3.294220836$$

17^{th} *Approximation* .

$$x_{17} = \frac{1}{-2.4}[-27.18 + 8(2.716013735) + 0.2(-3.294220836)] = \frac{1}{-2.4}[-6.110734289] = 2.546139287$$

$$y_{17} = \frac{1}{-8}[9.27 - 3.2(2.546139287) + 8.8(-3.294220836)] = \frac{1}{-8}[-27.866789079] = 3.483348635$$

$$z_{17} = \frac{1}{-12.2}[27.64051 + 2.79702(2.546139287) + 7(3.483348635)] = \frac{1}{-12.2}[59.145552953] = -4.847996144$$

18^{th} Approximation

$$x_{18} = \frac{1}{-2.4}[-27.18 + 8(3.483348635) + 0.2(-4.847996144)] = \frac{1}{-2.4}[-0.28281015] = 0.117837562$$

$$y_{18} = \frac{1}{-8}[9.27 - 3.2(0.117837562) + 8.8(-4.847996144)] = \frac{1}{-8}[-33.769446264] = 4.221180783$$

$$z_{18} = \frac{1}{-12.2}[27.64051 + 2.79702(0.117837562) + 7(4.221180783)] = \frac{1}{-12.2}[57.5183695] = -4.714620451$$

19^{th} Approximation

$$x_{19} = \frac{1}{-2.4}[-27.18 + 8(4.221180783) + 0.2(-4.714620451)] = \frac{1}{-2.4}[5.646522174] = -2.352717572$$

$$y_{19} = \frac{1}{-8}[9.27 - 3.2(-2.352717572) + 8.8(-4.714620451)] = \frac{1}{-8}[-24.689963735] = 3.086245467$$

$$z_{19} = \frac{1}{-12.2}[27.64051 + 2.79702(-2.352717572) + 7(3.086245467)] = \frac{1}{-12.2}[42.663630164] = -3.497018866$$

20^{th} *Approximation*

$$x_{20} = \frac{1}{-2.4}[-27.18 + 8(3.086245467) + 0.2(-3.497018866)] = \frac{1}{-2.4}[-3.189440038] = 1.328933349$$

$$y_{20} = \frac{1}{-8}[9.27 - 3.2(1.328933349) + 8.8(-3.497018866)] = \frac{1}{-8}[-25.756352737] = 3.219544092$$

$$z_{20} = \frac{1}{-12.2}[27.64051 + 2.79702(1.328933349) + 7(3.219544092)] = \frac{1}{-12.2}[53.894371801] = -4.417571459$$

21^{st} Approximation

$$x_{21} = \frac{1}{-2.4}[-27.18 + 8(3.219544092) + 0.2(-4.417571459)] = \frac{1}{-2.4}[-2.307161555] = 0.961317315$$

$$y_{21} = \frac{1}{-8}[9.27 - 3.2(0.961317315) + 8.8(-4.417571459)] = \frac{1}{-8}[-32.680844247] = 4.085105531$$

$$z_{21} = \frac{1}{-12.2}[27.64051 + 2.79702(0.961317315) + 7(4.085105531)] = \frac{1}{-12.2}[58.925072471] = -4.829923973$$

.

22^{nd} Approximation

$$x_{22} = \frac{1}{-2.4}[-27.18 + 8(4.085105531) + 0.2(-4.829923973)] = \frac{1}{-2.4}[4.534859452] = -1.889524772$$

$$y_{22} = \frac{1}{-8}[9.27 - 3.2(-1.889524772) + 8.8(-4.829923973)] = \frac{1}{-8}[-27.186851693] = 3.398356462$$

$$z_{22} = \frac{1}{-12.2}[27.64051 + 2.79702(-1.889524772) + 7(3.398356462)] = \frac{1}{-12.2}[46.143966655] = -3.782292349$$

23^{rd} Approximation

$$x_{23} = \frac{1}{-2.4}[-27.18 + 8(3.398356462) + 0.2(-3.782292349)] = \frac{1}{-2.4}[-0.749606776] = 0.312336157$$

$$y_{23} = \frac{1}{-8}[9.27 - 3.2(0.312336157) + 8.8(-3.782292349)] = \frac{1}{-8}[-25.013648371] = 3.126706046$$

$$z_{23} = \frac{1}{-12.2}[27.64051 + 2.79702(0.312336157) + 7(3.126706046)] = \frac{1}{-12.2}[50.401062802] = -4.131234656$$

24^{th} Approximation

$$x_{24} = \frac{1}{-2.4}[-27.18 + 8(3.126706046) + 0.2(-4.131234656)] = \frac{1}{-2.4}[-2.99259856] = 1.246916067$$

$$y_{24} = \frac{1}{-8}[9.27 - 3.2(1.246916067) + 8.8(-4.131234656)] = \frac{1}{-8}[-31.074996386] = 3.884374548$$

$$z_{24} = \frac{1}{-12.2}[27.64051 + 2.79702(1.246916067) + 7(3.884374548)] = \frac{1}{-12.2}[58.318781014] = -4.780227952$$

25^{th} Approximation

$$x_{25} = \frac{1}{-2.4}[-27.18 + 8(3.884374548) + 0.2(-4.780227952)] = \frac{1}{-2.4}[2.938950795] = -1.224562831$$

$$y_{25} = \frac{1}{-8}[9.27 - 3.2(-1.224562831) + 8.8(-4.780227952)] = \frac{1}{-8}[-28.877404917] = 3.609675615$$

$$z_{25} = \frac{1}{-12.2}[27.64051 + 2.79702(-1.224562831) + 7(3.609675615)] = \frac{1}{-12.2}[49.483112572] = -4.055992834$$

.

Solution by Gauss Seidel method.
$x = -1.224562831 \simeq -1$
$y = 3.609675615 \simeq 3.6$
$z = -4.05592834 \simeq -4.1$

Iteration are tabulated as below .

Table (4.3.6): Solutions of Gauss Seidel

Iteration	x	y	z
1	11.408333333	4.504583333	-7.465731954
2	-3.068133448	5.82630177	-4.90516326
3	-7.687242294	1.162032668	-1.169948216
4	7.549053457	3.147814421	-5.802464299
5	1.315823956	5.750290311	-5.866633451
6	-7.353748249	2.353047496	-1.929775536
7	3.642322974	2.42093228	-4.489728374
8	3.629369765	5.231699118	-6.099493741
9	-5.605705914	3.30841075	-2.878697844
10	0.536855654	2.22255989	-3.663936494
11	4.221795075	4.560298174	-5.850085449
12	-3.388486792	3.920949277	-3.738482755
13	-1.433290694	2.380264753	-3.302739388
14	3.666012439	3.940668302	-5.367139199
15	-1.363299408	4.199783356	-4.362771949
16	-2.310713524	2.716013735	-3.294220836
17	2.546139287	3.483348635	-4.847996144
18	0.117837562	4.221180783	-4.714620451
19	-2.352717572	3.086245467	-3.497018866
20	1.328933349	3.219544092	-4.417571459
21	0.961317315	4.085105531	-4.829923973
22	-1.889524772	3.398356462	-3.782292349

.

23	0.312336157	3.126706046	-4.131234656
24	1.246916067	3.884374548	-4.780227952
25	-1.224562831	3.609675615	-4.055992834

.

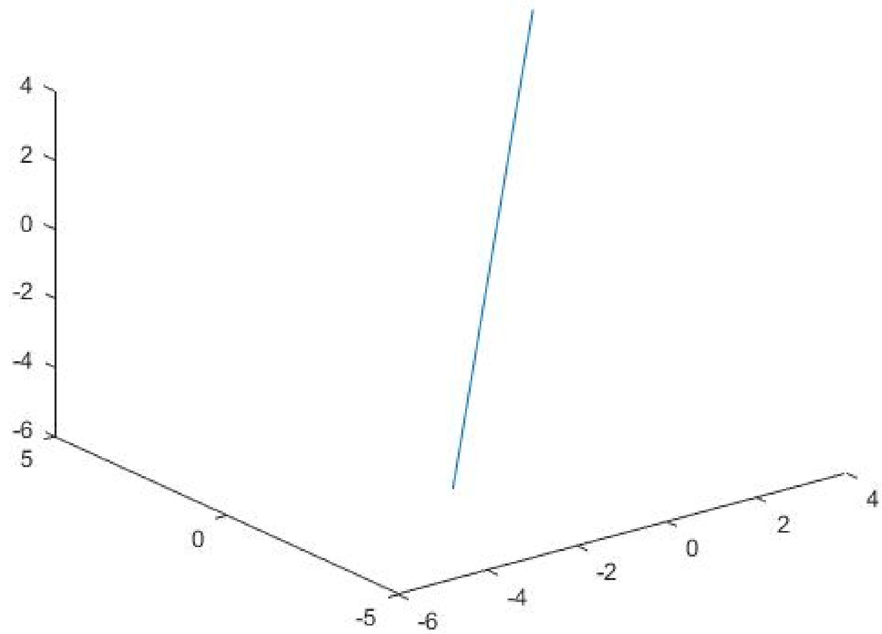

Figure (4.3.6): Solutions of Gauss Seidel .

Third, SOR method.

Now we will solve the system :

$$-2.4x - 8y - 0.2z = -27.18$$

$$3.2x - 8y - 8.8z = 9.27$$

$$-2.79702x - 7y - 12.2z = 27.64051$$

With initial guess

$$(x, y, z) = (0, 0, -1), \quad and \quad w = 0.604$$

By using Gauss SOR (Successive Over-Relaxation) method?

Solution :

We know that, for symmetric positive definite matrix the SOR method converges for values of the relaxation parameter w from the interval (0,2).

The iterations of the SOR method :

1. Total equations are 3.

$$-2.4x - 8y - 0.2z = -27.18$$

$$3.2x - 8y - 8.8z = 9.27$$

$$-2.79702x - 7y - 12.2z = 27.64051$$

2. From the above equations, first write down the equations for Gauss Seidel.

$$x_{k+1} = \frac{1}{-2.4}(-27.14 + 8y_k + 0.2z_k)$$

$$y_{k+1} = \frac{1}{-8}(9.27 - 3.2x_{k+1} + 8.8z_k)$$

$$z_{k+1} = \frac{1}{-12.2}(27.64051 + 2.79702x_{k+1} + 7y_{k+1})$$

3. Now multiply the right hand side by the parameter W and add to it the vector x_k . From the previous iteration multiplied by the factor of (1-w).

$$x_{k+1} = (1-w).x_k + w.\frac{1}{-2.4}(-27.14 + 8y_k + 0.2z_k)$$

$$y_{k+1} = (1-w).y_k + w.\frac{1}{-8}(9.27 - 3.2x_{k+1} + 8.8z_k)$$

$$z_{k+1} = (1-w).z_k + w.\frac{1}{-12.2}(27.64051 + 2.79702x_{k+1} + 7y_{k+1})$$

Solution steps are :

1^{st} *Approximation*

$$x_1 = (1-0.604).1+(0.604)\frac{1}{-2.4}(-27.18+8(0)+0.2(-1)) = (0.396).0$$

$$+0.604.\frac{1}{-2.4}(-27.38) = 0 + 6.89063333 = 6.89063333$$

$$y_1 = (1-0.604).1+(0.604)\frac{1}{-8}(9.27-3.2(6.89063333)+8.8(-1)) = (0.396).0+$$

$$0.604.\frac{1}{-8}(-21.580026667) = 0 + 1.629292013 = 1.629292013$$

$$z_1 = (1-0.604).1+(0.604)\frac{1}{-12.2}(27.64051+2.79702(6.89063333)+7(1.629292013))$$

$$= (0.396).-1+0.604.\frac{1}{-12.2}(58.318793339) = -0.396-2.88725893 = -3.28325829$$

$2^{nd} Approximation$

$$x_2 = (1-0.604).6.89063333 + (0.604)\frac{1}{-2.4}(-27.18 + 8(1.629292013) +$$

$$0.2(-3.28325893)) = 6.453940214$$

$$y_2 = (1-0.604).1.629292013 + (0.604)\frac{1}{-8}(9.27 - 3.2(6.453940214) +$$

$$8.8(-3.3283258293)) = 3.685983403$$

$$z_2 = (1-0.604).-3.28325829 + (0.604)\frac{1}{-12.2}(27.64051 + 2.79702(6.453940214) +$$

$$7(3.685983403)) = -4.839718889$$

$3^{rd} Approximation$

$$x_3 = (1-0.604).6.453940214 + (0.604)\frac{1}{-2.4}(-27.18 + 8(3.685983403) +$$

$$0.2(-4.839718889)) = 2.218546258$$

$$y_3 = (1-0.604).3.685983403 + (0.604)\frac{1}{-8}(9.27 - 3.2(2.218546258) +$$

$$8.8(-4.839718889)) = 4.511274433$$

$$z_3 = (1-0.604).-4.839718889 + (0.604)\frac{1}{-12.2}(27.64051 + 2.79702(2.218546258) +$$

$$7(4.511274433)) = -5.155590038$$

In the same way, we calculate the rest of the iterations...

Where the tenth iteration is :
$10^{th} Approximation$

$$x_{10} = (1-0.604).0.312006306+(0.604)\frac{1}{-2.4}(-27.18+8(3.548630869)+$$

$$0.2(-4.31966493)) = 0.036700816$$

$$y_{10} = (1-0.604).3.548630869 + (0.604)\frac{1}{-8}(9.27 - 3.2(0.036700816)+$$

$$8.8(-4.31966493)) = 3.584225121$$

$$z_{10} = (1-0.604).-4.31966493+(0.604)\frac{1}{-12.2}(27.64051+2.79702(0.036700816)+$$

$$7(3.584225121)) = -4.326240936$$

And the twentieth repetition is :

$20^{th} Approximation$

$$x_{20} = (1-0.604).0.06014094+(0.604)\frac{1}{-2.4}(-27.18+8(3.520218694)+$$

$$0.2(-4.272644348)) = -0.055833016$$

$$y_{20} = (1-0.604).3.520218694+(0.604)\frac{1}{-8}(9.27-3.2(-0.055833016)+$$

$$8.8(-4.27264438)) = 3.519377251$$

$$z_{20} = (1 - 0.604). - 4.272644348 + (0.604)\frac{1}{-12.2}(27.64051 + 2.79702$$

$$(-0.055833016) + 7(3.519377251)) = -4.272333617$$

And the twenty-two repetition is :
$22^{th} Approximation$

$$x_{22} = (1-0.604).0.052448614+(0.604)\frac{1}{-2.4}(-27.18+8(3.519655261)+$$

$$0.2(-4.272775571)) = -0.051645874$$

$$y_{22} = (1-0.604).3.519655261+(0.604)\frac{1}{-8}(9.27-3.2(-0.051645874)+$$

$$8.8(-4.272775571)) = 3.52025293$$

$$z_{22} = (1-0.604).-4.272775571+(0.604)\frac{1}{-12.2}(27.64051+2.79702$$

$$(-0.051645874)+7(3.52025293)) = -4.2723268871$$

Solution by SOR (Successive Over-Relaxation) method.
$x = -0.051645874 \simeq -0.05$
$y = \quad 3.52025293 \simeq 3.52$
$z = -4.2723268871 \simeq -4.27$

Iteration are tabulated as below .

Table (4.3.7): Solutions of SOR

Iteration	x	y	z
1	6.890633333	1.629292013	-3.283258293
2	6.453940214	3.685983403	-4.839718889
3	2.218546258	4.511274433	-5.155590038
4	-1.104356843	4.245141084	-4.728304147
5	-1.90591805	3.662206344	-4.246082078
6	-1.073966189	3.311975414	-4.048951907
7	-0.049303864	3.289869097	-4.105117793
8	0.403796817	3.427900735	-4.237938673
9	0.312006306	3.548630869	-4.31966493

10	0.036700816	3.584225121	-4.326240936
11	-0.143652259	3.55911624	-4.2951689
12	-0.16608349	3.523109477	-4.26727979
13	-0.103876392	3.505350509	-4.258695354
14	-0.043919743	3.507099985	-4.264204734
15	-0.023421883	3.516405493	-4.27244979
16	-0.033624818	3.52310346	-4.276623207
17	-0.050940359	3.524345238	-4.276308451
18	-0.060313269	3.522363363	-4.274199056
19	-0.06014094	3.520218694	-4.272644348
20	-0.055833016	3.519377251	-4.272333617
21	-0.052448614	3.519655261	-4.272775571
22	-0.051645874	3.52025293	-4.273268871

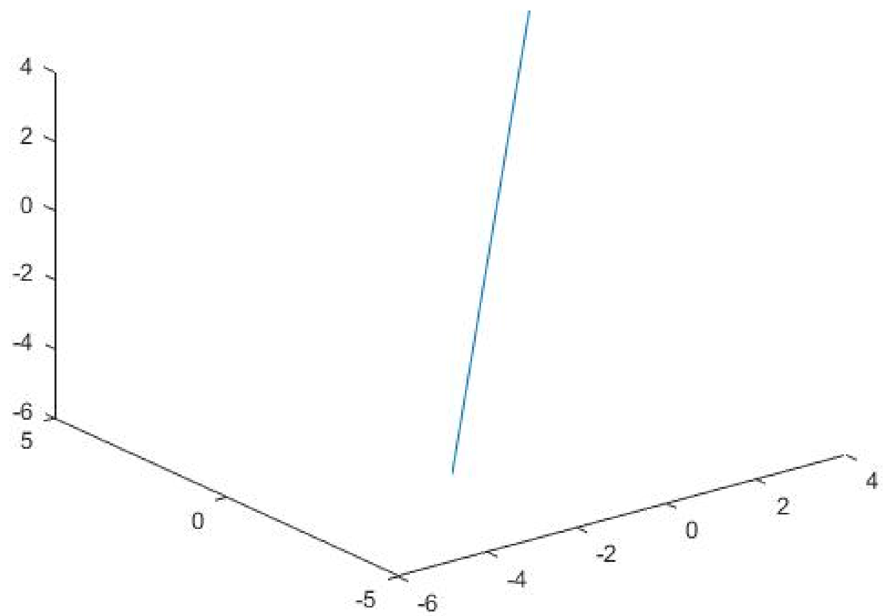

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Figure (4.3.7): Solutions of SOR .

A certain point may be determined by only three satellites. This new system can be resolved in the same way as before. Let's take this example 3: the position and ranges of three satellites are given in the following table.

Table (4.3.8): Satellite data.

Satellite	Position	Range
1	(-2,1,2)	4
2	(2,-3,1)	5
3	(3,2,0)	3

In the same way as the solution in the previous example, this example can be simplified and the following equations can be obtained:

i) $x^2 + y^2 + z^2 + 4x - 2y - 4z = 7$
ii) $x^2 + y^2 + z^2 - 4x + 6y - 2z = 11$
iii) $x^2 + y^2 + z^2 - 6x - 4y = -4$

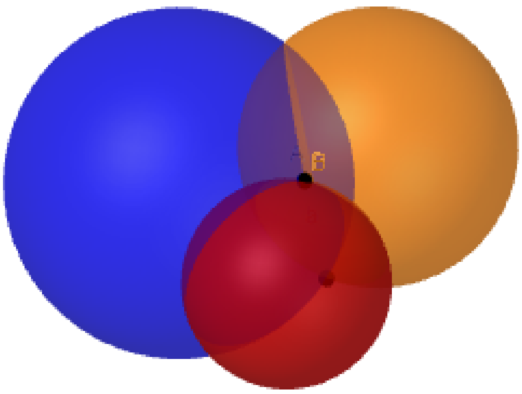

Figure (4.3.8): The intersection of three spheres in two points.

These three spheres intersect at the following two points :

(1.87 , 1.68 , 2.76)
(0.38 , 1.18 , -1.21)

GPS chooses the point closest to the center of the earth.

It is clear from the solutions and by comparing with the exact solution that SOR method is the best and most accurate way to solve.

Iterative methods are a class of methods for solving the linear system Ax = b. Unlike direct methods, based on Gauss elimination that involve a fixed number of iterations, iterative methods however do not have a fixed number of operations. They start from an initial assumption that is successively improved until a sufficiently accurate solution is obtained. Direct methods if applied to the solution of sparse systems often destroy the sparseness. Thus direct methods often cannot take advantage of the sparseness of the system in terms of reduced storage and operations. Large sparse systems are particularly suited to iterative methods since iterative methods do not introduce any additional non-zero elements to the coefficient matrix.

Comparison of the three methods in the solution: Jacobi and Gauss Seidel and SOR.

Jacobi :

The Jacobi method is based on solving for every variable locally with respect to the other variables; one iteration of the method corresponds to solving for every variable once. The resulting method is easy to understand and implement, but convergence is slow.[20]

Gauss-Seidel :

The Gauss-Seidel method is like the Jacobi method, except that it uses updated values as soon as they are available. In general, if the Jacobi method converges, the Gauss-Seidel method will converge faster than the Jacobi method, though still relatively slowly.[20]

SOR :

Successive Over Relaxation (SOR) can be derived from the Gauss-Seidel method by introducing an extrapolation parameter (w). For the optimal choice of (w), SOR may converge faster than Gauss-Seidel by an order of magnitude. The SOR method is an improvement to the

Gauss-Seidel algorithm. It substantially increases the rate of convergence of the Gauss Seidel method.[20]

The Successive-Over Relaxation (SOR) could be considered more efficient of the other methods because it is requires less iteration than the Gauss-Seidel and Jacobi methods.[23]

When dealing with the GPS may produce over determined system of equations, three equations with two unknowns (x,y). Which requires method least-squares to solve the system.

Least-Squares .

When the GPS receiver gets 4 satellites, we use Newton-Raphson method to determine and approximate the location of the receiver, but if there are more than 4 satellites, we use least square method to reach the exact location and ignore any additional observations from the other satellites.

Theorem

Let A be an $m \times n$ matrix and let b be a vector in R^m. The Least-Squares solution of $Ax = b$ are the solutions of the matrix equation $A^T Ax = A^T b$ [9]

In particular, finding a least-squares solution means solving a consistent system of linear equations. We can translate the above theorem into a recipe A.

Recipe : compute a least-squares solution. Let A be an $m \times n$ matrix and let b be a vector in R^m. Here is a method for computing a least-squares solution of $Ax = b$.

1) Compute the matrix $A^T A$ and the vector $A^T b$.

2) From the argumend matrix for the matrix equation $A^T Ax = A^T b$, and row reduce .

3) This equation is always constant, and any solution x is a least -squares solution .[9]

Example 4 (Over determined system of equations)

The position and ranges of four satellites are given in the following table. We calculate the user's position (x,y) two-dimensional.

Table (4.3.9): Satellite data

Satellite	Position	Range
1	(-2,2)	4.53103974
2	(-1,1)	$\sqrt{11.90963143}$
3	($\frac{-3}{2}$,1)	$\sqrt{15.7458383}$
4	($\frac{-3}{2}$,$\frac{5}{2}$)	$\sqrt{17.7889417}$

To locate the user's position two formulas are needed :

$$R = (t_r - t_e)c$$

$$R = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

The satellite's position at (-2, 2) can also be expressed as

$$R = \sqrt{(x + 2)^2 + (y - 2)^2}$$

$$(x + 2)^2 + (y - 2)^2 = 4.53103974^2$$

The positions of the other three satellites can be derived using the same logic. These are not linear equations, but can be linearized algebraically.

Table (4.3.10): Equation's of Satelites

Satellite	Position	Range	Equations
1	(-2,2)	4.53103974	$(x+2)^2+(y-2)^2=20.5303211$
2	(-1,1)	$\sqrt{11.90963143}$	$(x+1)^2+(y-1)^2=11.90963143$
3	$(\frac{-3}{2}$,1)	$\sqrt{15.7458383}$	$(x+\frac{3}{2})^2+(y-1)^2=15.7458383$
4	$(\frac{-3}{2},\frac{5}{2})$	$\sqrt{17.7889417}$	$(x+\frac{3}{2})^2+(y-\frac{5}{2})^2=17.7889417$

1. Simplify the four equations.

$$i)x^2+4x+y^2-4y=12.5303211$$

$$ii)x^2+2x+y^2-2y=9.90963143$$

$$iii)x^2+3x+y^2-2y=12.4958383$$

$$iv)x^2+3x+y^2-5y=9.28894171$$

2. Subtract the first equation from each of the other three.

i) First equation - Second equation :

$$2x-2y=2.62068966$$

ii) First equation - Third equation :

$$x-2y=0.172413791$$

iii) First equation - Fourh equation :

$$x+y=3.5862069$$

To find an approximate solution to this system we will use Least Squares method. In this method, the best unique solution (i.e. user position) for unknown parameters can be obtained by ignoring additional observations (equation).

$$2x - 2y = 2.62068966$$

$$x - 2y = 0.172413791$$

$$x + y = 3.5862069$$

From equation $Ax = b$ we have:

$$\text{A=} \begin{bmatrix} 2 & -2 \\ 1 & -2 \\ 1 & 1 \end{bmatrix} \qquad \text{and } b = \begin{bmatrix} 2.62068966 \\ 0.172413791 \\ 3.5862069 \end{bmatrix}.$$

we put

$$A^T A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ -5 & 9 \end{bmatrix}$$

Then

$$A^T b = \begin{bmatrix} 2 & 1 & 1 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2.62068966 \\ 0.172413791 \\ 3.5862069 \end{bmatrix} = \begin{bmatrix} 8.999958623 \\ -1.999958614 \end{bmatrix}$$

From $A^T A x = A^T b$ we have, we have

$$\begin{bmatrix} 6 & -5 \\ -5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8.999958623 \\ -1.999958614 \end{bmatrix}$$

By Gauss-Jordan elimination method :

$$\left[\begin{array}{cc|c} 6 & -5 & 8.999958623 \\ -5 & 9 & -1.999958614 \end{array}\right]$$

$$(RREF) \quad \left[\begin{array}{cc|c} 1 & 0 & 2.44827015644827586l9 \\ 0 & 1 & 1.13793246313793l0344 \end{array}\right]$$

So

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.4482701564482758619 \\ 1.1379324631379310344 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{71}{29} \\ \frac{33}{29} \end{bmatrix}$$

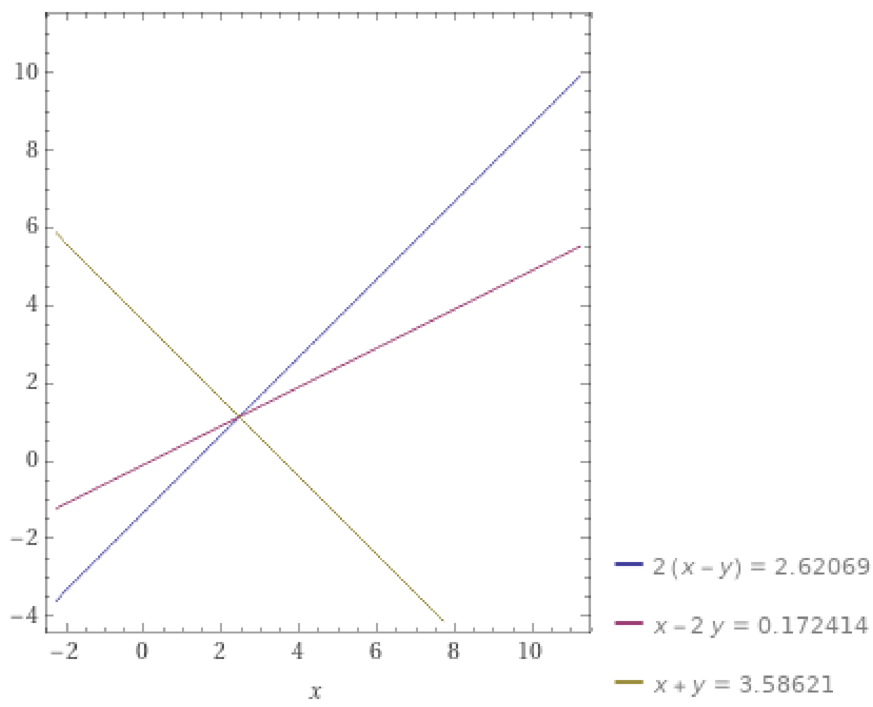

Figure (4.3.9): Solutions of Equations.

4.4 Non Linear Examples of GPS

When dealing with GPS systems, it may produce nonlinear systems, so we will work with these systems using Newton's method and Quasi Newton's method.

The systems of equations generated by satellite ranging measurements will most likely be inconsistent. Due to this reason, we have to use the Newton-Raphson method to estimate the user's position. The Newton-Raphson method is used in GPS receivers when ranging from four satellites with four unknowns; $(X, Y, Z) and \triangle T$.

Newton Raphson Methods

It is one of the best methods for finding successive approximations to the roots of a real-valued function. Requires only one initial guess and can converge quickly.

The iteration equation is $F(x_{i+1}) = x_i - \frac{F(x_i)}{F'(x_i)}$.[34]

There are advantages and disadvantages to Newton-Raphson method.

Two advantages are:
(1) It converges quickly, if it converges.
(2) It requires only one guess.

However, there are four disadvantages to this method :

(1)The initial guess can't be zero (dividing by zero).
If $F'(x_i)$ is equal to zero, then x_{i+1} cannot be defined.

(2) Root jumping.

For functions having periodic behavior and many roots , the method ends up giving the root closest to zero and skip the other ones . Example function f(x)= sin(x)

(3) Oscillations near local maxima or minima.

The iterative solution will start bouncing around and oscillates closer or around x = 0. This happens because the parabolic function does not have real roots.[36]

(4) Inflection points that are close to the roots.

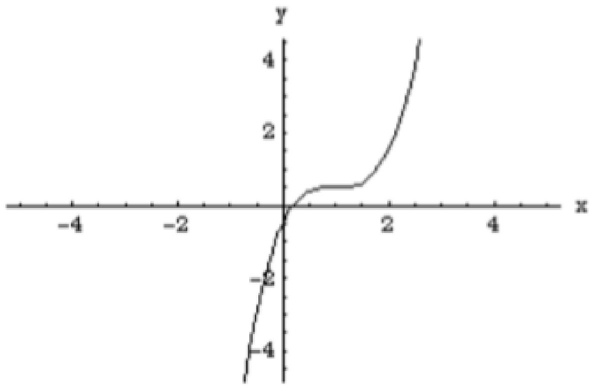

Figure (4.4.1): Plot of $F(x) = (x-1)^3 + 0.512$

Steps to Newton-Raphson iterations.

1. Set all functions equal to zero.

2. Start with an initial guess.

3. Calculate the Jacobian and Residual Matrix.

4. Find the next approximation to the desired solution by solving the linear system.

5. Calculate the new initial values.

6. If error is sufficiently small, stop the iteration, else repeat step 3 using the new initial values.[36]

Example (System of non linear equation)

Find root of the equations :

$$x^2 + y^2 - 4 = 0, x^2 - y + 1 = 0$$

with initial guess

$$(1, 2)$$

We will solve this system using Newton's method and Quasi Newtons method.

At first we will find the exact solution .

From the equation $x^2 - y = -1$ produces a value of x^2 :

$$x^2 = y - 1$$

Substitution in the first equation

$$y^2 + y - 1 - 4 = 0$$

$$y^2 + y - 5 = 0$$

We solve the quadratic equation and get :

y = 1.79128784 and y = -2.79128784

Now we find the values of x by substitution, when $x^2 = y - 1$ produces the exact solution in the next table.

Table(4.4.1):The exact solution

x	y
0.8895436133208984	1.79128784
- 0.8895436133208984	1.79128784
1.947122964786765 i	-2.79128784
- 1.947122964786765 i	-2.79128784

Now we will find solution the nonlinear system by using modified Newton Raphson method:

$$f_1 = x^2 + y^2 - 4 \longrightarrow (1)$$

$$f_2 = x^2 - y + 1 \longrightarrow (2)$$

In order to use modified Newton Raphson method (Multivariate Newton Raphson method), we must first determine the functional form of the partial derivatives.

$$J_{1,1} = \frac{\partial f_1}{\partial x} = 2x$$

$$J_{1,2} = \frac{\partial f_1}{\partial y} = 2y$$

$$J_{2,1} = \frac{\partial f_2}{\partial x} = 2x$$

$$J_{2,2} = \frac{\partial f_2}{\partial y} = -1$$

$$J = \begin{bmatrix} 2x & 2y \\ 2x & -1 \end{bmatrix}$$

Formula

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}^{-1} \times f(\begin{bmatrix} x_n \\ y_n \end{bmatrix})$$

1^{st} : let $n = 0, x = 1, y = 2$

Substitute into the previous equation

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1} \times f(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix}$$

2^{nd} : let $n = 1, x = 0.9, y = 1.8$

$$\begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix} - \begin{bmatrix} 1.8 & 3.6 \\ 1.8 & -1 \end{bmatrix}^{-1} \times f(\begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix})$$

$$= \begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix} - \begin{bmatrix} 1.8 & 3.6 \\ 1.8 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8896 \\ 1.7913 \end{bmatrix}$$

3^{rd} : let $n = 2, x = 0.8896, y = 1.7913$

$$\begin{bmatrix} 0.8896 \\ 1.7913 \end{bmatrix} - \begin{bmatrix} 1.7792 & 3.5826 \\ 1.7792 & -1 \end{bmatrix}^{-1} \times f(\begin{bmatrix} 0.8896 \\ 1.7913 \end{bmatrix})$$

$$= \begin{bmatrix} 0.8896 \\ 2.9333 \end{bmatrix} - \begin{bmatrix} 1.7792 & 3.5826 \\ 1.7792 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} 0.0002 \\ 0.0001 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8895 \\ 1.7913 \end{bmatrix}$$

Approximate root using Newton Raphson mehtod is x=0.8895,y=1.7913

Table (4.4.2): Solutions of Newtopn Raphson

Iteration	x	y
1	0.9000	1.8000
2	0.8896	1.7913
3	0.8895	1.7931
4	0.8895	1.7931

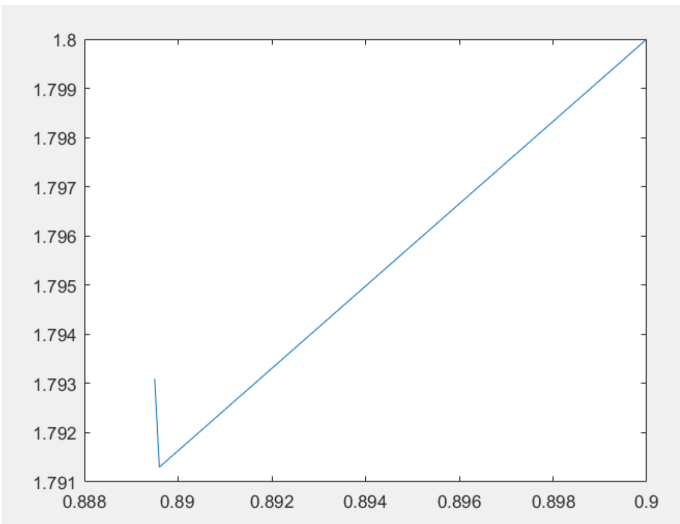

Figure (4.4.2): Solution of Newtopn Raphson.

Quasi Newton.

Find root of the equations using Quasi Newton method :[3]

$$F_1 = x^2 + y^2 - 4$$

$$F_2 = x^2 - y + 1$$

with initial guess

$$(1, 2)$$

Solution:

$$f_1 = x^2 + y^2 - 4 \longrightarrow (1)$$

$$f_2 = x^2 - y + 1 \longrightarrow (2)$$

$$J_{1,1} = \frac{\partial f_1}{\partial x} = 2x$$

$$J_{1,2} = \frac{\partial f_1}{\partial y} = 2y$$

$$J_{2,1} = \frac{\partial f_2}{\partial x} = 2x$$

$$J_{2,2} = \frac{\partial f_2}{\partial y} = -1$$

$$J(x, y) = \begin{bmatrix} 2x & 2y \\ 2x & -1 \end{bmatrix}$$

Formula

1^{st} : let $n = 0, x = 1, y = 2$

Find F_1, F_2 and $F(x^{(0)})$

$$F_1(x, y) = 1^2 + 2^2 - 4 = 1$$

$$F_2(x, y) = 1^2 - 2 + 1 = 0$$

$$F(x^{(0)}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_0 = J(x^{(0)}, y^{(0)}) = \begin{bmatrix} 2(1) & 2(2) \\ 2(1) & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}$$

First iteration:

$$X^{(1)} = X^{(0)} - (A_0)^{-1} F(X^{(0)})$$

$$X^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1} \times f(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix}$$

2^{nd} : n=1,x=0.9,y=1.8

$$F_1(x, y) = 0.9^2 + 1.8^2 - 4 = 0.05$$

$$F_2(x, y) = 0.9^2 - 1.8 + 1 = 0.01$$

$$F(x^{(1)}) = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}$$

$$y_1 = F(x^{(1)}) - F(x^{(0)}) = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.95 \\ 0.01 \end{bmatrix}$$

$$S_1 = x^{(1)} - x^{(0)} = \begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix} - \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} -0.1 \\ -0.2 \end{bmatrix}$$

$$S_1^t = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}$$

$$S_1^t A_0^{-1} Y_1 = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} -0.95 \\ 0.01 \end{bmatrix}$$

$$S_1^t A_0^{-1} Y_1 = 0.0475$$

$$A_1^{-1} = A_0^{-1} + (\tfrac{1}{S_1^t A_0^{-1} Y_1})[(S_1 - (A_0^{-1} Y_1)) S_1^t A_0^{-1})]$$

$$A_1^{-1} = \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1} +(21.0526)[(\begin{bmatrix} -0.1 \\ -0.2 \end{bmatrix} \text{-} \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -0.95 \\ 0.01 \end{bmatrix}) (\begin{bmatrix} -0.1 & -0.2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1})]$$

$$A_1^{-1} = \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1} +(21.0526) \begin{bmatrix} -0.009 \\ \\ -0.008 \end{bmatrix} \begin{bmatrix} -0.05 & 0 \end{bmatrix}$$

$$A_1^{-1} = \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}^{-1} + \begin{bmatrix} 0.00947 & 0 \\ \\ 0.00842 & 0 \end{bmatrix} = \begin{bmatrix} 0.00045 & 0.4 \\ \\ 0.2004 & -0.2 \end{bmatrix}$$

$$X^{(2)} = X^{(1)} - A_1^{-1} F(X^{(1)})$$

$$X^{(2)} = \begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix} - \begin{bmatrix} 0.00045 & 0.4 \\ \\ 0.2004 & -0.2 \end{bmatrix} \times \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} 0.8909 \\ 1.79798 \end{bmatrix}$$

It also yields $$X^{(3)} = \begin{bmatrix} 0.8902 \\ 1.7916 \end{bmatrix}$$

Table (4.4.3): Solutions of Quasi-Newton

Iteration	x	y
1	0.9000	1.8000
2	0.8915	1.7924
3	0.8902	1.7916

Approximate root using Quasi-Newton mehtod is x=0.8902,y=1.7916

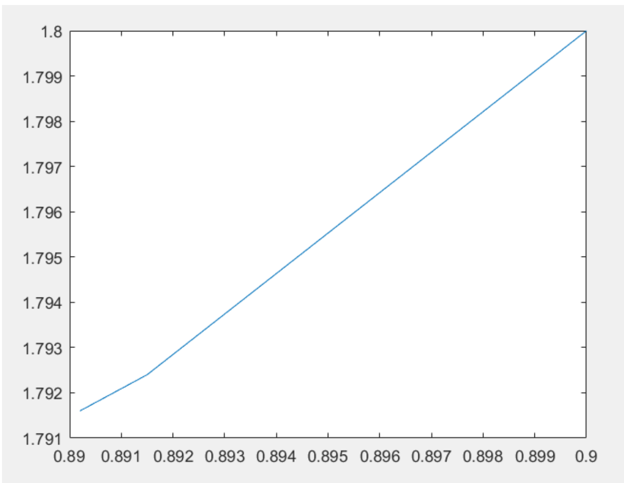

Figure (4.4.3): Solutions of Quasi-Newton.

By comparison between Newton's method and Quasi Newton method with the first exact solution at with initial guesses(1,2) , it turns out that Newton's method is the most accurate.

Chapter 5

System Modeling for Phase-Locked Loop of GPS

5.1 Introduction

Phase-Locked Loop (PLL) is a control system that produces an out-phase signal, linked to a phase of a reference signal input into the system.[25]

It is an electronic circuit consisting of an oscillator or generator and a phase detector. This circuit compares the phase of the incoming signal to the phase of the signal abstracted from the oscilloscope generator and adjusts the frequency of the oscilloscope to keep the phase in line with the reference. These loops can extract small signals that are present in disturbed communication channels. It can also produce frequency multiplexing or reference pulses.[25]

The basic (PLL) configuration contains:

• Phase detector (PD).
• Loop filter.
• Voltage-controlled oscillator (VCO).[38]

The most important characteristics of closed-locked loop:

Phase detector (PD):
• Analog signal multiplier.
•It is used to deal with the phase error, as it works to create a specific error signal that matches the phase error. Thus, the differences in the phases of signal input to the loop and output and the differences between the errors are determined.[13]

Loop filter:

• Dealing with conjugations (tranfer function F(s)).
• It has a distinct and influential role in the phase-closed loop, as it eliminates errors, noise and unwanted outputs, especially from (PD), and determines the best suitable dynamics for each phase - closed loop.[13]

Voltage-controlled oscillator (VCO):
• A sinusoidal signal is generated by it.
• The input voltage controls the frequency of the (VCO).[13]

Tuning the reference signal by synchronizing its frequency with the frequency of the local oscillator is the main purpose of the loop. The resulting bits are obtained from the raw data by processing GPS signals using acquisition and tracking algorithms. As this data in the form of bits is the basis for knowing the difference between the timing of the receiver and the satellite, thus helping to determine the false range between them.[25]

During the tracking process:

1- Fixed the center frequency of the narrow-band filter.
2- Generated locally signal follows the frequency of the input signal.
3- Both signals are compared through a phase compactor of which passes through a narrow band filter.[28]

To avoid reference input noise reference source noise, data jitter, phase noise on FM input signal, etc, the narrow bandwidth loop filter will help to suppress high frequency noise coming into the (PLL) from the reference port. (VCO) jitter is suppressed by the (PLL) within the loop bandwidth. It has a high-pass transfer function. Thus, to suppress (VCO) noise, we want a large loop bandwidth.[13]

To track a GPS signal two tracking loops are needed: a loop to track the carrier frequency referred as carrier loop and the other one to track the C/A code known as the code loop.[28]

Dealing with the C/A code requires going through a set of stages, which leads to the removal of the code, which leads to obtaining the navigation data through the signal tracking process, which leads to the continuous stimulation of the loop. But it must be considered that when the GPS receiver is stable, the refresh rate of the closed loop will be slow because the locally generated signal will be of slow frequency variation, as a result of the required basic frequency will be very slow due to the movement of the satellite.[28]

5.2 Modeling for Phase-Locked Loop

VCO (Voltage Controlled Oscillator) is an electronic oscillator whose frequency is controlled by the input voltage and this voltage determines the instantaneous oscillation frequency. Thus, (VCO) can be used for frequency modulation or phase modulation.

The (VCO) oscillates at an angular frequency, w_2(out). Its frequency is set to a nominal w_0 when the control voltage is zero. Frequency is assumed to be linearly proportional to the control voltage with a gain coefficient Q_1, so w_2(t)= w_0 + Q_1 u(t).[28]

A basic phase-locked loop:

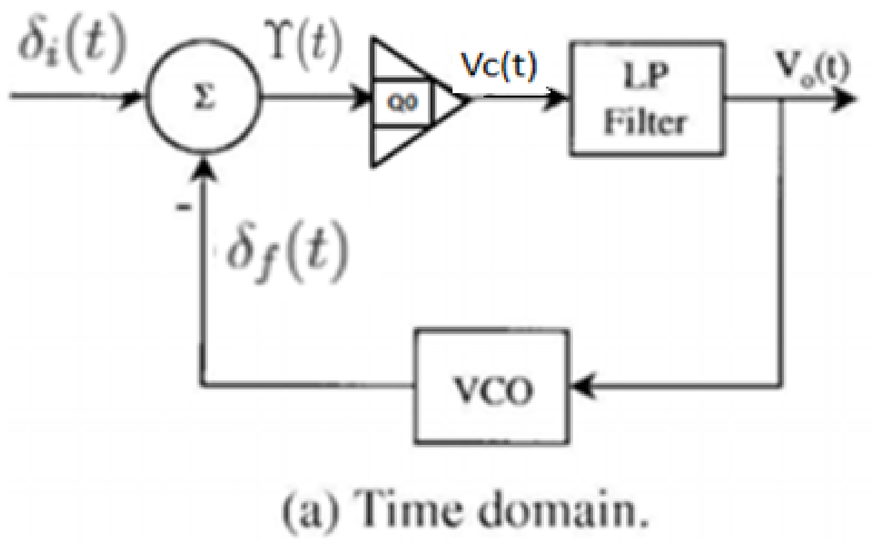

Figure (5.2.1): A basic Phase-Locked Loop (Time domain).

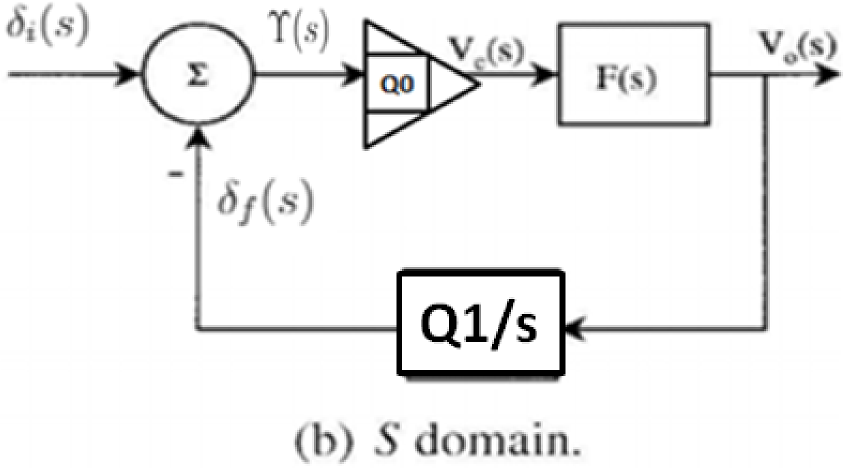

Figure (5.2.2): A basic Phase-Locked Loop (S domain).

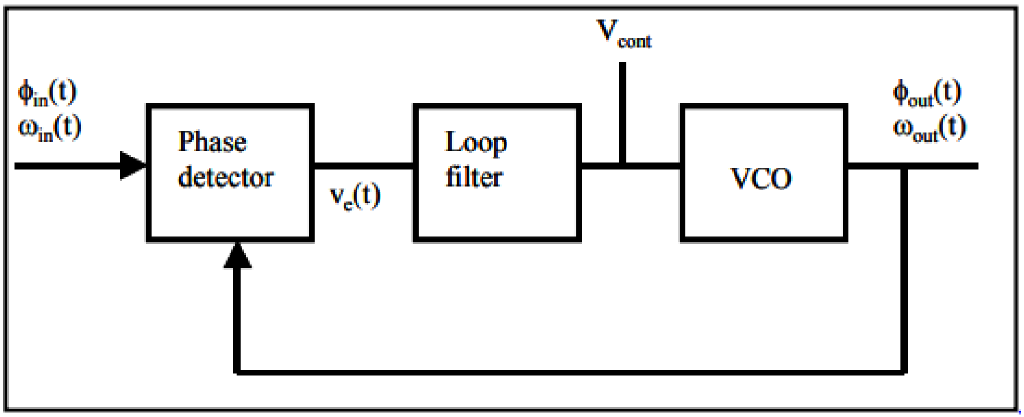

Figure (5.2.3): Phase Locked Loop Circuits .

Operation principle of phase-locked loop.

Phase detector (PD) compares the phase of the input signal δ_i(t) against the phase of the (VCO) output δ_f(t) and produces an error signal Υ(t). This error signal is then filtered, in order to remove noise and other unwanted components of the input spectrum, the sum of filter output V_0 and an additive external control voltage $V_c(t)$ controls the instantaneous (VCO) frequency.[13]

Important (PLL) characteristic.

The Phase-Locked Loop(PLL) tracks the phase of the input signal. But before a (PLL) can track, it must first reach the phase-locked condition In general, the (VCO) center frequency w_0 differs from the frequency w_i of the input signal. Therefore, first the (VCO) frequency has to be tuned to the input frequency by the loop this process is called frequency pull-in. Then the (VCO) phase has to be adjusted according to the input phase. This process is known as phase lock-in.[13]

Both the frequency pull-in and phase lock-in processes are parts of acquisition which is a highly nonlinear process and is very hard to analyze after acquisition the (PLL) achieves the phase- locked condition, where the (PLL) tracks the input phase. Under this phase-locked condition, the (VCO) frequency is equal to the input frequency.[13]

The output frequency, which can be expressed as:

$$w_2(t) = w_0 + Q_1 u(t)$$

The phase angle of the (VCO) can be obtained by integrating :

$$\int_0^t w_2(\tau)\, d\tau = w_0 t + \delta_f(t) = w_0 t + \int_0^t Q_1 u(\tau)\, d\tau$$

where:

$$\delta_f(t) = \int_0^t Q_1 u(\tau)\, d\tau$$

To find the Laplace transform of δ_f(t) we will use the definition of Laplace transform as the following:

$$L[f(t)] = f(s) = \int_0^\infty f(t) e^{-st}\, dt$$

where

$$f(t) = Q_1$$

So ,

$$L[f(t)] = f(s) = \int_0^\infty Q_1 e^{-st}\, dt$$

$$= \frac{-Q_1}{s}(e^{-\infty t} - e^{-0t}) = \frac{-Q_1}{s}(e^{-\infty} - e^{0}) = \frac{-Q_1}{s}(0-1) = \frac{Q_1}{s}$$

So,

$$\delta_f(s) = \frac{Q_1}{s}$$

From Figure (5.2.2):

$$V_c(s) = \Upsilon(s)Q_0 = (\delta_i(s) - \delta_f(s))Q_0$$

$$V_0(s) = V_c(s)F(s)$$

$$\delta_f(s) = V_0(s)\frac{Q_1}{s}[28]$$

Hence result the following:

The error function ($\Upsilon(s)$):[28]

$$\Upsilon(s) = \delta_i(s) - \delta_f(s)$$

$$\Upsilon(s) = \frac{V_c(s)}{Q_0}$$

$$\Upsilon(s) = \frac{V_0(s)}{Q_0F(s)}$$

$$\Upsilon(s) = \frac{1}{Q_0F(s)}(\frac{\delta_f(s)s}{Q_1})$$

And we have

$$\Upsilon(s) = \delta_i(s) - \delta_f(s) = \frac{1}{Q_0 F(s)} (\frac{\delta_f(s)s}{Q_1})$$

So,

$$\delta_i(s) = \delta_f(s)(1 + \frac{s}{Q_1 Q_0 F(s)})$$

The transfer function ($\hbar$(s)):

$$\hbar(s) = \frac{\delta_f(s)}{\delta_i(s)}$$

$$\hbar(s) = \frac{\delta_f(s)}{\delta_f(s)(1 + \frac{s}{Q_1 Q_0 F(s)})}$$

$$\hbar(s) = \frac{Q_0 Q_1 F(s)}{s + Q_0 Q_1 F(s)}$$

The error transfer function ($\hbar_e$(s)):[28]

$$\hbar_e(s) = \frac{\Upsilon(s)}{\delta_i(s)}$$

$$\hbar_e(s) = \frac{\delta_i(s) - \delta_f(s)}{\delta_i(s)}$$

$$\hbar_e(s) = 1 - \hbar(s)$$

$$\hbar_e(s) = 1 - \frac{Q_0 Q_1 F(s)}{s + Q_0 Q_1 F(s)}$$

$$\hbar_e(s) = \frac{s}{s + Q_0 Q_1 F(s)}$$

So, parameters of closed-loop transfer and error functions are determined by:

- Loop gain Q = $Q_0 Q_1$.

- Transfer function of loop filter F(s).[13]

The equivalent noise bandwidth (β_n):

$$\beta_n = \int_0^{\infty} |\hbar_{(jw)}|^2 \, df$$

The angular frequency:

$$w = 2\pi f$$

In order to study the properties of the phase-locked loops, two types of input signals are usually studied.[28]

The first type is a unit step function as:

$$\delta_i(t) = u_t \quad or \quad \delta_i(s) = \frac{1}{s}$$

The second type is a frequency-modulated signal as:

$$\delta_i(t) = \triangle w(t) \quad or \quad \delta_i(s) = \frac{\triangle w}{s^2}$$

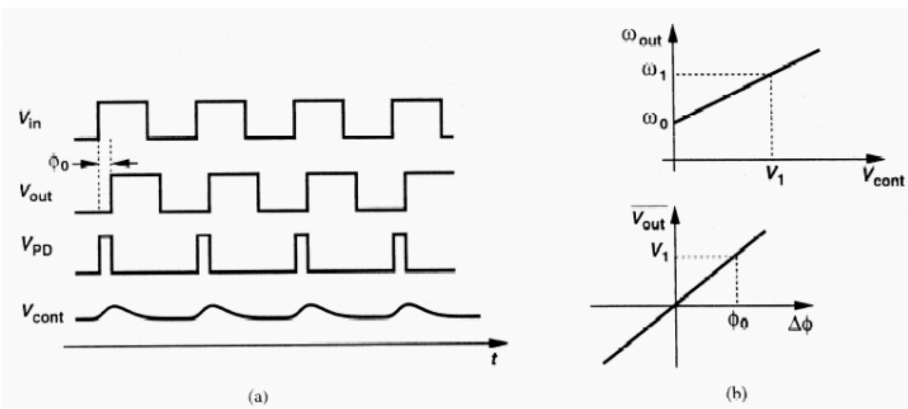

Figure (5.2.4): Inputs waves to the phase detector .

The two inputs to the phase detector are depicted as square waves. Transfer function $\hbar$(s) will produce an output pulse whenever there is a phase misalignment.[17]

Suppose that an output frequency w_1 is needed. From the upper right figure, a control voltage V_1 will be necessary to produce this output frequency.[25]

The phase detector can produce this V_1 only by maintaining a phase offset δ_i at its input. In order to minimize the required phase offset or error, the phase-locked loop gain, Q_1, Q_0, should be maximized, a high loop gain is beneficial for reducing phase errors.

The steady state error is a characteristic of feedback control systems, this is the error remaining in the loop at the phase detector output after all transients have died out. And the large loop gain Q(s) leads to a small phase error.

Order and type of the closed loop system is determined by the choice of loop filter F(s) which is more a controller than simply a filter.

5.3 First-Order Phase-Locked Loop

A first-order phase-locked loop implies that the denominator of the transfer function $\hbar$(s) is a first-order function of s.

The filter function in the case of the first order phase-locked loop with feedback is:

$$F(s) = 1$$

Therefore, input and output frequencies are identical. The input and output phase should track one another, but there may be a fixed offset depending on the phase detector implementation.[28]

In the phase-locked loop in the case of a unit step function input.

The equivalent transfer function $\hbar$(s):

$$\hbar(s) = \frac{Q_0 Q_1 F(s)}{s + Q_0 Q_1 F(s)} = \frac{Q_0 Q_1}{s + Q_0 Q_1}$$

and

$$\hbar(w) = \frac{Q_0 Q_1}{w + Q_0 Q_1}$$

The equivalent noise bandwidth β_n:[28]

$$\beta_n = \int_0^\infty |\hbar_{(w)}|^2 \, df$$

where

$$w = 2\pi f$$

and

$$dw = 2\pi df \quad so \quad df = \frac{dw}{2\pi}$$

$$\beta_n = \int_0^\infty \frac{(Q_0 Q_1)^2}{w^2 + (Q_0 Q_1)^2} \, dw$$

$$= \frac{(Q_0 Q_1)^2}{2\pi} \int_0^\infty \frac{dw}{w^2 + (Q_0 Q_1)^2}$$

By compensation

$$\beta_n = \frac{Q_0 Q_1}{2\pi}(\tan^{-1}\infty - \tan^{-1} 0)$$

$$\beta_n = \frac{Q_0 Q_1}{2\pi}(\frac{\pi}{2} - 0) = \frac{Q_0 Q_1}{4}$$

From the error transfer function $\hbar_e$(s):

$$\hbar_e(s) = \frac{\Upsilon(s)}{\delta_i(s)}$$

$$\Upsilon(s) = \hbar_e(s)\delta_i(s)$$

$$\Upsilon(s) = \frac{s}{s + Q_0 Q_1 f(s)}\delta_i(s)$$

$$\Upsilon(s) = \frac{s}{s + Q_0 Q_1}\delta_i(s)$$

First input signal $\delta_i(s) = \frac{1}{s}$

The error function:

$$\Upsilon(s) = \frac{s}{s + Q_0 Q_1}\frac{1}{s}$$

$$= \frac{1}{s + Q_0 Q_1}$$

Use the Laplace transform to find the steady-state error:

$$\lim_{t\to\infty} \Upsilon(t) = \lim_{s\to 0} s\Upsilon(s)[25]$$

The final value of $\Upsilon(t)$:

$$\lim_{t\to\infty} \Upsilon(t) =$$

By Laplace form

$$= \lim_{s\to 0} s\Upsilon(s) = \lim_{s\to 0} s\frac{1}{sQ_0Q_1} = 0.$$

Which implies that no matter how big the phase step, the PLL will eventually track it out and there will be no steady state phase error resulting from a phase step.

Second input signal $\delta_i(s) = \frac{\triangle w}{s^2}$

The error function:

$$\Upsilon(s) = \frac{s}{s+Q_0Q_1}\delta_i(s)$$

$$= \frac{s}{s+Q_0Q_1}\frac{\triangle w}{s^2}$$

$$= \frac{\triangle w}{s(s+Q_0Q_1)}$$

The steady-state error:

$$\lim_{t\to\infty} \Upsilon(t) = \lim_{s\to 0} s\Upsilon(s)$$

$$= \lim_{s\to 0} s\frac{\triangle w}{s(s+Q_0Q_1)} = \frac{\triangle w}{Q_0Q_1}$$

There is a static "error", but it can be made small by increasing Q_0Q_1. This is consistent with the idea that a shift in control voltage is needed to give a step in frequency. The steady-state phase error is not equal to zero . Therefore for a large value of Q_0Q_1 makes the error small. Thus a small value of $\Upsilon(t)$ also means large bandwidth, that is likely to contain more noise.[28]

5.4 Second-Order Phase-Locked Loop

The denominator of the transfer function $\hbar$(s) is a second-order function of s.

PLL implemented with active loop filter:

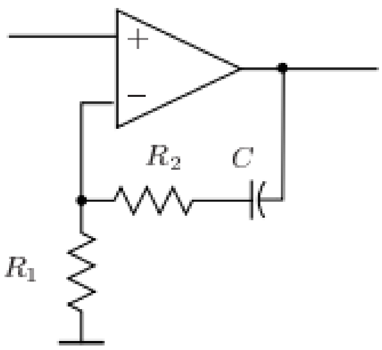

Figure (5.4.1): Circuit diagram of active loop filter .

If an ideal operational amplifier (op amp) is used then the transfer function of the filters to make such a second-order phase-locked loop is:

$$F(s) = \frac{sr_2 + 1}{sr_1} \quad [13]$$

where $r_1 = R_1C \quad and \quad r_2 = (R_1 + R_2)C$

The transfer function $\hbar$(s) becomes:

$$\hbar(s) = \frac{Q_0Q_1F(s)}{s + Q_0Q_1F(s)}$$

$$= \frac{Q_0Q_1(\frac{sr_2+1}{sr_1})}{s + Q_0Q_1(\frac{sr_2+1}{sr_1})}$$

$$= \frac{Q_0Q_1(sr_2 + 1)}{s^2r_1 + Q_0Q_1(sr_2 + 1)}$$

$$= \frac{Q_0Q_1sr_2 + Q_0Q_1}{s^2r_1 + Q_0Q_1sr_2 + Q_0Q_1}$$

$$= \frac{\frac{Q_0Q_1sr_2}{r_1} + \frac{Q_0Q_1}{r_1}}{s^2 + \frac{Q_0Q_1sr_2}{r_1} + \frac{Q_0Q_1}{r_1}}$$

let

$$w_n = \sqrt{\frac{Q_0Q_1}{r_1}}, \quad and \quad \zeta = \frac{w_n r_2}{2}$$

so

$$\hbar(s) = \frac{sr_2w_n^2 + w_n^2}{s^2 + sr_2w_n^2 + w_n^2}$$

where

$$r_2 = \frac{2\zeta}{w_n}$$

therefore

$$\hbar(s) = \frac{s(\frac{2\zeta}{w_n})w_n^2 + w_n^2}{s^2 + (\frac{2\zeta}{w_n})sw_n^2 + w_n^2}$$

$$= \frac{2s\zeta w_n + w_n^2}{s^2 + 2s\zeta w_n + w_n^2}[13]$$

Based on the transfer function of a second-order prototype system, a characteristic equation of the system is defined as :

$$\triangle(s) = s^2 + 2\zeta w_n s + w_n^2$$

By solving the roots of the characteristic equation, two poles of the system, S_0 and S_1, can be derived :[38]

$$S_0 = \frac{(-2\zeta w_n + \sqrt{(2\zeta w_n)^2 - 4w_n^2}}{2}$$

$$= -\zeta w_n + jw_n\sqrt{1-\zeta^2}$$

$$S_0 = -\alpha + jw$$

$$S_1 = \frac{(-2\zeta w_n - \sqrt{(2\zeta w_n)^2 - 4w_n^2}}{2}$$

$$= -\zeta w_n - jw_n\sqrt{1-\zeta^2}$$

$$S_1 = -\alpha - jw$$

Where α is defined as damping factor and w is defined as damped frequency. And the natural frequency $w_n = \sqrt{\frac{Q_0 Q_1}{r_1}}$, and the damping factor $\zeta = \frac{w_n r_2}{2}$ are standard control system terminology for a second order system.

Those two parameters are usually used to specify performance requirements of a system. As a matter of fact, most transient-response performances of a system can be determined based on these two parameters. The following is a list of performance parameters defined based on ζ and w_n.[38]

Damping factor α:

$$\alpha = \zeta w_n$$

Damped frequency w:

$$w = w_n\sqrt{1-\zeta^2}$$

Settling time:

$$t_s = \frac{4}{\zeta w_n}$$

Until this point, a second-order system has been defined in s-domain, and this system will meet performance requirements specified by ζ and w_n.

Therefore the noise bandwidth is the relationship between the natural frequency and damping frequency:[28]

$$\beta_n = \int_0^\infty |\hbar_{(w)}|^2\, df$$

$$= \frac{w_n}{2}(\zeta + \frac{1}{4\zeta})$$

The error transfer function:

$$\hbar_e(s) = 1 - \hbar(s)$$

$$= 1 - \frac{2\zeta w_n s + w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

$$\hbar_e(s) = \frac{s^2}{s^2 + 2\zeta w_n s + w_n^2}$$

First input signal $\delta_i(s) = \frac{1}{s}$

The error function

$$\Upsilon(s) = \hbar_e(s)\delta_i(s)$$

$$= \frac{s^2}{s^2 + 2\zeta w_n s + w_n^2}\frac{1}{s}$$

$$\Upsilon(s) = \frac{s}{s^2 + 2\zeta w_n s + w_n^2}$$

At steady state the error is given by:[28]

$$\lim_{t\to\infty} \Upsilon(t)$$

$$= \lim_{s\to 0} s\Upsilon(s)$$

$$= \lim_{s\to 0} s\frac{s}{s^2 + 2\zeta w_n s + w_n^2} = 0$$

Second input signal $\delta_i(s) = \frac{\triangle w}{s^2}$

The error function

$$\Upsilon(s) = \hbar_e(s)\delta_i(s)$$

$$= \frac{s^2}{s^2 + 2\zeta w_n s + w_n^2}\frac{\triangle w}{s^2}$$

$$\Upsilon(s) = \frac{\triangle w}{s^2 + 2\zeta w_n s + w_n^2}$$

At steady state the error is given by:[28]

$$\lim_{t\to\infty} \Upsilon(t) = \lim_{s\to 0} s\Upsilon(s)$$

$$= \lim_{s\to 0} s \frac{\triangle w}{s^2 + 2\zeta w_n s + w_n^2} = 0$$

Due to the infinite dc gain of ideal op amp.

The steady-state error is zero for the frequency modulated signal. Meaning that, the second-order loop tracks a frequency modulated signal and returns the phase comparator characteristic to the null point. Therefore, the conventional phase-locked loop in a GPS receiver is usually a second-order one.[28]

A narrow bandwidth loop filter will help to suppress high frequency noise coming into the PLL from the reference port. VCO jitter is suppressed by the PLL within the loop bandwidth. It has a high-pass transfer function. Thus, to suppress VCO noise, we want a large loop bandwidth.

5.5 Z-Transform of Continuous Systems

The analog control (continuous time) system must be changed into a digital control (discrete time), in order to build a phase-locked loop in software for digitized data.[28]

Why digital countrol?

Digital control offers distinct advantages over analog control that explain its popularity. Here are some of its many advantages:

- Accuracy:

* Digital signals are represented in terms of zeros and ones with typically 12 bits or more to represent a signal number. This involves a very small error.

* Analog signals contain noise and power supply drift.

- Implementation errors:

* Digital processing of control signals involves addition and multiplication by stored numerical values. The errors that result from digital representation and arithmetic are negligible.

* Processing of analog signals is preformed using components such as resistors and capacitors with actual values that vary significantly from the nominal design values.

- Flexibility:

* A digital controller is implemented in firmware or software and its modification is possible without a complete replacement of the original controller. Furthermore, the structure of the digital controller need not follow one of the simple forms that are typically used in analog control. More complex controller structure involves a few extra arithmetic operations and are easily realizable, because it represents the signals with a set of bitts and uses the computer as a controller.

* An analog controller is difficult to modify or redesign once implemented in hardware.

- Speed:

* The speed of computer hardware has increased exponentially since the 1980s. This increase in processing speed has made it possible to sample and process control signals at very high speeds. Because the interval between samples, the sampling period, can be made very small.

* Digital controllers achieve performance that is essentially the same as that based on continuous monitoring of the controlled variable.

- Cost:

Advances in technology, software and hardware components have led to potential the construction of a circuit using Digital Control, with high performance and reasonable cost. This has made the use of digital controllers more economical even for small, low-cost applications.

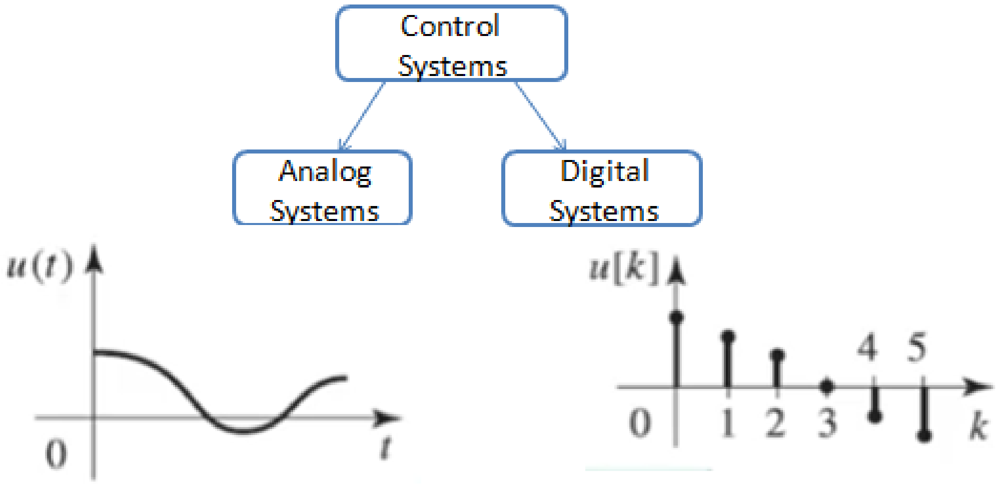

Figure (5.5.1): Continuous time and discrete time systems.

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-Continuous time (analog control signal): The signal has a value at any time value.

-Discrete time (digital control signal): The signal has values at separate specified time intervals.[31]

Advantages and constraints of digital signal processing.

Advantages:

- Accuracy can be controlled by choosing word length, number of bits.
- Many systems inherently digital.
- Repeated.
- Control algorithms easily modified.

- Minimal sensitivity to noises.
- Flexibility can be achieved with software implementations.
- Non-linear and time-varying operations are easier to implement.
- Digital storage is cheap.
- The price compared to the performance is good.[7]

Constraints:

- Sampling causes loss of information.
- The system under controller deal with analog signal and the computer deal with digital signal. Therefore, the signal enter to the computer must be in shape of digital, and the signal coming out from the computer and inside the system must be shaped analog signal. Which requires work translation between digital and analog signal, Analog to Digital (A/D) and Digital to Analog (D/A), which leads to a problem mixing hardware signals.
- Limited speed of processors, as a result of using the number of bits inappropriately with processors speed.
- Quantization and round-off errors.[7]

Discrete-time system and Z-transform .

Z-transform converts a discrete time-domain signal, which is a sequence of real or complex numbers, into a complex frequency-domain representation. Use of the Laplace transform gives rise to the basic concept of the transfer function of a continuous (or analog) system.[16]

To illustrate the idea of a discrete-time system, consider the digital control system shown in the next figure.

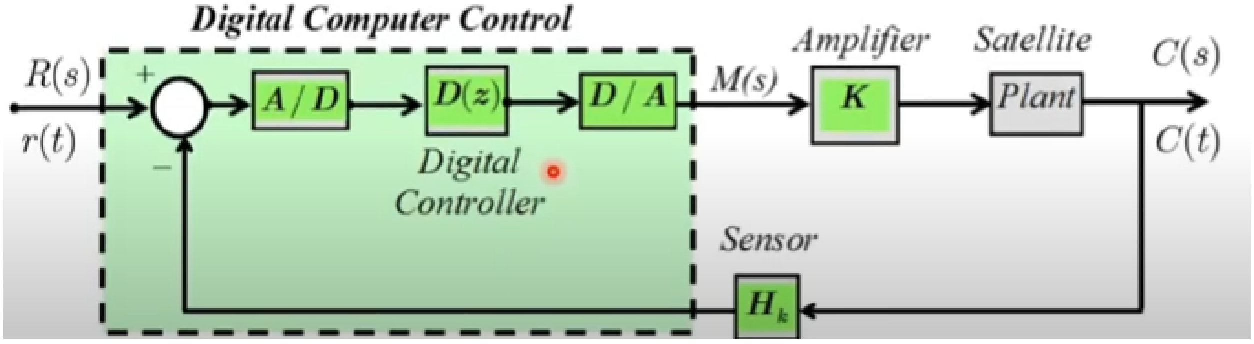

Figure (5.5.2): Diagram of a discrete control system .

- The digital computer performs the controller function within the system.
- The interface at the input of the computer is an analog-to-digital (A/D) converter.
- At the computer output, a digital-to-analog (D/A) converter is required, because the satellite and the amplifier engaged with analog signal.
- A/D converter: Is required to convert the error signal, which is a continuous-time signal, into a form that can be readily processed by the computer (digital form).
- D/A converter: Is required to convert the binary signals of the computer into a form necessary to drive the plant.

The transfer from the continuous s-domain into the discrete z-domain is through bilinear and boxcar transforms:

- Bilinear:

$$s = \frac{2(1 - z^{-1})}{t_s(1 + z^{-1})}$$

where t_s is settling time.

- Boxcar:

$$s = \frac{1 - z^{-1}}{Tz^{-1}} \quad [35]$$

where T is the integration time for correlation which is also the update interval of the loop (sampling interval).[35]

where the filter function $F(s) = \frac{sr_2+1}{sr_1}$

The filter is transformed to:

$$F(z) = \frac{\frac{2r_2(1-z^{-1})}{t_s(1+z^{-1})} + 1}{\frac{2r_1(1-z^{-1})}{t_s(1+z^{-1})}}$$

$$= \frac{\frac{2r_2(z-1)}{t_s(z+1)} + 1}{\frac{2r_1(z-1)}{t_s(z+1)}}$$

multiply by $\frac{t_s}{2r_1}$

$$F(z) = \frac{(\frac{2r_2(z-1)}{t_s(z+1)} + 1)\frac{t_s}{2r_1}}{(\frac{2r_1(z-1)}{t_s(z+1)})\frac{t_s}{2r_1}}$$

$$F(z) = \frac{\frac{r_2(z-1)}{r_1(z+1)} + \frac{t_s}{2r_1}}{\frac{z-1}{z+1}}$$

$$= \frac{\frac{r_2(z-1)}{r_1} + \frac{t_s(z+1)}{2r_1}}{z-1}$$

divide by z

$$F(z) = \frac{\frac{\frac{r_2(z-1)}{r_1} + \frac{t_s(z+1)}{2r_1}}{z}}{\frac{z-1}{z}}$$

$$F(z) = \frac{\frac{r_2}{r_1} - \frac{r_2}{zr_1} + \frac{t_s}{2r_1} + \frac{t_s}{2zr_1}}{\frac{z-1}{z}}$$

$$= \frac{\frac{2r_2}{2r_1} - \frac{r_2}{zr_1} + \frac{t_s}{r_1} - \frac{t_s}{2r_1} + \frac{t_s}{2zr_1}}{\frac{z-1}{z}}$$

$$= \frac{\frac{2r_2-t_s}{2r_1} + \frac{t_s}{r_1} - \frac{r_2}{zr_1} + \frac{t_s}{2zr_1}}{\frac{z-1}{z}}$$

$$= \frac{\frac{2r_2-t_s}{2r_1} + \frac{t_s}{r_1} - \frac{2r_2-t_s}{2zr_1}}{\frac{z-1}{z}}$$

let

$$C_1 = \frac{2r_2 - t_s}{2r_1}$$

and

$$C_2 = \frac{t_s}{r_1}$$

So, the filter is transformed to:

$$F(z) = \frac{C_1 + C_2 - C_1 z^{-1}}{1 - z^{-1}}[28]$$

The VCO in the phase-locked loop is replaced by a direct digital frequency synthesizer and its transfer function

$$N(z) = \frac{\delta_f}{V_0}$$

where

$$\delta_f(s) = V_0(s)\frac{Q_1}{s}$$

so

$$N(z) = \frac{Q_1}{s}$$

Using Boxcar rule which results in:

$$N(z) = \frac{Q_1}{\frac{1-z^{-1}}{Tz^{-1}}} = \frac{TQ_1 z^{-1}}{1 - z^{-1}} = \frac{TQ_1}{z - 1}$$

when

$$\hbar(s) = \frac{Q_0 Q_1 F(s)}{s + Q_0 Q_1 F(s)}$$

The transfer function $\hbar$(z) can be written as:

$$\hbar(s) = \frac{Q_0 Q_1 F(s)}{s + Q_0 Q_1 F(s)}$$

$$= \frac{\frac{Q_0 Q_1 F(s)}{s}}{1 + \frac{Q_0 Q_1 F(s)}{s}}$$

Using Boxcar rule

$$s = \frac{1 - z^{-1}}{Tz^{-1}}$$

So, the transfer function

$$\hbar(z) = \frac{\frac{F(z)Q_0TQ_1}{z-1}}{1+\frac{TQQ_0Q_1F(z)}{z-1}}$$

$$= \frac{Q_0F(z)N(z)}{1+Q_0F(z)N(z)}$$

$$= \frac{\frac{Q_0Q_1(C_1+C_2-C_1z^{-1})z}{(z-1)(z-1)}}{1+\frac{Q_0Q_1(C_1+C_2-C_1z^{-1})z}{(z-1)(z-1)}}$$

divide by z^2:

$$\hbar(z) = \frac{\frac{Q_0Q_1(C_1+C_2)z^{-1}-Q_0Q_1z^{-2}}{(z-1)^2}}{\frac{z^{-2}(z^2-2z+1)+Q_0Q_1(C_1+C_2)z^{-1}-Q_0Q_1C_1z^{-2}}{(z-1)^2}}$$

$$= \frac{Q_0Q_1(C_1+C_2)z^{-1}-Q_0Q_1C_1z^{-2}}{1+(Q_0Q_1(C_1+C_2)-2)z^{-1}+(1-Q_0Q_1C_1)z^{-2}}$$

$$\hbar(z) = \frac{[4\zeta w_n+(w_nt_s)^2]+2(w_nt_s)^2z^{-1}+[(w_nt_s)^2-r\zeta w_nt_s]z^{-2}}{[4+4\zeta w_n+(w_nt_s)^2]+[2(w_nt_s)^2-8]z^{-1}+[4-4\zeta w_n+(w_nt_s)^2]z^{-2}}$$

$$C_1 = \frac{1}{Q_0Q_1}\frac{8\zeta w_nt_s}{4+4\zeta w_nt_s+(w_nt_s)^2}$$

$$C_2 = \frac{1}{Q_0Q_1}\frac{4\zeta w_nt_s}{4+4\zeta w_nt_s+(w_nt_s)^2}[28]$$

This transfer function describes an infinite impulse response system exhibiting a low-pass behavior in the frequency domain. Notice that the filtering characteristics of the system depend on the chosen parameters Q_0Q_1 and C_1, C_2 of the PI controller. In particular, such values can be optimized experimentally in order to minimize the output period fluctuations on the basis of the spectral features of the input jitter.

Chapter 6

Conclusions and Comments

In this work we aimed to learn about the mechanism that the GPS receiver uses to determine our location based on the satellite information and to know what factors cause a different result for determining the fixed position from one device to another. We also sought to understand the details of signal transmission, the mathematical logic in its transmission, and how the process of signal protection and encryption takes place. How much we dealt with how the receivers deal with the incoming signal, how to solve them as mathematical systems, how the signal enters the receiver, and how to improve it and reduce errors from it.

In the first chapter, the focus was on the main components of a GPS receiver (hardware) and some useful applications of GPS . We examined how GPS hardware components strengthen, maintain, and purify the signal. Also, we focused on the significant role of the antenna in receiving and interpreting the signal and how to use an external antenna to increase accuracy. We determined the most important considerations to take into account when buying a GPS receiver.

Whereas, in the second chapter, the focus was on the accuracy of transmission using the appropriate code, and the understanding of the Exclusive-OR (XOR) logic in sending and receiving signals and clarifying the way to avoid signal interference. The C/A codes are characterized as It has the best advantage of cross-linking. Each satellite is distinguished from another by this code. But the P code is used to distinguish between data sent from a satellite to another satellite. As a result, it was found that the accuracy of the transmission using P code will be better than the accuracy of C/A code due to the fact that the difference in the chip rate of P code is small as the difference in the energy levels of C/A code is large. The process of merging the code with the original signal is done through the logic of XOR , as the transmitted signal is a process of calculating logic the Exclusive-OR (XOR) between the original signal and the locally generated random numbers. The Doppler effect will help to receive the signal sent at a specific frequency. As the Doppler phenomenon is represented in GPS through the change in the speed of sending various signals and the data carried on them, where this change is affected by changing the distance between the transmitter and the receiver of signal. The change in velocity of transmission is directly proportional to the shift in frequency of the transmitted signals, as the speed of the transmitted signal coming into the future differs from its speed while moving away from it, it gradually decreases when moving away. To avoid the problem of signal and information interference, because some satellites may transmit certain signals on the same frequency at the same time, code division multiple access (CDMA) method is used.

In the third chapter, we have studied the carrier wave in its three-dimensional form and tried to connect it with the GPS system . After deriving the three-dimensional wave equation using Maxwell's physical equations, and when applying this equation mathematically and solving it, it was found that the best way to solve the integration to reach the solution of this equation is integration by Simpson's Rule.

Then, in the fourth chapter, the study of the mathematical methods behind the Global Positioning System (GPS) was discussed, and it was found that Successive-Over Relaxation (SOR) method is the best and most accurate method for dealing with a linear system compared to Jacobi method and Gauss-Seidel method, because it requires less iteration than the Gauss-Seidel and Jacobi methods. But, when using information from 4 or more satellites, a system of equations can be produced consisting of more than two equations with two unknowns(x, y), and to deal with this system and to get to the exact location, we use the Least Square method. In addition, non-linear systems may result from receiving information from satellites, so we will work with these systems using Newton's method and Quasi Newton's method. By comparison between the two methods with the first exact solution, it turns out that Newton's method is the most accurate.

After studying some mathematical models that better describe the phase closed loop needed to roughly determine the GPS tracking signal in fifth chapter. We found that using First-Order Phase-Locked will result in a steady-state phase error not equal to zero and therefore more noise is likely. But using the Second -Order Phase-Locked Loop will cause the steady state error to be zero for the modulated modulation, which means that the second-order loop tracks the frequency-modulated signal and returns the phase comparison property to the full point, and it suppress the noise of the VCO. So the GPS device usually a second-class. Finally, the continuous system was changed into a discrete system by z-transform of continuous systems, in order to build a phase locked loop in software for digitized data. As the filtering characteristics of the system depend on the chosen parameters Q_0Q_1 and C_1, C_2 of the PI controller. Such values can be optimized experimentally in order to minimize the out-put period fluctuations.

Problem for further research directions based on this results. Since, this is limited and the topic in huge, further potential research is preferred to continue this first step of research on GPS.

Some of the problems that we encountered in this work are the lack of information and research, and consequently the lack of the following:

- More exact examples.
- More accurate resources.
- More investigation for the mathematical technique : Numerical, Z-Transform loops , Maxwell's equations.

These problems could prompt other GPS researchers to seek new results.

Chapter 7

Matlab codes

First :

Solving the integral of 3-D wave equation by matlab .

(1)

```
clc
clear all
fun = @(x,y,z)
(8/(10*15*20)).*(sin((x*1*pi)/10).*sin((y*2*pi)/
15).*sin((z*3*pi)/20).*x.*y.*z);
q = integral3(fun,0,20,0,15,0,10)
q= -27.3760
```

(2) Simpson's Rule :

For x :

```
clear all
clc
syms y z
f = @(x) (8/(10*15*20)).*(sin((x*1*pi)/10).
*sin((y*2*pi)/15).*sin((z*3*pi)/20).*x.*y.*z);
a=0;
b=20;
n=10;
h=(b-a)/n;
s=f(a)+f(b);
for i=1:2:n-1
s=s+4*f(a+i*h);
end
for i=2:2:n-2
s=s+2*f(a+i*h);
end
```

I=h/3*s
Answer : -(107613178791100501*y*z*sin((2*pi*y)/15)
*sin((3*pi*z)/20))/633318697598976000

For y :

```
clear all
clc
syms z
f = @(y) -(107613178791100501*y*z*sin((2*pi*y)/15)
*sin((3*pi*z)/20))/633318697598976000; a=0;
b=15;
n=10;
h=(b-a)/n;
s=f(a)+f(b);
for i=1:2:n-1
s=s+4*f(a+i*h);
end
for i=2:2:n-2
s=s+2*f(a+i*h);
end
I=h/3*s
```

Answer:
(1357101122991253391008036715661 97*z*
sin((3*pi*z)/20))/2228292070713684494818423603 2000

For z :

```
clear all
clc
f = @(z) (135710112299125339100803671566197*z*sin
((3*pi*z)/20))/22282920707136844948184236032000;
a=0;
b=10;
n=10;
h=(b-a)/n;
s=f(a)+f(b);
for i=1:2:n-1
s=s+4*f(a+i*h);
```

```
end
for i=2:2:n-
s=s+2*f(a+i*h);
end
I=h/3*s
Answer = -27.4021
```

(3) Trapezoidal Rule :

```
clear all
clc
syms y z
f = @(x) (8/(10*15*20)).*(sin((x*1*pi)/10).
*sin((y*2*pi)/15).*sin((z*3*pi)/20).*x.*y.*z);
a=0;
b=20;
n=10;
h=(b-a)/n;
s=0.5*((a)+f(b));
for i=1:n-1
s=s+f(a+i*h);
end
I=h*s
Answer:
-(208076127104680031472086600685921*y*z*sin((2*pi*y)/15)
*sin((3*pi*z)/20))/1267650600228229401496703205376000
```

For y :

```
clear all
clc
syms z
f = @(y) -(208076127104680031472086600685921*y*z*sin((2*pi*y)/15)*
sin((3*pi*z)/20))/1267650600228229401496703205376000;
a=0;
b=15;
```

```
n=10;
h=(b-a)/n;
s=0.5*(f(a)+f(b));
for i=1:n-1
s=s+f(a+i*h);
end
I=h*s
```

Answer :
(126741414250971745097225125972565295522308493123
*z*sin((3*pi*z)/20))/22300745198530623141535718272
64836150598041600

For z :

```
clear all
clc
f = @(z) (126741414250971745097225125972565295522
308493123*z*sin((3*pi*z)/20))/22300745198530623141
535718272648361505980416000;
a=0;
b=10;
n=10;
h=(b-a)/n;
s=0.5*(f(a)+f(b));
for i=1:n-1
s=s+f(a+i*h);
end
I=h*s
```

Answer :
I = -26.0717

Second :

Solution By MATLAB" Gauss Jacopi" Method [15]:

```
clc;
clear;
format short
A=[-2.4 -8 -0.2;3.2 -8 -8.8;-2.7970245 -7 -12.2];
b=[-27.18;9.27;27.6240512];
l=tril(-A,-1);
u=triu(-A,1);
d=diag(diag(A));
tj=inv(d)*(l+u);
cj=inv(d)*b;
xj= [0;0;-1];
N=84;
for i=1:N
xj=tj*xj+cj;
[i xj']
plot3(xj',xj',xj')
```

Third :

Solution By MATLAB "Gauss Seidel" Method [15] :

```
clc
clear
format short
A=[-2.4 -8 -0.2;3.2 -8 -8.8;-2.7970245 -7 -12.2];
b=[-27.18;9.27;27.6240512];
l=tril(-A,-1);
u=triu(-A,1);
d=diag(diag(A));
tg=inv(d-l)*u;
cg=inv(d-l)*b;
xg=[0;0;-1];
N=25;
for i=1:N
xg=tg*xg+cg;
```

```
[i xg']
end
plot3(xg',xg',xg')
```

Fourth :

Solution By MATLAB "SOR (successive over-relaxation)" Method [14] :

```
clc clear
format short
A=[-2.4 -8 -0.2;3.2 -8 -8.8;-2.7970245 -7 -12.2];
b=[-27.18;9.27;27.6240512];
n=size(A,1);
w=0.604;
x0=[0;0;-1];
kmax=7;
X=x0; at=zeros(n,n);
for k=1:kmax
X(1,:) = x0(1,:)+w*((b(1,:)-A(1,1:n)*x0(1:n,:)))/A(1,1);
for i = 2:n-1
tmp = w*(b(i,:)-A(i,1:i-1)*X(1:i - 1,:)-A(i,i:n)*x0(i:n,:));
X(i,:) = x0(i,:)+tmp/A(i,i);
end
X(n,:) = x0(n,:)+w*((b(n,:)-A(n,1:n)*X(1:n,:)))/A(n,n);
x0 = X;
[k X'];
end plot3(X',X',X')
```

Fifth :

Solution By MATLAB" Least-Squares" Method :

```
A=[2,-2; 1,-2; 1,1]
b=[2;1;4]
Z=transpose(A)*A
W=transpose(A)*b
S=[Z,W] RB=rref(S)
```

Results :

```
A=
   2     -2
   1     -2
   1      1

b=
   2
   1
   4

Z=
   6     -5
  -5      9

W=
   9
  -2
S=
   6     -5      9
  -5      9     -2

RB=
   1.0000        0     2.4483
   0        1.0000     1.1379
```

Sixth:

Solution By MATLAB" Newton Raphson " Method [4]:

clc

$f = @(x, y)[x^2 + y^2 - 4; x^2 - y + 1];$

$$f_p = @(x, y)\begin{bmatrix} 2 * x & 2 * y \\ 2 * x & -1 \end{bmatrix};$$

$x_0 = [1, 2];$

$N = 4;$

$x(1) = 1;$

$y(1) = 2;$

$n = 2;$

$nfinal = N + 1;$

for i=1:N

$f_e = f(x(n-1), y(n-1));$

$f_{pe} = f_p(x(n-1), y(n-1));$

$FF = inv(f_{pe}) * f_e;$

$z = [x(n-1); y(n-1)] - FF;$

x(n)=z(1) ;

y(n)=z(2);

n=n+1;

end

x

y

Seventh :

Solution By MATLAB” Quasi-Newton ” Method :

```
clc
```
$f = @(x, y)[x^2 + y^2 - 4; x^2 - y + 1];$

$fp = @(x, y)[2 * x2 * y2 * x - 1];$

```
x0 = [1;2];
N = 4 ;
x(1) = 1;
y(1)=2;
v= f(x0(1),x0(2))
A0= fp(x(1),y(1));
A= inv(A0);
s= -A*v;
xx= [x(1);y(1)] + s;
x(2)= xx(1);
y(2)= xx(2);
k=3;
TOL= 0.001;
while (k¡= N)
w=v;
v= f(x(k-1),y(k-1));
b= v-w;
z= -A*b;
u= -transpose(s)*A;
p= -transpose(s)*z;
A= A+ (1/p)*(s+z)*u;
s= -A*v;
xxx= [x(k-1);y(k-1)] + s;
x(k)= xxx(1);
y(k)= xxx(2);
k=k+1;
end
x
y
```

References

(1) Abba I.,Masri1 T., Abidin W.A.W.Z., Ping K.H., Bong V.P., Hussaini N.B., and Abdullah M. (2015), External antenna design for GPS signal reception enhancement. Department of Electrical and Electronic Engineering, Faculty of Engineering, university Malaysia Sarawak, 17-20.

(2) Alan O.(2017), L1 frequency currently on the L1 frequency, both the SPS and PPS services are offered, whereas on the L2 frequency only the PPS service is offered, Bahrain Polytechnic, Kingdom of Bahrain, 71-80.

(3) Anton R., and Yurii N.(2021), New results on superlinear convergence of classical Quasi-Newton methods. Journal of optimization theory and applications volume, 744–769.

(4) Ayokunle A.A., Olawale P., and Abdulkareem A., and Abdulkareem A.(2019), Nonlinear dynamic simulation of a magnetic levitation system using matlab/Simulink. International journal of engineering research and technology. ISSN 0974-3154, Volume 12, Number 12, pp. 2143-2158.

(5)Carl H., and Bob V.W. (1991), P-Code Versus C/A-Code GPS For Range Tracking Applications, Proceedings of the 1991 National Technical Meeting of The Institute of Navigation, 119-121.

(6) Charles G.T. (2014), 09 The Wave Equation in 3 Dimensions, Department of Physics, Utah State University, 85-91.

(7) Diksha P., (2021), Advantages and Disadvantages of Digital signal processing (DSP) system, Knowledge Base.

(8) Dorota G.B. (2002), GPS instrumentation issues. Published in Manual of Geospatial Science and Technology, J. Bossler (Ed.), Taylor and Francis.

(9) Endre S., and David F.M.(2012), An Introduction to Numerical Analysis. University of Oxford ,74-80.

(10) Florian I.T.P. (2012), A New Doppler Effect. Technical Sciencz Academy Romania.

(11) Geography. (2021), GPS Accuracy: HDOP, PDOP, GDOP, Multipath & the Atmosphere.

(12) George B.A., and Hans J.W. (2005), Mathematical Methods for Physicists, Sixth Edition. Miami University-Oxford, OH. University of Virginia Charlottesville, VA, 49-55.

(13) Geza K., Phase-Locked Loop, Budapest University of Technology and Economics Laboratory 2 - Experiment 9.

(14) Gismalla D.A. (2014), Matlab software for iterative methods and algorithms to solve a linear system. International journal of engineering and technical research (IJETR) ISSN 2321-0869, vol -ume 2, issue 2.

(15) Hazem I.A., Mai M.Z., Atif M.A., and Manar S.F. (2019), Guide to the practical side of the Principles of Numerical Analysis 1281 courseusing Matlab software. Al-Quds Open University.

(16) HELM. (2008), Basics of z-Transform Theory, Section 21.2.

(17) Ian C. (2018), Phase-Locked Loop (PLL) Fundamentals, Analog dialogue 52-07.

(18) Ignacio G.F.(December 2010), GPS space segment monitoring, Technical University of Lisbon, 4-6.

(19) ILcev D.S. (December 2020), Analyses of Code Division Multiple Access (CDMA) schemes for global mobile satellite communications (GMSC), Durban university of technology, Durban, South Africa, 805 - 807.

(20) Jack D. (1995), Overview of the Methods.

(21) Jacobus D.V. (October, 2014), Detection of Direct Sequence Spread Spectrum Signals, School of engineering university of Tasmania Hobart.

(22) James R.C. (2001), GPS Accuracy Levels Major Accuracy Divisions, Naval Postgraduate School.

(23) Karunanithi S., Gajalakshmi N., Malarvizhi M., and Saileswari M. (2018), A study on comparison of jacobi, gauss seidel and SOR methods for the solution in system of linear equations. International journal of mathematics trends and technology (IJMTT)volume 56 Issue 4.6.

(24) Lin Z., Shuaihe G.,and Xuechuan H.(2010), The design and validation of RF front-end platform for GPS receiver, Institute of electrical and electronics engineers.

(25) Long S. (2005), Phase locked loop circuits, UCSB/ECE department, 1-7.

(26) Marhauser F.(2018), Maxwell's Equations, U.S particle accelerator school.

(27) Michael J.D. (2013), Global Positioning System directorate systems engineering & integration, Interface specification IS-GPS-705.

(28) Mounchili M. (2008), Mathematical Modeling of The Global Positioning System Tracking Signals, School of engineering balking Institute of technology Sweden, 5-17.

(29) Rajan S., Mukesh C.,and B.K. Singh, (2015), Solution of 3-dimensional wave equation by method of separation of variables, Department of Mathematics University Moradabad India, 54-56.

(30) Ravi S.,and Vaishali A. (2012), GPS Signal Structure: A Look Behind the Scene. Department of Civil Engineering, Indian Institute of Technology Kanpur, 56-59.

(31) Richard B.E.A. (2021), Discrete-time processing of continuous-time signals, Electrical and Computer Engineering at Rice University.

(32) Richard B.L. (1998), A primer on GPS antenna , University of New Brunswick.

(33) Richard L.B., and Douglas J.F.(2010), Numerical Analysis, 9th Edition.Youngstown State University, 464-467.

(34) Saba A., and Qurrat U.A. (2015), Newton Raphson Method. International journal of scientific and engineering research, 1748-1750.

(35) Tahir R., Khurram U.M., and Salman A. (2015), GNSS Carrier Tracking Loop : Stability Analysis and Some Improvements, Department of Electrical Engineering, Syed Baber Ali School of Science and Engineering, Lahore University of Management Sciences, Lahore, Pakistan.

(36) Ta J.L. (2011), Global Positioning System, California State University Channel Islands, 33-50.

(37) Ting L.,and Roy K. (2019), An optimized design of Serial Input-Serial-Output (SISO) and Parallel-Input-Parallel-Output (PIPO) shift registers based on quantum dot cellular automata nanotechnology. International journal of theoretical physics.

(38) Wen L., and Jason M. (2000), Introduction to phase-locked loop system modeling. Texas Instruments Incorporated, 5-10.

الملخص

نظام تحديد المواقع العالمي (GPS) هو نظام ملاحة راديو قائم على الاقمار الصناعية يوفر خدمات متسقة لتحديد الموقع والتوقيت للمستخدمين بشكل مستمر. الهدف من هذا البحث هو اعطاء القراء معرفة علمية بالجوانب العملية والنظرية لعمل النظام. في هذا البحث نتناول اهم القضايا المتعلقة بنظام تحديد المواقع العالمي (GPS). سوف نقسم هذا البحث لخمسة أجزاء, في الجزء الأول نتناول المكونات الرئيسية لجهاز الاستقبال وأهم التطبيقات العملية للنظام. ثانيا, نناقش طبيعة الاشارة والالية المستخدمة في انتقالها, وكيفية يتم تشفير التردد باستخدام (PRN code) . ثالثا, نناقش انتقال الموجة بشكلها ثلاثي الابعاد وكيفية التعامل معها رياضيا. رابعا, نقدم الطرق الرياضية الاساسية المستخدمة لحساب موقع المستخدم باستخدام برنامج (Matlab). أخيرا نركز على كيفية نمذجة تعبير رياضي لتتبع الاشارة باستخدام حلقة مغلقة الطور من أجل الوصول لخطأ مرحلة مساوي لصفر, ونناقش طريقة التعامل مع البينات الرقمية باستخدام تحويل (Z-transform) لبناء حلقة طور مغلقة للتعامل مع البيانات الرقمية .