A Mathematical Model of Regional Socio-Economic Development of Palestine

By
Nour Jamal Haj Ali

Supervisor
Dr. Ata Abu As'ad

This thesis was submitted in partial fulfillment of the requirements for the Master's Degree of Science in Mathematical Modeling

Faculty of Graduate Studies

Palestine Technical University - Kadoorie (PTUK)

March, 2022

نموذج رياضي للتنمية الاجتماعية والاقتصادية الإقليمية لفلسطين

إعداد

نور جمال موسى حج علي

إشراف

د. عطا أسعد

قدمت هذه الرسالة استكمالاً لمتطلبات الحصول على درجة الماجستير في برنامج النمذجة الرياضية

كليّة الدراسات العليا

جامعة فلسطين التقنية – خضوري

مارس–2022

COMMITTEE DECISION

This Thesis Titled: "A Mathematical Model of Regional Socio-Economic Development of Palestine"

Was successfully defended and approved on 30/3/2022.

Examination Committee	Signature
Dr. Ata Abu-As'ad, (Supervisor) Assistant Professor. of Mathematics	....
Dr. Abdulqader Mustafa, (Internal Examiner) Assistant Professor. of Mathematics	....
Dr. Mohammad Yassin, (External Examiner) Assistant Professor. (An-Najah National University)	....

Palestine Technical University – Khadoorie

PTUK

"Authorization Form"

I, Nour Jamal Mousa Haj Ali, authorize PTUK to supply copies of my Thesis to Libraries or Establishments or Individuals upon request, according to the PTUK regulations.

Signature:....................................

Date:..

جامعة فلسطين التقنية –خضوري

"نموذج تفويض"

أنا نور جمال موسى حج علي، أفوض جامعة فلسطين التقنية –خضوري بتزويد نسخ من رسالتي للمكتبات أو المؤسسات أو الهيئات أو الأشخاص عند طلبهم حسب التعليمات النافذة في الجامعة.

التوقيع: ....

التاريخ: ....

Declaration

I hereby declare that this thesis is the product of my own efforts, except what has been referred to, and this thesis as a whole or any part of it has not been submitted as a requirement for attaining a degree or scientific or research title to any other educational or research institution.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

"A Mathematical Model of Regional Socio-Economic Development of Palestine"

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص، باستثناء ما تم الإشارة إليه حيثما ورد، وأنّ هذه الرسالة ككل، أو أي جزء منها لم يُقدم من قبل لنيل درجة أو لقب علمي أو بحثي لدى مؤسسة تعليميّة أو بحثيّة أخرى.

Nour Jamal Mousa Haj Ali	**نور جمال موسى حج علي**
Signature:--------	**التوقيع: ---------**
Date: ---------------	**التاريخ: ---------**

Dedication

I would like to dedicate my thesis to my father who advised me to work hard and taught me to believe and trust in myself. To my mother whose prayers were with me all the way to success and to confide in myself. To my husband who supported and encouraged me. To my lovely children. To everyone who always inspire me.

Acknowledgement

I would like to indicate my appreciation for everyone who helped me in this thesis. First and foremost praise and thanks to Allah for giving me the courage and determination as well as guidance in my thesis despite all difficulties. I would like to thank my supervisor Dr. Ata Asa'd for his supervision and for his instructions throughout this work. My sincere thanks to my teachers from the Departement of Mathematics who tought me how to excel and improve my knowelege in Mathematical Modeling. I wish to thank my friends and my colleagues in the department of Mathematics for the comments that helped me clarify and formulate my work in this form. Finally to my family, parents, husband , brothers, sister and my lovely sons whose supported and encouraged me during this journy.

Table of Contents

Contents **Page**

List of Tables

List of Figures

Abstract

In this study, we describe the socio-economic development in Palestine. Mathematical modeling is used as a tool to explain the main factors that effect on the development of Palestine economy. We use two mathematical models to perform the study. The first model is in three dimension and the second model is an extension to the first model and is in four dimensions. The suggested models are constructed from a set of non-linear continuous differential equations which introduce the main study variables (population, jobs, energy and occupation) and coefficients (control vector) that is joint between the two models. At each stage, the proposed models are analyzed mathematically by finding general and numerical fixed points. The stability is examined by analyzing Jacobian matrix and the eigenvalues. We clarify the type of bifurcation, direction field diagrams, the numerical solution and the behavior of the system during time. In addition, MATLAB program is used to find solutions and to demonestrate the results graphically and numerically. On the other hand, Newton method is used to obtain different values for the basic variables in each stage through some values and observations to give an impression for a progress and improvement in the field of regional economic and social development. Moreover, the value of coefficients are computed based on Palestinian Central Bureau of Statistics (PCBS) by using percentage rate method (growth rate for values). Furthermore, the study cover the occupation period by choosing different base and comparison years for each parameter. The results of numerical simulation and analysis verify the system is chaotic with unstable fixed points.

Keywords: Socio-economic, mathematical model, stability, bifurcation, numerical solution, Newton method, MATLAB program.

Chapter 1

Introduction

Mathematical modeling was not an important research tool until the second decade of the twentieth century. But its uses began to appear in various scientific and practical fields in the middle of the nineteenth century. The language of models or comparisons appeared in the late nineteenth century among physicists, then biologists who needed to represent Many scientific and cosmological phenomena using mathematics [1].

The use of mathematical modeling exceeded the previous scientific aspects. It was used in the literature, guiding policy decisions, and the decision-making process by collecting many diverse evidence, examining and testing them mathematically, to avoid risk, time and cost. In addition to its character of abstraction and simplification [2].

Many economic mathematical models that are based on time-dependent equations developed and illustrated the relationship between important economic variables such as, production, capital, investment, consumption and others. Usually, the analysis of this type of models works greatly to clarify the concept of economic growth rate for all variables and the economic contained in the equations [3].

In this section, we will focus on the impact of mathematical modeling and its uses in the economic aspect. It is not possible to imagine the modern economy without the use of mathematical and specialist tools. As many mathematical methods have been developed to describe, analyze and predict economic phenomena. The use of models in economic aspect back to many reasons such as, models provide a clear summary of the incoming economic problem, provide a clear explanation of the variables and symbols in the model, transform the complex reality to a simple form includes the main concepts which make the study easier, transform from an abstract concept to a quantitative approach capable of showing results and thus tested, finally these models become one tool of communication between all economists and other scientists and researchers from other disciplines [4].

Modern economics established in the two marginal and Keynesian revolutions. Which led to the emergence of basic concepts and methods in economic theory. It also allowed the use of calculus to describe various phenomena and changes. In addition, a new revolution appeared in economics called the memory revolution which aims to treatment of amnesia for economic theory, and the consequent use of mathematical tools such as theorems of calculus, derivatives, sums and variations in incorrect orders. The first phase of this revolution is connected with the work published by Clive W. G. Granger in the years 1966 and 1980. Nobel Prize winner and Memorial Prize In Economic Sciences in 2003 [5].

The development of regional economic models that imitate the macroeconomic model began in the fifties by the Nobel Prize winner Pan Tinbergen, and the aim was to translate spatial economic mechanisms in a quantitative way in order to make an appropriate decision. In the sixties, two types of economic models were developed the first model known as the traditional Keynesian model, and the second known as Regional input and output model. The expression of these models was largely using matrix algebra in order to obtain a simple economic model that is easy to study [4].

Mathematics used in economic applications passed through several stages, which presented as follows [5] :

1-ARFIMA (Autoregressive Fractionally Integrated Moving Average econometrics) stage: This stage is characterized by models with discrete time and application of the Grunwald–Letnikov fractional differences.

2-Fractional Brownian Motion Stage: This stage is characterized by financial models and the application of stochastic calculus methods and stochastic differential equations.

3-Econophysics Stage: This stage is characterized by financial models and the application of physical methods.

4-Deterministic Chaos Stage: This stage is characterized by financial and economic models and application of methods of nonlinear dynamics. Wei-Ching Chen proposed the first financial model with deterministic chaos in 2008. He is studied the behavior of the system, described fixed points, periodic movements, chaotic movements, determined the doubling and intermittent periods of chaos in the financial process and described it through a system of three equations with the Caputo fractional derivatives.

5-Mathematical Economics Stage: This stage is characterized by macroeconomic and microeconomics models with continuous time and generalization of basic economic concepts and notions.

There are many examples of studies of economic growth and development models that used mathematical modeling as a basis for analyzing the model and obtaining accurate and correct results. As such [6] aimed to determine the long-term economic growth in developing countries such as, Cote d'Ivoire, Bangladesh, or Uganda. The study relied on the use of the Public Choice Growth Model (PCGM). Where this model seeks to combine two basic elements in economic theory. Microeconomics, summarize the principles of the theory of public choice. Macroeconomics, include the factors of the slow growth model. The basic equation of the model depends on time and it is a function product that contains five basic variables, slow growth variables, government lending, and the inflation rate. Furthermore, in [7] socio-economic growth in the Russian Arctic Zone described. A mathematical model consists of three nonlinear differential equations devised. These equations built based on three basic variables: population, business, and energy. Besides, a control vector used to connect these variables and explain the relationship between them with nine components, each of which has a specific significance. In addition, the researcher was using the stochastic model to test the precisely of these components, and they are reached to need more parameters to make the model more accurate. On the other hand, the nature of the relationship between macroeconomic indicators and growth through capital accumulation in India in the period between 1983-2007 described in [8]. The pattern of market capitalization and GDP growth, and the provision of local labor to understand the future direction of the stock market verified. Kumar used the Gompertz model which is a growth equation that applies the Gompertz curve, the equation predicts the growth of any phenomenon. The curve is asymmetric and analyzes time series. [9] presented three mathematical models for forecasting the GDP in Mexico for the period 1935-2016 by using the regression. Three models of regression were used linear regression, exponential regression, and parabolic regression. It was found that the parabolic regression is the most appropriate and accurate. [10] purposed to scout about the causal relationship between public expenditure and GDP growth in Palestine in the period 1994-2013 using Wagner's Law and formulas. Data stability checked using Augmented-Dickey Fuller (ADF) test. Engle-Granger cointegration test used for test the long-term relationship between public expenditure and GDP growth. Eventually, the study illustrated a proportional relation between public development expenditure and GDP in Palestine.

Previous studies illustrated several measures to study the economic and social development in a country. Whereas, it based on the measure of the gross domestic product and public spending of the region, number of population, jobs and energy, government lending, inflation, employment, the rate of general income for population, labor, technology, stock, etc. However, mathematical modeling presented as a basic tool in demonstrating these problems, analyzing it and obtaining the results.

The economic growth is generally defined as "The increase in the capacity of an economy to produce goods and services", it is also "Thought of not only as an increase in productive capacity but also an improvement in the quality of life to the people of that economy"[11]. Moreover, there are several aspects that harmony together lead to economic development, such as the agricultural, tourism and industrial aspects, natural resources, population growth and other influencing factors, as well as their contribution to the gross domestic product (GDP).

Economic growth in the State of Palestine as in many other countries is heavily influence by political, demographical situation and the natural resources of the region. The most influential is the political portion, in particular the Israeli occupation in Palestine.

Since Oslo Accords of 1993, the Palestinian economy faced a series of ongoing shocks; the division of West bank, Paris protocol of the same year, the first and second Intifadas, and the siege of Gaza resultant three wars, this combined with freedom of movement restrictions enforced by the occupation security apparatus, limited access to basic services and resources, restrictions on investment and trade, and poor access to finance, has created an unsustainable economy artificially propped up by donor aid [11], [12].

The main characteristics of the Palestinian economy are depending on occupation policies and donors community and a lack of self-sufficiency. However, it is important to note that the fragmentation of the Palestinian territory into three areas, the West bank, Gaza strip, and east Jerusalem. As a result differing levels of governance and autonomy led to an unviable economy, suffers from economic dependence and limited growth potential. In fact, is subject to the occupation policies of economic strangulation [11], [12]. In this study, we present the description for the socio-economic development in Palestine, considering four main basic variables that have the greatest impact on the Palestinian economic and social reality. In this research, we mainly rely on mathematical modeling in representing the problem, analyzing it and obtaining results by showing the influence of the main variables at each other and the parameter we introduce in each variable. Furthermore, we illustrate two mathematical models in this study. The first model consists of three non-linear differential equations based on three main variables performed by the vector function $n(t) = [n_1(t), n_2(t), n_3(t)]$. Where $n_1(t)$ is the number of population, $n_2(t)$ is the number of jobs, and $n_3(t)$ is the available of energy in the area. Hence if $n_3 > 0$, there is an energy, but if $n_3 < 0$ there is lacke of energy. In addition, we have a control vector $\alpha(t) = [\alpha_1(t), ..., \alpha_9(t)]$ with nine components, connecting main variables to each other. The second model constructed by expanding the first model and adding the fourth main variable, which is the occupation $n_4(t)$. Finally, the model we have consists of four non-linear differential equations containing four basic variables $n(t) = [n_1(t), n_2(t), n_3(t), n_4(t)]$. Similarly to the first one with a control vector of sixteen components $\alpha(t) = [\alpha_1(t), ..., \alpha_{16}(t)]$ which connect these variables and explain the relationship.

Chapter 2

Main concepts

Before we study our models we are going to present a brief introduction of the main concepts we need in our study in the definitions of economic and math.

2.1 Economic preliminaries

In this section, we want to demonstrate some of the basic concepts in side of economic development as follows.

Definition 2.1.1 *[13] Socioeconomic Development includes general concerns in the development of social policies and economic initiatives.The ultimate goal of social development is to bring about a sustainable improvement in the well-being of the individual, groups, family, society and society as a whole. It implies a sustainable increase in the economic standard of living of the population of the country, and is usually achieved by increasing its stock of physical and human capital and thus improving its technology.*

Definition 2.1.2 *[14], [15], [16] Economic Development is defined as the level of diversity of economic resources in a country and the equitable distribution of these resources, whereas they cover the needs of most of the population. In addition, it's known as diversity of goods and services and the increase in the purchasing power* *of consumers. Furthermore, it means changes in income, savings and investment along with gradual changes in the social and economic structure of the country (institutional and technological changes). The result of this, is to increase the level of education, job opportunities, the level of income, reduce migration abroad and others. Moreover, it reflects the able of community to afford various disasters. In general, there are many methods are used to measure economic development such as HDI (Human Development Index), gender related index, Human poverty index (HPI), infant mortality, literacy rate, etc. Also, economic development it's a broad concept includes economic growth, it leads to qualitative and quantitative changes in the economy.*

Definition 2.1.3 *[16] Economic Growth is an increase in the real output of goods and services in the country. The economic growth can be measure by increasing real GDP or per capita income. It leads to quantitative changes in the economy.*

Definition 2.1.4 *[9] Gross Domestic Product is an aggregate measure of production equal to the sum of the gross values added to all resident and institutional units engaged in production (plus any taxes, and minus any subsidies, on products not included in the value of their outputs. As defined by the Organization for Economic Co-operation and Development (OECD) .*

Criterion 2.1.5 *[9] There are three ways to find GDP. Production approach, expenditure approach, income approach, the most common one is expenditure approach. Whereas, $GDP = C + I + G + (X - M)$. C is private consumption, I is gross private investment, G is government investment, X is exports and M is imports.*

2.2 Mathematical preliminaries

In this section, we illustrate some of basic notions in math such as definitions, remarks, theorem and corollary that we need in our study.

Definition 2.2.1 *[17] Continous Dynamic Model is a model contains dependant and independant variables since one of these variables or all of them are changing with time. That is, it depends on time. However, continous models indicates to partial differential equations.*

Now, we illustrate the definition of nonlinear system and the two connected remarks of the definition from [18] as the following

Definition 2.2.2 *Nonlinear Function is any function doesn't satisfy superposition and homogeneity proporities.*

Remark 2.2.3 *The principle of superposition states that for two different inputs, x and y in the domain of the function $f, f(x+y) = f(x) + f(y)$.*

Remark 2.2.4 *The property of homogeneity states that for a given input, x, in the domain of the function f,and for any real number k, $f(kx) = kf(x)$.*

Definition 2.2.5 *[23] Nonlinear System is a system of nonlinear equations has a form:*

$$f_1(x_1, x_2, ..., x_n) = 0$$
$$f_2(x_1, x_2, ..., x_n) = 0$$
$$\vdots$$
$$\vdots$$
$$f_n(x_1, x_2, ..., x_n) = 0,$$

where each function f_i can be thought as mapping a vector $x = (x_1, x_2, ..., x_n)^t$ of the n- dimensional space $\mathbb{R}^n$into the real line $\mathbb{R}$.

Definition 2.2.6 *[18] Equilibrium point is a point x_0 in the state space is an equilibrium point of the autonomous system $x' = f(x)$ if when the state x reaches x_0, it stays at x_0 for all future time. That is, for a nonlinear continous system, the equilibrium points are the solutions of the equation: $f(x) = 0$.*

Definition 2.2.7 *[19] Path-Connected set is a set A is path-connected if any pair of points of A can be joined by a path of A: $\forall(w,z) \in A^2, \exists\gamma \in C^0([0,1],A)$, such that $\gamma(0) = w, \gamma(1) = z$.*

Definition 2.2.8 *[20] Open set is a subset $G \subseteq \mathbb{R}$ is open in $\mathbb{R}$ if for each $x \in G$ there exists a neighborhood V of x such that $V \subseteq G$.*

Corollary 2.2.9 *[19] An open set is connected if and only if it is path-connected.*

Now, we present the theorem that check the existence and uniqueness for the nonlinear system from [21], after we explained the definition (2.2.8) and the corollary (2.2.9).

Theorem 2.2.10 *Existence and Uniqueness Theorem: Consider the initial value problem $\acute{x} = f(x), x(0) = x_0$.Suppose that f is continuous and that all its partial derivatives $\frac{df_i}{dx_j}, i,j = 1,...,n$,are continuous for x in some open connecte set $D \subset \mathbb{R}^n$.Then for $x_0 \in D$, the initial value problem has a solution $x(t)$on some time interval $(-\tau,\tau)$ about $t = 0$,and the solution is unique.*

Definition 2.2.11 ***Remark 2.2.12 Definition 2.2.13*** *[22] Jacobian Matrix is a matrix of frst order partial derivatives*

$$J(x) = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \cdots & \frac{df_1}{dx_n} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \cdots & \frac{df_2}{dx_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{df_n}{dx_1} & \frac{df_n}{dx_2} & \cdots & \frac{df_n}{dx_n} \end{bmatrix}$$

Definition 2.2.14 *[18] The inverse of Jacobian Matrix $J \in \mathbb{R}^{n*n}$ is $J^{-1} \in \mathbb{R}^{n*n}$ if*

$$J * J^{-1} = J^{-1} * J = I$$

Definition 2.2.15 *[22] Newton's Method is the most common numerical method, and powerful method that is used to solve a system of nonlinear equations. Since, the general form of this equation is*

$$x^{(k)} = x^{(k-1)} - J^{-1}(x^{(k-1)})F(x^{(k-1)}),$$

where $k = 1, 2, ..., n$ *represents the iteration, and*

$x \in \mathbb{R}^n$ *is represented the vector*

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where $x_i \in \mathbb{R},$ *and* $i = 1, 2, ..., n.$

$F \in \mathbb{R}^n \rightarrow \mathbb{R}^n$ *is a vector map function*

$$F(x_1, x_2, ..., x_n) = \begin{bmatrix} f_1(x_1, x_2, ..., x_n) \\ f_2(x_1, x_2, ..., x_n) \\ \vdots \\ f_n(x_1, x_2, ..., x_n) \end{bmatrix},$$

where $f_i \in \mathbb{R}^n \rightarrow \mathbb{R},$ *and* J^{-1} *is the inverse of the Jacobian matrix.*

Definition 2.2.16 *[24] Precentage rate(Growth rate for values) is a tool uses to measure the change in parameters or variables between two different years (base year and current year). Which is also one of formulas to measure growth rate in economi as GDP growth. Moreover, the results give an impression if there is an increase (+sign) or decrease (−sign) in values. The general formula of this tool is*

$$\Delta x = \frac{x_t - x_{t-1}}{x_{t-1}},$$

whereas, x_t *is the value in current year, and* x_{t-1} *is the value in base year. Both values must be with the same units.*

Definition 2.2.17 *[25] Chaos is a long-term, sustainable, disorganized-looking development that meets some special mathematical criteria and occurs in a nonlinear deterministic system. Whereas, Chaos theory is the principles and mathematical operations underlying chaos.*

2.3 Stability in two dimensional nonlinear dynamic systems

In this section, we will illustrate the stability in two dimensional nonlinear systems. Consider the system of two nonlinear differential equations

$$\begin{aligned} x_1^{'} &= f_1(x_1, x_2) \\ x_2^{'} &= f_2(x_1, x_2). \end{aligned}$$

To find equilibrium points for the system we must to solve the system of differential equations by set the equations in standstill. That is, equal the derivatives to zero. The set of equilibrium points will be (x_1^*, x_2^*). After that, the system of differential equations can be linearized by calculating the Jacobian matrix for the system at (x_1^*, x_2^*). The eigenvalues $(\lambda' s)$ of Jacobian matrix can determine the state of stability of the system at equilibrium points. The Jacobian matrix for this system is

$$J = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix}.$$

The characteristic polynomial equation for this system is

$$P(\lambda) = \lambda^2 - T\lambda + D,$$

where, $T = \frac{df_1}{dx_1} + \frac{df_2}{dx_2}$ is the trace of Jacobian matrix. $D = \frac{df_1}{dx_1} * \frac{df_2}{dx_2} - \frac{df_1}{dx_2} * \frac{df_2}{dx_1}$ is the determinant of Jacobian matrix.

In this type of systems we have two eigenvalues which we present their types and stability case in the following table [21].

Type of eigenvalues($\lambda's$)	State of stability
Two $\lambda's$ are real and possitive	Unstable node
Two $\lambda's$ are real and negative	Stable Node
one of $\lambda's$ is real positive but the other one is real negative	Saddle Point
one of $\lambda's$ is positive but the other is zero	Unstable point
Two of $\lambda's$ are complex with negative real part	Stable spiral
Two of $\lambda's$ are complex with positive real part	Unstable spiral
Two of $\lambda's$ are complex with zero real part	Center point
Two of $\lambda's$ are equal repeated eigenvalues and positive	Unstable point
Two of $\lambda's$ are equal repeated eigenvalues and negative	Stable point
one of $\lambda's$ is negative but the other is zero	Non isolated fixed point Stability indeterminate

Table(2.1): Stability for two dimensional systems in $\mathbb{R}^2$

2.4 Stability and Bifurcation of three dimensional systems

In this section, we will explain the stability and bifurcation for third dimension systems.

2.4.1 Stability

Consider the system of three nonlinear differential equations

$$x_1^{'} = f_1(x_1, x_2, x_3)$$
$$x_2^{'} = f_2(x_1, x_2, x_3)$$
$$x_3^{'} = f_3(x_1, x_2, x_3).$$

To find equilibrium points for the system we must to solve the system of differential equations by set the equations in standstill. That is, equal the derivatives to zero. The set of equilibrium points will be (x_1^*, x_2^*, x_3^*) [26]. After that, the system of differential equations can be linearized by calculating the Jacobian matrix for the system at (x_1^*, x_2^*, x_3^*).The eigenvalues $(\lambda' s)$ of Jacobian matrix determine the state of stability of the system at equilibrium points.The Jacobian matrix for this system is

$$J = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{bmatrix}.$$

The characteristic polynomial equation for this system is

$$P(\lambda) = \lambda^3 - T\lambda^2 + A\lambda - D,$$

where, $T = \frac{df_1}{dx_1} + \frac{df_2}{dx_2} + \frac{df_3}{dx_3}$ is the trace of Jacobian matrix.

$A = J_{11} + J_{22} + J_{33}$ is the sum of principal minors 2*2 matrices.

$D = \frac{df_1}{dx_1} \begin{vmatrix} \frac{df_2}{dx_2} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{vmatrix} - \frac{df_1}{dx_2} \begin{vmatrix} \frac{df_2}{dx_1} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_3} \end{vmatrix} + \frac{df_1}{dx_3} \begin{vmatrix} \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} \end{vmatrix}$ is the determinant of Jacobian matrix.

Clearly, this type of systems have three eigenvalues the following table illustrate it and type of stability [39]:

Type of eigenvalues($\lambda's$)	State of stability
Three $\lambda's$ are real and possitive	Repeller Node(unstable)
Three $\lambda's$ are real and negative	Stable Node
Two of $\lambda's$ are positive real but third one is real and negative	Saddle Point
Two of $\lambda's$ are negative real but third one is real and positive	Saddle Point
Two of $\lambda's$ are complex with negative real part but third one is real and negative	Spiral Node
Two of $\lambda's$ are complex with positive real part but third one is real and positive	Spiral Repeller
Two of $\lambda's$ are complex with negative real part but third one is real and positive	Spiral saddle point
Two of $\lambda's$ are complex with positive real part but third one is real and negative	Spiral saddle point

Table(2.2): Stability for three dimensional systems in $\mathbb{R}^3$

2.4.2 Bifurcation

Generally, there are two types of bifurcations that happen in dynamic systems which depends on continous time. The first type is local bifurcation which divides to two main parts are depending on the case of eigenvalues ($\lambda = a+bi$) of Jacobian matrix. The first part, if all eigenvalues are only real ($\lambda = a, bi = 0$), then we have three forms of bifurcations depend on the number of fixed points undergoes the name of saddle-node bifurcation. 1) Saddle node in this case the system may have one or two fixed points (one is stabe,the other is unstable) or no any fixed point. 2)Transcritical bifurcation the system have at least one fixed points and at most two fixed points (one stable and the other is unstable). 3)Pitchfork bifurcation in this case we have three fixed points whereas, if two fixed points are stable(unstable) but third one is unstable (stable) we have supercritical (subcritical) pitchfork. The second part, if we have at least two complex eigenvalues with zero real parts ($\lambda_{1,2} = bi, a \neq 0$) but third one is real ($\lambda_3 = a$). This type of bifurcation is Hopf bifurcation (limit cycle). However, this type divides for two parts. The first is Supercritcal hopf bifurcation (attractive limit cycle). The second is Subcritical hopf bifurcation (repulsive limit cycle) [26].

Proposition 2.4.1 *(Saddle node) A saddle-node arises if and only if* $D = 0$.

Proposition 2.4.2 *(Hopf) In this case, a Hopf bifurcation generically arises if and only if* $D = AT$ *and* $A > 0$.

The second type of bifurcation is global bifurcation when two of local bifurcation are coalesce to each other as Bogdanov-Takens bifurcation (two zeros eigenvalues) since here we have saddle node surrounded by limit cycle. Moreover, Gavrilov-Guckenheimer bifurcation ($\lambda_1 = 0, \lambda_2 = bi = -\lambda_3$). In this type of bifurcation, the intersection happen between hopf bifurcation and saddle node. Whereas, this type of bifurcation can lead to richer dynamics including the cases of invariant torus and local chaos.

2.5 Stability and Bifurcation in Four dimensional systems

In this section, we will explain the stability and bifurcation for fourth dimension systems.

2.5.1 Stability

Consider the system of four nonlinear differential equations

$$\begin{aligned}
x_1^{'} &= f_1(x_1, x_2, x_3, x_4) \\
x_2^{'} &= f_2(x_1, x_2, x_3, x_4) \\
x_3^{'} &= f_3(x_1, x_2, x_3, x_4) \\
x_4^{'} &= f_4(x_1, x_2, x_3, x_4).
\end{aligned}$$

To find equilibrium points for the system we must solve the system of differential equations by set the equations in standstill.That's it, equal the derivatives to zero. The set of equilibrium points will be $(x_1^*, x_2^*, x_3^*, x_4^*)$. After that, the system of differential equations can be linearized by calculating the Jacobian matrix for the system at $(x_1^*, x_2^*, x_3^*, x_4^*)$ [26]. The eigenvalues $(\lambda's)$ of Jacobian matrix can determine the state of stability of the system at equilibrium points. The Jacobian matrix for this system is

$$J = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} & \frac{df_1}{dx_4} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_3} & \frac{df_2}{dx_4} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} & \frac{df_3}{dx_4} \\ \frac{df_4}{dx_1} & \frac{df_4}{dx_2} & \frac{df_4}{dx_3} & \frac{df_4}{dx_4} \end{bmatrix}.$$

The characteristic polynomial equation for this system is

$$P(\lambda) = \lambda^4 - T\lambda^3 + A_2\lambda^2 - A_3\lambda + D,$$

where,$T = \frac{df_1}{dx_1} + \frac{df_2}{dx_2} + \frac{df_3}{dx_3} + \frac{df_4}{dx_4}$ is the trace of Jacobian matrix.

$A_2 = J_{1212} + J_{1313} + J_{1414} + J_{2323} + J_{2424} + J_{3434}$ is the sum of principal minors 2*2 matrices. Since J_{ijkl} is $2 * 2$ principal minor matrix. i, j are rows $\in [1, 4]$, and k, l are columns $\in [1, 4]$, with condition $i = k,\ j = l$.

$A_3 = J_{11} + J_{22} + J_{33} + J_{44}$.

$$D = \frac{df_1}{dx_1}\begin{vmatrix} \frac{df_2}{dx_2} & \frac{df_2}{dx_3} & \frac{df_2}{dx_4} \\ \frac{df_3}{dx_2} & \frac{df_3}{dx_3} & \frac{df_3}{dx_4} \\ \frac{df_4}{dx_2} & \frac{df_4}{dx_3} & \frac{df_4}{dx_4} \end{vmatrix} - \frac{df_1}{dx_2}\begin{vmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_3} & \frac{df_1}{dx_4} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_3} & \frac{df_3}{dx_4} \\ \frac{df_4}{dx_1} & \frac{df_4}{dx_3} & \frac{df_4}{dx_4} \end{vmatrix} + \frac{df_1}{dx_3}\begin{vmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_4} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_4} \\ \frac{df_4}{dx_1} & \frac{df_4}{dx_2} & \frac{df_4}{dx_4} \end{vmatrix} - \frac{df_1}{dx_4}\begin{vmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{vmatrix}$$

is the determinant of Jacobian matrix.

Generally, in higher dimensional systems we judge on stability of the system based on the state of real part of eigenvalues (real and complex). However, if all real parts of eigenvalues are positive then the equilibrium point is unstable, on the other hand if all real parts of eigenvalues are positive then the equilibrium point is stable. In addition, if at least one of real part of eigenvalues has different sign from others then the point is saddle node. Otherwise, if one of these eigenvalues is equal zero, in this case the nonlinearities determine the stability of the system, other eigenvalues will determine stability by different methods as lyapunov function [27].

2.5.2 Bifurcation

The bifurcation in four dimensional systems is similarly to the bifurcation in three dimensional systems, but in four dimensional systems we have additional type of global bifurcation which is a Double-Hopf bifurcation it is an intersection of two hopf bifurcations ($\lambda_1 = b_i, \lambda_2 = -b_i, \lambda_3 = d_i, \lambda_4 = -d_i$) [26].

Proposition 2.5.1 *A Hopf bifurcation generically arises if and only if*

$$A_2 = \frac{A_3}{T} + \frac{DT}{A_3}$$

and T and A_3 have the same sign.

Proposition 2.5.2 *A Bogdanov-Takens bifurcation generically occurs if and only if $A_3 = D = 0$.*

Chapter 3

Methodology of three dimensional systems

This chapter presents the mathematical model of socio-economic development in Palestine, we find the equilibrium points of the model, then show type of stability, bifurcation. In addition, MATLAB program is used to visualize the system and Newton Method is used to obtain the best values of three variables. The suggested model is a system of nonlinear differential equations are describing the regional economic development in palestine. The model is represented by a vector function of three components which are population of the region n_1, number of jobs in the real sector of the regional economy n_2 and energy supply in the region n_3.

3.1 The mathematical model of the system

The proposed model we use in this chapter is limited to three variables before adding the fourth variable of the occupation factor, which we will disscuss in chapter four. Our first model consists of three nonlinear differential equatons with three main indicators $n(t) = [n_1(t), n_2(t), n_3(t)]$ and nine components of controlling vector $\alpha(t) = [\alpha_1(t), ..., \alpha_9(t)]$. The model is represented by the following equations:

$$\begin{aligned}
\frac{dn_1}{dt} &= \alpha_1 n_1 - \alpha_2 n_1 n_2 + \alpha_3 n_1 n_3 \\
\frac{dn_2}{dt} &= \alpha_5 n_2 + \alpha_6 n_1 n_2 + \alpha_7 n_2 n_3 \\
\frac{dn_3}{dt} &= \alpha_9 n_3 - \alpha_{10} n_1 n_3 - \alpha_{11} n_2 n_3
\end{aligned} \tag{3.1}$$

We notice that the above model 3.1 is a system of nonlinear differential equations describing the changes of three main variables the regional population n_1, the number of jobs in real sector of the regional economy n_2 and the regional energy supply n_3. Whereas, $\alpha's$ are the controlling parameters that determine the required values of main variables. The previous system is analyzed based on local Palestinian social and economic reality, as stated in Palestinian data and statistics.

Now, we illustrate the first equation of the system (3.1) and explain the interaction between environments as follows

$$\frac{dn_1}{dt} = \alpha_1 n_1 - \alpha_2 n_1 n_2 + \alpha_3 n_1 n_3 \tag{3.2}$$

In this equation, $\frac{dn_1}{dt}$ is the rate of change in population during time. Clearly, the rate of change is proportional to the regional population n_1 itself. Since the higher in the population is the greater the growth rate by $\alpha_1$which is the demographic activity coefficient. In addition, we note the negative impact of job opportunities n_2 on the population growth rate. As there are a large number of holders of bachelor degree and postgraduate degrees greater than a number of job opportunities available in the region. As such, an increased focus of the demand for work places on experiences, skills and competencies, which lead to an increase in the unemployment rate in the young and working age community, we see that population growth rate decreases under the effect of number of jobs by α_2 which is the coefficient of academic and professional level for population [28], [29]. In third term of equation, we notice proportional relation between energy supply n_3 and population growth rate, hence more amount of energy enters the region will encourage population to increase and growth. Energy supply indicator increases the population growth rate by α_3 which is the energy supply coefficient [30].

Now, we discuss the second equation of the system (3.1) and present the effect of three variables on the rate of change of the economic development.

$$\frac{dn_2}{dt} = \alpha_5 n_2 + \alpha_6 n_1 n_2 + \alpha_7 n_2 n_3 \tag{3.3}$$

In this equation, $\frac{dn_2}{dt}$ is the rate of change of the economic development during time. The rate of change of economic development is proportional to the number of jobs in real sector n_2, such that when the number of jobs increases in region the rate of economic development will be rise by $\alpha_5$which is the coefficient of the real sector economic development, it was measured through the contribution of the different sectors to the gross domestic product (GDP) [31]. Moreover there is a positive relation between the rate of change of the economic development and regional population n_1, hence when the number of population of the region rises, the people are intrested in economic development by increasing their contribution to the workforce by a factor α_6 which is the coefficient of people interesting in economic development [12]. In the third part, the energy supply n_3 will increase the economic growth rate by the factor α_7 which is the coefficient of energy supply per internal workplace, that is when the total energy supplied to the region increases at reasonable prices and at a low cost, it leads to an increase and expansion in the economic sector in various fields and thus leads to economic growth [32], [33].

Now, we demonstrate the third equation of the system (3.1) and present the effect of three variables on the rate of change in the region's energy supply.

$$\frac{dn_3}{dt} = \alpha_9 n_3 - \alpha_{10} n_1 n_3 - \alpha_{11} n_2 n_3 \tag{3.4}$$

In this equation,$\frac{dn_3}{dt}$ is the rate of change in the region's energy supply. There is positive relation between the rate of change in energy supply $\frac{dn_3}{dt}$ and the energy consumption n_3, so when the amount of energy consumed decrease the energy supply rate also will decline by α_9 which is the energy supply coefficient [34], on the other hand the effect of population n_1 on the rate of change in energy supply is negative. Hence, when the number of population increases the rate of change in energy supply will decrease by α_{10} which is the conformity ratio of the population with energy supply. Moreover, there is an inverse relation between number of jobs n_2 and the rate of change in energy supply, that is when the number of jobs decreases the rate of change in energy supply will decrease by α_{11} which is the conformity ratio of the economic development with the energy supply [35, 36, 37].

The summary of factors (α_i) in previous system 3.1 are as the following

α_1 : is the demographic activity coefficient.

α_2 : is the coefficient of academic and professional level for population.

α_3 : is the energy supply coefficient.

α_5 : is the coefficient of the real sector economic development

α_6 : is the coefficient of people interest in economic development

α_7 : is the coefficient of energy supply per internal workplace.

α_9 : is the energy supply coefficient.

α_{10} : is the conformity ratio of the population with energy supply.

α_{11} : is the conformity ratio of the economic development with the energy supply.

Before we are going to study the fixed points of the system we want to check if system 3.1 has a solution by theorem 2.2.10.

3.2 Existence and Uniqeness for the System

Consider the system 3.1 with initial condition $n(0) = n_0$. Suppose $f_{1,}f_2, f_3$ are continous and differentiable with respect to n_i. In other words, the functions are smooth enough to ensure the existence and uniqeness of solutions starting from any point in phase space. The partial derivatives of the function are as follows

$$\frac{df_1}{dn_1} = \alpha_1 - \alpha_2 n_2 + \alpha_3 n_3$$

$$\frac{df_1}{dn_2} = n_1(\alpha_1 - \alpha_2 + \alpha_3 n_3)$$

$$\frac{df_1}{dn_3} = n_1(\alpha_1 - \alpha_2 n_2 + \alpha_3)$$

$$\frac{df_2}{dn_1} = n_2(\alpha_5 + \alpha_6 + \alpha_7 n_3)$$

$$\frac{df_2}{dn_2} = \alpha_5 + \alpha_6 n_1 + \alpha_7 n_3$$

$$\frac{df_2}{dn_3} = n_2(\alpha_5 + \alpha_6 n_1 + \alpha_7)$$

$$\frac{df_3}{dn_1} = n_3(\alpha_9 - \alpha_{10} - \alpha_{11} n_2)$$

$$\frac{df_3}{dn_2} = n_3(\alpha_9 - \alpha_{10} n_1 - \alpha_{11})$$

$$\frac{df_3}{dn_3} = \alpha_9 - \alpha_{10} n_1 - \alpha_{11} n_2$$

hence, all above partial derivatives are continous for n_i, $i = 1, .., 3$, in some open connected set $D \subset \mathbb{R}^3$, then for $n_0 \in D$ the system 3.1 with initial condition $n(0) = n_0$ which is an I.V.P has a solution $n(t)$ on some time interval around $t = 0$, and the solution is uniqe.

3.3 Fixed points of the system (equilibrium points)

In this section,we want to find the fixed points of the system in general case. After that, we want to subtitute values of our factors based on figures and statistics obtained from the Palestinian Central Bureau of Statistics (PCBS) [38], to get the values of fixed points numerically.

3.3.1 General fixed points and stability for the system

The proposed system 3.1 is continuous dynamic system. To find the general equilibrium points for the system we put $\acute{n}_1 = 0, \acute{n}_2 = 0, \acute{n}_3 = 0$. As a result we have the following equations

$$\alpha_1 n_1 - \alpha_2 n_1 n_2 + \alpha_3 n_1 n_3 = 0 \tag{3.5}$$

$$\alpha_5 n_2 + \alpha_6 n_1 n_2 + \alpha_7 n_2 n_3 = 0 \tag{3.6}$$

$$\alpha_9 n_3 - \alpha_{10} n_1 n_3 - \alpha_{11} n_2 n_3 = 0 \tag{3.7}$$

To find the fixed point for equation (3.5), we have

$$\alpha_1 n_1 - \alpha_2 n_1 n_2 + \alpha_3 n_1 n_3 = 0,$$

hence,

$$n_1(\alpha_1 - \alpha_2 n_2 + \alpha_3 n_3) = 0,$$

so $n_1^* = 0$, or

$$\alpha_1 - \alpha_2 n_2^* + \alpha_3 n_3^* = 0.$$

This implies

$$n_2^* = \frac{-\alpha_1 - \alpha_3 n_3^*}{-\alpha_2} = \frac{\alpha_1 + \alpha_3 n_3^*}{\alpha_2} \tag{3.8}$$

From equation (3.6)

$$\alpha_5 n_2^* + \alpha_6 n_1^* n_2^* + \alpha_7 n_2^* n_3^* = 0,$$

so,

$$n_2^*(\alpha_5 + \alpha_6 n_1^* + \alpha_7 n_3^*) = 0.$$

Then $n_2^* = 0$, or

$$\alpha_5 + \alpha_6 n_1^* + \alpha_7 n_3^* = 0.$$

So, we get

$$n_1^* = \frac{-\alpha_5 - \alpha_7 n_3^*}{\alpha_6}. \tag{3.9}$$

Finally from equation (3.7) we have

$$\alpha_9 n_3^* - \alpha_{10} n_1^* n_3^* - \alpha_{11} n_2^* n_3^* = 0,$$

so,

$$n_3^*(\alpha_9 - \alpha_{10} n_1^* - \alpha_{11} n_2^*) = 0.$$

Therefore, we get $n_3^* = 0$, or

$$\alpha_9 - \alpha_{10} n_1^* - \alpha_{11} n_2^* = 0. \tag{3.10}$$

Substitute equations (3.8) and (3.9) in equation (3.10), we have

$$\alpha_9 + \frac{\alpha_{10}\alpha_5 + \alpha_{10}\alpha_7 n_3^*}{\alpha_6} - \frac{\alpha_{11}\alpha_1 + \alpha_{11}\alpha_3 n_3^*}{\alpha_2} = 0$$

This equation takes the form

$$\frac{\alpha_2\alpha_{10}\alpha_5 + \alpha_9\alpha_6\alpha_2 - \alpha_{11}\alpha_6\alpha_1}{\alpha_6\alpha_2} + \left(\frac{-\alpha_6\alpha_{11}\alpha_3 + \alpha_2\alpha_{10}\alpha_7}{\alpha_6\alpha_2}\right) n_3^* = 0.$$

Thus, we obtain

$$n_3^* = -\frac{\alpha_2\alpha_{10}\alpha_5 + \alpha_9\alpha_6\alpha_2 - \alpha_{11}\alpha_6\alpha_1}{\alpha_7\alpha_{10}\alpha_2 - \alpha_{11}\alpha_6\alpha_3}.$$

From equation (3.9) we have

$$n_1^* = \frac{\alpha_7\alpha_2\alpha_{10}\alpha_5 + \alpha_7\alpha_9\alpha_6\alpha_2 - \alpha_7\alpha_{11}\alpha_6\alpha_1}{\alpha_6\alpha_7\alpha_{10}\alpha_2 - \alpha_{11}\alpha_6^2\alpha_3} - \frac{\alpha_5}{\alpha_6}.$$

From equation (3.8)we have

$$n_2^* = \frac{\alpha_3\alpha_{11}\alpha_6\alpha_1 - \alpha_3\alpha_2\alpha_{10}\alpha_5 - \alpha_2\alpha_3\alpha_{10}\alpha_5}{\alpha_7\alpha_{10}\alpha_2^2 + \alpha_{11}\alpha_6\alpha_3\alpha_2} + \frac{\alpha_1}{\alpha_2}.$$

Consider $\eta = \frac{\alpha_2\alpha_{10}\alpha_5 + \alpha_9\alpha_6\alpha_2 - \alpha_{11}\alpha_6\alpha_1}{\alpha_7\alpha_{10}\alpha_2 - \alpha_{11}\alpha_6\alpha_3}$. The fixed points will be $F_{p_1} = (0,0,0)$, $F_{p_2} = (\frac{\alpha_7\eta - \alpha_5}{\alpha_6}, \frac{\alpha_1 - \alpha_3\eta}{\alpha_2}, -\eta)$. These fixed points to be a solution for the system, this condition must be satisfy $\alpha_7\alpha_{10}\alpha_2 - \alpha_{11}\alpha_6\alpha_3 \neq 0$, that is $\alpha_7\alpha_{10}\alpha_2 \neq \alpha_{11}\alpha_6\alpha_3$ and $\alpha_2, \alpha_6 \neq 0$. Otherwise, the system doesn't have fixed points.

Now, we want to analyze the stability. We are going to find the eigenvalues of Jacobian matrix at any fixed point $F_p(n_1^*, n_2^*, n_3^*)$:

$$J(n_1^*, n_2^*, n_3^*) = \begin{bmatrix} \alpha_1 - \alpha_2 n_2^* + \alpha_3 n_3^* & -\alpha_2 n_1^* & \alpha_3 n_1^* \\ \alpha_6 n_2^* & \alpha_5 + \alpha_6 n_1^* + \alpha_7 n_3^* & \alpha_7 n_2^* \\ -\alpha_{10} n_3^* & -\alpha_{11} n_3^* & \alpha_9 - \alpha_{10} n_1^* - \alpha_{11} n_2^* \end{bmatrix} \tag{3.11}$$

Now, for the fixed point $F_{p_1} = (0, 0, 0)$,we have

$$J(n_1^*, n_2^*, n_3^*) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_5 & 0 \\ 0 & 0 & \alpha_9 \end{bmatrix} \tag{3.12}$$

Since, (3.12) is diagonal matrix, the eigenvalues of this matrix are $\lambda_1 = \alpha_1$, $\lambda_2 = \alpha_5$, $\lambda_3 = \alpha_9$, if they are positive real numbers, then this point is unstable fixed point. In addition, if they are negative real numbers, then this point is stable fixed point. But, if at least one of them has different sign, then the point unstable saddle node.

Now, we illustrate the analysis of general fixed point $F_{p_2} = (\frac{\alpha_7\eta - \alpha_5}{\alpha_6}, \frac{\alpha_1 - \alpha_3\eta}{\alpha_2}, -\eta)$.The jacobian matrix of F_{p_2} is

$$J(n_1^*, n_2^*, n_3^*) = \begin{bmatrix} \alpha_1 - \frac{\alpha_2\alpha_1 - \alpha_2\alpha_3\eta}{\alpha_2} - \alpha_3\eta & -\frac{\alpha_2\alpha_7\eta - \alpha_2\alpha_5}{\alpha_6} & \frac{\alpha_3\alpha_7\eta - \alpha_3\alpha_5}{\alpha_6} \\ \frac{\alpha_6\alpha_1 - \alpha_6\alpha_3\eta}{\alpha_2} & \begin{matrix} \alpha_5 + \frac{\alpha_6\alpha_7\eta - \alpha_6\alpha_5}{\alpha_6} \\ -\alpha_7\eta \end{matrix} & \frac{\alpha_7\alpha_1 - \alpha_7\alpha_3\eta}{\alpha_2} \\ \alpha_{10}\eta & \alpha_{11}\eta & \begin{matrix} \alpha_9 - \frac{\alpha_{10}\alpha_7\eta - \alpha_{10}\alpha_5}{\alpha_6} \\ -\frac{\alpha_{11}\alpha_1 - \alpha_{11}\alpha_3\eta}{\alpha_2} \end{matrix} \end{bmatrix} \tag{3.13}$$

Simplifying the matrix in (3.13) we get

$$J(\frac{\alpha_7\eta - \alpha_5}{\alpha_6}, \frac{\alpha_1 - \alpha_3\eta}{\alpha_2}, -\eta) = \begin{bmatrix} 0 & -\frac{\alpha_2(\alpha_7\eta - \alpha_5)}{\alpha_6} & \frac{\alpha_3(\alpha_7\eta - \alpha_5)}{\alpha_6} \\ \frac{\alpha_6(\alpha_1 - \alpha_3\eta)}{\alpha_2} & 0 & \frac{\alpha_7(\alpha_1 - \alpha_3\eta)}{\alpha_2} \\ \alpha_{10}\eta & \alpha_{11}\eta & \frac{\chi + \psi\eta}{\alpha_6\alpha_2} \end{bmatrix}.$$

Let $\chi = \alpha_9\alpha_6\alpha_2 + \alpha_{10}\alpha_5\alpha_2 - \alpha_{11}\alpha_1\alpha_6$, and $\psi = \alpha_{11}\alpha_3\alpha_6 - \alpha_{10}\alpha_7\alpha_2$.

The charactrestic polynomial $\lambda^3 - tr(J)\lambda^2 + (J_{11} + J_{22} + J_{33})\lambda - \det(J) = 0$. So we have

$$\begin{aligned}
tr(J) &= \frac{\chi + \psi\eta}{\alpha_6\alpha_2}, \\
J_{11} &= \frac{\alpha_7\alpha_3\alpha_{11}\eta^2 - \alpha_7\alpha_1\alpha_{11}\eta}{\alpha_2}, \\
J_{22} &= \frac{\alpha_3\alpha_5\alpha_{10}\eta - \alpha_3\alpha_7\alpha_{10}\eta^2}{\alpha_6}, \\
J_{33} &= \frac{-\alpha_6\alpha_1\alpha_2\alpha_5 + (\alpha_6\alpha_1\alpha_2\alpha_7 + \alpha_2\alpha_5\alpha_6\alpha_3)\eta - \alpha_6\alpha_3\alpha_2\alpha_7\eta^2}{\alpha_2\alpha_6}, \\
\det(J) &= \frac{\alpha_2\alpha_7\eta - \alpha_2\alpha_5}{\alpha_6}J_{12} + \frac{\alpha_3\alpha_7\eta - \alpha_3\alpha_5}{\alpha_6}J_{13}.
\end{aligned}$$

We analyze the stability of the system at F_{p_2} by finding the eigenvalues of (3.13). As such, if all eigenvalues are real and positive the equilibrium point is unstable. Furthermore, if all eigenvalues are real and negative the fixed point is stable, but if at least one of eigenvalues has different sign then the point is saddle and unstable. On the other hand, if one of eigenvalues is real and the other two eigenvalues are complex , then the real part will determine the case of stability. Since, the real part is positive and the real eigenvalue is positive also, then the point is spiral repeller (unstable). But if the real part is negative and real eigenvalue negative too, then the point is stable spiral node. Moreover, if the real part has different sign from the real eigenvalue, then the point is spiral saddle node (unstable) as in table (2.2). Finally, if one of real eigenvalues is zero and the others are real positive then the point is unstable, but if the others are negtive the stability is uncertained. Also, if the real part of complex eigenvalues are zeros the point will be a center.

In addition, we have three cases of fixed points emanate from general fixed point F_{p_2}. Significantly, the analyze of stability of these cases, similarly to F_{p_2} fixed point depending on sign and type of eigenvalues for jacobian matrix .

We consider the following cases

Case (1) : when $\frac{\alpha_7\eta - \alpha_5}{\alpha_6} = 0$, that is $\eta = \frac{\alpha_5}{\alpha_7}$ then the fixed point becomes F_{p_3} $= (0, \frac{\alpha_1 - \alpha_3\eta}{\alpha_2}, -\eta)$, F_{p_3} is a solution for the system if $\alpha_7 \neq 0$ and $\alpha_2 \neq 0$. The

Jacobian matrix

$$J(n_1^*, n_2^*, n_3^*) = \begin{bmatrix} \alpha_1 - \alpha_2\frac{\alpha_1-\alpha_3\eta}{\alpha_2} - \alpha_3\eta & 0 & 0 \\ \alpha_6\frac{\alpha_1-\alpha_3\eta}{\alpha_2} & \alpha_5 - \alpha_7\eta & \alpha_7\frac{\alpha_1-\alpha_3\eta}{\alpha_2} \\ \alpha_{10}\eta & \alpha_{11}\eta & \alpha_9 - \alpha_{11}\frac{\alpha_1-\alpha_3\eta}{\alpha_2} \end{bmatrix}. \quad (3.14)$$

Simplifying the matrix in (3.14) we get

$$J(0, \frac{\alpha_1 - \alpha_3\eta}{\alpha_2}, -\eta) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\alpha_6(\alpha_1-\alpha_3\eta)}{\alpha_2} & \alpha_5 - \alpha_7\eta & \frac{\alpha_7(\alpha_1-\alpha_3\eta)}{\alpha_2} \\ \alpha_{10}\eta & \alpha_{11}\eta & \frac{\omega+\alpha_3\alpha_{11}\eta}{\alpha_2} \end{bmatrix}$$

Let $\omega = \alpha_9\alpha_2 - \alpha_{11}\alpha_1$ and the charactrestic polynomial equation is

$\lambda^3 - tr(J)\lambda^2 + (J_{11} + J_{22} + J_{33})\lambda - \det(J) = 0.$

So we have

$$\begin{aligned} tr(J) &= \frac{\alpha_5\alpha_2 + \omega + (\alpha_{11}\alpha_3 - \alpha_7\alpha_2)\eta}{\alpha_2}, \\ J_{11} &= \frac{\alpha_5\omega + (\alpha_3\alpha_{11}\alpha_5 - \alpha_7\alpha_9\alpha_2)\eta}{\alpha_2}, \\ J_{22} &= 0, \\ J_{33} &= 0, \\ \det(J) &= 0. \end{aligned}$$

In this case, let $tr(J) = \gamma$, and $J_{11} = \delta$. Then the charactrestic equation is

$$\lambda^3 - \gamma\lambda^2 + \delta\lambda = 0$$

The solutions are

$$\lambda_1 = 0, \ \lambda_{2,3} = \frac{\gamma \pm \sqrt{\gamma^2 - 4\delta}}{2}$$

λ_2 and λ_3 subject to eigenvalues rules in two dimensional nonlinear systems as are informed in table (2.1) . Whereas, if $\gamma^2 - 4\delta > 0$ both eigenvalues are real. If they are positive then the point is unstable, but if both are negative we cann't assest the stability of the point; it may be stable or saddle node. Now, if $\gamma^2 - 4\delta = 0$

we have repeated eigenvalues. The sign of these eigenvalues determines the case of stability. On the other hand if $\gamma^2 - 4\delta < 0$, we have two complex eigenvalues. If the real part of both these eigenvalues is positive we have unstable spiral node, but if real part of both eigenvalues are negative we have uncertained stable spiral. Finally, if both real parts are zeros we have a center point.

Case (2) : when $n_2^* = 0$ then $\frac{\alpha_1 - \alpha_3\eta}{\alpha_2} = 0$, that is $\eta = \frac{\alpha_1}{\alpha_3}$,the fixed point equals $F_{p_4} = (\frac{\alpha_7\eta - \alpha_5}{\alpha_6}, 0, -\eta)$, this point is a solution if $\alpha_3 \neq 0$ and $\alpha_6 \neq 0$. Then the jacobian matrix is

$$J(n_1^*, n_2^*, n_3^*) = \begin{bmatrix} \alpha_1 - \alpha_3\eta & -\alpha_2\frac{\alpha_7\eta-\alpha_5}{\alpha_6} & \alpha_3\frac{\alpha_7\eta-\alpha_5}{\alpha_6} \\ 0 & \alpha_5 + \alpha_6\frac{\alpha_7\eta-\alpha_5}{\alpha_6} - \alpha_7\eta & 0 \\ \alpha_{10}\eta & \alpha_{11}\eta & \alpha_9 - \alpha_{10}\frac{\alpha_7\eta-\alpha_5}{\alpha_6} \end{bmatrix}. \tag{3.15}$$

Simplifying this matrix, we have

$$J(\frac{\alpha_7\eta - \alpha_5}{\alpha_6}, 0, -\eta) = \begin{bmatrix} \alpha_1 - \alpha_3\eta & \frac{\alpha_2(\alpha_5-\alpha_7\eta)}{\alpha_6} & \frac{\alpha_3(\alpha_7\eta-\alpha_5)}{\alpha_6} \\ 0 & 0 & 0 \\ \alpha_{10}\eta & \alpha_{11}\eta & \frac{u-\alpha_7\alpha_{10}\eta}{\alpha_6} \end{bmatrix}.$$

The charactrestic polynomial equation is

$\lambda^3 - tr(J)\lambda^2 + (J_{11} + J_{22} + J_{33})\lambda - \det(J) = 0,$

so we have

$$\begin{aligned} tr(J) &= \frac{\alpha_6\alpha_1 + \epsilon - (\alpha_6\alpha_3 + \alpha_7\alpha_{10})\eta}{\alpha_2}, \\ J_{11} &= 0, \\ J_{22} &= \frac{\alpha_1\epsilon - (\alpha_3\alpha_9\alpha_6 + \alpha_7\alpha_9\alpha_1)\eta}{\alpha_6}, \\ J_{33} &= 0, \\ \det(J) &= 0, \end{aligned}$$

where $\epsilon = \alpha_9\alpha_6 + \alpha_{10}\alpha_5$.

Let $tr(J) = \gamma_1$, and $J_{22} = \gamma_2$. The charactrestic equation takes the form

$\lambda^3 - \gamma_1\lambda^2 + \gamma_2\lambda = 0.$

Then the eigenvalues are

$$\lambda_1 = 0,\ \lambda_{2,3} = \frac{\gamma_1 \pm \sqrt{\gamma_1^2 - 4\gamma_2}}{2}$$

Similarly to Case (1) one of the eigenvalues is zero, so we can judge on other two eigenvalues as in $2*2$ nonlinear dynamic systems. We illustrated this case in details in Case (1).

Case (3) : when $n_3^* = 0$. That is, $\eta = 0$. Then the fixed point equals $F_{p_5} = (\frac{-\alpha_5}{\alpha_6}, \frac{\alpha_1}{\alpha_2}, 0)$, this point is a solution if $\alpha_6 \neq 0$ and $\alpha_2 \neq 0$. Then the jacobian matrix of this point is

$$J(\frac{-\alpha_5}{\alpha_6}, \frac{\alpha_1}{\alpha_2}, 0) = \begin{bmatrix} 0 & \frac{\alpha_2\alpha_5}{\alpha_6} & \frac{-\alpha_3\alpha_5}{\alpha_6} \\ \frac{\alpha_6\alpha_1}{\alpha_2} & 0 & \frac{\alpha_7\alpha_1}{\alpha_2} \\ 0 & 0 & \frac{\zeta}{\alpha_2\alpha_6} \end{bmatrix}.$$

The characteristic polynomial equation for Jacobean matrix is

$\lambda^3 - tr(J)\lambda^2 + (J_{11} + J_{22} + J_{33})\lambda - \det(J) = 0.$

So we have

$$\begin{aligned} tr(J) &= \frac{\zeta}{\alpha_2\alpha_6}, \\ J_{11} &= 0, \\ J_{22} &= 0, \\ J_{33} &= -\alpha_1\alpha_5, \\ \det(J) &= \frac{-\alpha_2\alpha_5}{\alpha_6} J_{12}, \end{aligned}$$

where $\zeta = \alpha_9\alpha_6\alpha_2 + \alpha_{10}\alpha_5 - \alpha_{11}\alpha_1$.

3.3.2 Numerical fixed points

We obtain the values of parameters for the control vector $\alpha(t)$ by using the Palestinian Central Bureau of Statistics (PCBS) [38] bassed on different years for each parameter. Whereas, the base year differed from one factor to another according to the available information. Moreover, the values of parameters are calculated by using precentage rate method. In our calculations, we don't depend on one base year for all parameters, we take different base and comparsion years for each parameter, that's back for many reasons such as, the lack of sufficient informations in certain years, the informations may be available for the base year but not available for the comparsion year too. On the other hand, if the informations are available, then there is a large discrepancy in the results, because palestine suffers from instabillity case as a result of the surrounding conditions like of occupation factor. For example, if we look at the effects of the occupation on the energy sector and how it imposes their control on large desert lands which can be exploited in the cultivation of solar cells and then satisfy self-sufficiency.

Now, we present the values of parameters, the mechanism is used for calculating their values, the base and the comparison year for each parameter in the following table

Parameters	Mechanism of calculate their values
α_1	Number of population[41]
α_2	Percentage of population who completed education with abachelor's degree or higher[42]
α_3	Quantity of available electricity in Palestine[43]
α_5	The percentage of labor force contribution to the Palestinian economy [45],[44]
α_6	Gross Domestic Product in palestine(GDP)[46]
α_7	The total energy supplied[47],[48]
α_9	The total energy purchases in economic activities[49],[50]
α_{10}	Percentage of the population with access to electricity[51]
α_{11}	Energy consumption by the economic sector[52],[53]

parameters	Base year	current year	values
α_1	2000	2021	0.7127
α_2	1997	2017	1.909
α_3	2010	2019	0.5242
α_5	1999	2018	0.0457
α_6	2000	2017	2.739
α_7	2001	2017	1.422
α_9	1997	2006	0.0506
α_{10}	2007	2019	0.1012
α_{11}	2000	2017	0.01325

Table(3.1): Control vector(parameters)values in 3-D system.

Based on the previous values, we have the following five equilibrium points for the previous system

$p_1(0,0,0)$, $p_2(0.5,0,-1.3596)$, $p_3(0,3.8189,-0.0321)$, $p_4(-0.0167,0.3733,0)$ and $p_5(0.4859,0.1075,-0.9681)$

3.4 Stability for the system

In this section, the stability of the system is demonstrated at each previous fixed point by using jacobian matrix and Matlab program. In order to know the stability of any system consisting of ordinary differential equations (ODE's) regardless of the degree of the system, it is necessary to know the Jacobian Matrix of the system and then find the eigenvalues of this matrix by substituting the equilibrium points of the system in the matrix. However,The stability (stable, unstable, node, spiral, center, etc.) is determined by knowing the form of eigenvalues either real or complex and the sign of the real part of these eigenvalues [39], [40].

Now, we have the following numerical system

$$\frac{dn_1}{dt} = 0.7127\ n_1 - 1.909\ n_1 n_2 + 0.5242\ n_1 n_3, \tag{3.16}$$

$$\frac{dn_2}{dt} = 0.0457\ n_2 + 2.739\ n_1 n_2 + 1.422\ n_2 n_3, \tag{3.17}$$

$$\frac{dn_3}{dt} = 0.0506\ n_3 - 0.1012\ n_1 n_3 - 0.01325\ n_2 n_3. \tag{3.18}$$

The Jacobian matrix for above system is

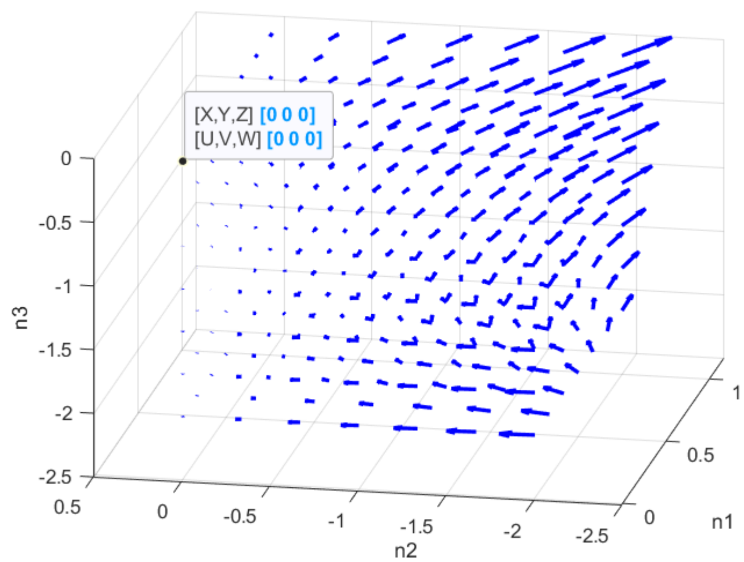
$$J = \begin{bmatrix} 0.7127 - 1.909n_2 + 0.5242n_3 & -1.909n_1 & 0.5242n_1 \\ 2.739n_2 & 0.0457 + 2.739n_1 + 1.422n_3 & 1.422n_2 \\ -0.1012n_3 & -0.01325n_3 & \begin{array}{c} 0.0506 - 0.1012n_1 - \\ 0.01325n_2 \end{array} \end{bmatrix}.$$

Since, we have equilibrium points and jacobian matrix for our system, we get the following results

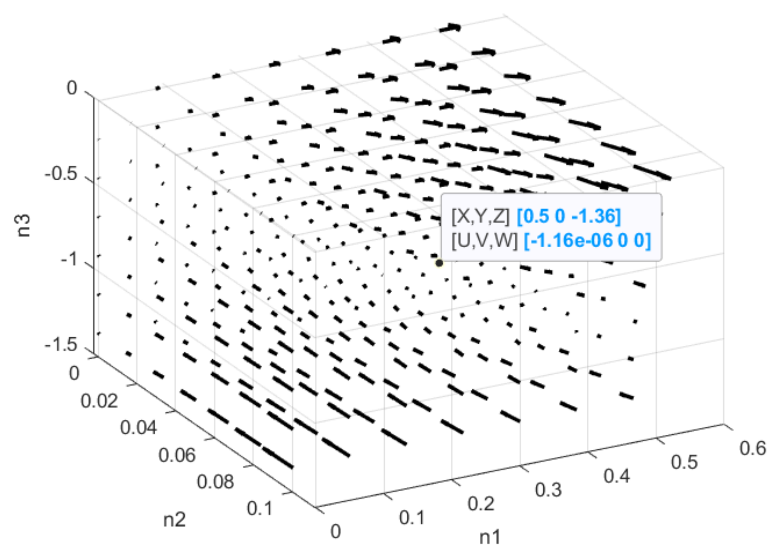
equilibrium points	eigenvalues(λ's)	sign of eigenvalues(λ's)	stability
$p_1(0,0,0)$	$\lambda_1 = 0.7127$ $\lambda_2 = 0.0457$ $\lambda_3 = 0.0506$	all eigenvalues are real and positive	unstable node
$p_2(0.5, 0, -1.3596)$	$\lambda_1 = 0.1899$ $\lambda_2 = -0.1899$ $\lambda_3 = -0.5182$	all eigenvalues are real and with differernt sign	saddle node
$p_3(0, 3.8189, -0.0321)$	$\lambda_1 = -0.0466$ $\lambda_2 = 0.0467$ $\lambda_3 = -6.5944$	all eigenvalues are real and with different sign	saddle node
$p_4(-.0167, 0.3733, 0)$	$\lambda_1 = 0.1807$ $\lambda_2 = -0.1806$ $\lambda_3 = 0.0473$	all eigenvalues are real and with different sign	saddle node
$p_5(0.4859, 0.1075, -0.9681)$	$\lambda_1 = 0.0259 + .4982i$ $\lambda_2 = 0.0259 - .4982i$ $\lambda_3 = -0.05194$	real eigenvalue has a sign differ than real part of complex eigenvalues	spiral saddle point (unstable)

Table(3.2): Equilibrium points and stability for 3-D system in $\mathbb{R}^3$

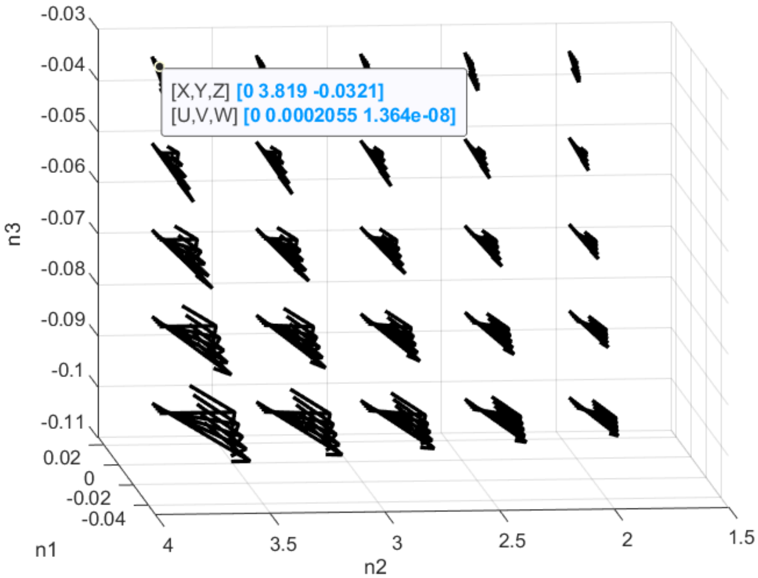
Now, we illustrate the graphical simulation of the stability for the fixed points in the following figures



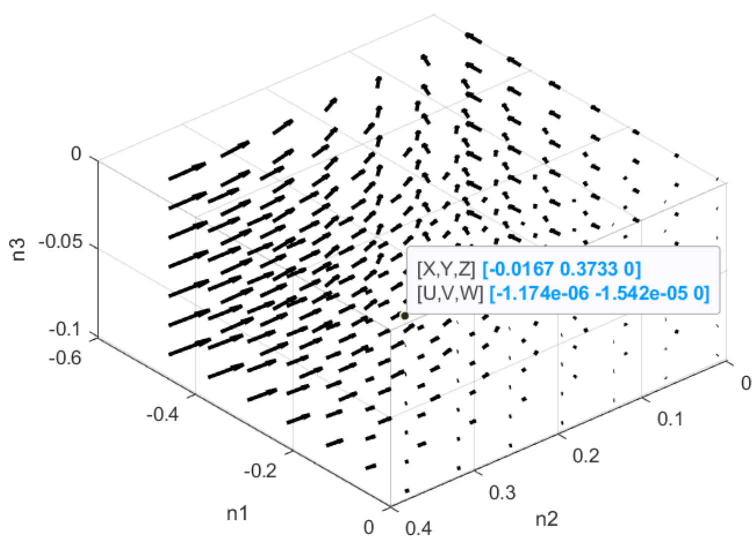
(a) Direction field around $p_1(0, 0, 0)$, (unstable node)



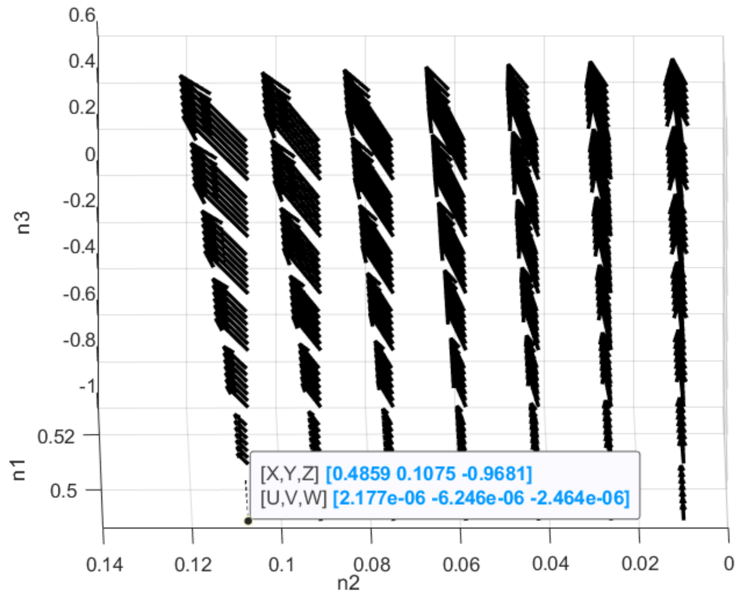
(b) Direction field around $p_2(0.5, 0, -1.3596)$, (saddle node).



(c) Direction field around $p_3(0, 3.8189, -0.0321)$, (saddle node)



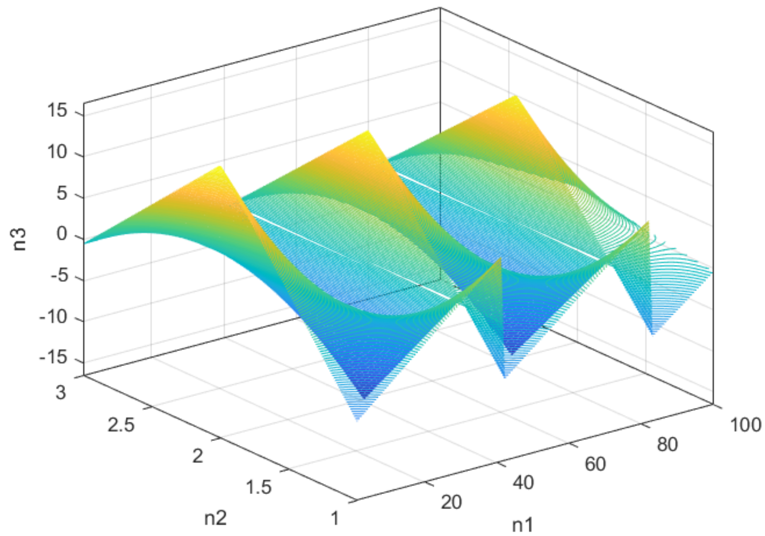
(d) Direction field around $p_4(-0.0167, 0.3733, 0)$, (saddle node)



(e) Direction field around $p_5(0.4859, 0.1075, -0.9681)$, (spiral saddle point).

Figure(3.1): Stability diagram for the five fixed points in 3-D system.

Now, we present the movement of the system around their five fixed points graphically



Figure(3.2):Movement of 3-D system around the five fixed points

This figure explains the movement of the system around their five fixed points. Since, the graph shows different values of n_1 and n_2 at different peaks and bottoms are represented by n_3. For example, when the value of n_1 is 82 and n_2 is 2 at this point, the level is at the top which is around $n_3 = 10.79$. On the other side, when the value of n_1 is 81 and n_2 is 1 the level is at the bottom, which is $n_3 = -8.886$. The fluctuation in the values and the levels constructs this graph.

3.5 Bifurcation and Numerical solution

In this section, the trajectory of the system is illustrated at different initial conditions $(0.3, 0.3, -0.3), (0.3, 0.3, -.6)$. The relation between variables is presented in different spaces, sensitivity of the system is illustrated when the initial conditions are changed and we show the behavior of the system during time. In addition, the type of bifurcation is explained at different cases. We have more than one local bifurcation (saddle-node, transcritical, pitchfork) coalesce to each other which lead to complicated bifurcation (global bifurcation) or chaotic system.

Now, we take the initial condition $n_1(0) = 0.3, n_2(0) = 0.3, n_3(0) = -0.3$ and time period $t \in [0, 100]$ and with parameters mentined in Table (2). The simulation results are as the following

Figure(3.3) : 3-D view for 3-D system on $n_1 - n_2 - n_3$ space with $(0.3, 0.3, -0.3)$

Figure (3.3) demonstrates the relation between population, jobs and energy. At the begining the number of population and energy are decreasing for a certain period, while the number of jobs is increasing. After approximately 12 years, the values of number of population and number of jobs begin to increase and the energy is continuing in decrease. However, the values keep on fluctuate. After a long period of time, the energy fades away, while the number of jobs nears to zero, but the population are increasing in a very small amount. Finally,we notice the emergence of a chaotic dynamic system, due to the fluctuation (increasing and decreasing) in the values of variables from time to time.

The following table illustrates the numerical solution for 3-D system with initial condition (0.3,0.3,-0.3)

t	n_1	n_2	n_3
0	0.3	0.3	-0.3
1.167253889	0.243364167	0.46790615	-0.306091505
5.22586431	0.05799919	0.314442631	-0.349399953
10.28915908	0.159691791	0.079763872	-0.42731429
15.26198654	0.399791491	0.438285843	-0.458155719
20.23530016	0.091205248	0.147883534	-0.530691693
25.44418168	0.399965084	0.043075755	-0.619229875
30.1396762	0.193197712	0.563436728	-0.608331174
35.01955553	0.165199999	0.037729754	-0.721916415
40.02468338	0.651149022	0.027341557	-0.775454838
45.20795488	0.104086363	0.275424248	-0.746119678
50.00947674	0.209308564	0.007335516	-0.887988766
55.31442784	0.668393983	0.002736919	-0.937393908
60.10060685	0.208584678	0.870275903	-0.772580156
65.01760053	0.098538946	0.009259217	-0.935979467
70.26299505	0.238811601	5.71e-05	-1.118395621
75.21769555	0.38029341	1.21e-06	-1.229301604
80.35060443	0.504128112	9.83e-08	-1.266296644
86.11810866	0.711599912	5.80e-08	-1.199922801
90.20538246	1.17873228	2.99e-06	-1.018271438
95.00323035	0.00984566	1.67818711	-0.629481132
99.14431621	0.001198964	0.03629627	-0.756626738
100	0.00149958	0.014789043	-0.789799551

Table (3.3): Numerical solution for 3-D system with (0.3,0.3,-0.3)

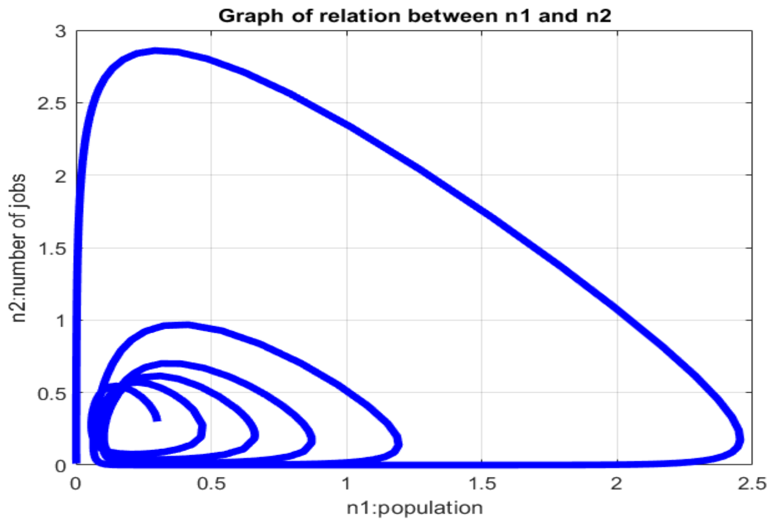


Figure (3.4): Projection for 3-D system on $n_1 - n_2$ space.

Figure (3.4) presents the nature of the relationship between the number of population and number of jobs. Clearly, it is similar to the predator-prey model. Whereas, there's an increase in the values of one variable but a decrease in the other variables during time. The number of jobs is increasing with moderate values, compared to a nature of decrease on the number of population to a certain extent. After that, the values of jobs decrease, corresponding to an increase in number of population. However, this relation continue until the values of n_2 disappear and n_1 increases dramatically after a very long period of time, then the values return to stabilize again.

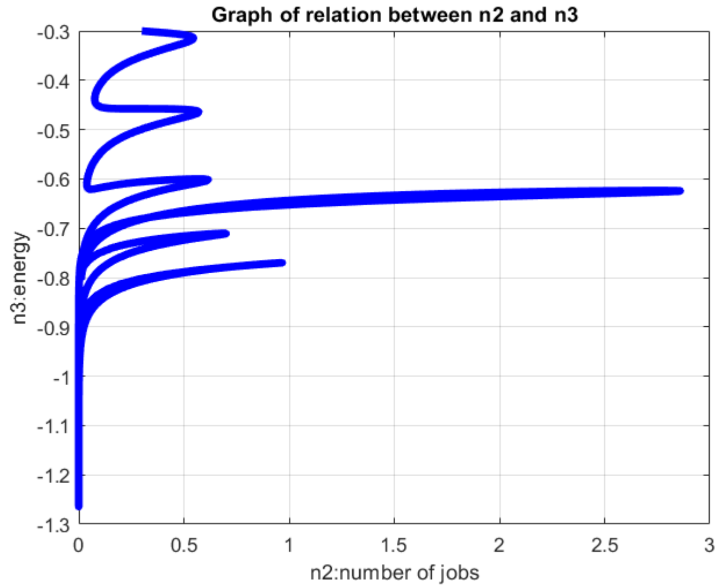


Figure (3.5): Projection for 3-D system on $n_2 - n_3$ space

Figure (3.5) explains an aperiodic relation between number of jobs and energy. There's an increse in the values of number of jobs but a decrease in energy. When the values of n_2 increase it reache to a certain point (peak), but the values of n_3 decrease by a very small amount. After that, the number of jobs and energy are decreasing until they reach bottom. After a long period of time, the number of jobs vanishs, and the energy continues to decline then becomes constant. As for population, jobs and energy increase and decrease during this period according to the available conditions.

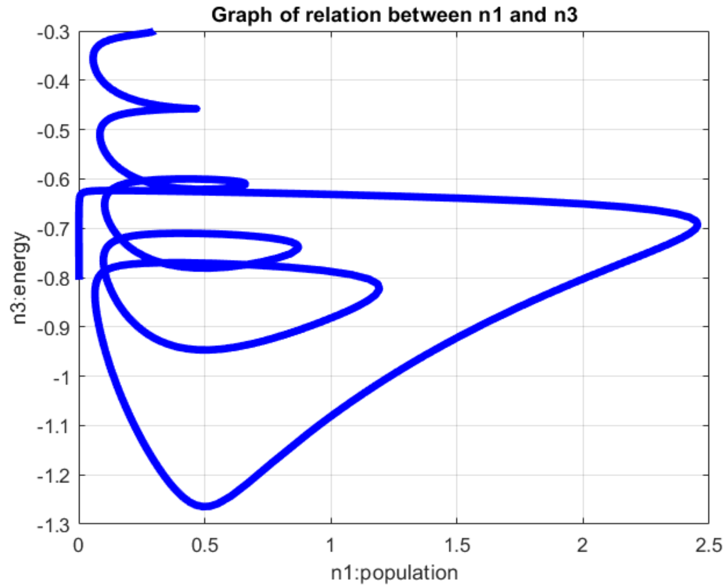


Figure (3.6): Projection of 3-D system on $n_1 - n_3$ space.

In figure (3.6) there's seem a periodic relation between population and energy. First, the number of population and energy are decreasing at limit point. After that, $n_1$increases whereas n_3 maintains decrease in very small percentage, that is almost negligible. During the time, the amount of increase in population and decrease in energy are becoming more. After approximately 65 years, there is a large increase in both of them. After this period, decrease in population and increase in energy continue until the population disappears and energy almost becoming stable.We explain also the behavior of the system along time by useing previous initial condition $(0.3, 0.3, -0.3)$. We get these results

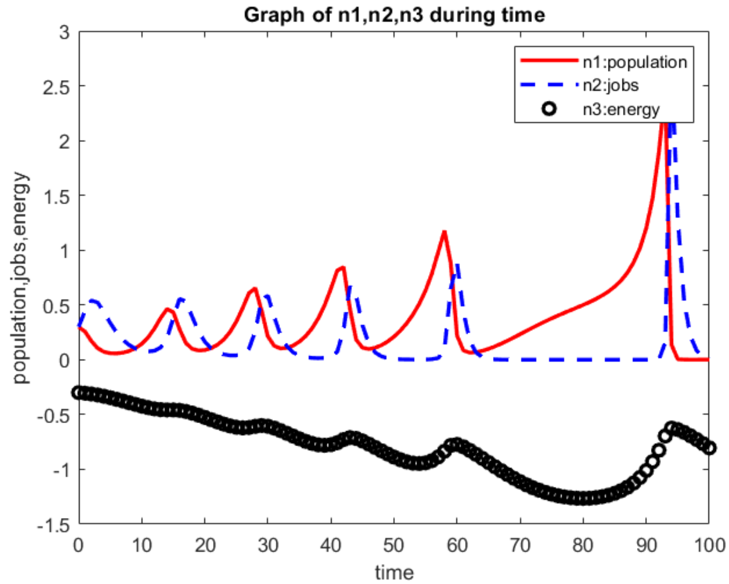


Figure (3.7): Behaviour of (n_1,n_2,n_3) during $t \in [0, 100]$ in 3-D system

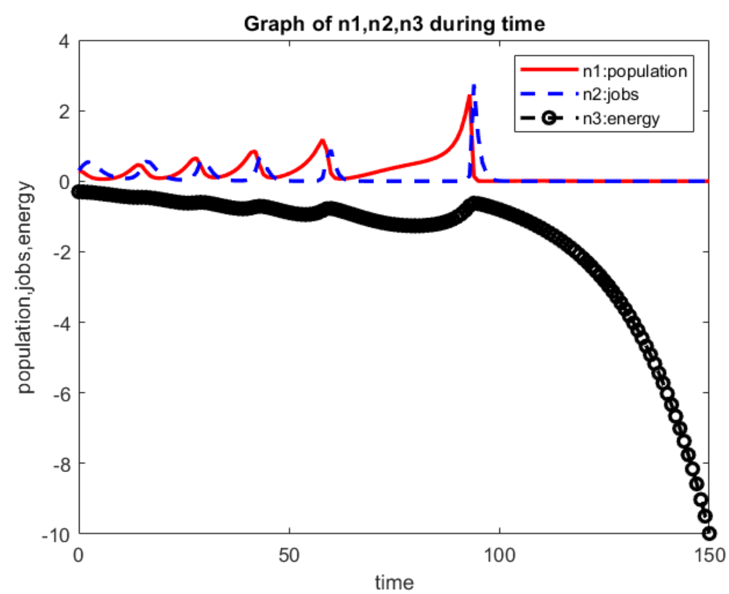


Figure (3.8): Behaviour of (n_1, n_2, n_3) after long time $t \in [0, 150]$ in 3-D system

Figure (3.7) and Figure (3.8) demonstrate the behaviour of three variables during time. Since, population and jobs are decreasing and increasing periodically untill almost 60 years. In the same period of time, energy is decreasing and increasing a little. After sixty years, the behaviour of the three functions are different, the population will increase rapidly but the number of jobs still constant then increases, but energy dereases and increases alternately. However, in the last five years all variables are decreasing like figure (3.7). After very long period of time, $n_1$and n_2 will be almost zero, n_3 decreases and goes to $(-\infty)$ like figure (3.8), that is there is no energy the society will disapear.

Now, we illustrate the sensitivity of the system when the initial condition is changed to $(0.3, 0.3, -0.6)$, during time $t \in [0, 100]$, with the same values of parameters, we get these results

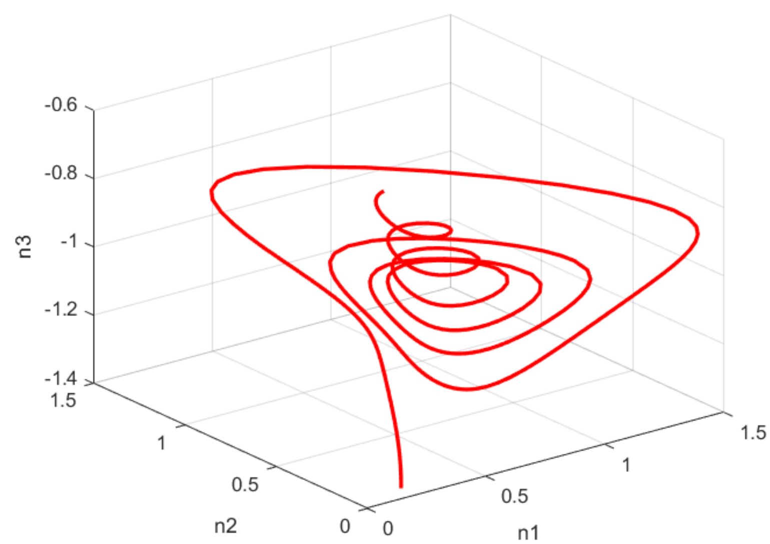


Figure (3.9): 3-D graph for 3-D system on $n_1 - n_2 - n_3$ space with $(0.3, 0.3, -0.6)$

Figure (3.9) presents the periodic relation between population, jobs and energy. At the beginning, three varibles are decreasing. After a short period of time, the population increases but number of jobs decreases little bit and energy continues in decrease. After long period of time, population and energy are decreasing deeply, but the number of jobs is almost disappearing. The following table presents the numerical solution for the figure (3.9) in different periods

t	n_1	n_2	n_3
0	0.3	0.3	-0.6
0.287664502	0.285238309	0.299286053	-0.602944182
0.575329004	0.27146145	0.294867749	-0.606161101
3.259643972	0.224967223	0.173148884	-0.645835733
4.317983651	0.244616923	0.133447523	-0.663135755
5.360893201	0.280502481	0.109184071	-0.678923064
9.848768361	0.473482416	0.192573737	-0.707829044
10.66588542	0.444014867	0.246988435	-0.708363546
15.27653347	0.258999274	0.129456506	-0.762839309
20.72680637	0.546626963	0.101702544	-0.809758744
25.29746779	0.284499558	0.231200786	-0.811907213
30.06078707	0.387482002	0.042431803	-0.889824354
50.21024124	0.361116164	0.416151118	-0.870862545
60.3358426	0.636794149	0.006077863	-1.020384617
70.02583965	0.175984086	0.016054731	-0.992818161
80.21742678	0.68850501	0.000177046	-1.110791204
90.06110837	0.047360076	0.034074213	-0.917353511
95.00656643	0.106653253	8.80e-05	-1.134218715
96.31019491	0.12187614	1.63e-05	-1.193397169
98.66086757	0.139822581	6.71e-07	-1.302543496
98.9975832	0.141025553	4.95e-07	-1.318599981
99.7493958	0.142272789	1.64e-07	-1.355028217
100	0.142233219	1.16e-07	-1.367377486

Table (3.4): Numerical solution for 3-D system with (0.3,0.3,-0.6)

On the other hand, we determine the behaviour of the system when we change one of α parameters but others stay constant. We consider α_2 as a parameter of bifurcation and we illustrate the type or the form of bifurcation. We replace α_2 by $m \in [-8, 5]$ and explain each case with both initial conditions $(0.3, 0.3, -0.3)$, $(0.3, 0.3, -0.6)$.

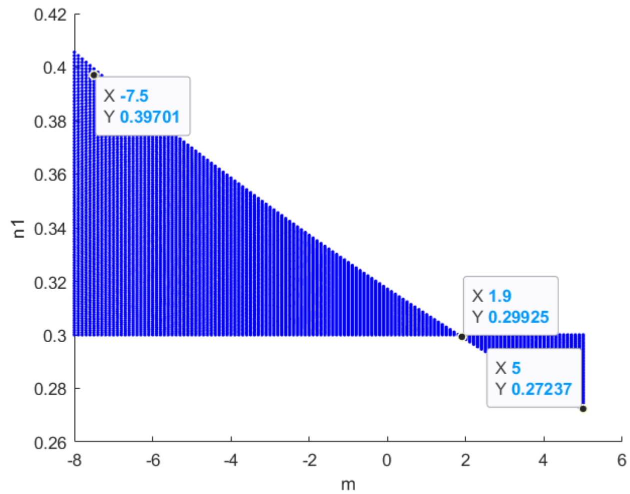


Figure (3.10): Bifurcation diagram of $n_1$and $m \in [-8, 5]$ with $(0.3, 0.3, -0.3)$ in 3-D system

Figure (3.10) explains the bifurcation happens at $m = 1.9$, and $n_1 = 0\ .29925$ it seems transcritical bifurcation. When $m < 1.9$, the population values are between the period $[0.3, 0.41]$, but when $m > 1.9$, the population values are between the period $[0.272, 0.3]$. However, there is an inverse relation between bifurcation parameter and number of population, since when the value of m is increasing the number of populatin is decreasing. The highest value of population is when m near to -8. That is, lower values of parameter (α_2) give more growth in population (n_2) values and more stable system.

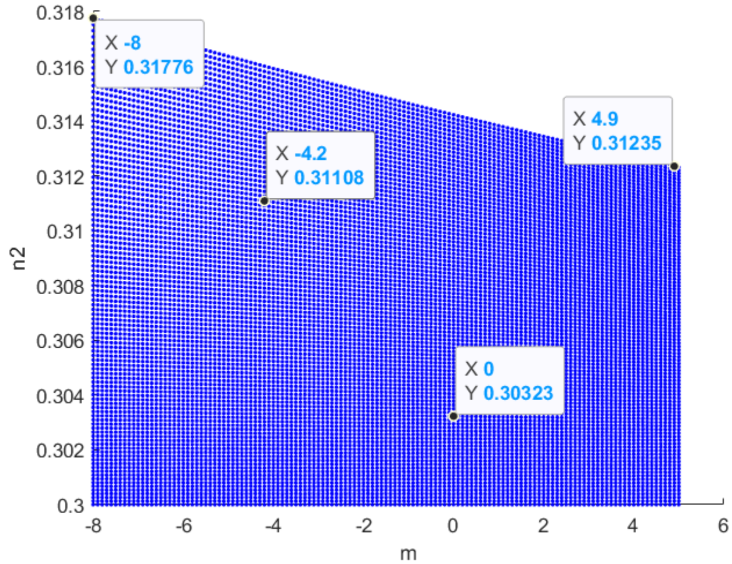


Figure (3.11): Bifurcation diagram of n_2 and $m \in [-8, 5]$ with $(0.3, 0.3, -0.3)$ in 3-D system

Figure (3.11) illustrartes the bifurcation relation between the number of jobs n_2 and the bifurcation parameter m (α_2) is saddle node bifurcation. When the value of m increases, the number of jobs decreases. The difference in the values of n_2 is very small unnoticed over time with large change in the values of m. Moreover, the solution exists during the values of jobs in interval $[0.3, 0.318]$.

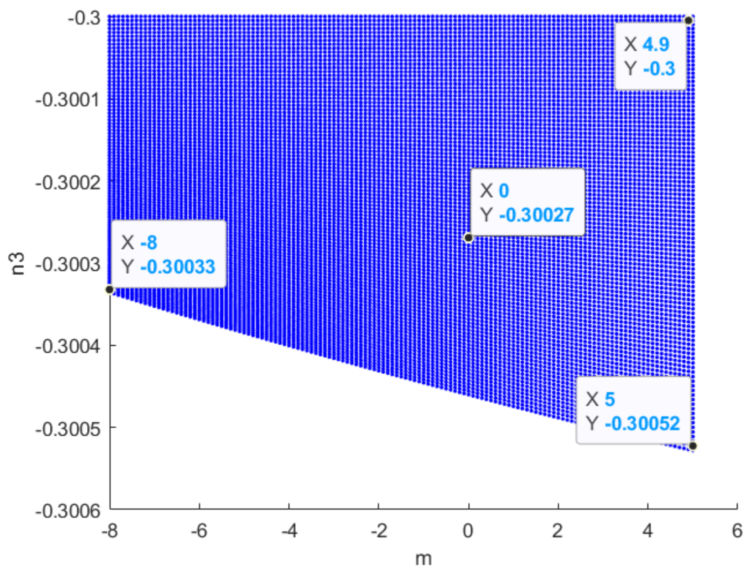


Figure (3.12): Bifurcation diagram of n_3 and $m \in [-8, 5]$ with $(0.3, 0.3, -0.3)$ in 3-D system

Figure (3.12) presents the bifurcation relation between energy n_3 and the bifurcation parameter m (α_2) is saddle node bifurcation. The values of n_3 are between $[-0.3005, -0.3]$ when $m \in [-8, 5]$, that is the system has a solution during these values. It is clear that, there is a positive relation between n_3 and m (α_2) with small variation in their values.

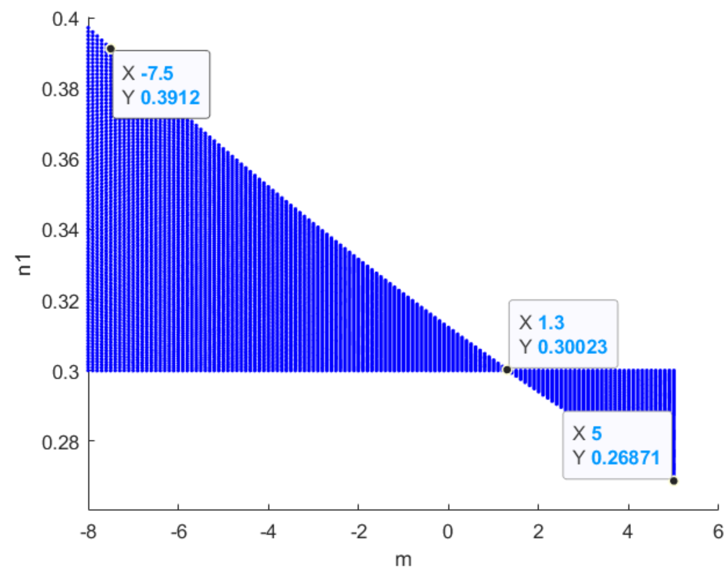


Figure (3.13): Bifurcation diagram of n_1 and m with $(0.3, 0.3, -0.6)$ in 3-D system.

Figure(3.13) presents the bifurcation happens at $m = 1.3$ and $n_1 = 0.30023$ it seems transcritical bifurcation. There is an inverse relation between bifurcation parameter and the number of population, since when the value of m is increasing the number of populatin is decreasing. However, the small different in intial conditions, causes change in bifurcation node like figures (3.10) and (3.13), and change in the relation between variables and behavior of the system as figures (3.3) and (3.9).

3.6 Newton Method and fifth fixed point(F_{p_5})

We use Newton Method in definition (2.2.15) to predict the best value of the three variables (n_1, n_2, n_3) which give growth and increase in all variables, that leads to a more stable system. In this part, we change the values of some parameters and define it on an interval and take different initial conditions to get required results. We use Matlab programme for applying Newton Method with error value $10\hat{}^{-6}$ for all parameters. We consider $\alpha_2 \in [-8, 5]$ with initial condition $(0.3, 0.3, -0.3)$, the best value we have $F_{p_5}(0.4817, 0.1396, -0.9600)$ at $\alpha_2 = 1.5$. We notice an increase in both values of n_1 and n_2, but decrease in n_3. Moreover, we change the initial condition to $(0.3, 0.3, -0.6)$ at $\alpha_2 = .3$ we get $F_{p_5}(0.3150, 1.1383, -0.70082)$ which means an increase in the first two variables but a small decrease in the third one, here we get a fabulous results. Since, this point is stable, that is the behavior of the system around this point is stable. The following figure illustrates this result

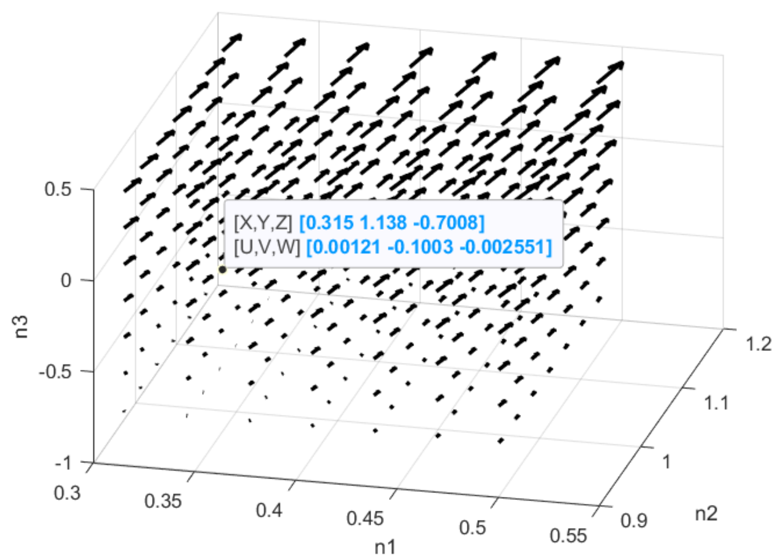


Figure (3.14): The behaviour of 3-D system around $F_{p_5}(0.3150, 1.1383, -0.70082)$ with $\alpha_2 = 0.3$

We explain the change in α_5 during the interval $[-8, 5]$ with initial condition $(0.3, 0.3, -0.3)$. We have the great fixed point $F_{p_5}(0.4358, 0.4904, 0.4264)$ at $\alpha_5 = -1.8$. Since, increase and growth happen in population, jobs and energy. On the other hand, when we change two parameters together (α_2, α_5) with intial condition $(0.3, 0.3, -0.3)$, we have this point $F_{p_5}(0.4547, 0.3457, -0.1727)$ at $\alpha_2 = 1.8$ and $\alpha_5 = -1$. We get increase in all variables. In addition, another values as $(0.8, -0.6)$ give the point $F_{p_5}(0.4159, 0.6425, -0.3791)$, that is highly increase in n_1 and n_2 but lowly decrease in n_3. Eventually, we concentrate on these parameters, because we can do any change of them in future to get growth and more stable system.

Chapter 4

Analysis of four dimensional dynamic system of socio-economic model

This chapter demonstrates the new mathematical model of socio-economic development in Palestine with adding the occupation variable to show the influnce of this variable on the socioeconomic life for palestine population. So we study and analyze the system by predicting the equilibrium points solution, illustrate the type of stability and find the bifurcation of this proposed system, and we are using the Matlab program to visualize the system. Finally, we use Newton method to find the best values of four variables which give more growth and stable for the new system.

4.1 The mathmatical model for the new system

In this chapter, we illustrate the new system which is constructed of four nonlinear differential equations with four main variables.

$n(t) = [n_1(t), n_2(t), n_3(t), n_4(t)]$ and 16-component of controlling vector $\alpha(t) = [\alpha_1, \alpha_2(t), ...\alpha_{15}(t), \alpha_{16}(t)]$. The model is represented by the following equations

$$
\begin{aligned}
\frac{dn_1}{dt} &= \alpha_1 n_1 - \alpha_2 n_1 n_2 + \alpha_3 n_1 n_3 + \alpha_4 n_1 n_4 \\
\frac{dn_2}{dt} &= \alpha_5 n_2 + \alpha_6 n_1 n_2 + \alpha_7 n_2 n_3 - \alpha_8 n_2 n_4 \\
\frac{dn_3}{dt} &= \alpha_9 n_3 - \alpha_{10} n_1 n_3 - \alpha_{11} n_2 n_3 - \alpha_{12} n_3 n_4 \\
\frac{dn_4}{dt} &= \alpha_{13} n_4 + \alpha_{14} n_1 n_4 - \alpha_{15} n_2 n_4 - \alpha_{16} n_3 n_4
\end{aligned} \tag{4.1}
$$

The above model is a system of fourth order nonlinear differential equations is describing the changes of four main indicators which are the regional population n_1, the number of jobs in real sector of the regional economy n_2, the regional energy supply n_3 and the occupation n_4. Whereas, $\alpha' s$ are the controlling parameters that determine the required values for main variables. However, we analyze our system based on local Palestinian social and economic reality, as state in Palestinian data and statistics [38]. We add the occupation variable for the equations (3.2),(3.3),(3.4) and illustrate the interaction between the three variables (n_1, n_2, n_3) with the new one (n_4). Moreover, we add a new equation which is described the change of occupation during time and present the effect of the three variables on occupation.

Now, we illustrate the first equation of the system (4.1) and explain the interaction between environments as follows

$$
\frac{dn_1}{dt} = \alpha_1 n_1 - \alpha_2 n_1 n_2 + \alpha_3 n_1 n_3 + \alpha_4 n_1 n_4 \tag{4.2}
$$

In this equation, we add to equation (3.2) the occupation variable n_4 and show the effect of this variable on the rate of change in population during time. The interaction between population and occupation explains the proportional relatioship between them. Hence, when the occupation is increased the number of population is increased too by α_4 which is the coefficient of people's motivation to childbearing. Moreover, there are many causes for this increase. First, to resist the occupier as well as to compensate the shortfall in the number of people's who pass away because of the war and this is very clear in the Gaza Strip. Second, to recompense the absence due to the imprisonment [54].

Now, we discuss the second equation of the system (4.1) and present the effect of the four variables on the rate of change of the economic development.

$$\frac{dn_2}{dt} = \alpha_5 n_2 + \alpha_6 n_1 n_2 + \alpha_7 n_2 n_3 - \alpha_8 n_2 n_4 \tag{4.3}$$

In this equation, we add to equation(3.3) n_4 and present the effect of this variable on the rate of change of the economic development. Hence, the interaction occur between occupation and the number of jobs. There is an inverse relation between them, since when the occupation increases the number of jobs decreases by α_8 which is the coefficient of occupation per internal workplace. However, the occupation costs Palestinian economy heavy losses, such as it prevents exports on abroad. In addition, demolish the economic infrastructure of many economic institutions, establishments, places of job, and employment opportunities for many labors. Furthermore, the occupation imposes heavy taxes on imports which aren't except through them, thus the debts accumulate on the Palestinian National Authority [55] , [56].

Now, we demonstrate the third equation of the system (4.1) and present the effect of the four variables on the rate of change in the region's energy supply.

$$\frac{dn_3}{dt} = \alpha_9 n_3 - \alpha_{10} n_1 n_3 - \alpha_{11} n_2 n_3 - \alpha_{12} n_3 n_4 \tag{4.4}$$

In this equation, we add to equation (3.4) n_4 and explain the effect of this variable on the rate of change in the region's energy supply. Hence, the interaction happen between occupation and energy. There is an inverse relation between both of them. That is, when the occupation is increased the energy supply is decreased by α_{12} which is the conformity ratio of the occupation with the energy supply. Though, the occupation imposes its control on the natural resources, natural gas, oil and petroleum wells in Palestine by occupying the lands in 1967 and prevents the Palestinians to access own resources. Moreover, the occupation affects on electricity sector and the percentage of electricity that reaches to the Palestinian lands. In addition, palestinians are importing and buying electricity through the occupation.

The debts and taxes are accumulated on Palestinian as a result of occupation practices. In addition to theft and loot of natural resources that help in the production of energy, such as water and the vast desert areas that can be exploit by creating solar cell projects. Unfortunatly, these circumstances give negative effect on the energy sector [57].

Now, we discuss the fourth equation of the system (4.1) and explain the effect of the four variables on the rate of change of occupation.

$$\frac{dn_4}{dt} = \alpha_{13}n_4 + \alpha_{14}n_1n_4 - \alpha_{15}n_2n_4 - \alpha_{16}n_3n_4 \tag{4.5}$$

In this equation, we present the rate of change of occupation during time $\frac{dn_4}{dt}$ and the factors that affect on it. However, there is a proportional relation between occupation (n_4) itself and rate of change of occupation $\frac{dn_4}{dt}$. Naturally, when the number of population increases the rate of change of occupation increases by α_{13}which is the occupation demographic activity coefficient. Meanwhile, at the beginning of occupation on Palestine, the number of Jewish population are increase, either with the increasing immigration to Palestine or with the increase in the number of births. As such, the Jewish number becomes more than the number of Palestinians. Consequently, an increase in the construction of settlements major at the expense of the Palestinian lands, and the confiscation of their lands [58]. The second term of equation, illustrates proportional relationship between number of population in palestine (n_1) and occupation (n_4). When the number of population increases, the occupation increases by α_{14} which is the coefficient of forced displacement of residents from Palestinian lands. At the beginning of occupation, large number of Palestinians are displaced forcibly from their lands, and there are confiscated for the benefit of settlements and the Jewish population in Palestine. On the other hand, the increase in the number of Palestinians is limited for many reasons such as imigration and travel to obtain best alternatives for the standard of living, and to get rid of the distress imposed on them, which give the opportunity for Jews to increase [59], [60]. In the third term of equation, we notice an inverse relation between the number of jobs (n_2) and occupation by α_{15} which is the coefficient of relationship between palestinian goverment, private sector and civil society. However, when the partnership between the private and government sector increase, economic development and the rate of employment increases too. On the other side, support the government by the private sector is reducing the debts and taxes. That is, reduction in the impact of occupation and its extortion over time [61]. In the fourth term of equation, the interaction happen between energy (n_3) and occupation. Clearly, there is an inverse relation since when the use of renewable energy sources increases the effect of occupation decreases by α_{16} which is the coefficient of creating new energy alternatives for palestinians. When the use of renewable energy alternatives increases that activate and exploit more projects, the available manpower in this field lead to negative impact on the occupation. Eventualy, Palestinian needs will fill and reduce the depending on occupation resourses [62]. In side of the factors that we indicated in chapter three, we have additional factors in this system as the following.

α_4: is the coefficient of people's motivation to childbearing.

α_8: is the coefficient of occupation per internal workplace.

α_{12}: is the conformity ratio of the occupation with energy supply.

α_{13}: is the occupation demographic activity coefficient.

α_{14}: is the coefficient of forced displacement of residents from palestinian lands.

α_{15}: is the coefficient of relationship between Palestinian government, private sector and civil society.

α_{16}: is the coefficient of creating new energy alternatives for Palestinians.

Before we are going to study the fixed points of the system we want to check if the system 4.1 has a solution by 2.2.10.

4.2 Existence and Uniqueness for the system

Consider the system 4.1 with initial condition $n(0) = n_0$. Suppose f_1, f_2, f_3, f_4 are continous and differentiable with respect to n_i. In other words, the functions are smooth enough to ensure the existence and uniqeness of solutions starting from any point in phase space. The partial derivatives of the function are as follows.

$$\frac{df_1}{dn_1} = \alpha_1 - \alpha_2 n_2 + \alpha_3 n_3 + \alpha_4 n_4$$

$$\frac{df_1}{dn_2} = n_1(\alpha_1 - \alpha_2 + \alpha_3 n_3 + \alpha_4 n_4)$$

$$\frac{df_1}{dn_3} = n_1(\alpha_1 - \alpha_2 n_2 + \alpha_3 + \alpha_4 n_4)$$

$$\frac{df_1}{dn_4} = n_1(\alpha_1 - \alpha_2 n_2 + \alpha_3 n_3 + \alpha_4)$$

$$\frac{df_2}{dn_1} = n_2(\alpha_5 + \alpha_6 + \alpha_7 n_3 - \alpha_8 n_4)$$

$$\frac{df_2}{dn_2} = \alpha_5 + \alpha_6 n_1 + \alpha_7 n_3 - \alpha_8 n_4$$

$$\frac{df_2}{dn_3} = n_2(\alpha_5 + \alpha_6 n_1 + \alpha_7 - \alpha_8 n_4)$$

$$\frac{df_2}{dn_4} = n_2(\alpha_5 + \alpha_6 n_1 + \alpha_7 n_3 - \alpha_8)$$

$$\frac{df_3}{dn_1} = n_3(\alpha_9 - \alpha_{10} - \alpha_{11} n_2 - \alpha_{12} n_4)$$

$$\frac{df_3}{dn_2} = n_3(\alpha_9 - \alpha_{10} n_1 - \alpha_{11} - \alpha_{12} n_4)$$

$$\frac{df_3}{dn_3} = \alpha_9 - \alpha_{10}n_1 - \alpha_{11}n_2 - \alpha_{12}n_4$$

$$\frac{df_3}{dn_4} = n_3(\alpha_9 - \alpha_{10}n_1 - \alpha_{11}n_2 - \alpha_{12})$$

$$\frac{df_4}{dn_1} = n_4(\alpha_{13} + \alpha_{14} - \alpha_{15}n_2 - \alpha_{16}n_3)$$

$$\frac{df_4}{dn_2} = n_4(\alpha_{13} + \alpha_{14}n_1 - \alpha_{15} - \alpha_{16}n_3)$$

$$\frac{df_4}{dn_3} = n_4(\alpha_{13} + \alpha_{14}n_1 - \alpha_{15}n_2 - \alpha_{16})$$

$$\frac{df_4}{dn_4} = \alpha_{13} + \alpha_{14}n_1 - \alpha_{15}n_2 - \alpha_{16}n_3$$

hence, all above partial derivatives are continous for n_i , $i = 1, .., 4$, in some open connected set $D \subset \mathbb{R}^4$, then for $n_0 \in D$ the system 4.1 with initial condition $n(0) = n_0$ which is an I.V.P has a solution $n(t)$ on some time interval around $t = 0$, and the solution is uniqe.

4.3 Fixed points for the new system(equilibrium points)

In this section, we find the fixed points of the new system in general case. After that, we subtitute the value of parameters in equations and get the value of fixed points numerically. However, we calculate the value of parameters by using precentage rate (growth rate of values) based on the informations and statistics obtained from Palestinian Central Bureau of Statistics (PCBS). In additin, we donot depend on one year for all parameters in calculations, we use different base and comparsion years for parameters according to the availability of informations for these years in

PCBS [38]. Moreover, if the informations are avaliable there is a big difference in the results, this is because the instability case that Palestinian territories are suffered from and many factors that affect on them. The proposed system is continous dynamic system, to find the general equilibrium points for the system we put ($\acute{n}_1 = 0, \acute{n}_2 = 0, \acute{n}_3 = 0, \acute{n}_4 = 0$). As a result we have the following equations:

$$\alpha_1 n_1 - \alpha_2 n_1 n_2 + \alpha_3 n_1 n_3 + \alpha_4 n_1 n_4 = 0 \tag{4.6}$$

$$\alpha_5 n_2 + \alpha_6 n_1 n_2 + \alpha_7 n_2 n_3 - \alpha_8 n_2 n_4 = 0 \tag{4.7}$$

$$\alpha_9 n_3 - \alpha_{10} n_1 n_3 - \alpha_{11} n_2 n_3 - \alpha_{12} n_3 n_4 = 0 \tag{4.8}$$

$$\alpha_{13} n_4 + \alpha_{14} n_1 n_4 - \alpha_{15} n_2 n_4 - \alpha_{16} n_3 n_4 = 0 \tag{4.9}$$

4.3.1 General fixed points and stability for the system

To find the fixed points for equation (4.6), we have

$$\alpha_1 n_1^* - \alpha_2 n_1^* n_2^* + \alpha_3 n_1^* n_3^* + \alpha_4 n_1^* n_4^* = 0,$$

hence,

$$n_1^*(\alpha_1 - \alpha_2 n_2^* + \alpha_3 n_3^* + \alpha_4 n_4^*) = 0$$

So $n_1^* = 0$, or

$$\alpha_1 - \alpha_2 n_2^* + \alpha_3 n_3^* + \alpha_4 n_4^* = 0.$$

This implies,

$$n_2^* = \frac{\alpha_1 + \alpha_3 n_3^* + \alpha_4 n_4^*}{\alpha_2}. \tag{4.10}$$

From equation (4.7)

$$\alpha_5 n_2^* + \alpha_6 n_1^* n_2^* + \alpha_7 n_2^* n_3^* - \alpha_8 n_2^* n_4^* = 0,$$

So,

$$n_2^*(\alpha_5 + \alpha_6 n_1^* + \alpha_7 n_3^* - \alpha_8 n_4^*) = 0.$$

Then $n_2^* = 0$, or

$$\alpha_5 + \alpha_6 n_1^* + \alpha_7 n_3^* - \alpha_8 n_4^* = 0.$$

So, we get

$$n_1^* = \frac{-\alpha_5 - \alpha_7 n_3^* + \alpha_8 n_4^*}{\alpha_6}. \tag{4.11}$$

From equation (4.8)

$$\alpha_9 n_3^* - \alpha_{10} n_1^* n_3^* - \alpha_{11} n_2^* n_3^* - \alpha_{12} n_3^* n_4^* = 0,$$

hence,

$$n_3^*(\alpha_9 - \alpha_{10} n_1^* - \alpha_{11} n_2^* - \alpha_{12} n_4^*) = 0.$$

So $n_3^* = 0$, or

$$n_4^* = \frac{\alpha_9 - \alpha_{10} n_1^* - \alpha_{11} n_2^*}{\alpha_{12}}. \tag{4.12}$$

Finally from equation (4.9), we have

$$\alpha_{13} n_4^* + \alpha_{14} n_1^* n_4^* - \alpha_{15} n_2^* n_4^* - \alpha_{16} n_3^* n_4^* = 0,$$

So,

$$n_4^*(\alpha_{13} - \alpha_{14} n_1^* - \alpha_{15} n_2^* - \alpha_{16} n_4^*) = 0.$$

Then $n_4^* = 0$, or

$$n_3^* = \frac{\alpha_{13} - \alpha_{14} n_1^* - \alpha_{15} n_2^*}{\alpha_{16}}. \tag{4.13}$$

Subtitute equations (4.10) and (4.11) in equations (4.12) and (4.13), we have

$$\alpha_2\alpha_9\alpha_6 - \alpha_{11}\alpha_1 + \alpha_{10}\alpha_5 + (\alpha_{10}\alpha_7 - \alpha_{11}\alpha_3)n_3^* - (\alpha_{10}\alpha_8 + \alpha_{11}\alpha_4 + \alpha_2\alpha_6\alpha_{12})n_4^* = 0 \tag{4.14}$$

This equation takes the form

$$\alpha_2\alpha_6\alpha_{13} - \alpha_5\alpha_4 - \alpha_1\alpha_5 + (\alpha_{14}\alpha_8 - \alpha_4\alpha_{15})n_4^* - (\alpha_7\alpha_{14} + \alpha_3\alpha_{15} + \alpha_2\alpha_6\alpha_{16})n_3^* = 0 \tag{4.15}$$

solve equations (4.14) and (4.15) by elimination and substitution we get these results:

$$n_3^* = -\frac{\begin{array}{c}(\alpha_{14}\alpha_8 - \alpha_4\alpha_{15})(\alpha_2\alpha_9\alpha_6 - \alpha_{11}\alpha_1 + \alpha_{10}\alpha_5)+ \\ (\alpha_{10}\alpha_8 + \alpha_{11}\alpha_4 + \alpha_2\alpha_6\alpha_{12}) \\ (\alpha_2\alpha_6\alpha_{13} - \alpha_5\alpha_4 - \alpha_1\alpha_5)\end{array}}{\begin{array}{c}(\alpha_{10}\alpha_7 - \alpha_{11}\alpha_3)(\alpha_{14}\alpha_8 - \alpha_4\alpha_{15}) - (\alpha_{10}\alpha_8 + \alpha_{11}\alpha_4 + \alpha_2\alpha_6\alpha_{12}) \\ (\alpha_7\alpha_{14} + \alpha_3\alpha_{15} + \alpha_2\alpha_6\alpha_{16})\end{array}}.$$

Let, $\eta = \frac{(\alpha_{14}\alpha_8-\alpha_4\alpha_{15})(\alpha_2\alpha_9\alpha_6-\alpha_{11}\alpha_1+\alpha_{10}\alpha_5)+(\alpha_{10}\alpha_8+\alpha_{11}\alpha_4+\alpha_2\alpha_6\alpha_{12})(\alpha_2\alpha_6\alpha_{13}-\alpha_5\alpha_4-\alpha_1\alpha_5)}{(\alpha_{10}\alpha_7-\alpha_{11}\alpha_3)(\alpha_{14}\alpha_8-\alpha_4\alpha_{15})-(\alpha_{10}\alpha_8+\alpha_{11}\alpha_4+\alpha_2\alpha_6\alpha_{12})(\alpha_7\alpha_{14}+\alpha_3\alpha_{15}+\alpha_2\alpha_6\alpha_{16})}$.

So,

$$n_3^* = -\eta.$$

The other result we have,

$$n_4^* = -\frac{-\alpha_2\alpha_9\alpha_6 + \alpha_{11}\alpha_1 - \alpha_{10}\alpha_5 - (\alpha_{10}\alpha_7 - \alpha_{11}\alpha_3)\eta}{(\alpha_{10}\alpha_8 + \alpha_{11}\alpha_4 + \alpha_2\alpha_6\alpha_{12})}.$$

Let $\beta = \frac{-\alpha_2\alpha_9\alpha_6+\alpha_{11}\alpha_1-\alpha_{10}\alpha_5-(\alpha_{10}\alpha_7-\alpha_{11}\alpha_3)\eta}{(\alpha_{10}\alpha_8+\alpha_{11}\alpha_4+\alpha_2\alpha_6\alpha_{12})}$.

So,

$$n_4^* = -\beta.$$

As a result, we have these fixed points $F_{p_1} = (0, 0, 0, 0)$,

$F_{p_2} = (\frac{-\alpha_5+\alpha_7\eta-\alpha_8\beta}{\alpha_6}, \frac{\alpha_{1-}\alpha_3\eta-\alpha_4\beta}{\alpha_2}, -\eta, -\beta)$, F_{p_2} is a solution if these conditions are satisfied

(1) $-(\alpha_{10}\alpha_7 - \alpha_{11}\alpha_3)(\alpha_{14}\alpha_8 - \alpha_4\alpha_{15}) \neq (\alpha_{10}\alpha_8 + \alpha_{11}\alpha_4 + \alpha_2\alpha_6\alpha_{12})(\alpha_7\alpha_{14} + \alpha_3\alpha_{15} + \alpha_2\alpha_6\alpha_{16})$

(2) $-(\alpha_{10}\alpha_8 + \alpha_{11}\alpha_4 + \alpha_2\alpha_6\alpha_{12}) \neq 0$

(3) $-\alpha_2 \neq 0$ and $\alpha_6 \neq 0$.

Clearly, F_{p_1} and F_{p_2} are the two general fixed points for the system (4.1). In addition, we have ten general cases emanate from the general point F_{p_2} , thus we have twelve equilibrium points in general situation. However, we analyze the stability for the system at each fixed point by finding the general jacobin matrix for the system 4.1, then we explain the eigenvalues of jacobian matrix to determine the case of stability.

The general jacobin matrix at any fixed point $F_p(n_1^*, n_2^*, n_3^*, n_4^*)$

$$J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} \begin{matrix}\alpha_1 - \alpha_2 n_2^* + \\ \alpha_3 n_3^* + \alpha_4 n_4^*\end{matrix} & -\alpha_2 n_1^* & \alpha_3 n_1^* & \alpha_4 n_1^* \\ \alpha_6 n_2^* & \begin{matrix}\alpha_5 + \alpha_6 n_1^* + \\ \alpha_7 n_3^* - \alpha_8 n_4^*\end{matrix} & \alpha_7 n_2^* & -\alpha_8 n_2^* \\ -\alpha_{10} n_3^* & -\alpha_{11} n_3^* & \begin{matrix}\alpha_9 - \alpha_{10} n_1^* - \\ \alpha_{11} n_2^* - \alpha_{12} n_4^*\end{matrix} & -\alpha_{12} n_3^* \\ \alpha_{14} n_4^* & -\alpha_{15} n_4^* & -\alpha_{16} n_4^* & \begin{matrix}\alpha_{13} + \alpha_{14} n_1^* - \\ \alpha_{15} n_2^* - \alpha_{16} n_3^*\end{matrix} \end{bmatrix}$$

Now, for the fixed point $F_{p_1} = (0, 0, 0, 0)$, we have

$$\det J(0,0,0,0) = \begin{vmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_5 & 0 & 0 \\ 0 & 0 & \alpha_9 & 0 \\ 0 & 0 & 0 & \alpha_{13} \end{vmatrix} \tag{4.16}$$

Hence (4.16) is diagonal matrix, the eigenvalues of this matrix are $\lambda_1 = \alpha_1, \lambda_2 = \alpha_5, \lambda_3 = \alpha_9, \lambda_4 = \alpha_{13}$. If these eigenvalues are real and positive, the point is unstable. In addition, if all eigenvalues are real and negative the point is stable. But, if at least one of these eigenvalues has different sign the point is unstable saddle node.

Now, we illustrate the analysis of general fixed point $F_{p_2} = (\frac{-\alpha_5+\alpha_7\eta-\alpha_8\beta}{\alpha_6}, \frac{\alpha_1-\alpha_3\eta-\alpha_4\beta}{\alpha_2}, -\eta, -\beta)$, where $n_1^*, n_2^*, n_3^*, n_4^* \neq 0$.

The jacobin matrix of F_{p_2} is

$$J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} 0 & \frac{\alpha_2(\alpha_5-\alpha_7\eta+\alpha_8\beta)}{\alpha_6} & \frac{-\alpha_3(\alpha_5+\alpha_7\eta+\alpha_8\beta)}{\alpha_6} & \frac{-\alpha_4(\alpha_5-\alpha_7\eta+\alpha_8\beta)}{\alpha_6} \\ \frac{\alpha_6(\alpha_1-\alpha_3\eta-\alpha_4\beta)}{\alpha_2} & 0 & \frac{\alpha_7(-\alpha_1-\alpha_3\eta-\alpha_4\beta)}{\alpha_2} & \frac{\alpha_8(-\alpha_1+\alpha_3\eta+\alpha_4\beta)}{\alpha_2} \\ \alpha_{10}\eta & \alpha_{11}\eta & \frac{\zeta+\theta\eta+\vartheta\beta}{\alpha_2\alpha_6} & \alpha_{12}\eta \\ -\alpha_{14}\beta & \alpha_{15}\beta & \alpha_{16}\beta & \frac{\iota+\tau\eta+\kappa\beta}{\alpha_2\alpha_6} \end{bmatrix} \tag{4.17}$$

Let $\theta = \alpha_{11}\alpha_3 - \alpha_{10}\alpha_7$, $\vartheta = \alpha_2\alpha_6\alpha_{12} + \alpha_8\alpha_{10} + \alpha_{11}\alpha_4$, $\iota = \alpha_2\alpha_6\alpha_{13} - \alpha_5\alpha_{14} - \alpha_1\alpha_{15}$, $\tau = \alpha_2\alpha_6\alpha_{16} + \alpha_7\alpha_{14} + \alpha_{15}\alpha_3$ and $\kappa = \alpha_4\alpha_{15} - \alpha_8\alpha_{14}$

To demonstrate the stability of this fixed point, we need to find the eigenvalues by using charactrestic polynomial equation for jacobin matrix as the following. The charactrestic polynomial equation for $4D$ system is

$\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda + \det(J) = 0$. Whereas, $A_2 = J_{1212} + J_{1313} + J_{1414} + J_{2323} + J_{2424} + J_{3434}$. Hence J_{ijkl} is $2*2$ principal minor matrix. i, j are rows $\in [1, 4]$, and k, l are columns $\in [1, 4]$, with condition $i = k$, $j = l$. $A_3 = J_{11} + J_{22} + J_{33} + J_{44}$. So we have

$$
\begin{aligned}
tr(J) &= \frac{\zeta+\iota+(\theta+\tau)\eta+(\vartheta+\kappa)\beta}{\alpha_2\alpha_6}, \\
J_{1212} &= \frac{[((\zeta+\theta)\eta+\vartheta\beta)((\iota+\tau)\eta+\kappa\beta)]-\alpha_{16}\alpha_{12}\eta\beta}{\alpha_2^2\alpha_6^2}, \\
J_{1313} &= \frac{\alpha_{15}\alpha_8(\alpha_1\beta-\alpha_3\eta\beta-\alpha_4\beta^2)}{\alpha_2}, \\
J_{1414} &= \frac{\alpha_7\alpha_{11}(-\alpha_1\eta+\alpha_4\eta\beta+\alpha_3\eta^2)}{\alpha_2}, \\
J_{2323} &= \frac{\alpha_4\alpha_{14}(-\alpha_5\beta+\alpha_7\eta\beta-\alpha_8\beta^2)}{\alpha_6}, \\
J_{2424} &= \frac{\alpha_{10}\alpha_3(\alpha_5\eta+\alpha_8\eta\beta+\alpha_7\eta^2)}{\alpha_6}, \\
J_{3434} &= -(\alpha_1-\alpha_3\eta+\alpha_4\beta)(\alpha_5-\alpha_7\eta+\alpha_8\beta), \\
J_{11} &= -\frac{\begin{array}{c}\alpha_7(\alpha_1-\alpha_3\eta-\alpha_4\beta)[\alpha_{11}(\iota\eta+\tau\eta^2)+(\alpha_{11}\kappa-\alpha_2\alpha_6\alpha_{16}\alpha_{12})\eta\beta] \\ +\alpha_8(-\alpha_1+\alpha_3\eta+\alpha_4\beta)[\alpha_{15}(\zeta\beta-\alpha_{15}\vartheta\beta^2)+(\alpha_{15}\theta+\alpha_{11}\alpha_{16})\eta\beta]\end{array}}{\alpha_2^2\alpha_6}, \\
J_{22} &= \frac{\begin{array}{c}\alpha_3(-\alpha_5-\alpha_7\eta-\alpha_8\beta)[\alpha_{10}(\iota\eta+\tau\eta^2)+(\alpha_{10}\kappa+\alpha_{14}\alpha_{12}\alpha_2\alpha_6)\eta\beta]+ \\ \alpha_4(-\alpha_5+\alpha_7\eta-\alpha_8\beta)[\alpha_{14}(\zeta\beta+\vartheta\beta^2)+(\alpha_{14}\theta+\alpha_{10}\alpha_{16}\alpha_2\alpha_6)\eta\beta]\end{array}}{\alpha_2\alpha_6^2}, \\
J_{33} &= [\frac{\begin{array}{c}-(\alpha_5\alpha_{12}-\alpha_7\alpha_2\eta+\alpha_8\alpha_2\beta)[\alpha_6(\alpha_1-\alpha_3\eta-\alpha_4\beta) \\ (\iota+\tau\eta+\kappa\beta)+\alpha_2\alpha_6\alpha_{14}\alpha_8(-\alpha_1\beta+\alpha_3\eta\beta+\alpha_4\beta^2)]\end{array}}{\alpha_2^2\alpha_6^2}] \\
&\quad -\frac{\alpha_4(-\alpha_5+\alpha_7\eta-\alpha_8\beta)\alpha_{15}\alpha_6(\alpha_1\beta-\alpha_3\eta\beta-\alpha_4\beta^2)}{\alpha_2\alpha_6}, \\
J_{44} &= [\frac{\begin{array}{c}-\alpha_2(\alpha_5-\alpha_7\eta-\alpha_8\beta)[\alpha_6(\alpha_1-\alpha_3\eta-\alpha_4\beta) \\ (\zeta+\theta\eta+\vartheta\beta)-\alpha_2\alpha_6\alpha_{10}\alpha_7(\alpha_1\beta-\alpha_3\eta\beta+\alpha_4\beta^2)]\end{array}}{\alpha_6^2\alpha_2^2}] \\
&\quad +\frac{\alpha_3(-\alpha_5-\alpha_7\eta-\alpha_8\beta)\alpha_6\alpha_{11}(\alpha_1\eta-\alpha_4\eta\beta-\alpha_3\eta^2)}{\alpha_2\alpha_6}, \\
\det(J) &= -\frac{\alpha_2(\alpha_5-\alpha_7\eta+\alpha_8\beta)}{\alpha_6}J_{12}-\frac{\alpha_3(-\alpha_5-\alpha_7\eta-\alpha_8\beta)}{\alpha_6}J_{13}+ \\
&\quad \frac{\alpha_4(-\alpha_5+\alpha_7\eta-\alpha_8\beta)}{\alpha_6}J_{14}.
\end{aligned}
$$

The system have a charactrestic equation of order four, we are going to make numerical analysis for it to explain the stability of the system at general fixed point

F_{p_2} by finding the eigenvalues of (4.17). As such, if all eigenvalues are real and positive the equilibrium point is unstable. Furthermore, if all eigenvalues are real and negative the fixed point is stable, but if at least one of eigenvalues has different sign then the point is saddle and unstable. Moreover, if one of eigenvalues is zero the other eigenvalues determine the case of stability subject to the eigenvalues rules in three dimensional nonlinear systems as are informed in table (2.2), but if two of eigenvalues are zeros the other eigenvalues determine the case of stability subject to eigenvalues rules in two dimensional nonlinear systems as are informed in table (2.1). On the other hand, if eigenvalues are complex here we have two cases. The first one if two of eigenvalues are real and the others are complex. The second one if four of eigenvalues are complex. In the first case, if real eigenvalues are positive and the real part of complex eigenvalues positive too the point is unstable, but if the real eigenvalues are negative and the real part of complex eiegenvalues negative too the point is stable, finally if the signs of real eigenvalues differ from the sign of real part of complex eigenvalues the point is unstable. In the second case, the real part of eigenvalues determine the case of stability since if all are positive then the point is unstable, and if all are negative then the point is stable, but if the real parts have different signs the point is unstable too. However, these rules of stability are applied on the ten cases that are emanated from the general fixed point F_{p_2}. Now, we present the ten cases as the following:

Case (1): when $n_1^* = 0$, but $n_2^*, n_3^*, n_4^* \neq 0$. The fixed point is $F_{p_3}(0, \frac{\alpha_1 - \alpha_3\eta - \alpha_4\beta}{\alpha_2}, -\eta, -\beta)$, in this case $\eta = \frac{\alpha_5 + \alpha_8\beta}{\alpha_7}$ or $\beta = \frac{\alpha_7\eta - \alpha_5}{\alpha_8}$.This point is a solution for the system if $\alpha_2 \neq 0$ and $\alpha_7 \neq 0$ or $\alpha_8 \neq 0$. The jacobin matrix at this point $F_{p_3}(0, \frac{\alpha_1 - \alpha_3\eta - \alpha_4\beta}{\alpha_2}, -\eta, -\beta)$

$$J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\alpha_6(\alpha_1 - \alpha_3\eta - \alpha_4\beta)}{\alpha_2} & \alpha_5 - \alpha_7\eta - \alpha_8\beta & \frac{\alpha_7(\alpha_1 - \alpha_3\eta - \alpha_4\beta)}{\alpha_2} & \frac{\alpha_8(-\alpha_1 + \alpha_3\eta + \alpha_4\beta)}{\alpha_2} \\ \alpha_{10}\eta & \alpha_{11}\eta & \frac{\omega + \alpha_{11}\alpha_3\eta + \nu\beta}{\alpha_2} & \alpha_{12}\eta \\ -\alpha_{14}\beta & \alpha_{15}\beta & \alpha_{16}\beta & \frac{\varphi + \xi\eta + \alpha_4\alpha_{15}\beta}{\alpha_2} \end{bmatrix}$$

Let $\nu = \alpha_2\alpha_{12} + \alpha_{11}\alpha_4$, $\varphi = \alpha_2\alpha_{13} - \alpha_1\alpha_{15}$ and $\xi = \alpha_2\alpha_{16} + \alpha_{15}\alpha_3$

The charactrestic polynomial equation is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda + \det(J) = 0$, so we have

$$\begin{aligned}
tr(J) &= \frac{(\alpha_2\alpha_5 + \omega + \varphi) + (\alpha_{11}\alpha_3 - \alpha_7\alpha_2 + \chi)\eta + (\nu + \alpha_4\alpha_{15} - \alpha_2\alpha_8)\beta}{\alpha_2}, \\
J_{1212} &= \frac{(\omega + \alpha_{11}\alpha_3\eta + \kappa\beta)(\varphi + \chi\eta + \alpha_4\alpha_{15}\beta) - \alpha_2^2\alpha_{16}\alpha_{12}\eta\beta}{\alpha_2^2}, \\
J_{1313} &= \frac{(\alpha_5 - \alpha_7\eta - \alpha_8\beta)(\varphi + \chi\eta + \alpha_4\alpha_{15}\beta) - \alpha_8\alpha_{15}(-\alpha_1\beta + \alpha_3\eta\beta + \alpha_4\beta^2)}{\alpha_2}, \\
J_{1414} &= \frac{(\alpha_5 - \alpha_7\eta - \alpha_8\beta)(\omega + \nu\beta + \alpha_3\alpha_{11}\eta) - \alpha_7\alpha_{11}(\alpha_1\eta - \alpha_4\eta\beta - \alpha_3\eta^2)}{\alpha_2}, \\
J_{2323} &= 0, \\
J_{2424} &= 0, \\
J_{3434} &= 0, \\
J_{11} &= \frac{\begin{array}{c}(\alpha_5 - \alpha_7\eta - \alpha_8\beta)[(\omega + \nu\beta + \alpha_3\alpha_{11}\eta)(\varphi + \chi\eta + \alpha_4\alpha_{15}\beta) \\ -\alpha_2^2\alpha_{12}\alpha_{16}\eta\beta] - \alpha_7(\alpha_1 - \alpha_3\eta - \alpha_4\beta)[\alpha_{11}\varphi\eta + \alpha_{11}\chi\eta_2^2 + \alpha_{15}\nu\eta\beta] \\ +\alpha_8(-\alpha_1 + \alpha_3\eta + \alpha_4\beta)[\alpha_{11}\xi\eta\beta - \alpha_{15}\omega\beta - \alpha_{15}\nu\beta^2]\end{array}}{\alpha_2^2}, \\
J_{22} &= 0, \\
J_{33} &= 0, \\
J_{44} &= 0, \\
\det(J) &= 0.
\end{aligned}$$

Hence, the charactrestic equation is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda = 0$, it's clearly one of eigenvalues is zero, the other eigenvalues determine the case of stability subject to the eigenvalues rules in three dimensional nonlinear systems as are informed in table (2.2).

Case (2): when $n_2^* = 0$, but $n_1^*, n_3^*, n_4^* \neq 0$. The fixed point is $F_{p_4} = (\frac{-\alpha_5+\alpha_7\eta-\alpha_8\beta}{\alpha_6}, 0, -\eta, -\beta)$, in this case $\eta = \frac{\alpha_1-\alpha_4\beta}{\alpha_3}$ or $\beta = \frac{\alpha_1-\alpha_3\eta}{\alpha_4}$. This point is a solution for the system if $\alpha_6 \neq 0$ and $\alpha_3 \neq 0$ or $\alpha_4 \neq 0$. The jacobin matrix at this point $F_{p_4}(\frac{-\alpha_5+\alpha_7\eta-\alpha_8\beta}{\alpha_6}, 0, -\eta, -\beta)$ is

$$J(n_1^*,n_2^*,n_3^*,n_4^*)=\begin{bmatrix} \alpha_1-\alpha_3\eta-\alpha_4\beta & \frac{\alpha_2(\alpha_5-\alpha_7\eta+\alpha_8\beta)}{\alpha_6} & \frac{\alpha_3(-\alpha_5+\alpha_7\eta-\alpha_8\beta)}{\alpha_6} & \frac{\alpha_4(-\alpha_5+\alpha_7\eta-\alpha_8\beta)}{\alpha_6} \\ 0 & 0 & 0 & 0 \\ \alpha_{10}\eta & \alpha_{11}\eta & \frac{\epsilon-\alpha_{10}\alpha_7\eta+\sigma\beta}{\alpha_6} & \alpha_{12}\eta \\ -\alpha_{14}\beta & \alpha_{15}\beta & \alpha_{16}\beta & \frac{\varsigma+\upsilon\eta-\alpha_8\alpha_{14}\beta}{\alpha_6} \end{bmatrix}$$

Let $\sigma=\alpha_8\alpha_{10}+\alpha_{12}\alpha_6$, $\varsigma=\alpha_6\alpha_{13}-\alpha_5\alpha_{14}$ and $\upsilon=\alpha_6\alpha_{16}+\alpha_{14}\alpha_7$

The charactrestic polynomial is $\lambda^4-tr(J)\lambda^3+A_2\lambda^2-A_3\lambda+\det(J)=0$, so we have

$$\begin{aligned}
tr(J) &= \frac{(\alpha_1\alpha_6+\epsilon+\varsigma)+(\upsilon-\alpha_7\alpha_{10}-\alpha_6\alpha_3)\eta+(\sigma-\alpha_8\alpha_{14}-\alpha_6\alpha_4)\beta}{\alpha_6},\\
J_{1212} &= 0,\\
J_{1313} &= 0,\\
J_{1414} &= 0,\\
J_{2323} &= \frac{(\alpha_1-\alpha_3\eta-\alpha_4\beta)(\varsigma+\upsilon\eta-\alpha_8\alpha_{14}\beta)+\alpha_4\alpha_{14}(-\alpha_5\beta+\alpha_7\eta\beta-\alpha_8\beta^2)}{\alpha_6},\\
J_{2424} &= \frac{(\alpha_1-\alpha_3\eta-\alpha_4\beta)(\epsilon-\alpha_{10}\alpha_7\eta+\sigma\beta)-\alpha_{10}\alpha_3(-\alpha_5\eta-\alpha_8\eta\beta+\alpha_7\eta^2)}{\alpha_6},\\
J_{3434} &= 0,\\
J_{11} &= 0,\\
J_{22} &= \frac{(\alpha_1-\alpha_3\eta-\alpha_4\beta)[(\varsigma+\upsilon\eta-\alpha_8\alpha_{14}\beta)(\epsilon-\alpha_{10}\alpha_7\eta+\sigma\beta)-\alpha_6^2\alpha_{12}\alpha_{16}\eta\beta]}{\alpha_6^2},\\
J_{33} &= 0,\\
J_{44} &= 0,\\
\det(J) &= 0.
\end{aligned}$$

Similarly to the case (1) here we have one zero eigenvalue, the other eigenvalues determine the case of stability subject to the eigenvalues rules in three dimensional nonlinear systems as are informed in table (2.2).

Case (3): when $n_3^*=0$, but $n_1^*,n_2^*,n_4^*\neq 0$. The fixed point is $F_{p_5}=(\frac{-\alpha_5-\alpha_8\beta}{\alpha_6},\frac{\alpha_1-\alpha_4\beta}{\alpha_2},0,-\beta)$, in this case $\eta=0$ and $\beta=\frac{-\alpha_2\alpha_6\alpha_9+\alpha_{11}\alpha_1-\alpha_{10}\alpha_5}{\alpha_{10}\alpha_8+\alpha_{11}\alpha_4+\alpha_2\alpha_6\alpha_{12}}$. This point is a solution for the system if $\alpha_6 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_{10}\alpha_8 + \alpha_{11}\alpha_4 + \alpha_2\alpha_6\alpha_{12} \neq 0$. The jacobin matrix at this point $F_{p_5}(\frac{-\alpha_5-\alpha_8\beta}{\alpha_6}, \frac{\alpha_1-\alpha_4\beta}{\alpha_2}, 0, -\beta)$ is

$$J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} 0 & \frac{\alpha_2(\alpha_5+\alpha_8\beta)}{\alpha_6} & \frac{\alpha_3(-\alpha_5-\alpha_8\beta)}{\alpha_6} & \frac{\alpha_4(-\alpha_5-\alpha_8\beta)}{\alpha_6} \\ \frac{\alpha_6(\alpha_1-\alpha_4\beta)}{\alpha_2} & 0 & \frac{\alpha_7(\alpha_1-\alpha_4\beta)}{\alpha_2} & \frac{\alpha_8(-\alpha_1+\alpha_4\beta)}{\alpha_2} \\ 0 & 0 & \frac{\phi+\varpi\beta}{\alpha_6\alpha_2} & 0 \\ -\alpha_{14}\beta & \alpha_{15}\beta & \alpha_{16}\beta & \frac{\rho+\varkappa\beta}{\alpha_6\alpha_2} \end{bmatrix}$$

Let $\phi = \alpha_2\alpha_6\alpha_9 + \alpha_2\alpha_{10}\alpha_5 + \alpha_1\alpha_6\alpha_{11}$, $\varpi = \alpha_2\alpha_8\alpha_{10} + \alpha_2\alpha_{12}\alpha_6 + \alpha_4\alpha_6\alpha_{11}, \rho = \alpha_2\alpha_6\alpha_{13} - \alpha_2\alpha_5\alpha_{14} - \alpha_1\alpha_6\alpha_{15}$ and $\varkappa = \alpha_6\alpha_{15}\alpha_4 - \alpha_2\alpha_8\alpha_{14}$.

The charactrestic polynomial is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda + \det(J) = 0$, so we have

$$\begin{aligned}
tr(J) &= \frac{(\phi+\rho)+(\varpi+\varkappa)\beta}{\alpha_6\alpha_2}, \\
J_{1212} &= \frac{(\phi+\varpi\beta)(\rho+\varkappa\beta)}{\alpha_2^2\alpha_6^2}, \\
J_{1313} &= \frac{\alpha_{15}\alpha_8(\alpha_1\beta-\alpha_4\beta^2)}{\alpha_2}, \\
J_{1414} &= 0, \\
J_{2323} &= \frac{\alpha_4\alpha_{14}(-\alpha_5\beta-\alpha_8\beta^2)}{\alpha_6}, \\
J_{2424} &= 0, \\
J_{3434} &= -\frac{(\alpha_6\alpha_1-\alpha_6\alpha_4\beta)(\alpha_5\alpha_2+\alpha_8\alpha_2\beta)}{\alpha_2\alpha_6}, \\
J_{11} &= -\frac{(-\alpha_8\alpha_1+\alpha_8\alpha_4\beta)(\alpha_{15}\phi\beta+\alpha_{15}\varphi\beta^2)}{\alpha_2^2\alpha_6}, \\
J_{22} &= -\frac{(-\alpha_4\alpha_5-\alpha_4\alpha_8\beta)(\alpha_{14}\phi\beta+\alpha_{14}\varphi\beta^2)}{\alpha_2\alpha_6^2}, \\
J_{33} &= -\frac{(\alpha_2\alpha_5+\alpha_2\alpha_8\beta)}{\alpha_6}\left[\frac{(\alpha_6\alpha_1-\alpha_6\alpha_4\beta)(\rho+\varkappa\beta)+\alpha_8\alpha_{14}\alpha_6\alpha_2(-\alpha_1\beta+\alpha_4\beta^2)}{\alpha_2^2\alpha_6}\right] - \\
&\quad \frac{\alpha_4\alpha_{15}(-\alpha_5-\alpha_8\beta)(\alpha_1\beta-\beta^2)}{\alpha_2}, \\
J_{44} &= -\frac{(\alpha_5+\alpha_8\beta)(\alpha_1-\alpha_4\beta)(\phi+\varphi\beta)}{\alpha_2\alpha_6}, \\
\det(J) &= -\left(\frac{\alpha_2\alpha_5+\alpha_2\alpha_8\beta}{\alpha_6}\right)J_{12} + \left(\frac{-\alpha_3\alpha_5-\alpha_3\alpha_8\beta}{\alpha_6}\right)J_{13} - \left(\frac{-\alpha_4\alpha_5-\alpha_4\alpha_8\beta}{\alpha_6}\right)J_{14}.
\end{aligned}$$

Case (4): when $n_4^*=0$, but $n_1^*,n_2^*,n_3^*\neq 0$. The fixed point is $F_{p_6}=(\frac{-\alpha_5+\alpha_7\eta}{\alpha_6},\frac{\alpha_1-\alpha_3\eta}{\alpha_2},-\eta,0)$, in this case $\beta=0$ and $\eta=\frac{\alpha_2\alpha_6\alpha_9+\alpha_{11}\alpha_1-\alpha_{10}\alpha_5}{\alpha_{10}\alpha_7-\alpha_{11}\alpha_3}$. This point is a solution for the system if $\alpha_6\neq 0,\alpha_2\neq 0$ and $\alpha_{10}\alpha_7=\alpha_{11}\alpha_3$. The jacobin matrix at this point $F_{p_6}(\frac{-\alpha_5+\alpha_7\eta}{\alpha_6},\frac{\alpha_1-\alpha_3\eta}{\alpha_2},-\eta,0)$ is

$$J(n_1^*,n_2^*,n_3^*,n_4^*)=\begin{bmatrix} 0 & \frac{\alpha_2(\alpha_5-\alpha_7\eta)}{\alpha_6} & \frac{\alpha_3(-\alpha_5+\alpha_7\eta)}{\alpha_6} & \frac{\alpha_4(-\alpha_5+\alpha_7\eta)}{\alpha_6} \\ \frac{\alpha_6(\alpha_1-\alpha_3\eta)}{\alpha_2} & 0 & \frac{\alpha_7(\alpha_1-\alpha_3\eta)}{\alpha_2} & \frac{-\alpha_8(-\alpha_1+\alpha_3\eta)}{\alpha_2} \\ \alpha_{10}\eta & \alpha_{11}\eta & \frac{y+\mu\eta}{\alpha_6\alpha_2} & \alpha_{12}\eta \\ 0 & 0 & 0 & \frac{\rho+\varepsilon\eta}{\alpha_6\alpha_2} \end{bmatrix}$$

Let $\mu=-\alpha_2\alpha_7\alpha_{10}+\alpha_3\alpha_6\alpha_{11}$ and $\varepsilon=\alpha_6\alpha_{15}\alpha_3+\alpha_2\alpha_7\alpha_{14}+\alpha_2\alpha_6\alpha_{16}$

The charactrestic polynomial is $\lambda^4-tr(J)\lambda^3+A_2\lambda^2-A_3\lambda+\det(J)=0$, so we have

$$\begin{aligned}
tr(J) &= \frac{\phi+\rho+(\mu+\varepsilon)\eta}{\alpha_6\alpha_2},\\
J_{1212} &= \frac{(\phi+\mu\eta)(\rho+\varepsilon\eta)}{\alpha_2^2\alpha_6^2},\\
J_{1313} &= 0,\\
J_{1414} &= \frac{\alpha_{11}\alpha_7(-\alpha_1\eta+\alpha_3\eta^2)}{\alpha_2},\\
J_{2323} &= 0,\\
J_{2424} &= \frac{\alpha_{10}\alpha_3(\alpha_5\eta-\alpha_3\eta^2)}{\alpha_6},\\
J_{3434} &= -(\alpha_1-\alpha_3\eta)(\alpha_5-\alpha_7\eta^2),\\
J_{11} &= -\frac{(\alpha_7\alpha_1-\alpha_7\alpha_3\eta)(\alpha_{11}\rho\eta+\alpha_{11}\varepsilon\eta^2)}{\alpha_2^2\alpha_6},\\
J_{22} &= -\frac{(-\alpha_3\alpha_5+\alpha_3\alpha_7\eta)(\alpha_{10}\rho\eta+\alpha_{10}\varepsilon\eta^2)}{\alpha_2\alpha_6^2},\\
J_{33} &= -\frac{(\alpha_2\alpha_5-\alpha_2\alpha_7\eta)(\alpha_6\alpha_1-\alpha_6\alpha_3\eta)(\rho+\varepsilon\eta)}{\alpha_2^2\alpha_6^2},\\
J_{44} &= -(\alpha_5-\alpha_7\eta)[\frac{(\alpha_1-\alpha_3\eta)(\phi+\mu\eta)-\alpha_2\alpha_{10}\alpha_7(\alpha_1-\alpha_3\eta)}{\alpha_2\alpha_6}]+\\
&\quad \frac{\alpha_3\alpha_{11}(-\alpha_5+\alpha_7\eta)(\alpha_1\eta-\alpha_3\eta^2)}{\alpha_2},\\
\det(J) &= -(\frac{\alpha_2\alpha_5-\alpha_2\alpha_7\eta}{\alpha_6})J_{22}+(\frac{-\alpha_3\alpha_5+\alpha_3\alpha_7\eta}{\alpha_6})J_{33}-(\frac{-\alpha_4\alpha_5+\alpha_4\alpha_7\eta}{\alpha_6})J_{44}.
\end{aligned}$$

Case (5): when $n_1^*, n_2^* = 0$, but $n_3^*, n_4^* \neq 0$. The fixed point is $F_{p_7} = (0, 0, -\eta, -\beta)$, in this case we have two values of η and β. Hence $\eta = \frac{\alpha_1 - \alpha_4\beta}{\alpha_3}$, $\beta = \frac{\alpha_1 - \alpha_3\eta}{\alpha_4}$. Or $\eta = \frac{\alpha_5 + \alpha_8\beta}{\alpha_7}$, $\beta = \frac{\alpha_7\eta - \alpha_5}{\alpha_8}$. This point is a solution if $\alpha_3 \neq 0$ and $\alpha_4 \neq 0$ or $\alpha_7 \neq 0$ and $\alpha_8 \neq 0$. The jacobin matrix at this point $F_{p_7}(0, 0, -\eta, -\beta)$ is

$$J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} \alpha_1 - \alpha_3\eta - \alpha_4\beta & 0 & 0 & 0 \\ 0 & \alpha_5 - \alpha_7\eta + \alpha_8\beta & 0 & 0 \\ \alpha_{10}\eta & \alpha_{11}\eta & \alpha_9 + \alpha_{12}\beta & \alpha_{12}\eta \\ -\alpha_{14}\beta & \alpha_{15}\beta & \alpha_{16}\beta & \alpha_{13} + \alpha_{16}\eta \end{bmatrix}$$

The charactrestic polynomial is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda + \det(J) = 0$, so we have

$$\begin{aligned}
tr(J) &= (\alpha_1 + \alpha_5 + \alpha_9 + \alpha_{13}) + (\alpha_{16} - \alpha_7 - \alpha_3)\eta + (\alpha_8 + \alpha_{12} - \alpha_4)\beta, \\
J_{1212} &= \alpha_9\alpha_{13} + \alpha_9\alpha_{16}\eta + \alpha_{13}\alpha_{12}\beta, \\
J_{1313} &= (\alpha_5 - \alpha_7\eta + \alpha_8\beta)(\alpha_{13} + \alpha_{16}\eta), \\
J_{1414} &= (\alpha_5 - \alpha_7\eta + \alpha_8\beta)(\alpha_9 + \alpha_{12}\beta), \\
J_{2323} &= (\alpha_1 - \alpha_3\eta - \alpha_4\beta)(\alpha_{13} + \alpha_{16}\eta), \\
J_{2424} &= (\alpha_1 - \alpha_3\eta - \alpha_4\beta)(\alpha_9 + \alpha_{12}\beta), \\
J_{3434} &= (\alpha_1 - \alpha_3\eta - \alpha_4\beta)(\alpha_5 - \alpha_7\eta + \alpha_8\beta), \\
J_{11} &= (\alpha_5 - \alpha_7\eta + \alpha_8\beta)(\alpha_9\alpha_{13} + \alpha_9\alpha_{16}\eta + \alpha_{13}\alpha_{12}\beta), \\
J_{22} &= (\alpha_1 - \alpha_3\eta - \alpha_4\beta)(\alpha_9\alpha_{13} + \alpha_9\alpha_{16}\eta + \alpha_{13}\alpha_{12}\beta), \\
J_{33} &= (\alpha_1 - \alpha_3\eta - \alpha_4\beta)(\alpha_5 - \alpha_7\eta + \alpha_8\beta)(\alpha_{13} + \alpha_{16}\eta), \\
J_{44} &= (\alpha_1 - \alpha_3\eta - \alpha_4\beta)(\alpha_5 - \alpha_7\eta + \alpha_8\beta)(\alpha_9 + \alpha_{12}\beta), \\
\det(J) &= (\alpha_1 - \alpha_3\eta - \alpha_4\beta)J_{11}.
\end{aligned}$$

Case (6): when $n_1^*, n_3^* = 0$, but $n_2^*, n_4^* \neq 0$. The fixed point is $F_{p8} = (0, \frac{\alpha_1 - \alpha_4\beta}{\alpha_2}, 0, -\beta)$, in this case $\eta = 0$ and $\beta = -\frac{\alpha_5}{\alpha_8}$. This point is a solution for the system if $\alpha_2 \neq 0$ and $\alpha_8 \neq 0$. The jacobin matrix at this point $F_{p_8}(0, \frac{\alpha_1 - \alpha_4\beta}{\alpha_2}, 0, -\beta)$ is

$$J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\alpha_6(\alpha_1-\alpha_4\beta)}{\alpha_2} & \alpha_5+\alpha_8\beta & \frac{\alpha_7(\alpha_1-\alpha_4\beta)}{\alpha_2} & \frac{\alpha_8(-\alpha_1+\alpha_4\beta)}{\alpha_2} \\ 0 & 0 & \frac{q+s\beta}{\alpha_2} & 0 \\ -\alpha_{14}\beta & \alpha_{15}\beta & \alpha_{16}\beta & \frac{r+(\alpha_4\alpha_{15})\beta}{\alpha_2} \end{bmatrix}$$

The charactrestic polynomial is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda + \det(J) = 0$, so we have

$$\begin{aligned}
tr(J) &= \frac{(\alpha_2\alpha_5+\omega+\varphi)+(\nu+\alpha_4\alpha_{15}+\alpha_2\alpha_8)\beta}{\alpha_2}, \\
J_{1212} &= \frac{(\omega+\nu\beta)(\varphi+(\alpha_{15}\alpha_4)\beta)}{\alpha_2^2}, \\
J_{1313} &= \frac{\alpha_5 r+(\alpha_2\alpha_{13}\alpha_8+\alpha_{15}\alpha_4\alpha_5)\beta}{\alpha_2}, \\
J_{1414} &= (\alpha_5+\alpha_8\beta)\frac{\omega+\nu\beta}{\alpha_2}, \\
J_{2323} &= 0, \\
J_{2424} &= 0, \\
J_{3434} &= 0, \\
J_{11} &= \frac{(\alpha_5+\alpha_8\beta)[(\omega+\nu\beta)(\varphi+\alpha_{15}\alpha_4\beta)]+\alpha_{15}\alpha_8(\alpha_1\beta-\alpha_4\beta^2)(\omega+\nu\beta)}{\alpha_2^2}, \\
J_{22} &= 0, \\
J_{33} &= 0, \\
J_{44} &= 0, \\
\det(J) &= 0.
\end{aligned}$$

Similarly to the case (1) and case (2), we have one zero eigenvalue. Hence, the charactrestic equation is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda = 0$.

Case (7): when $n_1^*, n_4^* = 0$, but $n_2^*, n_3^* \neq 0$.The fixed point is $F_{p_9} = (0, \frac{\alpha_1-\alpha_3\eta}{\alpha_2}, -\eta, 0)$, in this case $\beta = 0$ and $\eta = \frac{\alpha_5}{\alpha_7}$. This point is a solution for the system if $\alpha_2 \neq 0$ and $\alpha_7 \neq 0$.The determinant of jacobin matrix at this point $F_{p_9}(0, \frac{\alpha_1-\alpha_3\eta}{\alpha_2}, -\eta, 0)$ is

$$J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\alpha_6(\alpha_1-\alpha_3\eta)}{\alpha_2} & \alpha_5 - \alpha_7\eta & \frac{\alpha_7(\alpha_1-\alpha_3\eta)}{\alpha_2} & \frac{\alpha_8(-\alpha_1+\alpha_3\eta)}{\alpha_2} \\ \alpha_{10}\eta & \alpha_{11}\eta & \frac{\omega+\alpha_3\alpha_{11}\eta}{\alpha_2} & \alpha_{12}\eta \\ 0 & 0 & 0 & \frac{(\alpha_2\alpha_{14}-\alpha_1\alpha_{15})+\chi\eta}{\alpha_2} \end{bmatrix}$$

The charactrestic polynomial is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda + \det(J) = 0$, so we have

$$\begin{aligned}
tr(J) &= \frac{(\alpha_2\alpha_5 + \omega + \alpha_2\alpha_{14} - \alpha_1\alpha_{15}) + (\chi + \alpha_3\alpha_{11} - \alpha_2\alpha_7)\eta}{\alpha_2}, \\
J_{1212} &= \frac{(\omega + \alpha_3\alpha_{11}\eta)((\alpha_2\alpha_{14} - \alpha_1\alpha_{15}) + \chi\eta)}{\alpha_2^2}, \\
J_{1313} &= (\alpha_5 - \alpha_7\eta)\frac{(\alpha_2\alpha_{14} - \alpha_1\alpha_{15}) + \chi\eta}{\alpha_2}, \\
J_{1414} &= \frac{\alpha_5\omega + (-\alpha_2\alpha_7\alpha_9 + \alpha_{11}\alpha_3\alpha_5)\eta}{\alpha_2}, \\
J_{2323} &= 0, \\
J_{2424} &= 0, \\
J_{3434} &= 0, \\
J_{11} &= \frac{\begin{array}{c}(\alpha_5 - \alpha_7\eta)[(\omega + \alpha_3\alpha_{11}\eta)(\alpha_2\alpha_{14} - \alpha_1\alpha_{15}) + \chi\eta] - \\ \alpha_{11}\alpha_7(\alpha_1\eta - \alpha_3\eta^2)((\alpha_2\alpha_{14} - \alpha_1\alpha_{15}) + \chi\eta)\end{array}}{\alpha_2^2}, \\
J_{22} &= 0, \\
J_{33} &= 0, \\
J_{44} &= 0, \\
\det(J) &= 0.
\end{aligned}$$

Similarly to the case (1), we have one zero eigenvalue. Hence, the charactrestic equation is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda = 0$.

Case (8): when $n_2^*, n_3^* = 0$,but $n_1^*, n_4^* \neq 0$. The fixed point is $F_{p_{10}} = (\frac{-\alpha_5-\alpha_8\beta}{\alpha_6}, 0, 0, -\beta)$, in this case $\eta = 0$ and $\beta = \frac{\alpha_1}{\alpha_4}$. This point is a solution for the system if $\alpha_6 \neq 0$ and $\alpha_4 \neq 0$. The jacobin matrix at this point $F_{p_{10}}(\frac{-\alpha_5-\alpha_8\beta}{\alpha_6}, 0, 0, -\beta)$ is

$$\det J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} \alpha_1 - \alpha_4\beta & \frac{\alpha_2(\alpha_5+\alpha_8\beta)}{\alpha_6} & \frac{\alpha_3(-\alpha_5-\alpha_8\beta)}{\alpha_6} & \frac{\alpha_4(-\alpha_5-\alpha_4\beta)}{\alpha_6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{u+v\beta}{\alpha_6} & 0 \\ -\alpha_{14}\beta & \alpha_{15}\beta & \alpha_{16}\beta & \frac{w-\alpha_8\alpha_{14}\beta}{\alpha_6} \end{bmatrix}$$

The charactrestic polynomial is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda + \det(J) = 0$, so we have

$$\begin{aligned}
tr(J) &= \frac{(\alpha_1\alpha_6 + \epsilon + \sigma) + (\sigma - \alpha_8\alpha_{14} - \alpha_4\alpha_6)\beta}{\alpha_6}, \\
J_{1212} &= \frac{((\varsigma - \alpha_8\alpha_{14}\beta)(\epsilon + \sigma\beta)}{\alpha_6^2}, \\
J_{1313} &= 0, \\
J_{1414} &= 0, \\
J_{2323} &= \frac{\alpha_1\varsigma - (\alpha_1\alpha_{14}\alpha_8 + \alpha_{13}\alpha_4\alpha_6)\beta}{\alpha_6}, \\
J_{2424} &= (\alpha_1 - \alpha_4\beta)\frac{\epsilon + \sigma\beta}{\alpha_6}, \\
J_{3434} &= 0, \\
J_{11} &= 0, \\
J_{22} &= \frac{(\alpha_1 - \alpha_4\beta)[(\varsigma - \alpha_8\alpha_{14}\beta)(\epsilon + \sigma\beta] + \alpha_4\alpha_{14}(-\alpha_5\beta - \alpha_8\beta^2)(\epsilon + \sigma\beta)}{\alpha_6^2}, \\
J_{33} &= 0, \\
J_{44} &= 0, \\
\det(J) &= 0.
\end{aligned}$$

Similarly to the case (1), we have one zero eigenvalue. Hence, the charactrestic equation is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda = 0$.

Case (9): when $n_2^*, n_4^* = 0$, but $n_1^*, n_3^* \neq 0$. The fixed point is $F_{p_{11}} = (\frac{-\alpha_5+\alpha_7\eta}{\alpha_6}, 0, -\eta, 0)$, in this case $\beta = 0$ and $\eta = \frac{\alpha_1}{\alpha_3}$. This point is a solution for the system if $\alpha_6 \neq 0$ and $\alpha_3 \neq 0$. The jacobin matrix at this point $F_{p_{11}}(\frac{-\alpha_5+\alpha_7\eta}{\alpha_6}, 0, -\eta, 0)$ is

$$J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} \alpha_1 - \alpha_3\eta & \frac{\alpha_2(\alpha_5-\alpha_7\eta)}{\alpha_6} & \frac{\alpha_3(-\alpha_5+\alpha_7\eta)}{\alpha_6} & \frac{\alpha_4(-\alpha_5+\alpha_7\eta)}{\alpha_6} \\ 0 & 0 & 0 & 0 \\ \alpha_{10}\eta & \alpha_{11}\eta & \frac{\epsilon-\alpha_7\alpha_{10}\eta}{\alpha_6} & \alpha_{12}\eta \\ 0 & 0 & 0 & \frac{\varsigma+\upsilon\eta}{\alpha_6} \end{bmatrix}$$

The charactrestic polynomial is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda + \det(J) = 0$, so we have

$$\begin{aligned}
tr(J) &= \frac{(\alpha_1\alpha_6 + \epsilon + \varsigma) + (\upsilon - \alpha_7\alpha_{10} - \alpha_3\alpha_6)\eta}{\alpha_6}, \\
J_{1212} &= \frac{(\epsilon - \alpha_7\alpha_{10}\eta)(\varsigma + \upsilon\eta)}{\alpha_6^2}, \\
J_{1313} &= 0, \\
J_{1414} &= 0, \\
J_{2323} &= (\alpha_1 - \alpha_3\eta)\frac{\varsigma + \upsilon\eta}{\alpha_6}, \\
J_{2424} &= \frac{\alpha_1\epsilon - (\alpha_1\alpha_7\alpha_{10} + \alpha_9\alpha_3\alpha_6)\eta}{\alpha_6}, \\
J_{3434} &= 0, \\
J_{11} &= 0, \\
J_{22} &= \frac{(\varsigma + \upsilon\eta)[(\alpha_1 - \alpha_3\eta)(\epsilon - \alpha_7\alpha_{10}\eta) - \alpha_3\alpha_{10}(-\alpha_5\eta + \alpha_7\eta^2)]}{\alpha_6^2}, \\
J_{33} &= 0, \\
J_{44} &= 0, \\
\det(J) &= 0.
\end{aligned}$$

Similarly to the case (1), we have one zero eigenvalue. Hence, the charactrestic equation is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda = 0$.

Case (10): when $n_3^*, n_4^* = 0$, but $n_1^*, n_2^* \neq 0$. The fixed point is $F_{p_{12}} = (\frac{-\alpha_5}{\alpha_6}, \frac{\alpha_1}{\alpha_2}, 0, 0)$, in this case $\beta = 0$ and $\eta = 0$. This point is a solution for the system if $\alpha_6 \neq 0$ and $\alpha_2 \neq 0$. The jacobin matrix at this point $F_{p_{12}}(\frac{-\alpha_5}{\alpha_6}, \frac{\alpha_1}{\alpha_2}, 0, 0)$ is

$$J(n_1^*, n_2^*, n_3^*, n_4^*) = \begin{bmatrix} 0 & \frac{\alpha_5\alpha_2}{\alpha_6} & \frac{-\alpha_5\alpha_3}{\alpha_6} & \frac{-\alpha_5\alpha_4}{\alpha_6} \\ \frac{\alpha_1\alpha_6}{\alpha_2} & 0 & \frac{\alpha_1\alpha_7}{\alpha_2} & \frac{-\alpha_1\alpha_8}{\alpha_2} \\ 0 & 0 & \frac{\phi}{\alpha_6\alpha_2} & 0 \\ 0 & 0 & 0 & \frac{\rho}{\alpha_6\alpha_2} \end{bmatrix}$$

The charactrestic polynomial is $\lambda^4 - tr(J)\lambda^3 + A_2\lambda^2 - A_3\lambda + \det(J) = 0$, so we have

$$\begin{aligned}
tr(J) &= \frac{(\phi+\rho)}{\alpha_2\alpha_6}, \\
J_{1212} &= \frac{\phi\rho}{\alpha_2^2\alpha_6^2}, \\
J_{1313} &= 0, \\
J_{1414} &= 0, \\
J_{2323} &= 0, \\
J_{2424} &= 0, \\
J_{3434} &= 0, \\
J_{11} &= 0, \\
J_{22} &= 0, \\
J_{33} &= -\alpha_1\alpha_5(\frac{\rho}{\alpha_2\alpha_6}), \\
J_{44} &= -\alpha_1\alpha_5(\frac{\phi}{\alpha_2\alpha_6}), \\
\det(J) &= -\frac{\alpha_5\alpha_2}{\alpha_6}J_{12} + \frac{-\alpha_5\alpha_3}{\alpha_6}J_{13} + \frac{\alpha_5\alpha_4}{\alpha_6}J_{14}.
\end{aligned}$$

4.3.2 Numerical fixed points for the system:

In this section, we want to find the value of fixed points numerically by calculating the value of parameters and using Matlab programme. We use precentagr rate method to calculate the value of parameters based on Palestinian Central Bureau of Statistics (PCBS) [38]. However, we found the most values of parameters in chapter three as are informed in table (3.1), the remaning parameters that we have in the new system are explained in the following table

parameters	Mechanism of calculate their values
α_4	Fertility rate [63]
α_8	Percentage of Palestinian lands controlled by the occupation [64]
α_{12}	The amount of electrical energy purchased from the Israel Electricity Company [65], [66]
α_{13}	The number of population of occupation [67]
α_{14}	The percentage of population are forcibly displaced from palestinian lands [68]
α_{15}	The percentage of joint projects between the public and private sectors [69]
α_{16}	primary production of solar energy [47], [70]

Parameters	Base year	current year	Values
α_4	2003	2017	-0.2483
α_8	1948	2019	0.85
α_{12}	2004	2017	1.1014
α_{13}	1997	2017	0.493
α_{14}	1967	2015	0.67
α_{15}	2001	2016	1.826
α_{16}	2001	2019	1.0162

Table (4.1): Additional parameters values in 4-D system

According to the previous parameters and table (3.1), we have twelve fixed points for the new system

$$p_1(-3.0194, 0, -1.5056, -0.3234),\ p_2(0, 0.2737, -0.0066, -0.0426),\ p_3(0, 0, 0, 0),$$
$$p_4(-0.7358, 0, 0, 3.0110),\ p_5(0, 0.2700, 0, -0.0538),\ p_6(0.5, 0, -1.35016, 0),$$

$$p_7(0, 0, 0.4851, -0.0459),\ p_8(0, 3.8189, -0.0321, 0),\ p_9(-0.0167, 0.3733, 0, 0)$$

$p_{10}(0.0506, 0.3484, -0.1075, -0.0371)$, $p_{11}(-3.3723, -0.9674, 0, 10.8131)$ and

$$p_{12}(0.4859, 0.1075, -0.9681, 0)$$

4.4 Stability for the system

In this part, we demonstrate the stability of the system around their twelve fixed points by finding the eigenvalues of the jacobian matrix. Moreover, we use Matlab programme in all calculations of this part because it is an easy and precise tool to get our results. The ordinary differential equation of the system (4.1) after substituting the values of pareameters are

$$\frac{dn_1}{dt} = .7127n_1 - 1.909n_1n_2 + .5242n_1n_3 - .2367n_1n_4 \tag{4.18}$$

$$\frac{dn_2}{dt} = .0457n_2 + 2.739n_1n_2 + 1.422n_2n_3 + .85n_2n_4 \tag{4.19}$$

$$\frac{dn_3}{dt} = .0506n_3 - .1012n_1n_3 - .01325n_2n_3 + 1.1014n_3n_4 \tag{4.20}$$

$$\frac{dn_4}{dt} = .493n_4 + .67n_1n_4 - 1.826n_2n_4 - 1.0162n_3n_4 \tag{4.21}$$

The jacobian matrix of the system is

$$J = \begin{bmatrix} \begin{array}{l} 0.7127 - 1.909n_2 \\ +0.5242n_3 - 0.2367n_4 \end{array} & -1.909n_1 & 0.5242n_1 & -0.2367n_1 \\ 2.739n_2 & \begin{array}{l} 0.0457 + 2.739n_1 \\ +1.422n_3 + 0.85n_4 \end{array} & 1.422n_2 & 0.85n_2 \\ -0.1012n_3 & -0.01325n_3 & \begin{array}{l} 0.0506 - 0.1012n_1 \\ -0.01325n_2 \\ +1.1014n_4 \end{array} & 1.1014n_3 \\ 0.67n_4 & -1.826n_4 & -1.0162n_4 & \begin{array}{l} 0.493 + 0.67n_1 \\ -1.826n_2 \\ -1.0162n_3 \end{array} \end{bmatrix}$$

After we determined the jacobian matrix of the system and calculated their fixed points, we illustrate the type of stability for the system at each fixed point as the following table

Fixed points	Eigenvalues($\lambda' s$)	Sign of eigenvalues($\lambda' s$)	Stability
$p_1(-3.0194, 0, -1.5056, -.3234)$	$\lambda_1 = .0802 + .7297i$ $\lambda_2 = .0802 - .7297i$ $\lambda_3 = -.1603$ $\lambda_4 = -10.6403$	two of eigenvalues are complex with positive real part(unstable spiral) andtwo of them are real with negative sign(stable)	Unstable
$p_2(0, .2737, -.0066, -.0426)$	$\lambda_1 = -.1583$ $\lambda_2 = .0792 + .052i$ $\lambda_3 = .0792 - .052i$ $\lambda_4 = .1968$	two of eigenvalues are complexwith positive real part(unstable spiral)and two of them are real with different signs(saddle)	Unstable
$p_3(0, 0, 0, 0)$	$\lambda_1 = .7127$ $\lambda_2 = .0457$ $\lambda_3 = .0506$ $\lambda_4 = .4930$	all eigenvalues are real and positive (unstable node)	Unstable
$p_4(-.7358, 0, 0, 3.0110)$	$\lambda_1 = -.3418$ $\lambda_2 = 1.8916 + 2.7406i$ $\lambda_3 = 1.8916 - 2.7406i$ $\lambda_4 = .5897$	two of eigenvalues are complex with positive real part(unstable spiral) and two of them are real with different signs(saddle)	Unstable
$p_5(0, .2700, 0, -.0538)$	$\lambda_1 = -.18$ $\lambda_2 = .0839 + .0621i$ $\lambda_3 = .0839 - .0621i$ $\lambda_4 = .2100$	two of eigenvalues are complex with positive real part(unstable spiral) and two of them are real with different signs (saddle)	Unstable

$p_6(.5, 0, -1.35016, 0)$	$\lambda_1 = .1917$ $\lambda_2 = -.1868$ $\lambda_3 = -.5047$ $\lambda_4 = 2.200$	all eigenvalues are real with different signs (saddle node)	Unstable
$p_7(0, 0, .4851, -.0459)$	$\lambda_1 = .0486i$ $\lambda_2 = -.0486i$ $\lambda_3 = .9779$ $\lambda_4 = .6965$	two of eigenvalues are complex with zero real part (center)but the other two are real and positive (unstable)	Unstable
$p_8(0, 3.8189, -.0321, 0)$	$\lambda_1 = -.0466$ $\lambda_2 = .0467$ $\lambda_3 = -6.5944$ $\lambda_4 = -6.4477$	all eigenvalues are real with different signs (saddle node)	Unstable
$p_9(-.0167, .3733, 0, 0)$	$\lambda_1 = .1807$ $\lambda_2 = -.1806$ $\lambda_3 = .0473$ $\lambda_4 = -.1998$	all eigenvalues are real with different signs(saddle node)	Unstable
$p_{10}(.0506, .3484, -.1075,$ $-.0371)$	$\lambda_1 = .0144 + .2625i$ $\lambda_2 = .0144 - .2625i$ $\lambda_3 = -.0145 + .0624i$ $\lambda_4 = -.0145 - .0624i$	all of eigenvalues are complex but two of them are with positive part (unstable spiral) and the others are with negative real part (stable spiral)	Unstable

$p_{11}(-3.3723, -.9674, 0, 10.8131)$	$\lambda_1 = -1.3016 + 5.4081i$ $\lambda_2 = -1.3016 - 5.4081i$ $\lambda_3 = 7.4587 + 8.8410i$ $\lambda_4 = 7.4587 - 8.8410i$	all of eigenvalues are complex but two of them are with positive part (unstable spiral) and the others are with negative real part(stable spiral)	Unstable
$p_{12}(.4859, .1075, -.9681, 0)$	$\lambda_1 = .0259 + .4982i$ $\lambda_2 = .0259 - .4982i$ $\lambda_3 = -.0520$ $\lambda_4 = 1.6060$	two of eigenvalues are complex with positive real part(unstable spiral) and two of them are real with different signs (saddle)	Unstable

Table (4.2): Equilibrium points and stability for 4-D system

4.5 Bifurcation and Numerical solution

In this section, we present fourth order chaotic system, and explain the sensitivity of the system when we change the initial conditions. On the other hand, we illustrate the behavior of the system when some of parameters are defined on an interval. In addition, we demonstrate the numerical solution between the main variables by presenting different figures to clarify the type of relation between these variables. Moreover, we explain the effect of the time on the trajectory of four main variables (n_1, n_2, n_3, n_4) in the system. In this section, we concentrate on two main initial conditions $(0.205, 0.7, -0.205, -0.7)$ and $(0.23, 0.7, -0.23, -0.7)$, with time period $t \in [0, 100]$, with value of parameters mentioned in table (3.1) and table (4.1). The simulation results are as the following

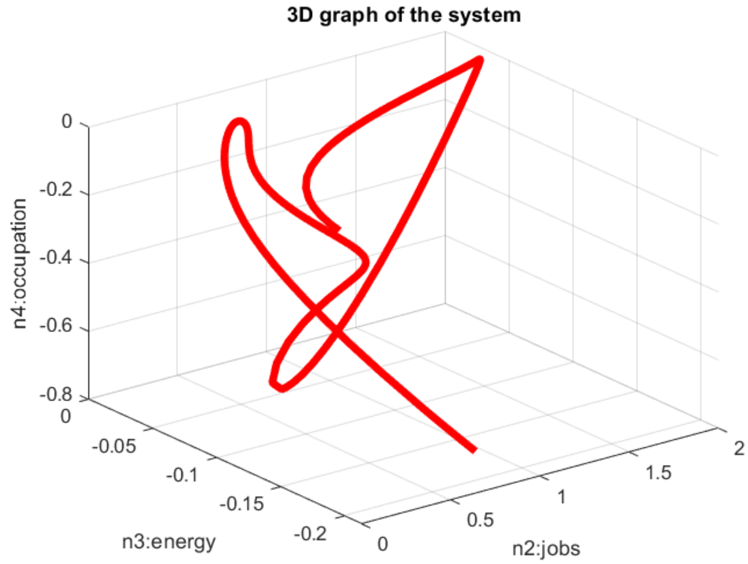


Figure (4.1): 3-D view for 4-D system on the $n_2 - n_3 - n_4$ space with $(0.205, 0.7, -0.205, -0.7)$

Figure (4.1) illustrates the relation between jobs, energy and occupation. At the begining, the number of jobs decreases but energy and occupation increase up to 5 years. Then, n_2, n_3, n_4 increase until 20 years. The fluctuation is continuous until 57 years. After this period, the three variables increase. However, when time reachs to 70 years the occupation decreases, similarly to the number of jobs but in small amount. In addition, energy still increases.

The following table present numerical solution for figure (4.1) in different periods

t	n_1	n_2	n_3	n_4
0	0.205	0.7	-0.205	-0.7
0.066947843	0.197550624	0.687280561	-0.195107398	-0.679889
5.389803406	0.052467622	0.50298455	-0.03667527	-0.123491436
10.39375141	0.009848042	0.580264424	-0.03364279	-0.01294078
15.35798294	0.001187604	0.582677131	-0.040512751	-0.000892559
20.34724983	0.00017536	0.534371619	-0.050143938	-7.87e-05
25.03253756	5.01e-05	0.457361136	-0.061609864	-1.46e-05
31.51325484	3.23e-05	0.318199613	-0.082697584	-5.69e-06
37.48850221	9.65e-05	0.185931407	-0.109683038	-1.25e-05
40.6095283	0.000291132	0.126971234	-0.12760726	-3.51e-05
45.16813678	0.002380069	0.062599081	-0.159654646	-0.000298618
50.0817123	0.03376899	0.025839741	-0.201022002	-0.005932399
55.23782275	0.621037167	0.09291958	-0.154423638	-0.327762494
60.01906898	0.000101078	1.888451053	-0.040958384	-0.000105737
65.06013393	-4.82e-07	1.737328694	-0.046812179	5.14e-07
70.04452493	1.02e-07	1.527538007	-0.054067317	-1.14e-07
75.40360154	2.63e-07	1.245626849	-0.064241352	-3.35e-07
80.71143883	-3.08e-08	0.931147454	-0.077832871	5.25e-08
85.48262882	2.27e-09	0.647604728	-0.094270231	-3.06e-09
90.55900861	-1.40e-09	0.381149077	-0.117765816	3.76e-09
95.91316745	2.36e-08	0.176059935	-0.151472039	-5.22e-08
98.63772248	6.24e-08	0.10649887	-0.172990234	-1.52e-07
100	1.15e-07	0.080271659	-0.185025938	-3.08e-07

Table (4.3): Numerical solution for 4-D system with $(0.205, 0.7, -0.205, -0.7)$

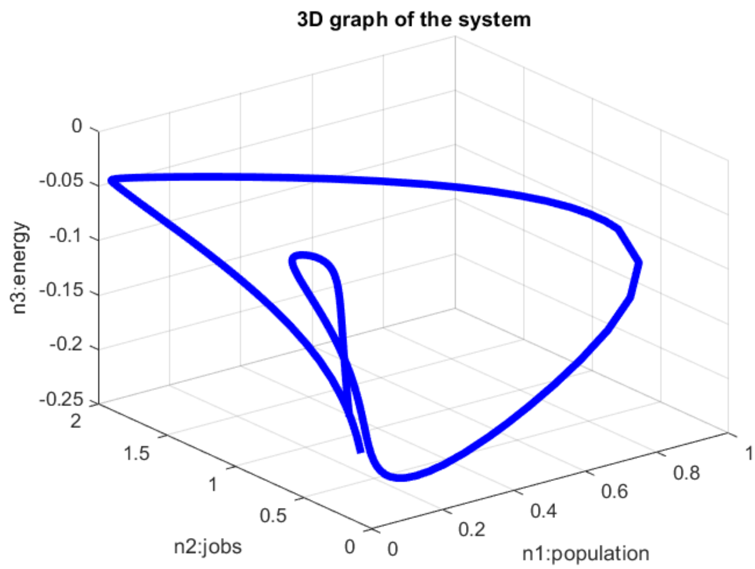


Figure (4.2): 3-D view for 4-D system on $n_1-n_2-n_3$ space with (0.205,0.7,-0.205,-0.7)

Figure (4.2) presents the relation between population, jobs and energy. At the first period, the number of population and jobs decrease but energy increases. This strategy continues until 10 years. After that, there is a negligible increase in jobs for 4 years. Moreover, all variables decrease untill approximately 39 years. After 54 years, the number of jobs increases rapidly for a long time. Finally, all variables decrease in appropriate amount.

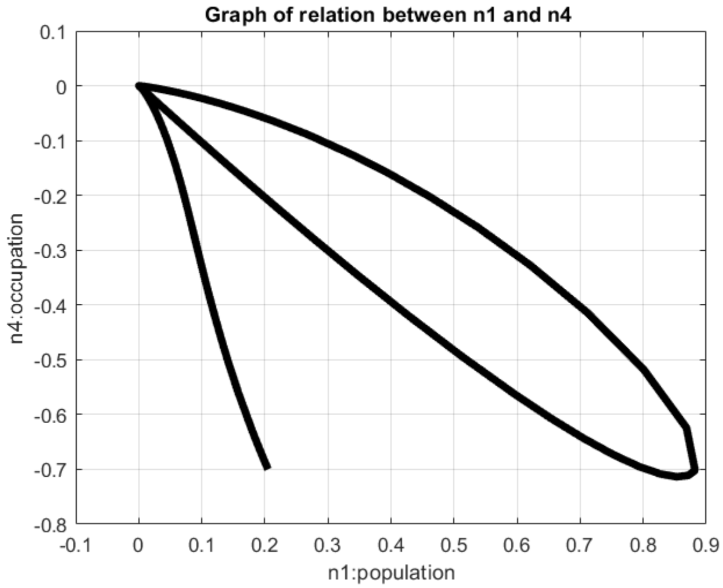


Figure (4.3): 2-D view for 4-D system on $n_1 - n_4$ space with (0.205,0.7,-0.205,-0.7)

The above figure presents an inverse relationship between population and occupation. At start of period, occupation increases but the number of population decreases until about 45 years. After this period, the occupation decreases but population increases to 57 years. In addition, this strategy go on for 60 years. Laterly, we notice a stability in both variables.

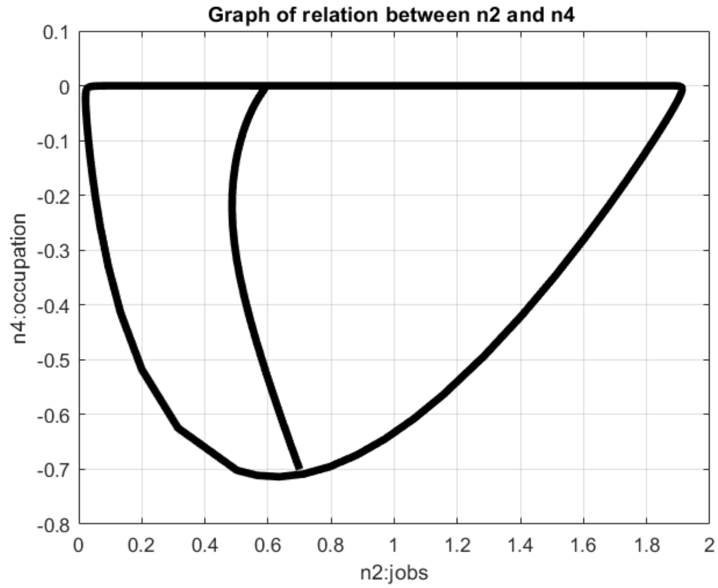


Figure (4.4): 2-D view for 4-D system on $n_2 - n_4$ space with (0.205,0.7,-0.205,-0.7)

Figure (4.4) explains the relation between jobs and occupation. At first, we notice an inverse relation between jobs (n_2) and occupation (n_4) to 5 years, since occupation increases but the number of jobs decreases. After that, there is a proportional relation between them. Whereas, both of them increase for 16 years. After this stage, the inverse relation back again where (n_2) increases rapidaly but (n_4) decreases in sensible amounts. Lately, the number of jobs decreases but occupation becomes stable. The figure we have look like predator-prey and cycle model.

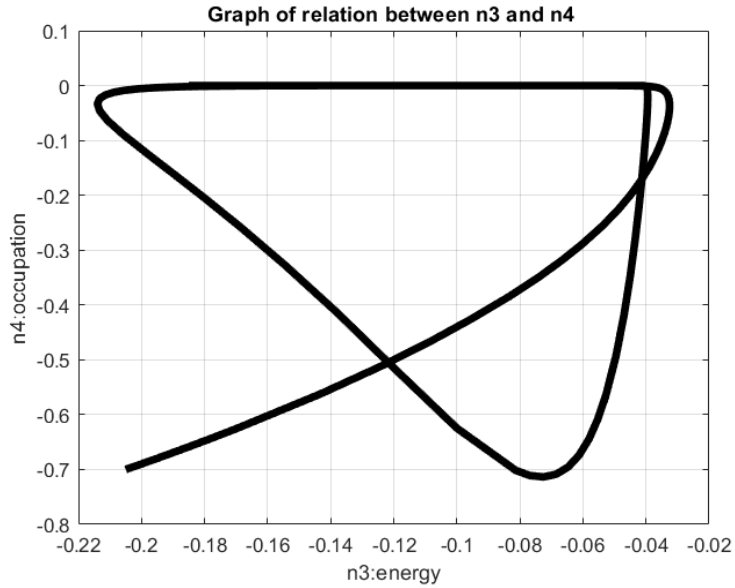


Figure (4.5): 2-D view for 4-D system on $n_3 - n_4$ space with (0.205,0.7,-0.205,-0.7)

The above figure demonstrates a proportional relation between energy and occupation. Since, both of them increase untill 15 years. On the other hand, energy and occupation decrease for 54.7 years. Furthermore, we notice an inverse relation go on for approximately two years and half. Eventualy, a proportional relation back again.

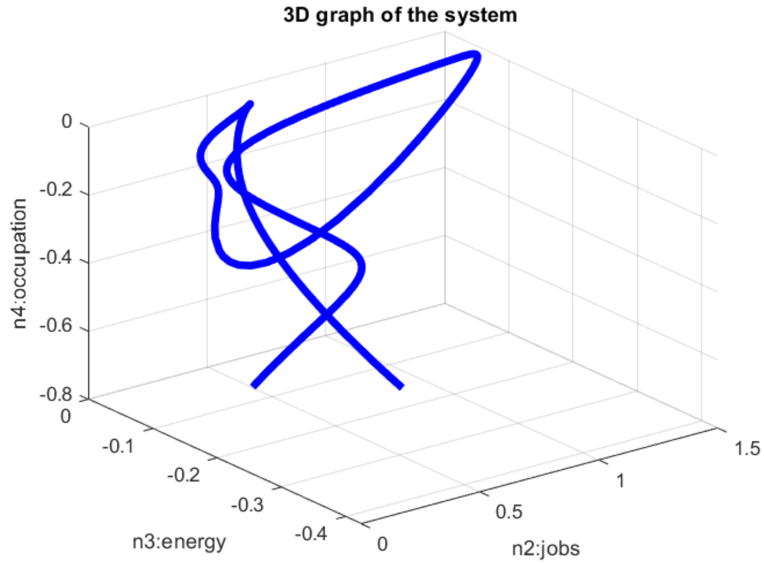


Figure (4.6): 3-D view for 4-D system on $n_2 - n_3 - n_4$ space with $(0.23, 0.7, -0.23, -0.7)$

As noticed from results, at the beginning the number of jobs decreases but energy and occupation increase for 8.2 years. Later, since energy start in decrease to 23.4 years, new change happen such that occupation decrease too. Based on such review, it is clear that all variables actually decrease. Few years later, we notice an increase in jobs and energy. In addition, occupation increases a little. Eventually, all variables are actually decrease such as in n_4.

The following table presents numerical solution for the figure (4.6) in different periods

t	n_1	n_2	n_3	n_4
0	0.23	0.7	-0.23	-0.7
0.066722885	0.221462765	0.688768154	-0.21888564	-0.681762633
5.23037344	0.052424187	0.513861012	-0.041497791	-0.123294084
10.16084332	0.009910648	0.570178072	-0.038141838	-0.013558594
15.00862313	0.001484322	0.554814123	-0.04564847	-0.001210499
20.52733466	0.000268763	0.480489033	-0.057913641	-0.000128526
26.06556937	1.26e-04	0.369752932	-0.074241635	-3.80e-05
30.24889498	1.58e-04	0.275425932	-0.090097395	-3.62e-05
35.87210715	6.43e-04	0.156677376	-0.117790025	-1.16e-04
40.33163778	4.12e-03	0.08518263	-0.146232298	-7.22e-04
45.25605791	0.051431169	0.042592373	-0.181500217	-0.011515809
50.11713641	0.595941713	0.265636893	-0.121869763	-0.340916193
55.0416982	0.000251614	1.36164781	-0.065594513	-0.000214089
60.13543153	1.05e-07	1.023263466	-0.078298949	-8.70e-08
65.35560211	1.85e-09	0.682249027	-0.096143716	-1.44e-09
71.23980315	4.01e-09	0.357260552	-0.124417524	-3.98e-09
75.17714217	1.08e-08	0.199087838	-0.149694068	-1.14e-08
80.2370134	7.25e-08	0.074009687	-0.191701396	-9.99e-08
85.5895111	1.17e-06	0.017759039	-0.250604028	-3.11e-06
90.31589612	1.55e-05	0.003298221	-0.318083671	-1.14e-04
95.36106756	2.15e-04	0.000308688	-0.406206786	-8.64e-03
98.09473759	8.48e-04	6.27E-05	-0.414765364	-1.07e-01
100	2.62e-03	1.69E-05	-0.256098987	-5.44e-01

Table (4.4): Numerical solution for 4-D system with $(0.23, 0.7, -0.23, -0.7)$

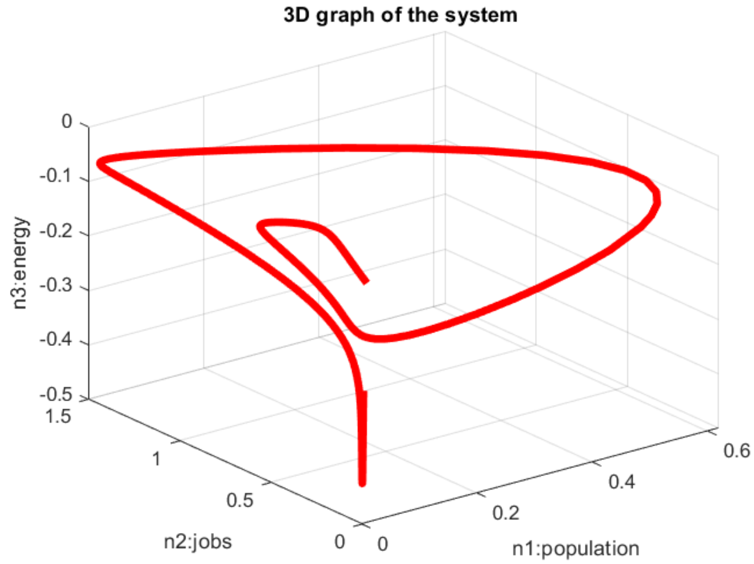


Figure (4.7): 3-D view for 4-D system on $n_1-n_2-n_3$ space with $(0.23, 0.7, -0.23, -0.7)$

Figure (4.7) illustrates the number of population and jobs decrease but energy increases, this continues for 11.4 years. After that, energy start to decrease. After 30.2 years, the number of population is beginning to increase. In addition, we have the new change which is the increase in both of jobs and energy, this improvement go on for approximately 3 years. However, the number of population is backs down again but during this period the number of jobs increases rapidly. Finally, all variables decrease. In last three years, there is a little increase in both of population and energy.

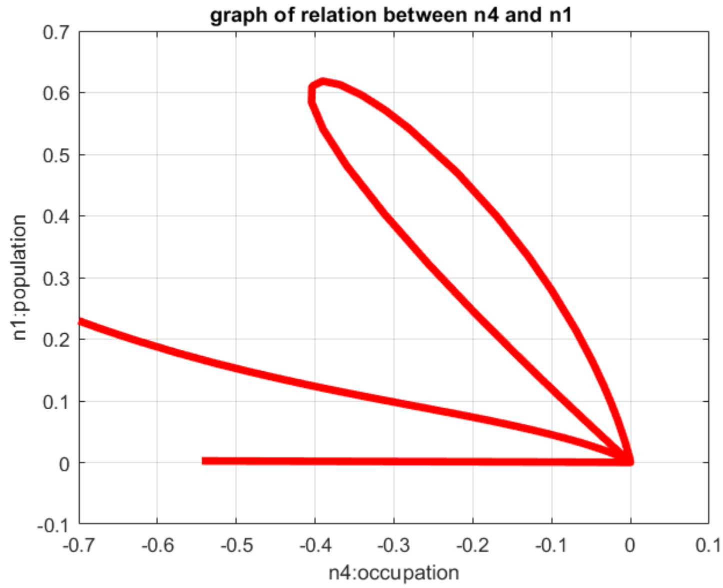


Figure (4.8): 2-D view for 4-D system on $n_4 - n_1$space with $(0.23, 0.7, -0.23, -0.7)$

Figure(4.8) presents an inverse relationship between population and occupation. Since, when occupation increases the number of population decreases, for a long time about 23.4 years. After this period, the number of population increases but occupation decreases. This an inverse relation fluctuates in different periods of time.

Figure (4.9): 2-D view for 4-D system on $n_2 - n_4$ space with $(0.23, 0.7, -0.23, -0.7)$

We have demonstrate the obtained resullts using the above figure. As such, at the beginning there is an inverse relation where occupation increases and the number of jobs decreases, this continue for 23.4 years. After this period, occupation becomes stable such that no any change in the values but the number of jobs decreases a little and such situation remaines for 8 years. Furthermore, there is a proportional relation between jobs and occupation where both are decrease, this go on approximately for 18 years. The Inverse relatiohship returns again and job opportunities increase, however the occupation decreases and later becomes stable.

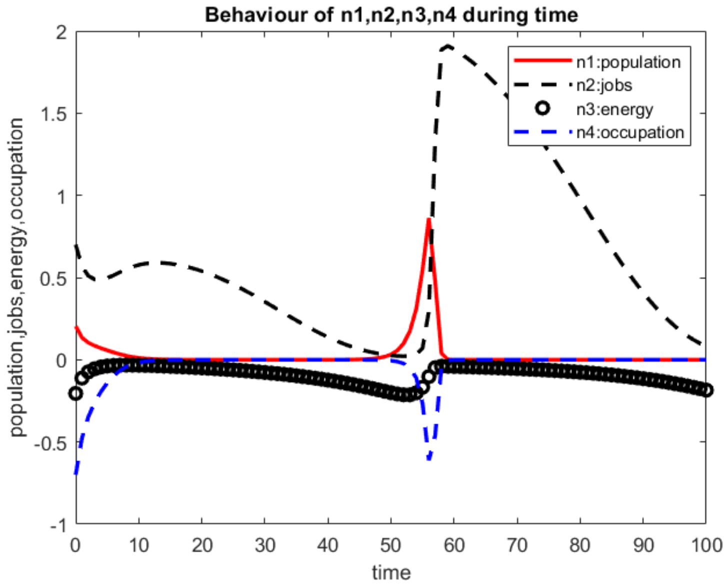


Figure (4.10): Behavior of n_1, n_2, n_3, n_4 during $t \in [0, 100]$ with $(0.205, 0.7, -0.205, -0.7)$

Figure (4.10) illustrates the behaviour of the four variables during time with initial condition $(.205, .7, -.205, -.7)$. At the beginning, for many years population decreases and becomes stable, it increase clearly at a half of period. The fluctuate in the population repeats itself in a second half of the period. Jobs and energy fluctuate along the period with deeply increase of jobs after 45 years. Finally, occupation increases and maintain their values, we notice clearly decrease at half of time to a limit point. In the second part of period, the curve is repeating itself without decreasing.

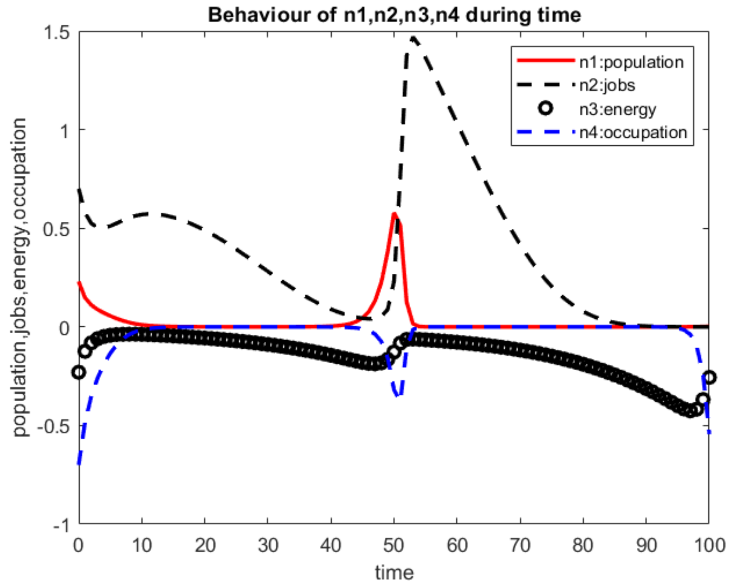


Figure (4.11): Behaviour of n_1, n_2, n_3, n_4 during $t \in [0, 100]$ with $(0.23, 0.7, -0.23, -0.7)$

Figure (4.11) illustrates the behaviour of the four variables during time with initial condition $(0.23, 0.7, -0.23, -0.7)$. Similar to figure (4.10) with small different. As such, the occupation decreases and energy increases in the last of period, there is no big effect of occupation on energy.

On the other hand, we demonstrate the behavior of the system when one of control vector parameters $(\alpha' s)$ is changed but the others stay constant. As such, we define α_{15} on an interval and replace it by$m \in [-3, 3]$, and consider the initial condition $(0.205, 0.7, -0.205, -0.7)$. However, we present the attitude of each main variable (population (n_1), jobs (n_2), energy (n_3), occupation(n_4)) with m (α_{15}) as the following.

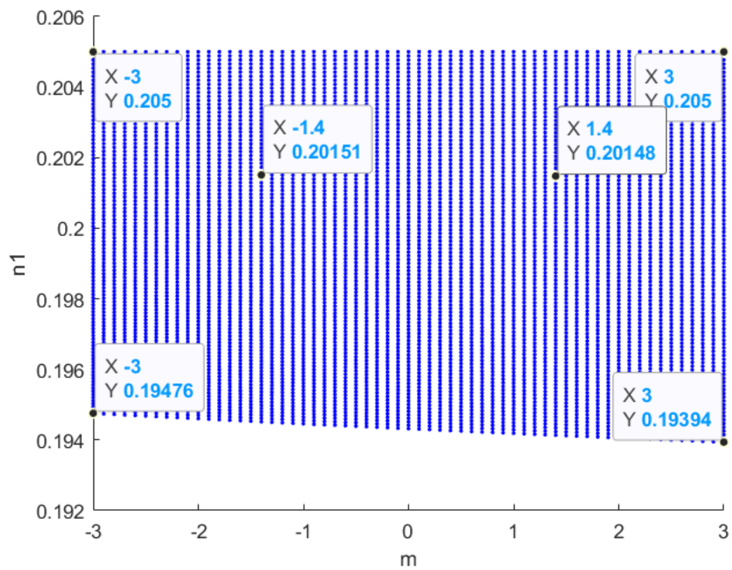


Figure (4.12) : Bifurcation diagram of n_1 vs $m \in [-3, 3]$ with $(0.205, 0.7, -0.205, -0.7)$ in 4-D system

Figure(4.12) illustrates the relation between n_1 and the parameter m it seems saddle node bifurcation. Since, when the value of m change there is no significant effect on the population. That is, lower and higher values of m give the same value of n_1 .

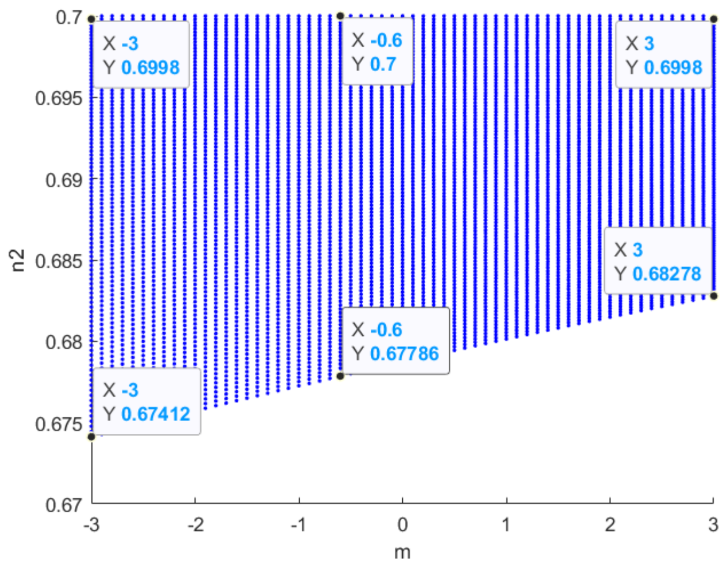


Figure (4.13) : Bifurcation diagram of n_2 vs $m \in [-3, 3]$ with $(0.205, 0.7, -0.205, -0.7)$ in 4-D system

Figure (4.13) explains the behaviour of jobs with m it's look like a saddle node bifurcation. Since, when the values of m increases, the number of jobs increases too. Moreover, more than one value of m give the same value of jobs. That is, flexibility in the values.

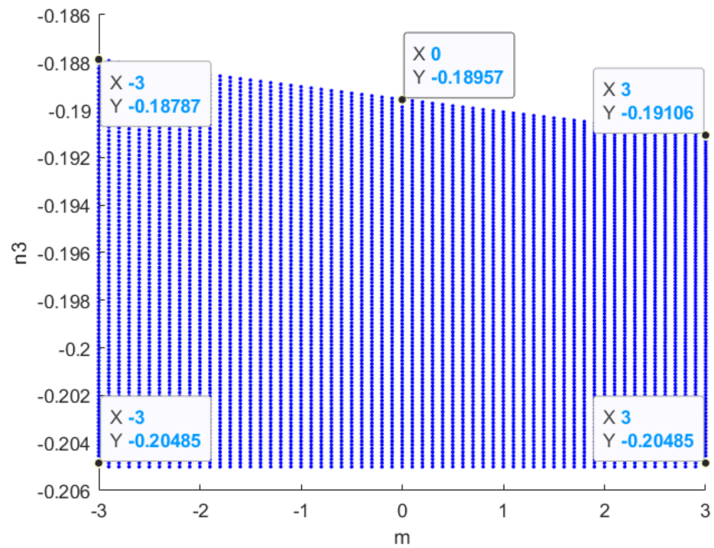


Figure (4.14) : Bifurcation diagram of n_3 vs $m \in [-3, 3]$ with $(0.205, 0.7, -0.205, -0.7)$in 4-D system

Figure (4.14) demonstrates the attitude of n_3 when the values of m are changing. Here, the bifurcation seems a saddle node bifurcation. It is clearly, different values of m give the same value of energy. Moreover, when the values of m increase the value of n_3 decreases. The greatest value of n_3 happens at small value of m.

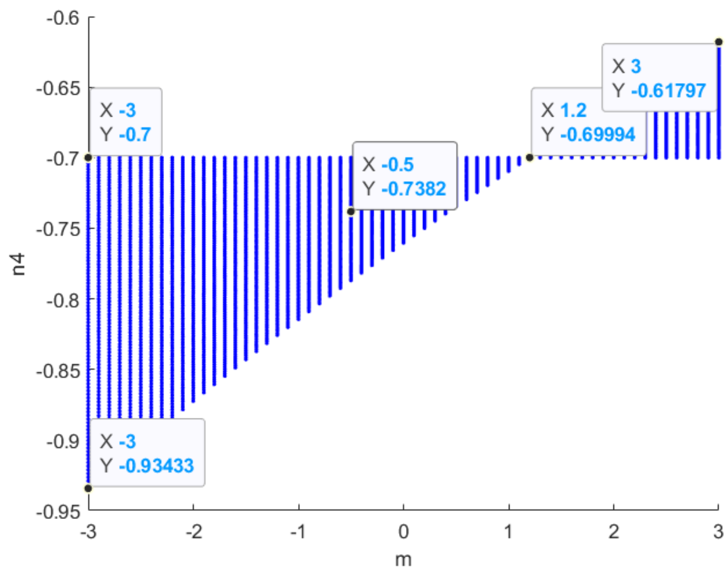


Figure (4.15) : Bifurcation diagram of n_4 vs $m \in [-3, 3]$ with $(0.205, 0.7, -0.205, -0.7)$in 4-D system

Figure (4.15) presents the relation between occupation and m, it seems a transcritical bifurcation. However, there is a proportional relation between them, when the values of m increase the values of occupation increase too. In additin, the bifurcation value is $m = 1.2$. After this value, the occupation is increasing highly and becoming more and more effecting in all variables.

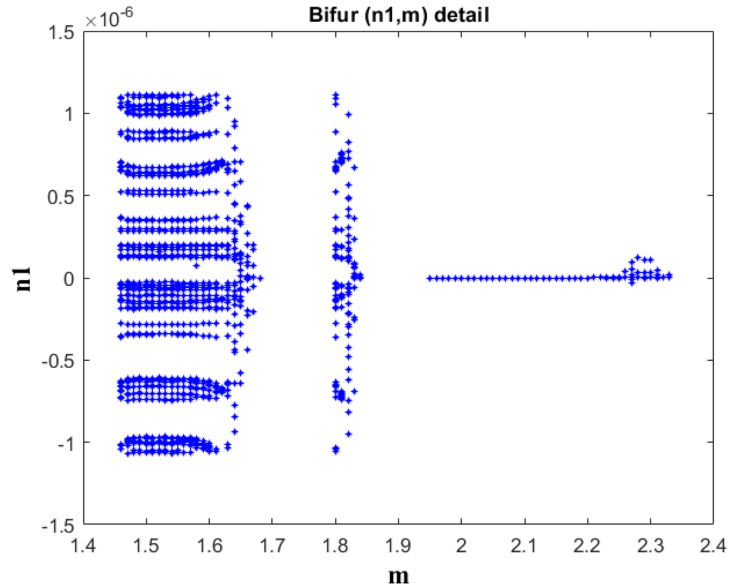


Figure (4.16): Direction field for 4-D system $n_1$vs $m \in [-3,3]$ with (0.205,0.7,-0.205,-0.7)

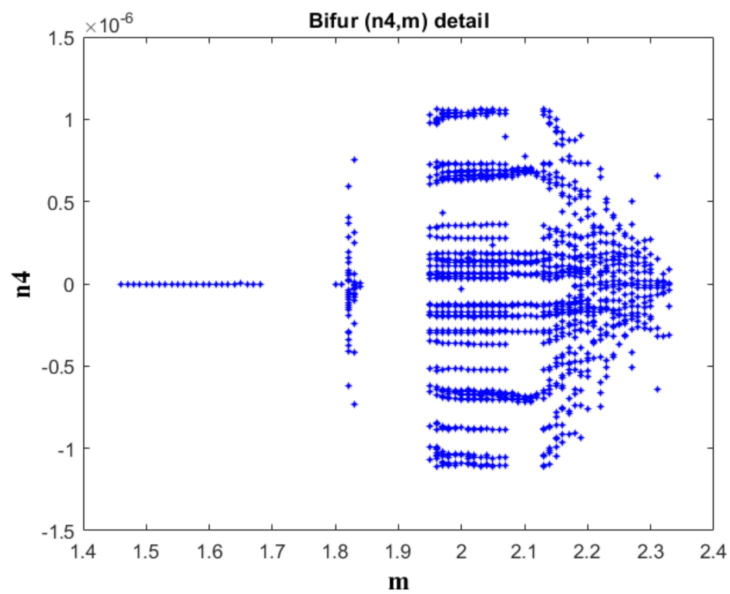


Figure (4.17): Direction field for 4-D system n_4 vs $m \in [-3,3]$with (0.205,0.7,-0.205,-0.7)

Figures (4.16) and (4.17) illustrate the direction field and bfurcation diagrams for the chaotic dynamic system, they are look like pitchfork bifurcation. However, these graphs depend on the initial condition, on the different values of the bifurcation parameter m (α_{15}) and the constant values of other parameters. In addition, they explain the sensitivty and relativity of the system. Clearly, the behaviour of the system changes also when the main variables are different. Since, the motion is stable at different values, otherwise the movement disappears. Whereas, the attitude of the system is very strong during some periods of time at certain values. It's clear, more than one type of local bifurcation coalesce to each other, that is we have the global bifurcation in the chaotic system.

4.6 Newton Method and the tenth fixed point($F_{p_{10}}$)

In this section, we use Newton Method in definition (2.2.15) to predict the best value of four variables (n_1, n_2, n_3, n_4), which give growth and increase in the first three variables but decrease in the fourth one. However, we change the values of some parameters and define it on an interval and take different initial conditions to get required results by using built-in Matlab code to apply Newton Method with error value $10\hat{}^{-6}$ for all parameters. First, we consider the change in the values of some parameters, such as (α_{10}) we define it on an interval $[-3, 3]$ with initial condition $(0.205, 0.7, -0.205, -0.7)$, the best value we have $F_{p_{10}}(0.3056, 0.4464, -0.1155, -0.8451)$ at $\alpha_{10} = -2.9$. We notice an increase in population and energy but a decrease in jobs and occupation. On the other hand, we change α_{10} and α_{16} together. Whereas, $\alpha_{10} \in [-3, 3]$ and $\alpha_{16} \in [-2, 2]$ with initial condition $(0.205, 0.7, -0.205, -0.7)$. The result we have $F_{p_{10}}(.2802, .4479, -.0914, -.8039)$ at $\alpha_{10} = -3$ and $\alpha_{16} = 1.8$. It's clear that, the regular increase in the population and energy, in parallel to the regular decrease in the jobs and occupation. In addition, we consider $\alpha_{10} \in [-3, 3]$ and $\alpha_2 \in [-2, 2]$. Since, at $\alpha_{10} = -2.8$ and $\alpha_2 = 1.9$, we get this point $F_{p_{10}}(.2669, .4326, -.1163, -.7191)$. Similarly, we analyze this point as previous one. Second, we explain the change in the initial condition with same parameters. For instance, we take $(.4, .4, -.4, -.4)$ and $\alpha_{10} \in [-3, 3]$, $\alpha_{16} \in [-2, 2]$.

The point we have $F_{p_{10}}(.2342, .4301, -.0903, -.6573)$, at $\alpha_{10} = -2.9$ and $\alpha_{16} = 1.5$. It's clearly, there is a balance in decrease of population and occupation, that is equivalent to the increase in jobs and energy. Moreover, we consider the changes in $\alpha_1 \in [-2, 2]$, $\alpha_{10} \in [-3, 3]$ with initial condition $(.4, .4, -.4, -.4)$. We get great result at $\alpha_{10} = -1.7$ and $\alpha_1 = 1.4$, which is $F_{p_{10}}(.4556, .6970, -.4669, -.7408)$. As such, we notice an increase in population and jobs with negligible decrease in energy, but fabulous decrease in occupation. Actually, we decide to choose these parameters, because it is possible to make any change on it as appropriate with palestinian circumference.

Chapter 5

Conclusion and Summary

The study works established the idea of using and applying mathematical modeling for regional socio-economic development in Palestine. A number of modeling equations are proposed and formed for the problem. The equilibrium points and bifurcation are calculated and measured for stability options by analyzing the Jacobin Matrix. Numerical methods as Newton method is applied to obtain numerical values for the main variables. In addition, MATLAB program is used to present different numerical solution figures and bifurcation diagrams. However, two mathematical models are constructed to demonstrate the problem. The first model was a three dimensional nonlinear continuous model consisting of three equations with three main variables (population, jobs and energy), that are connected to each other by nine parameters. The parameters control of the nature of the relation between the main variables in terms of increasing and decreasing or fluctuating in the status. There is a special significance for each parameter that shows the effect of the interaction of the main variables with each other. As we explained previously, we obtained the values of these variables based on the data and information available in the Palestinian Central Bureau of Statistics (PCBS) by adopting the percentage rate method in our calculations, which depends on the method of comparison between two years base and current years of the study phenomenon. We concluded the three dimensional system is a chaotic system. As such, we presented a new visualize of the chaotic system combining more than one system to each other such as circular, spiral, predator-prey model and others. Mainly, that depends on the change of the initial conditions and the interaction of environments for each of these variables. In addition, Newton Method is used to predict the values of three basic variables in which the development may occur. This method showed if there is a change in some values, it might lead the system to the state of stability that we are looking for. For example, we considered α_2 parameter, which is the coefficient of academic and professional level for population. The study indicated that when this value is reduced to .3 there would be an increase in the number of people and jobs in parallel to a simple decrease in energy. In this case, the system has a stable equilibrium point, that is the system reaches to a state of stability around this point. Practically, this can be by directing students towards vocational and technical education instead of academic education, and encouraging them to create projects that employ them instead of relying on governmental jobs. The second model was a non-linear four-dimensional model that resulted from the expansion of the first model and the addition of the influencing factor, which is the occupation. We illustrated the effect of each of these variables on the occupation, as well as the effect of the occupation on it. We joined between them by 16 parameters that control and clarify the nature of relation between them. Lately, the numerical calculation verified that the system looks like Butterfly models i.e. the system is chaotic system by using MATLAB program and analyzing the Jacobian matrix at each point. Moreover, we obtained twelve fixed points all of them are unstable. Similarly, to the first system, we made a change in some of control parameters by using Newton Method, we got the results that indicate the presence of improvement and development in these variables. For instance, α_{10} parameter is the conformity ratio of the population with energy supply. When α_{10} value increases, the population and jobs increase, with a negligible decrease in energy and a significant decrease in occupation. This is a good indicator since it reflects negatively on the occupation but positively on the Palestinian reality.

In future works, we suggest to optimize the system stability on specific variables such as jobs or energy. In addition, specific parameters can be defined and calculated based on specific years for precise and harmonic model, change in the mechanism of calculate the coefficients and add new variables to the system. Moreover, since we have obstacles in obtained the values of parameters which are effected on the stability of the system, we suggest to use generate random variable by monte carlo simulation to have most suitable data that can be analyzed more efficient. Despite of many problems that cannot be resolved definitively, such as the problem of occupation which is imposed on Palestinians, but we can create the alternatives that mitigate the negative impact on us and improve our future, live with the existing reality and overcome on the face difficulties. We are certain and full of hope the occupation will end soon and forever.

References

[1] Schank, J. C., & Twardy, C. (2009). Mathematical Models. The Modern Biological and Earth Sciences.

[2] Squires, H., & Tappenden, P. (2011). Mathematical modelling and its application to social care.

[3] Dassios, I. K., & Devine, M. T. (2016). A macroeconomic mathematical model for the national income of a union of countries with interaction and trade. Journal of Economic Structures, 5(1), 1-15.

[4] Nijkamp, P., & Poot, J. (2008). Mathematical models in regional economics.

[5] Tarasov, V. (2019). On history of mathematical economics: Application of fractional calculus. Mathematics, 7(6), 509.

[6] Van, G. (2020). A Mathematical Theory of Economic Growth: The Public Choice Growth Model.

[7] Boldyrev, Y., Chernogorskiy, S., Shvetsov, K., Zherelo, A., &Kostin, K. (2019). A mathematical model of regional socio-economic development of the Russian Arctic Zone. Resources, 8(1), 45.

[8] Kumar, N., & Rai, L. P. (2020). Use of Mathematical Models to Forecast Market Capitalization and Economic Growth in India.

[9] Montemayor, O. M. F., Rojas, A. L., Chavarria, S. L., Elizondo, M. M., Vargas, I. R., & Hernandez, J. F. G. (2018). Mathematical modeling for forecasting the

gross domestic product of Mexico. International Journal of Innovative Computing, Information and Control, 14(2), 423-36.

[10] Abu-Eideh, O. M. (2015). Causality between public expenditure and GDP growth in Palestine: An econometric analysis of Wagner's law. Journal of Economics and Sustainable Development, 6(2), 189-199.

[11] Government of Palestine and the United Nations Development Program. (2016). Building Economic Resiliencein the State of Palestine. Palestine resilience conference, 24-25 November 2016 . Amman, Jordan.

[12] Ridao-Cano, C., Rother, F. U., & Sanchez-Reaza, J. (2019). West Bank and Gaza-Jobs in West Bank and Gaza Project: Enhancing Job Opportunities for Palestinians.

[13] Rahman, H. (2009). Network deployment for social benefits in developing countries. In Encyclopedia of Multimedia Technology and Networking, Second Edition (pp. 1048-1054). IGI Global.

[14] Zakour, M. J., & Swager, C. M. (2018). Vulnerability-plus theory: the integration of community disaster vulnerability and resiliency theories. In Creating Katrina, Rebuilding Resilience (pp. 45-78). Butterworth-Heinemann.

[15] Engle, C. R., & Bohorquez, J. J. (2019). Mariculture, economic, and social impacts. Encyclopedia of Ocean Sciences, 2, 422–430. https://doi.org/10.1016/b978-0-12-409548-9.11400-9

[16] Economic growth and Economic development. (n.d.). Retrieved February1,2022, from

https://nios.ac.in/media/documents/SrSec318NEW/318_Economics

[17] Troch, I., & Breitenecker, F. (2009). Modeling and Simulation of Dynamic Systems. MECHANICAL ENGINEERING, ENERGY SYSTEMS AND SUSTAINABLE DEVELOPMENT-Volume II, 113.

[18] Hedrick, J. K., & Girard, A. (2005). Control of nonlinear dynamic systems: Theory and applications. Controllability and observability of Nonlinear Systems, 48.

[19] Boisgérault, S. (2017, April25). Connected sets. Retrieved February12,2022, from https://direns.mines-paristech. fr/Sites/ Complex-analysis/Connected%20Sets/#corollary–open-connected-sets.

[20] Royden, L. H., & Fitzpatrick, M. P. (2010). Real Analysis. Fourth Edition. p327.

[21] Strogast, S. H. (2015). Nonlinear Dynamics and Chaos. p. 149,150-154.

[22] Remani, C. (2013). Numerical methods for solving systems of nonlinear equations. Lakehead University Thunder Bay, Ontario, Canada.p 4-9.

[23] Burden, R. L., & Faires, J. D. (2001). Study guide for numerical analysis. Seventh edition. Brooks/Cole. p602.

[24] Cooper, R. W., & John, A. A. (2012). Macroeconomics: Theory through applications. Saylor Academy.

[25] Williams, G. P. (1997). Chaos Theory Tamed, all rights reserved. Published by arrangement with Taylor & Francis Ltd.

[26] Bosi, S., & Desmarchelier, D. (2017). A simple method to study local bifurcations of three and four-dimensional systems: characterizations and economic applications. FAERE Working Paper.

[27] Amon, A. (2007). Nonlinear dynamics (Doctoral dissertation, Université Rennes 1).

[28] Palestinian Central Bureau of Statistics. (2018). Population, Housing and Establishments Census 2017. Ramallah – Palestine.

[29] Courbage, Y., Abu Hamad, B., & Zagha, A. (2016). Palestine 2030-demographic change: opportunities for development. State of Palestine: United Nations Population Fund Palestine.

[30] World Bank Group. (2016). West Bank & Gaza Energy Efficiency Action Plan 2020–2030 Final Report June 2016.

[31] Palestinian Central Bureau of Statistics. (2014). National Accounts at Current and Constant Prices1994-2012. Ramallah – Palestine.

[32] Pcbs. (n.d.). Totalenergy supply by year and type of energy, 2009-2019. PCBS. Retrieved October 12,2021, from https://www.pcbs.gov.ps/statisticsIndicatorsTables.aspx?lang=en&table_id=531.

[33] palestine Resilience conference 2016. UNDP in Programme of Assistance to the Palestinian People. (n.d.). Retrieved October12,202, from file:///C:/Users/User/Downloads/UNDP-papp-research-PRC_Building.

[34] World Bank. (2007, May 1). West Bank and Gaza Energy Sector Review. Washington, DC. World Bank. Retrieved November 20,2021, from https://openknowledge.worldbank.org/handle/10986/19226.Bank, W. (2007). Gaza Energy Sector Review. World Bank report, Report, (39695-GZ).

[35] Wang, L., Chen, Y., & Zhu, W. (2018, May 1). In Proceedings of the 2018 7th International Conference on Energy, Environment and Sustainable Development (ICEESD 2018). Correlation analysis of the population and energy –the coexistence of promotion and inhibition. Atlantis Press. 1-7.

[36] Pcbs. (n.d.). Annual Energy Tables and Energy Balance. PCBS. Retrieved October 12, 2021, from https://www.pcbs.gov.ps/site/886/Default.aspx.

[37] Zabel, G. (2009, April 19). Peak people: The Interrelationship between population growth and energy resources. Resilience. Retrieved October 12, 2021, from https://www.resilience.org/stories/2009-04-20/

[38] Palestinian Central Bureau of Statistics. Pcbs. Retrieved February 20, 2022, from https://www.pcbs.gov.ps/site/lang__en/1/default.aspx.

[39] Stumpf, P., Sütő, Z., & Nagy, I. (2011). Research in Nonlinear Dynamics triggered by R&D Experiance.Budapest University of Technology and Economics. AACS 2011. Proceeding of Automation and Applied Computer Science Workshop.

[40] Izhikevich, E. M. (n.d.). Equilibrium. Scholarpedia. Retrieved November 20,2021, from http://www.scholarpedia.org/article/Equilibrium.

[41] Pcbs. (n.d.). Estimated population in the Palestine mid-year by governorate, 1997-2026. PCBS. Retrieved November 20,202, from https://www.pcbs.gov.ps/statisticsIndicatorsTables.aspx?lang=en& table_id=676.

[42] Pcbs. (n.d.). On theeve of internationalyouthday, 12/8/2019 PCBS: PCBS. Retrieved November 20,2021, from

https://www.pcbs.gov.ps/post.aspx?lang=en&ItemID=3528

[43] Pcbs. (n.d.). Quantity of available electricity in Palestine by year and source of Electrical Energy, 2010-2019. PCBS. Retrieved

November 20,2021,from https://www.pcbs.gov.ps/statisticsIndicatorsTables.aspx?

lang=en&table_id=528.

[44] Palestinian Central Bureau Statistics, 2019. Labour Force Survey(July- September, 2019) Round (Q3/2019). Press Report on the Labour Force Survey Results. Ramallah-Palestine.

[45] Palestinian Central Bureau Statistics, 2000. Labour Force Survey: Annual Report: 1999. Ramallah-Palestine.

[46] Energy Tables 2017. (n.d.). Retrieved November 18,2021,from https://www.pcbs.gov.ps/Portals/_Rainbow/Documents/Energy%20Tables%202017.pdf.

[47] Palestinian Central Bureau Statistics, 2005. Energy Balance in Palestine 2001,2002,2003. Ramallah-Palestine.

[48] Pcbs. (2018). Energy Tables 2017. PCBS. Retrieved November 6,2021, from https://www.pcbs.gov.ps/Portals/_Rainbow/Documents/Energy%20Tables%202017.pdf

[49] Palestinian Central Bureau Statistics, 2000. Energy Consumption in the Palestine Territory: Annual Report : 1997. Ramallah - Palestine.

[50] Palestinian Central Bureau Statistics, 2008. Energy Consumption in the Palestine Territory: Annual Report :2006 . Ramallah - Palestine.

[51] Pcbs. (n.d.). Proportion of population (%) with access to electricity by year, region and locality type, 2007-2019. PCBS.Retrieved November 18,2021,from https://www.pcbs.gov.ps/statisticsIndicatorsTables.aspx?lang=en&table_id=504.

[52] Palestinian Central Bureau Statistics, 2004. Table about Energy Consumption in the Palestinian Territory 2002. Ramallah-Palestine.

[53] Pcbs.(n.d.).Selected energy performance indicators in Palestine,2014-2019.PCBS.RetrievedNovember18,2021,from https://www.pcbs.gov.ps/statisticsIndicatorsTables.aspx?lang=en&table_id=529

[54] Palestinian Central Bureau Statistics, 2020. Sustainable Housing Development in Palestine 2007, 2017. Ramallah-Palestine.

[55] Arkadie, B. V. (1977). The Impact of the Israeli Occupation on the Economies of the West Bank and Gaza, Journal of Palestine Studies, 6:2, 103-129, DOI: 10.2307/2535505

[56] Misyaf, J. M., & Ashour, T. (2019). The economic costs of the Israeli occupation for the Palestinian people: Cumulative fiscal costs. Geneva: United Nations.52.

[57] United Nations. (2019). In The economic costs of the Israeli occupation for the Palestinian people: The unrealized oil and natural gas potential. Geneva.

[58] Pcbs. (n.d.). Dr. Awad, reviews the conditions of the Palestinian people via statistical figures and findings,on the72nd annual commemoration of the Palestinian. PCBS. RetrievedNovember29,2021,from https://www.pcbs.gov.ps/site/512/default.aspx?lang=en&ItemID=3734.

[59] Abu-Lughod, J. L. (1983). Demographic consequences of the occupation. MERIP Reports, (115), 13-17.

[60] Forced displacement-relief web. (n.d.). Retrieved November 29,2021, from https://reliefweb.int/sites/reliefweb.int/files/resources/Forced%20displacement.

[61] NGO Development Center. (2009). Palestinian Non Governmental Organizations (NGOs) and the Private Sector: Potentials for Cooperation and Partnerships.

[62] Abu Hamed, T., Flamm, H., & Azraq, M. (2012). Renewable energy in the Palestinian Territories: Opportunities and challenges. Renewable and Sustainable Energy Reviews, 16(1), 1082–1088. https://doi.org/10.1016/j.rser.2011.10.011

[63] Fertility rate, total (births per woman)-West Bank and Gaza. Data. (n.d.). Retrieved December 2,2021, from https://data. worldbank . org / indicator/SP.DYN.TFRT.IN?locations=PS.

[64] Pcbs. (n.d.). H.E. dr. Awad, highlights the forty four annual commemoration of Land Day in statistical figures. PCBS.Retrieved December 4, 2021, from https://www.pcbs.gov.ps/post.aspx?lang=en&.

[65] Palestinian Central Bureau Statistics, 2006. Table about Energy Consumption in the PalestinianTerritor 2004. Ramallah - Palestine.

[66] Quantity of electricity imported and purchased in Palestine. (n.d.). Retrieved December 4, 2021, from https://pcbs.gov. ps / Portals/_ Rainbow /Documents/Energy-IMP-elec-E2018.html.

[67] Israel population(live).Worldometer. (n.d.). Retrieved December 4,2021,from https://www.worldometers.info/world- population/ israel-population/.

[68] BADIL Resource Center for Palestinian Residency & Refugee Rights. 2013-2015. Survey of Palestinian Refugees and Internally Displaced Persons.Vol VIII 2013-2015.184 Pages.

[69] Bilal, M. (2017). Partnership between Local Government Units and Private Sector and its Role in the Developmentof Local Economy in Palestine. Faculty of graduate Studies, An-Najah National University, Nablus, Palestine.

[70] Pcbs.(n.d.). Energy balance of Palestinein Terajoul, 2019. PCBS. Retrieved December 5,2021,from https://www.pcbs.gov.ps/statisticsIndicatorsTables.aspx?.

appendix

In this part,we will present some of codes used to find the simulations and the numerical and graphical solutions for proposed systems in this thesis as the following:

1- Fixed points code for four dimension system:

```
% three equations used for three dimension system
clc
clear all
syms n1 n2 n3 n4
eq1=%equation1==0;
eq2=%equation2==0;
eq3=%equation3==0;
eq4=%equation4==0;
eqs=[eq1 eq2 eq3 eq4];
sol=solve(eqs,[n1 n2 n3 n4]);
n1sol=sol.n1;
n1sol=simplify(n1sol)
n2sol=sol.n2;
n2sol=simplify(n2sol)
n3sol=sol.n3;
n3sol=simplify(n3sol)
n4sol=sol.n4;
n4sol=simplify(n4sol)
```

2-Numerical solution code for four dimension system:

```
%this code used for relation between three variables
clear all
clc
f=@(t,n)[eq1;eq2;eq3;eq4];
```

```
[t,na]=ode45(f,[t0,t],[n10,n20,n30,n40]);
plot3(na(:,1),'r',na(:,2),'y',na(:,3),'black','linewidth',4),grid on
xlabel('n1')
ylabel('n2')
zlabel('n3')
title('Graph of relation between n1,n2 and n3 ')
na=[t,na]
```

To plot the relation between two variables we used %plot

3-Behavior of the system during time code:

```
clc
clear all
f=@(t,n)[eq1;eq2;eq3;eq4];
[t,na]=ode45(f,[t0 t],[n10,n20,n30,n40])
figure
plot(t,ya(:,1),'r','linewidth',[2])
hold on
plot(t,ya(:,2),'--b','linewidth',[2])
hold on
plot(t,ya(:,3),'black--o','linewidth',[2])
hold on
plot(t,ya(:,4),'--b','linewidth',[2])
xlabel('time')
ylabel('population,jobs,energy,occupation')
legend('n1:population','n2:jobs','n3:energy','n4:occupation')
title('Behaviour of n1,n2,n3,n4 during time')
```

4- Direction field for the system in three dimension:

```
clc
clear all
```

```
f=@(n)[eq1;eq2;eq3];
n1=linspace(-2,2,7);
n2=linspace(-2,2,7);
n3=linspace(-2,2,7);
[x,y,z]=meshgrid(n1,n2,n3);
size(x);
size(y);
size(z);
u=zeros(size(x));
v=zeros(size(y));
w=zeros(size(z));
for i=1:numel(x)
yprime=f([x(i);y(i);z(i)]);
u(i)=yprime(1);
v(i)=yprime(2);
w(i)=yprime(3);
end
quiver3(x,y,z,u,v,w,'g')
xlabel('n1')
ylabel('n2')
zlabel('n3')
```

5-Movement of the system around fixed points code (contour code):

```
clc
clear all
f=@(n)[eq1;eq2;eq3];
[n1,n2,n3]=meshgrid(-2:0.5:2);
size(n1);
size(n2);
```

```
size(n3);
u=zeros(size(n1));
v=zeros(size(n2));
w=zeros(size(n3));
for i=1:numel(n1)
yprime=f([n1(i);n2(i);n3(i)]);
u(i)=yprime(1);
v(i)= yprime(2);
w(i)=yprime(3);
end
x=[u(1:100);v(1:100);w(1:100)]
contour3(x,200)
xlabel('n1')
ylabel('n2')
zlabel('n3')
```

6-Bifurcation code for the system:

```
clear all;
clc;
%%Model parameters
%%Time parameters
dt = .001;
N = 100;
%%Set-up figure and axes
figure;
% ax(1) = subplot(3,1,1);
hold on
xlabel ('m');
ylabel ('n1');
```

```
% ax(2) = subplot(3,1,2);
% hold on
% xlabel ('m');
% ylabel ('n2');
% ax(3)=subplot(3,1,3);
% xlabel ('m');
%ylabel ('n3');
%%Main loop
for m=-a:.1:a                    % where a is an interval for parameter
n1 = zeros(N,1);
n2 = zeros(N,1);
n3=zeros(N,1);
n4=zeros(N,1);
t = zeros(N,1);
n1(1) = n10;
n2(1) = n20;
n3(1)=n30;
n4(1)=n40;
t(1) = 0;
for i=1:N
t(i+1) = t(i)+dt;
n1(i+1) = n1(i) + dt*(eq1)
n2(i+1) = n2(i) + dt*(eq2);
n3(i+1)=n3(i)+ dt*(eq3);
n4(i+1)=n4(i)+dt*(eq4);
end
plot(m,n1,'color','blue','marker','.')
% plot(m,n2,'color','blue','marker','.')
% plot(m,n3,'color','blue','marker','.')
```

end

7-Direction field for four dimensional system code:

```
clc
clear all
global m        % where m is one of parameters in equation
D1=[];
D2=[];
D3=[];
D4=[];
f=@(t,n,m)[eq1;eq2;eq3;eq4];
for m=-a:.01:a
[t,na]=ode45(@(t,n)f(t,n,m),[t0,t],[n10 n20 n30 n40]);
n1=na(:,1);
n2=na(:,2);
n3=na(:,3);
n4=na(:,4);
for i=round(length(n1)*3/4):length(n1)-1 % i=round(length(n)/2):length(n)-1
if((n1(i)>=n1(i-1))&&(n1(i)>=n1(i+1)))||((n1(i)<=n1(i-1))&&(n1(i)<=n1(i+1)))
D1=[D1;m n1(i)];
end
if((n2(i)>=n2(i-1))&&(n2(i)>=n2(i+1)))||((n2(i)<=n2(i-1))&&(n2(i)<=n2(i+1)))
D2=[D2;m n2(i)];
end
if((n3(i)>=n3(i-1))&&(n3(i)>=n3(i+1)))||((n3(i)<=n3(i-1))&&(n3(i)<=n3(i+1)))
D3=[D3;m n3(i)];
end
if((n4(i)>=n4(i-1))&&(n4(i)>=n4(i+1)))||((n4(i)<=n4(i-1))&&(n4(i)<=n4(i+1)))
D4=[D4;m n4(i)];
end
```

```
end
end
plot(D1(:,1),D1(:,2),'*','MarkerEdgeColor','b','MarkerSize',3)
set(gcf,'color','w')
title('Bifur (n1,m) detail')
xlabel('m','FontName', 'Times New Roman','FontSize',14,'FontWeight','bold','Color','k')
ylabel('n1','FontName', 'Times New Roman','FontSize',14,'FontWeight','bold','Color','k')
```

8- Newton Method code:

```
function p = sysNewton(f,J,x0,tol)
xold=x0;
xnew=x0-J(x0)\f(x0);
while norm(xnew-xold)>tol
xold=xnew;
xnew=xold-J(xold)\f(xold);
end
p=xnew;
end
clc
clear all
x=[];
m=[];
a=[];
for a=-c:.1:c          %where a is one of parameters in equations belong to an interval [-c,c]
for m=-d:.1:d          %where m is another parameter in equation belong to an interval [-d,d]
sysNewton(@(x) [eq1;eq2;eq3;eq4],@(x) [dfi/dxj],[x10;x20;x30;x40],eror)
m=[m,a,x]
end
```

الملخص

تهدف هذه الدراسة إلى وصف التنمية الاقتصادية الاجتماعية في فلسطين. حيث تُستخدم النمذجة الرياضية هنا كأداة رئيسية لعرض العوامل الرئيسية التي تؤثر على تطور الاقتصاد الفلسطيني. قمنا بتنفيذ هذه الدراسة على مرحلتين أساسيتين، تُمثَّلان بواسطة نموذجين رياضيين من ثلاثة وأربعة أبعاد. حيث أن النموذج الثاني هو عبارة عن توسيع للنموذج الأول. تُبنى النماذج الرياضية المقترحة من مجموعة من المعادلات التفاضلية المستمرة غير الخطية التي تقدم متغيرات الدراسة الرئيسية (السكان، الوظائف، الطاقة، الاحتلال) والمعاملات التي تربط بينهم. في كل مرحلة، تُحلَّل النماذج المقترحة رياضياً من خلال إيجاد نقاط الثبات العامة والعددية، ويحدد ثباتها من خلال تحليل مصفوفة الجاكوبيان وقيمها الذاتية، كما اننا في هذه الدراسة نوضح نقاط التشعب ونوعها، والحلول العددية وسلوك المتغيرات الرئيسية خلال الزمن لكل نظام من الانظمة المقترحة. بالإضافة إلى ذلك، يُستخدم برنامج الماتلاب لإيجاد الحلول والنتائج بيانيًا ورقميًا. من ناحية أخرى، تُستخدم طريقة نيوتن العددية للحصول على قيم مختلفة للمتغيرات الأساسية في كل مرحلة حيث أنه من خلالها يمكن الحصول على بعض القيم التي تعطي انطباعًا عن حدوث تطور او تنمية في المجال الاقتصادي الاجتماعي للمنطقة. علاوة على ذلك، يتم حساب قيمة المعاملات في المعادلات بناءً على قيم واحصائيات تم الحصول عليها من الجهاز المركزي للإحصاء الفلسطيني باستخدام طريقة معدل النسبة المئوية (معدل النمو للقيم). في هذا المجال تم استخدام سنوات اساس وسنوات مقارنة مختلفة لكل معامل والهدف من هذا محاولة شمل الاحتلال في فترات مختلفة من الزمن. في نهاية الدراسة، تؤكد نتائج المحاكاة والتحليل العددي أن النظام فوضوي وان نقاط التوازن لهذا النظام غير مستقرة.

الجمل المفتاحية: التنمية الاقتصادية الاجتماعية، نموذج رياضي، الثبات والاستقرار، التشعب، الحلول العددية، طريقة نيوتن، برنامج الماتلاب.